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Section A :
Physique théorique.

Gravitational radiation from a bounded source I

par

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ABSTRACT. — An obstruction is found to considering a solution of the linearized field equations in harmonic coordinates as a first approximation to a radiative solution of the exact field equations with bounded sources.

RÉSUMÉ. — On démontre l'impossibilité de considérer une solution des équations du champ linéarisées en coordonnées harmoniques comme première approximation à une solution radiative des équations exactes avec sources bornées.

INTRODUCTION

In this and a following paper we shall consider asymptotic expansions of retarded radiative solutions of the Einstein field equations with bounded sources.

Two general methods have been used to attempt to find these types of solutions. The oldest is the successive approximation proceedure with the harmonic coordinate condition imposed. The linear approximation equations are wave equations in Minkowski space and may be readily integrated. The second method which has been used is the formal power series approach initiated by Bondi [*1*]. The metric is assumed to have an asymptotic expansion along a family of forward null hypersurfaces $u =$ const. in inverse powers of a radial parameter r and the field equations are formally integrated.

The partial solutions obtained by these two methods have a property in common which will be of interest to us here.

A retarded solution of the linearized field equations in harmonic coordinates possesses an asymptotic expansion in powers of $1/r$ along any regular family of forward Minkowski null cones. This expansion is the multipole expansion.

The Bondi-type asymptotic solutions have a similar property. Consider in fact the coordinate system (x^i) obtained from a Bondi-type coordinate system (u, r, θ, ϕ) by the following coordinate transformation:

$$x^0 = u + r, \quad x^1 = r \sin\theta \cos\phi, \quad x^2 = r \sin\theta \sin\phi, \quad x^3 = r \cos\theta. \quad (1)$$

If we take a solution of the type obtained by Bondi and Sachs [1], [2] and express it in the coordinate system (x^i) we find that the components of the metric tensor approach the standard Minkowski metric asymptotically as r tends to infinity. We find also that the hypersurfaces $u =$ const. along which the components of the metric tensor are given as an asymptotic expansion in powers of $1/r$ form a family of foward Minkowski null cones of the coordinate system (x^i).

The same remarks apply to the asymptotic solutions found by Newman and Penrose [3] in a slightly different type of coordinate system. The coordinate systems obtained by (1) from Newman-Penrose-type coordinate systems have been used by Papapetrou [4] to study the asymptotic properties of exact solutions of the field equations.

We see then that all of these coordinate systems have the property that the components of the metric tensor possess an asymptotic expansion along a family of hypersurfaces which are either the Minkowski null cones (Papapetrou type) or which are the Minkowski null cones of a simply related coordinate system through the coordinate transformation (1) (Newman-Penrose type and Bondi-Sachs type).

What we shall do in Part I is consider all coordinate systems in which the components of a given radiative solution of the field equations with bounded sources possess an asymptotic expansion in inverse powers of a radial parameter r along a family of foward Minkowski null hypersurfaces. We shall find a necessary and sufficient condition which the components of the metric tensor must satisfy in order that such a expansion exist.

When we apply these considerations to the retarded solution of the linearized field equations in harmonic coordinates we find that the condition is not satisfied. We have then a condition which must be satisfied by a solution to the exact equations but which is not by the linear approximation solution. Using this fact we shall show that it is impossible to consider a retarded solution of the linearized field equations in harmonic coordinates as a first approximation to a retarded radiative solution of the exact field equations with bounded sources.

There are four sections. In Section I a decomposition is given of an arbitrary symmetric matrix with respect to two Minkowski null vectors.

In sections II and III we use the field equations

$$R_{ij} = 0,$$

which are valid outside of the world-tube of the sources. In Section II we do not assume an expansion in powers of $1/r$ along any set of hypersurfaces. We only assume that the field becomes Minkowskian as $1/r$ in the limit as r tends to infinity.

The main section is III. We here assume an asymptotic expansion in powers of $1/r$ along a family of forward Minkowski null cones and consider the conclusions which can be drawn from setting $R_{ij}^{(2)}$, the coefficient of $1/r^2$ in the asymptotic expansion of the Ricci tensor, equal to zero.

In the last section we consider the method of successive approximations using the harmonic coordinate condition.

I

Let V_4 be a Space-Time whose sources are restricted to some finite region of space. We shall assume that V_4 admits global coordinate systems

$$V_4 \underset{\sim}{\overset{\phi}{\longrightarrow}} \mathbb{R}^4$$

and we shall restrict our attention to the subset Φ'' of these ϕ such that the components of the metric tensor are of the form

$$g_{ij} = \eta_{ij} + \frac{h_{ij}(u, r, \theta, \phi)}{r} \tag{2}$$

where (r, θ, ϕ) is the polar coordinate system for $\mathbb{R}^3$ and $u = x^0 - r$ (We shall later define the sets of coordinate systems Φ' and Φ). Latin indices take the values (0, 1, 2, 3) and η_{ij} is the standard Minkowski metric of signature -2. In the following we shall use η_{ij} to raise and lower all indices. The coordinate functions (x^i) are related to the functions (u, r, θ, ϕ) by the transformation (1) of the Introduction. We use the latter only as expansion parameters; (u, r, θ, ϕ) considered as a coordinate system does not belong to the set Φ''. The hypersurfaces $u =$ const. are the family of forward Minkowski null cones whose vertices lie on the world-line of the origin.

The functions h_{ij} as well as their first and second derivatives are assumed to be bounded functions of r in the limit as r tends to infinity. Since the sources are bounded we may impose also that the time axes be such that the Ricci tensor vanishes outside of a timelike cylinder $r =$ const. Our group G$''$ of admissible coordinate transformations will be the set of maps $\mathbb{R}^4 \to \mathbb{R}^4$ such that $G'' \circ \Phi'' \subset \Phi''$. Notice that G$''$ contains only those Lorentz transformations which do not change the direction of the time axes.

We wish to consider retarded radiative solutions of the fields equations (We shall define later what we mean by a retarded solution). We shall investigate the conditions which the coefficient of $1/r$ on the right-hand side of equation (2) must satisfy in order that the left-hand side be a radiative solution of the field equations.

The principal tool of our investigations is a decomposition of an arbitrary symmetric matrix with respect to the pair of associated Minkowski null vectors ξ^i and ζ^i, given by

$$\xi^i = (1, x^\alpha/r),$$
$$\zeta^i = (-1, x^\alpha/r).$$

Greek indices take the values (1, 2, 3).

Define a symmetric projection operator π_{ij} by

$$\pi_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & \eta_{\alpha\beta} + \xi_\alpha \xi_\beta \end{bmatrix}.$$

π_{ij} satisfies the following relations:

$$\eta^{ij}\pi_{ij} = 2 \quad , \quad \pi_{ij}\xi^j = 0 \quad , \quad \pi_{ij}\zeta^i = 0 \quad , \quad \pi_{ij}\pi_{jk} = \pi_{ik}. \tag{3}$$

Let h_{ij} be an arbitrary symmetric matrix. Set

$$h = \eta^{ij}h_{ij} \quad , \quad h(\xi, \zeta) = h_{ij}\xi^i\zeta^j.$$

Define the quantities $\hat{h}$, K, H, $\hat{\mathrm{K}}$, $\hat{h}_i$, H_i by the equations

$$\hat{h} = \frac{1}{4}h(\zeta, \zeta) \quad , \quad \mathrm{K} = \frac{1}{4}h(\xi, \zeta) \quad , \quad \mathrm{H} = \frac{1}{4}h(\xi, \xi) \quad , \quad \hat{\mathrm{K}} = h + 4\mathrm{K}, \tag{4a}$$

$$\hat{h}_i = \frac{-1}{2} h_{ij}\zeta^j - \hat{h}\xi_i - \mathrm{K}\zeta_i, \tag{4b}$$

$$\mathrm{H}_i = \frac{-1}{2} h_{ij}\xi^j - \mathrm{K}\xi_i - \mathrm{H}\zeta_i. \tag{4c}$$

The quantities $\hat{h}_i$ and H_i satisfy the relations

$$\hat{h}_i\xi^i = \hat{h}_i\zeta^i = \mathrm{H}_i\xi^i = \mathrm{H}_i\zeta^i = 0. \tag{5a}$$

We have also

$$\hat{h}_0 = \mathrm{H}_0 = 0. \tag{5b}$$

Therefore we find

$$\pi_{ij}\hat{h}^j = \hat{h}_i \quad , \quad \pi_{ij}\mathrm{H}^j = \mathrm{H}_i. \tag{5c}$$

Define $\hat{h}_{ij}$ by the equation

$$h_{ij} = h_{ij} + 2\hat{h}_{(i}\xi_{j)} + 2\mathrm{H}_{(i}\zeta_{j)} + \frac{\hat{\mathrm{K}}}{2}\pi_{ij} + \hat{h}\xi_i\xi_j + 2\mathrm{K}\xi_{(i}\zeta_{j)} + \mathrm{H}\zeta_i\zeta_j. \tag{6}$$

Multiplying both sides of this equation respectively by ξ^i, ζ^i, η^{ij}, we find that $\hat{h}_{ij}$ satisfies the relations

$$\hat{h}_{ij}\xi^j = h_{ij}\zeta^j = \hat{h}^i_i = 0. \tag{7a}$$

We find also that

$$\hat{h}_{0i} = 0 \quad , \quad \pi_{ij}\hat{h}^j_k = \hat{h}_{ik}. \tag{7b}$$

Equation (6) is the decomposition of h_{ij} with respect to ξ^i and ζ^i. We have four functions $\hat{h}$, K, H, $\hat{\mathrm{K}}$, two Cartesian vectors $\hat{h}_i$, H_i each with two independent components, and a symmetric Cartesian tensor $\hat{h}_{ij}$ with two independent components, giving a total of ten components for h_{ij}.

The decomposition (6) is similar to the Fourier transform of the spin decomposition given by Arnowitt, Deser and Misner [5]. We are here working in the position variable space not in the momentum space. However, we are primarily interested in the leading term of a spherical wave far from the sources and in this wave zone a spin decomposition is equivalent to the algebraic decomposition we have given. For example if f^i is a vector field of the form

$$f^i = \frac{f^i_1(u, \theta, \phi)}{r} + O(r^{-2}),$$

then we have

$$f^i_{,i} = \frac{\partial f^i_1}{\partial u}\frac{\xi_i}{r} + O(r^{-2}).$$

We see that to within terms of order $0(r^{-2})$ the differential condition

$$f^i_{,i} = 0$$

is equivalent to the algebraic condition

$$\frac{\partial f^i_1}{\partial u}\xi_i = 0.$$

A similar decomposition may be given with respect to an arbitrary pair of associated vectors

$$p^i = (p^0, p^\alpha) \quad , \quad q^i = (-\,p^0, p^\alpha),$$

which are time-like or null. Formula (6) remains valid with (ξ^i, ζ^i) replaced by (p^i, q^i) but the expressions on the right-hand side of the defining relations (4) are more complicated.

II

In an admissible coordinate system $\phi \in \Phi''$ the components of the metric tensor are of the form (I-2).

Set

$$\dot{h}_{ij} = \frac{\partial h_{ij}}{\partial u} \quad , \quad h'_{ij} = \frac{\partial h_{ij}}{\partial r}.$$

Then, since

$$\frac{\partial r}{\partial x^\alpha} = -\,\xi_\alpha \quad , \quad \frac{\partial u}{\partial x^i} = \xi_i,$$

we have

$$g_{ij,kl} = \frac{\ddot{h}_{ij}}{r}\xi_k\xi_l - \frac{2\dot{h}'}{r}\xi_{(k}\delta^\alpha_{l)}\xi_\alpha + \frac{h''_{ij}}{r}\delta^\alpha_k\delta^\beta_l\xi_\alpha\xi_\beta + 0(r^{-2}), \tag{1a}$$

where we define the symmetric product of two vectors a_i and b_i as

$$a_{(i}b_{j)} = \frac{1}{2}(a_ib_j + a_jb_i).$$

Using the fact that

$$\xi_i + \zeta_i = 2\delta^\alpha_i\xi_\alpha,$$

$(1a)$ may be written as

$$g_{ij,kl} = \frac{1}{4r}[(4\ddot{h}_{ij} - 4\dot{h}'_{ij} + h''_{ij})\xi_k\xi_l - 2(2\dot{h}'_{ij} - h''_{ij})\xi_{(k}\zeta_{l)} + h''_{ij}\zeta_k\zeta_l] + 0(r^{-2}). \tag{1b}$$

Equation (I-2) defines a symmetric matrix of functions h_{ij} which we may decompose according to equation (I-6). Since the Ricci tensor vanishes for large values of r, the elements of this decomposition will not be arbitrary but will be forced to satisfy certain equations. To obtain those equations which will interest us here in Section II it is sufficient to consider the leading term of the Ricci tensor. That is, it is sufficient to impose

$$\mathrm{R}_{ij} = 0(r^{-2}). \tag{2}$$

From equation ($1b$) we find the following expression for the Riemann tensor:

$$\begin{aligned}
\mathrm{R}_{ijkl} = {} & \frac{1}{2r}(4\xi_{[i}\ddot{h}_{j][k}\xi_{l]} - 4\xi_{[i}\dot{h}'_{j][k}\xi_{l]} + \xi_{[i}h''_{j][k}\xi_{l]}) \\
& - \frac{1}{2r}(2\xi_{[i}\dot{h}'_{j][k}\zeta_{l]} + 2\zeta_{[i}\dot{h}'_{j][k}\xi_{l]} - \xi_{[i}h''_{j][k}\zeta_{l]} - \zeta_{[i}h''_{j][k}\xi_{l]}) \\
& + \frac{1}{2r}\zeta_{[i}h''_{j][k}\zeta_{l]} + 0(r^{-2}).
\end{aligned} \tag{3}$$

Therefore the Ricci tensor is given by

$$\begin{aligned}
2r\mathrm{R}_{jk} = {} & \ddot{h}\xi_j\xi_k - 2\ddot{h}_{(j}\xi_{k)} \\
& - 2\dot{h}'\xi_\alpha\delta^\alpha_{(j}\xi_{k)} + 2\dot{h}'_{(j}\xi^\alpha_{k)}\xi_\alpha + 2\xi^\alpha\dot{h}'_{\alpha(j}\xi_{k)} + 2\dot{h}'_{jk} \\
& + h''\delta^\alpha_j\delta^\beta_k\xi_\alpha\xi_\beta - 2\xi^\beta h''_{\beta(j}\delta^\alpha_{k)}\xi_\alpha - h''_{jk} + 0(r^{-1}),
\end{aligned} \tag{4}$$

where for simplicity we have set $h_{ij}\xi^j = h_i$. If we decompose h_{ij} according to equation (I-6), equation (4) becomes

$$\begin{aligned}
2r\mathrm{R}_{jk} = {} & 2\ddot{\hat{h}}_{jk} - \hat{h}''_{jk} \\
& + 4\ddot{\mathrm{H}}_{(j}\xi_{k)} - 4\dot{\mathrm{H}}'_{(j}\xi_{k)} + \mathrm{H}''_{(j}\xi_{k)} + 2\dot{\hat{h}}'_{(j}\xi_{k)} - \hat{h}''_{(l}\xi_{k)} \\
& + 2\dot{\mathrm{H}}'_{(j}\zeta_{k)} - \mathrm{H}''_{(j}\zeta_{k)} + \hat{h}''_{(j}\zeta_{k)} \\
& + (\dot{\hat{\mathrm{K}}}' - \hat{\mathrm{K}}''/2)\pi_{ij} \\
& + (\ddot{\hat{\mathrm{K}}} - \dot{\hat{\mathrm{K}}}' + \hat{\mathrm{K}}''/4)\xi_j\xi_k \\
& + \left(4\ddot{\mathrm{H}} - \dot{h}' - 4\dot{\mathrm{H}}' + \hat{h}'' + \mathrm{H}'' + \frac{h''}{2}\right)\zeta_{(j}\xi_{k)} \\
& + \frac{\hat{\mathrm{K}}''}{4}\zeta_j\zeta_k + 0(r^{-1}).
\end{aligned} \tag{5}$$

Therefore after a short calculation we see that equation (2) is equivalent to the following set of equations :

$$2\dot{\hat{h}}'_{ij} - \hat{h}''_{ij} = 0(r^{-1}), \tag{6a}$$
$$4\ddot{\mathrm{H}}_j - 4\dot{\mathrm{H}}'_j + \mathrm{H}''_j + 2\dot{\hat{h}}'_j - \hat{h}''_j = 0(r^{-1}), \tag{6b}$$
$$2\dot{\mathrm{H}}'_j - \mathrm{H}''_j + \hat{h}''_j = 0(r^{-1}), \tag{6c}$$
$$\ddot{\hat{\mathrm{K}}} = \dot{\hat{\mathrm{K}}}' = \hat{\mathrm{K}}'' = 0(r^{-1}) \quad , \quad 4\ddot{\mathrm{H}} - \dot{h}' - 4\dot{\mathrm{H}}' + \hat{h}'' + \mathrm{H}'' + \frac{h''}{2} = 0(r^{-1}). \tag{6d}$$

Using these equations, (3) reduces to the form

$$\mathrm{R}_{ijkl} = \frac{1}{2r}(4\xi_{[i}\ddot{\hat{h}}_{j][k}\xi_{l]} - 2\xi_{[i}\dot{\hat{h}}'_{j][k}\xi_{l]} + \zeta_{[i}\hat{h}''_{j][k}\zeta_{l]}) + 0(r^{-2}). \tag{7}$$

This yields us the following result: the leading term of the components of the Riemann tensor in an admissible coordinate system depends only on the part $\hat{h}_{ij}$ of the matrix h_{ij}. This is true without imposing an outgoing (or incoming) radiation condition. We see from the form of (7) that the leading term of the Riemann tensor vanishes if the matrix $\hat{h}_{ij}$ does not depend on u or r.

Let us consider now in more detail equation (6a). This equation has two particular classes of solutions which yield simplified expressions for the leading term of the components of the Riemann tensor. The first class consists of those matrices $h_{ij}^{(r)}$ which satisfy

$$\hat{h}'^{(r)}_{ij} = \frac{f_{ij}(u, r, \theta, \phi)}{r^1} \quad , \tag{8}$$

where f_{ij} is an arbitrary matrix of functions which are bounded along with their first derivatives with respect to r as r tends to infinity. A particular subset of these solution is given by

$$\hat{h}^{(r)}_{ij} = \hat{h}^{(1)}_{ij}(u, \theta, \phi) + \frac{\hat{h}^{(2)}_{ij}(u, \theta, \phi)}{r} + 0(r^{-2}), \tag{9}$$

where $\hat{h}^{(1)}_{ij}$ and $\hat{h}^{(2)}_{ij}$ are arbitrary functions of (u, θ, ϕ). These matrices $\hat{h}^{(1)}_{ij}$ and $\hat{h}^{(2)}_{ij}$ as well as the matrix f_{ij} given above in equation (8) must of course satisfy the algebraic conditions (I-7) at each point.

If $\hat{h}^{(r)}_{ij}$ is a solution of (6a) of the form (8) then the corresponding expression for the leading term of the components of the Riemann tensor given by equation (7) becomes

$$\mathrm{R}_{ijkl} = \frac{2}{r}\xi_{[i}\ddot{\hat{h}}^{(r)}_{j][k}\xi_{l]} + 0(r^{-2}). \tag{10}$$

The leading term is of type N with ξ_i the principal null vector (ξ_i are the components in the particular coordinate system considered of a vector whose norm tends to zero as $1/r$ in the limit as r tends to infinity).

Condition (8) may be considered as a weakened form of the Sommerfeld radiation condition.

Another particular class of solutions to equation (6a) consists of those matrices $\hat{h}_{ij}^{(a)}$ which satisfy

$$2\dot{\hat{h}}_{ij}^{(a)} - \hat{h}_{ij}^{\prime(a)} = \frac{f_{ij}(u, r, \theta, \phi)}{r^1}, \tag{11}$$

where f_{ij} is an arbitrary matrix of functions of the same type as in (8). A particular subset of these solutions is given by

$$\hat{h}_{ij}^{(a)} = \hat{h}_{ij}^{(1)}(u + 2r, \theta, \phi) + \frac{\hat{h}_{ij}^{(2)}(u + 2r, \theta, \phi)}{r} + 0(r^{-2}), \tag{12}$$

where $\hat{h}_{ij}^{(1)}$ and $h_{ij}^{(2)}$ are of the same type as in (9).

If $\hat{h}_{ij}^{(a)}$ is a solution of (6a) of the form (11) then the corresponding expression for the leading term of the components of the Riemann tensor given by equation (7) becomes

$$R_{ijkl} = \frac{2}{r}\zeta_{[i}\hat{h}_{j][k}^{\prime\prime(a)}\zeta_{l]} + 0(r^{-2}). \tag{13}$$

The leading term is again of type N but with principal null vector ζ_i.

Equation (7) yields the result that the $1/r$ term of the Riemann tensor is always of type N or I (if it does not vanish). No combination of outgoing or incoming radiation can yield an intermediate type. In fact, if we define A_{ij} and B_{ij} by

$$A_{ij} = 2(2\ddot{\hat{h}}_{ij} - \dot{\hat{h}}_{ij}^{\prime}) \quad , \quad B_{ij} = \hat{h}_{ij}^{\prime\prime}, \tag{14}$$

then equation (7) may be written as follows :

$$R_{ijkl} = \frac{1}{2r}(\xi_{[i}A_{j][k}\xi_{l]} + \zeta_{[i}B_{j][k}\zeta_{l]}) + 0(r^{-2}). \tag{15}$$

If either A_{ij} or B_{ij} vanishes as $r^{-\alpha}$ the leading term is obviously of type N. Suppose now that, for example, $\lim_{r\to\infty} A_{ij}$ does not vanish and that the leading term is algebraically special. We wish to show that this implies that B_{ij} must vanish as $r^{-\alpha}$ ($\alpha > 0$).

If the leading term is algebraically special then there exists a non-sero null vector field a_i which satisfies the following equation:

$$a_{[m}R_{j]ikl}a^i a^l = 0(r^{-(1+\alpha)}). \tag{16}$$

We may decompose a_i as follows:

$$a_i = -\frac{1}{2}(a\cdot\xi)\zeta_i - \frac{1}{2}(a\cdot\zeta)\xi_i + \hat{a}_i \tag{17}$$

where $\hat{a}_i$ satisfies the conditions (I-5). Using this and the expression (15) for the Riemann tensor we find that (16) becomes

$$\begin{aligned} a_{[m}R_{j]ikl}a^i a^l = & -\frac{1}{8r}\{a_{[m}A_{j]k}(a\cdot\xi)^2 + a_{[m}B_{j]k}(a\cdot\zeta)^2 \\ & - \hat{a}^i A_{ik} a_{[m}\xi_{j]}(a\cdot\xi) - \hat{a}^i A_{i[j}a_{m]}\xi_k(a\cdot\xi) \\ & - \hat{a}^i B_{ik} a_{[m}\zeta_{j]}(a\cdot\zeta) - \hat{a}^i B_{i[j}a_{m]}\zeta_k(a\cdot\zeta) \\ & + A_{il}\hat{a}^i\hat{a}^l\xi_{[j}a_{m]}\xi_k + B_{il}\hat{a}^i\hat{a}^l\zeta_{[j}a_{m]}\zeta_k\} + 0(r^{-(1+\alpha)}). \end{aligned} \tag{18}$$

If (16) is to be satisfied then the leading term of (18) must vanish as $r^{-\alpha}$. We may decompose this term with respect to the index k as we did a_i above in (17). This yields three equations

$$\begin{aligned} a_{[m}A_{j]k}(a\cdot\xi)^2 + a_{[m}B_{j]k}(a\cdot\zeta)^2 - a_{[m}\xi_{j]}(a\cdot\xi)\hat{a}^i A_{ik} \\ - a_{[m}\zeta_{j]}(a\cdot\zeta)a^i B_{ik} = 0(r^{-\alpha}), \end{aligned} \tag{19a}$$

$$A_{il}\hat{a}^i\hat{a}^l\xi_{[j}a_{m]} - \hat{a}^i A_{i[j}a_{m]}(a\cdot\xi) = 0(r^{-\alpha}), \tag{19b}$$

$$B_{il}\hat{a}^i\hat{a}^l\zeta_{[j}a_{m]} - \hat{a}^i B_{i[j}a_{m]}(a\cdot\zeta) = 0(r^{-\alpha}). \tag{19c}$$

If we denote asymptotic parallelism by the symbol $\sim$, we have from (19) the following conditions on a_i:

$$a_j \sim A_{jk}\lambda^k(a\cdot\xi)^2 + B_{jk}\lambda^k(a\cdot\zeta)^2 - \xi_j(a\cdot\xi)\hat{a}^i A_{ik}\lambda^k - \zeta_j(a\cdot\zeta)\hat{a}^i B_{ik}\lambda^k, \tag{20a}$$

$$a_j \sim A_{il}\hat{a}^i\hat{a}^l\xi_j - \hat{a}^i A_{ij}(a\cdot\xi), \tag{20b}$$

$$a_j \sim B_{il}\hat{a}^i\hat{a}^l\zeta_j - \hat{a}^i B_{ij}(a\cdot\zeta), \tag{20c}$$

where λ^i is an arbitrary vector field. We have from (17) the equation

$$\hat{a}^2 - (a\cdot\zeta)(a\cdot\xi) = 0(r^{-1}) \tag{21}$$

The following diagram summarizes the various possibilities. The numbers beside the arrows indicate the formulae which are involved in the implication. Note that from their definition (14), A_{ij} and B_{ij} satisfy the algebraic conditions (I.7).

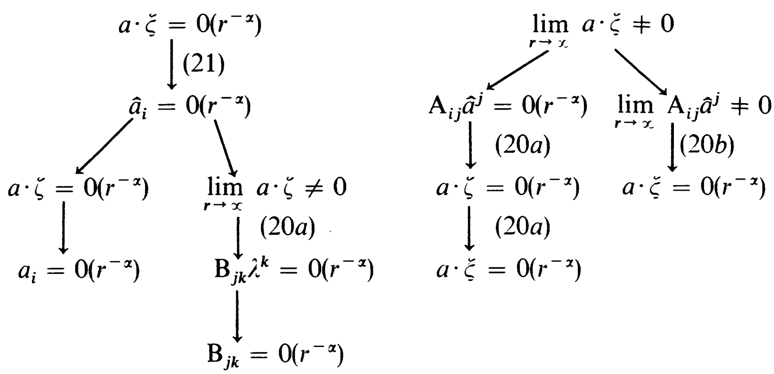

DIAGRAM 1.

The only consistant possibility is

$$B_{ij} = 0(r^{-\alpha}).$$

We see then that if the leading term is algebraically special it must be of type N.

III

We now assume that for a given bounded source configuration there is always a retarded solution of the field equations and from now on we shall restrict our considerations to these solutions. We define a retarded solution as a solution such that there exists a coordinate system $\phi \in \Phi''$ with the components g_{ij} of the metric tensor of the form

$$g_{ij} = \eta_{ij} + \frac{h^{(1)}_{ij}(u, \theta, \phi)}{r} + \frac{h^{(2)}_{ij}(u, \theta, \phi)}{r^2} + 0(r^{-3}). \tag{1a}$$

That is, such that

$$h_{ij}(u, r, \theta, \phi) = h^{(1)}_{ij}(u, \theta, \phi) + \frac{h^{(2)}_{ij}(u, \theta, \phi)}{r} + 0(r^{-2}). \tag{1b}$$

Since we shall no longer have occasion to use $h_{ij}(u, r, \theta, \phi)$, to alleviate the formulae we shall in the following drop the superscript on $h^{(1)}_{ij}(u, \theta, \phi)$. We define the coordinate set Φ' as the subset of Φ'' consisting of all ϕ such that (1) is satisfied.

Since all the information concerning radiation is contained in the leading term of the asymptotic expansion, one could reasonably hope to be able to decide whether one had a retarded solution or not simply by regarding this term. We know that if Φ' is non-void then the leading $(1/r)$ term of the Riemann tensor is of type N [2]. It would be nice to be able to prove a converse to this result, that is, to prove that if the leading term of the Riemann tensor is of type N with a forward principal null vector then Φ' is non-void.

For the rest of this section we fix a particular coordinate system $\phi \in \Phi'$. Write ($1a$) in the form

$$g_{ij} = \eta_{ij} + \frac{h_{ij}(u, \theta, \phi)}{r} + \frac{\tilde{h}_{ij}(u, r, \theta, \phi)}{r^2}. \tag{2}$$

It is important that the following inequality be satisfied:

$$| \tilde{h}_{ij}(u, r, \theta, \phi) | \leqslant \mathrm{K} \, | h_{ij}(u, \theta, \phi) | \tag{3}$$

where K is an arbitrary constant. It is this inequality that assures that the field variables tend uniformly to their Minkowski limit for all values of u as r tends to infinity. We shall not use (3) explicitly in what follows so we have not included it in the definition of Φ'. Strictly speaking however one cannot call a solution retarded unless (3) is satisfied for at least one coordinate system in Φ'. There is an interesting counter-example in electrodynamics furnished by the everywhere regular vaccum solutions considered for example by Synge [6]. These solutions are symmetric under time inversion and so cannot represent everywhere outgoing radiation. They possess however for all values of (u, θ, ϕ) an expansion of the form (1) in powers of $1/r$ (*). An inspection of the coefficients of $1/r$ and $1/r^2$ shows that for arbitrarily large but fixed r the second term becomes dominant over the first for u tending to $-\infty$ and condition (3) is not satisfied for any finite K.

We now proceed as in the previous section and decompose the matrices of functions h_{ij} and $h^{(2)}_{ij}$ with respect to ξ^i and ζ^i. We do not claim that the elements of the decomposition of $h^{(2)}_{ij}$, which we distinguish with a superscript, have any physical significance; we use them only for convenience of calculation.

Since we have an asymptotic expansion for the metric tensor we shall also have one for the Ricci tensor:

$$\mathrm{R}_{ij} = \frac{\mathrm{R}^{(1)}_{ij}}{r} + \frac{\mathrm{R}^{(2)}_{ij}}{r^2} + 0(r^{-3}). \tag{4}$$

(*) I wish to thank M. Papapetrou for bringing this to my attention.

Setting $R_{ij}^{(1)}$ equal to zero proceeds as in the previous section and we obtain the set of equations (II-6) simplified by the fact now $h'_{ij} = 0 - \dot{h}_{ij}$ does not depend on r. Equations (II-6) become

$$\ddot{H}_j = \ddot{H} = \ddot{\hat{K}} = 0, \tag{5}$$

and since we are here considering a particular solution of (II-8) the leading term of the Riemann tensor is given by equation (II-10). With equations (5) satisfied, the Ricci tensor is therefore of the form

$$R_{ij} = \frac{R_{ij}^{(2)}}{r^2} + O(r^{-3}). \tag{6}$$

We shall now impose the condition

$$R_{ij}^{(2)} = 0, \tag{7}$$

and consider what restrictions this places on the elements of the decomposition of h_{ij} and $h_{ij}^{(2)}$. A short calculation gives that the system (7) is equivalent to the following set of equations (The details are given in the Appendix):

$$\begin{gathered} \ddot{H}^{(2)} = \dot{K} \quad , \quad \dot{H} = 0 \quad , \quad \dot{\hat{K}} = 0 \quad , \quad \dot{H}_i = 0, \\ \ddot{\hat{K}}^{(2)} - 2\dot{\hat{h}} - 2\dot{\hat{h}}^{\alpha}_{|\alpha} - \ddot{\hat{h}}_{ij}\hat{h}^{ij} - \frac{\dot{\hat{h}}_{ij}}{2}\dot{\hat{h}}^{ij} - 4\dot{K} = 0, \\ \dot{\hat{h}}^{\alpha}_{j|\alpha} + 2\dot{\hat{h}}_j - 2\ddot{H}_j^{(2)} + 2\ddot{\hat{h}}_{jk}H^k + 2\dot{K}_{|\alpha}\delta^{\alpha}_j = 0, \end{gathered} \tag{8}$$

and

$$H\ddot{\hat{h}}_{ij} = 0. \tag{9}$$

The most important of these equations is equation (9). It states that for the field equations to be satisfied either $\ddot{\hat{h}}_{ij} = 0$ or $H = 0$.

From equation (II-10) we see that $\ddot{\hat{h}}_{ij} = 0$ if and only if the $1/r$ term of the Riemann tensor vanishes. We now consider under what conditions the function H vanishes. Let $u =$ const. be a fixed Minkowski coordinate null cone and let $u + \omega =$ const. be a (with respect to the Riemannian metric) null hypersurface. Because of the expansion (1) for the components of the metric tensor, the Minkowski cone will be asymptotically tangent to a null hypersurface if and only if we can choose ω to be asymptotically of the form

$$\omega = \frac{\omega_1}{r} + O(r^{-2}) \tag{10}$$

where ω_1 is a function of (u, θ, ϕ). Suppose this to be the case and let

p_i be the normal to the hypersurface $u + \omega = \text{const.}$ Then p_i is of the form

$$p_i = \xi_i + \frac{\dot{\omega}_1}{r}\xi_i + 0(r^{-2}).$$

Since p_i is a null vector we have

$$\left(\eta^{ij} - \frac{h^{ij}}{r}\right)\left(\xi_i + \frac{\dot{\omega}_1}{r}\xi_i\right)\left(\xi_j - \frac{\dot{\omega}_1}{r}\xi_j\right) = 0(r^{-2}), \tag{11}$$

and therefore $\mathrm{H} = 0$.

Conversely if $\mathrm{H} = 0$ an ω can be found of the form (10) by assuming an expansion in powers of $1/r$ and explicitly calculating the coefficients. If H does not vanish we may consider the hypersurfaces given by

$$\omega = 2\mathrm{H} \log r + 0(r^{-1}).$$

From the field equation (8*b*) we saw that H is independent of u. We shall see in Part II that H is a constant. Therefore one can show (as above for the case where $\mathrm{H}=0$ that there exists a null hypersurface asymptotically of the form $u + 2\mathrm{H} \log r + 0(r^{-1}) = \text{const.}$

We have therefore shown that a necessary and sufficient condition for the Minkowski cones $u = \text{const.}$ to be asymptotically tangent to a null hypersurface is that H vanish. If H does not vanish then the cones diverge from all null hypersurfaces as $2\mathrm{H} \log r$ in the limit as r tends to infinity.

Referring back to equation (9) we see that we have proven the following: the $1/r$ term in the asymptotic expansion of the Riemann tensor is of type N or vanishes according to whether the Minkowski cones $u = \text{const.}$ are asymptotically tangent to null hypersurfaces or not. Therefore if the

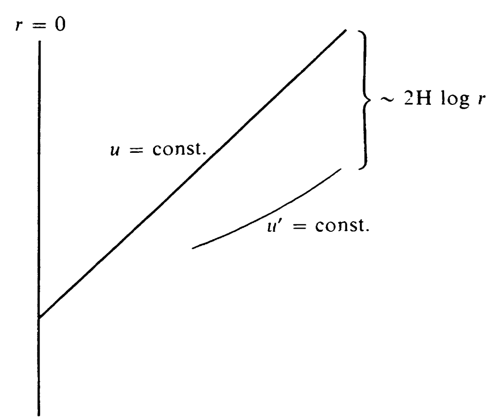

FIG. 1. — $u' = \text{const.}$ is a null hypersurface.

components of a metric tensor which is a radiative solution of the field equations possess an asymptotic expansion in powers of $1/r$ along the cones u = const., then these cones must be asymptotically tangent to null hypersurfaces. Define the subset Φ of Φ' as the set of ϕ such that H vanishes. One may formulate the consequence of (9) as follows: if the $1/r$ term of the Riemann tensor does not vanish then

$$\Phi = \Phi'.$$

We have considered so far asymptotic expansions of the components of the metric tensor only along the particular family of forward Minkowski null cones u = const. This restriction may be weakened. Suppose we have an asymptotic expansion of the components of the metric tensor in a coordinate system (x^i) along a regular family of hypersurfaces u' = const. which has the following property: we can choose functions (r', θ', ϕ') such that the coordinate system (x'^i) constructed by formula (1) of the Introduction is in Φ. For example any regular family of Minkowski null hypersurfaces of the coordinate system (x^i) is a family which has this property. Let G be the group of coordinate transformations which takes Φ into itself: $G \circ \Phi \subset \Phi$. The two coordinate systems (x^i) and (x'^i) are connected by an element of G. It is easy to see (and we shall show in Part II) that H is invariant under the group G. Since u' = const. is the family of forward Minkowski null cones of the coordinate system (x'^i) whose vertices lie on the world-line of the origin, we may conclude that

$$H = H' = 0.$$

We see therefore that the vanishing of H is a necessary and sufficient condition which the components of a metric tensor which is a radiative solution of the field equations with bounded sources, must satisfy in order that they possess an asymptotic expansion in inverse powers of r along a family of Minkowski null hypersurfaces.

IV

In conclusion we shall discuss the harmonic coordinate system in the first approximation. We shall discuss these coordinate systems in general in Part II [7]. Let g_{ij} be the components of the metric in a given harmonic coordinate system. We have then

$$g_{ij} = \eta_{ij} + k_{ij} \tag{1}$$

where $| k_{ij} | << 1$. Define

$$\psi_{ij} = k_{ij} - \eta^{pq}k_{pq}\eta_{ij}/2 \tag{2}$$

and let T_{ij} be the components of the matter tensor. If we neglect quadratic terms then the field equations

$$G_{ij} = - 8\pi\kappa T_{ij}$$

may be explicitly integrated to give the retarded solution

$$\psi_{ij}(x) = - 4\kappa \int_{C_x^-} \frac{T_{ij}(x^0 - R, y^\alpha)}{R} dy^1dy^2dy^3. \tag{3}$$

R is the Euclidean distance from the point of integration to the point where ψ_{ij} is being evaluated. C_x^- is the retarded Minkowski null cone with vertex x. The right-hand side of equation (3) may be expanded in powers of $1/r$ along the cones u = const.:

$$\psi_{ij} = \frac{- 4\kappa}{r} \int_{C_x^-} T_{ij}dx^1dx^2dx^3 + 0(r^{-2}). \tag{4}$$

The energy-momentum vector P^i is given by the integral

$$P^i = \int_{y^0=\mathrm{const}} T^{i0}dy^1dy^2dy^3.$$

This may be written as an integral over the retarded cone C_x^- with vertex x (Details are given for example in Part II [7]):

$$P^i = \int_{C_x^-} T^{ij}\xi_j dy^1dy^2dy^3 + 0(r^{-1}).$$

Recall that ξ_i is tangent to C_x^- at the point where it intersects the world-line of the origin. From equation (4) we have

$$\psi_{ij}\xi^j = - \frac{4\kappa}{r} P_i + 0(r^{-2}).$$

By our standard choice of time-axis the energy-momentum vector P^i has components

$$P^i = (P^0, 0, 0, 0).$$

Therefore we have

$$\psi_{ij}\xi^i\xi^j = - \frac{4\kappa}{r} P^0 + 0(r^{-2}),$$

and from (2)

$$g_{ij}\xi^i\xi^j = -\frac{4\kappa}{r}\mathrm{P}^0 + 0(r^{-2}).$$

We see then that H is given by

$$\mathrm{H} = -\kappa\mathrm{P}^0. \tag{5}$$

If (1) is radiative then

$$\ddot{\bar{h}}_{ij} \neq 0.$$

We have therefore a coordinate system in which the components of the metric tensor possess an asymptotic expansion in powers of $1/r$ along a family of forward Minkowski null cones and in which we have

$$\mathrm{H} \neq 0$$

and

$$\ddot{\bar{h}}_{ij} \neq 0.$$

There is however no contradiction with the results of the previous section since they were based on equation (III-9). This equation is quadratic and (3) is valid only in the linear approximation.

The results of Section III assure us however that it is impossible to use (3) as the first approximation in an iterative proceedure to obtain a second or higher approximation to an exact retarded solution. For suppose that we have a second approximation $g_{(2)ij}$ to a retarded radiative solution of the field equations. Then in some coordinate system (x'^i), the components of the metric tensor are of the form (III-1). Since

$$|\, g_{ij}(x) - g_{(2)ij}(x)\,| = 0(\kappa^2)$$

and since the components g_{ij} of the first approximation metric tensor are of the form (III-1) in the harmonic coordinate system (x^i), (x'^i) may be chosen to differ from (x^i) in the second order only:

$$|\, x'^i(x) - x^i(x)\,| = 0(\kappa^2).$$

We will then have from equation (III-9)

$$\mathrm{H}\ddot{\bar{h}}_{ij} = 0 \tag{6}$$

where H and $\ddot{\bar{h}}_{ij}$ are the first order contributions to $\mathrm{H}'_{(2)}$ and $\ddot{\bar{h}}'_{(2)ij}$ respectively. We have dropped the prime since the two coordinate systems (x^i) and (x'^i) are equal in the first approximation. But we saw above that neither H nor $\ddot{\bar{h}}_{ij}$ vanishes. Therefore (6) is not satisfied.

It is important to remark that it is not the harmonic coordinate condition which we criticize here but rather the use of (3) as a first approximation to a solution. (3) can be considered as a first approximation only in regions of space-time near the sources. This fact was recognized by Fock. He has attempted to find second order approximations to solutions of the field equations using the Schwarzschild metric as zeroth approximation instead of the Minkowski metric [8].

The author wishes to thank M. Papapetrou for a critical reading of the manuscript.

APPENDIX

We show here how the equation (III-7) implies the system of equations (III-8), (III-9). Retaining only the second order terms, we have

$$g_{ij} = \eta_{ij} + \frac{h_{ij}(u, \theta, \phi)}{r} + \frac{h^{(2)}_{ij}(u, \theta, \phi)}{r^2} + O(r^{-3}),$$

$$g^{ij} = \eta^{ij} - \frac{h^{ij}}{r} + O(r^{-2}). \quad h^{ij} = n^{ik}\eta^{jl}h_{kl}$$

The Christoffel symbols are given by

$$\Gamma^k_{ij} = \frac{1}{2r}(2\xi_{(i}\dot{h}^k_{j)} - \dot{h}_{ij}\xi^k) + \frac{1}{2r^2}(2\xi_{(i}\dot{h}^{(2)k}_{j)} - \dot{h}^{(2)}_{ij}\xi^k)$$
$$+ \frac{1}{2r^2}(2\delta^\alpha_{(i}h^k_{j)|\alpha} - h_{ij|\alpha}\eta^{\alpha k}) + \frac{1}{2r^2}(2\delta^\alpha_{(i}h^k_{j)}\xi_\alpha - h_{ij}\delta^k_\alpha\xi^\alpha)$$
$$- \frac{1}{2r^2}(2\xi_{(i}\dot{h}_{j)l}h^{lk} - \dot{h}_{ij}h^k) + O(r^{-3}).$$

We have set $h_{ij}\xi^j = h_i$ and for simplicity we have defined

$$h_{ij|\alpha} = r\left.\frac{\partial h_{ij}}{\partial x^\alpha}\right|_{u=\text{const}}$$

Differentiation gives

$$\Gamma^i_{jl,k} = \frac{1}{2r}(2\xi_{(j}\ddot{h}^i_{l)}\xi_k - \ddot{h}_{jl}\xi^i\xi_k) + \frac{1}{2r^2}(2\xi_{(j}\ddot{h}^{(2)i}_{l)}\xi_k - \ddot{h}^{(2)}_{jl}\xi^i\xi_k)$$
$$+ \frac{1}{2r^2}(2\xi_{(j}\dot{h}^i_{l)} - \dot{h}_{jl}\xi^i)\delta^\alpha_k\xi_\alpha$$
$$+ \frac{1}{2r^2}(2\pi_{k(j}\dot{h}^i_{l)} - \dot{h}_{jl}\pi^i_k + (2\xi_{(j}\dot{h}^i_{l)|\alpha} - \dot{h}_{jl|\alpha}\xi^i)\delta^\alpha_k)$$
$$+ \frac{1}{2r^2}(2\delta^\alpha_{(j}\dot{h}^i_{l)|\alpha} + 2\delta^\alpha_{(j}\dot{h}^i_{l)}\xi_\alpha - (\dot{h}_{jl|\alpha} + \dot{h}_{jl}\xi_\alpha)\eta^{\alpha i})\xi_k$$
$$- \frac{1}{2r^2}(2\xi_{(j}\ddot{h}_{l)m}h^{mi}\xi_k - \ddot{h}_{jl}h^i\xi_k)$$
$$- \frac{1}{2r^2}(2\xi_{(j}\dot{h}_{l)m}\dot{h}^{mi}\xi_k - \dot{h}_{jl}\dot{h}^i\xi_k) + O(r^{-3}).$$

We have therefore

$$\Gamma^i_{ji,k} = \frac{\ddot{h}}{2r}\xi_j\xi_k + \frac{\ddot{h}^{(2)}}{2r^2}\xi_j\xi_k + \frac{\dot{h}}{2r^2}\pi_{jk} + \frac{1}{2r^2}(2\dot{h}\xi_{(j}\delta^\alpha_{k)}\xi_\alpha + 2\xi_{(j}\delta^\alpha_{k)}\dot{h}_{|\alpha})$$
$$- \frac{1}{2r^2}(\ddot{h}_{il}h^{il} + \dot{h}_{il}\dot{h}^{il})\xi_j\xi_k + O(r^{-3}),$$

$$\Gamma^i_{jl,i} = \frac{1}{2r}2\xi_{(j}\ddot{h}_{l)} + \frac{1}{2r^2}\xi_{(j}\ddot{h}^{(2)}_{l)} + \frac{1}{2r^2}(2\xi_{(j}\dot{h}^\alpha_{l)}\xi_\alpha + 2\xi_{(j}\dot{h}^\alpha_{l)|\alpha})$$
$$+ \frac{1}{2r^2}(2\delta^\alpha_{(j}\dot{h}_{l)|\alpha} + 2\delta^\alpha_{(j}\dot{h}_{l)}\xi_\alpha) - \frac{1}{2r^2}(2\xi_{(j}\ddot{h}_{l)m}h^m - \ddot{h}_{jl}h(\xi, \xi))$$
$$- \frac{1}{2r^2}(2\xi_{(j}\dot{h}_{l)m}\dot{h}^m - \dot{h}_{jl}\dot{h}(\xi, \xi)) + O(r^{-3}),$$

$$\Gamma^m_{ji}\Gamma^i_{mk} = \frac{1}{4r^2}(2\dot{h}_j\dot{h}_k + \dot{h}^{im}\dot{h}_{im}\xi_j\xi_k) + O(r^{-3}),$$

$$\Gamma^m_{jk}\Gamma^i_{mi} = \frac{1}{2r^2}\dot{h}\xi_{(j}\dot{h}_{k)} + O(r^{-3}).$$

We have set $h^{(2)}_{ij}\eta^{ij} = h^{(2)}$ and $h^{(2)}_{ij}\xi^j = h^{(2)}_i$.

The coefficient of $1/r^2$ in the asymptotic expansion of the Ricci tensor is given by

$$\begin{aligned}2R^{(2)}_{jk} &= \ddot{h}^{(2)}\xi_j\xi_k - 2\xi_{(j}\ddot{h}^{(2)}_{k)} + \dot{h}\pi_{jk} + 2(\dot{h}\xi_{(j}\delta^\alpha_{k)} - \xi_{(j}\dot{h}^\alpha_{k)} - \delta^\alpha_{(j}\dot{h}_{k)})\xi_\alpha \\ &+ 2(\xi_{(j}\delta^\alpha_{k)}\dot{h}_{|\alpha} - \xi_{(j}\dot{h}^\alpha_{k)|\alpha} - \delta^\alpha_{(j}\dot{h}_{k)|\alpha}) - \left(\ddot{h}_{il}h^{il} + \frac{\dot{h}_{il}\dot{h}^{il}}{2}\right)\xi_j\xi_k \\ &+ 2\xi_{(j}\ddot{h}_{k)m}h^m + 2\xi_{(j}\dot{h}_{k)m}\dot{h}^m + \dot{h}_j\dot{h}_k - \dot{h}\xi_{(j}\dot{h}_{k)} - \ddot{h}_{jk}h(\xi,\xi) - \dot{h}_{jk}\dot{h}(\xi,\xi).\end{aligned}$$

Using equation (III-5) we have

$$2R^{(2)}_{jk}\xi^i\xi^k = 8\dot{H},$$

which gives us

$$\dot{H} = 0.$$

Using this result, we have

$$2R^{(2)}_{jk}\xi^k = -(4\ddot{H}^{(2)} + \dot{\hat{K}} + \xi^k\dot{h}^\alpha_{k|\alpha} - 4\dot{H}_i\dot{H}^i)\xi_j - 4\dot{H}_j,$$

which gives us

$$\dot{H}_i = 0.$$

Using these two results we have for the trace of $R^{(2)}_{jk}$

$$2R^{(2)} = -2(4\ddot{H}^{(2)} + \xi^j\dot{h}^\alpha_{j|\alpha}),$$

which yields us

$$\dot{\hat{K}} = 0.$$

Using this we have

$$2R^{(2)}_{jk}\xi^k = -4(\ddot{H}^{(2)} - \dot{K})\xi_j,$$

which gives us

$$\ddot{H}^{(2)} = \dot{K}.$$

Placing these relations in the expression for $R^{(2)}_{jk}$ gives

$$\begin{aligned}2R^{(2)}_{jk} &= \left(\ddot{K}^{(2)} - 2\dot{\hat{h}} - 2\dot{\hat{h}}^\alpha_{|\alpha} - \ddot{\hat{h}}_{il}\hat{h}^{il} - \frac{\dot{\hat{h}}_{il}}{2}\dot{\hat{h}}^{il} - 4\dot{K}\right)\xi_j\xi_k \\ &- 2(\xi_{(j}\dot{\hat{h}}^\alpha_{k)|\alpha} + 2\xi_{(j}\dot{\hat{h}}_{k)}) \\ &+ 4\xi_{(j}\ddot{H}^{(2)}_{k)} - 4\dot{K}_{|\alpha}\xi_{(j}\xi^\alpha_{k)} - 4\xi_{(j}\ddot{\hat{h}}_{k)m}H^m - 4H\ddot{\hat{h}}_{jk}.\end{aligned}$$

This yields us the rest of the system (III-8), (III-9).

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