ANNALES DE L'I. H. P., SECTION A

BOGDAN MIELNIK
JERZY PLEBAŃSKI

Combinatorial approach to Baker-Campbell-Hausdorff exponents

Annales de l'I. H. P., section A, tome 12, nº 3 (1970), p. 215-254
<http://www.numdam.org/item?id=AIHPA_1970__12_3_215_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. XII, n° 3, 1970, p. 215-254. *Physique théorique.*

Combinatorial approach to Baker-Campbell-Hausdorff exponents

par

Bogdan MIELNIK (*) and Jerzy PLEBAŃSKI (*)

ABSTRACT. — A survey of problems related to the Baker-Campbell-Hausdorff formula is given. The technique of effective « addition » of non-commuting exponents is outlined. The first part of the paper reviews the theory of the discrete BCH formula including its most recent developments. The second part presents a new effective approach to the continuous BCH-formula and yields a general algorithm for an arbitrary analytic function of the quantum theoretical evolution operator. The results are expressed in terms of new natural ordering operators containing the chronological and normal orderings as special cases.

1. INTRODUCTION

As is well known, the relation $e^x e^y = e^{x+y}$ does not hold for non-commuting quantities. If x and y are elements of an associative but non-commutative algebra, then $e^x e^y = e^{x \# y}$, where the exponent $x \# y$ is given by an infinite Baker-Campbell-Hausdorff series of multiple commutators with the rational coefficients:

$$(1.1) \quad x \# y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + \dots$$

This series was first studied by Campbell [*1*], Baker [*2*] and Hausdorff [*3*] who found an iterative method of computing $x \# y$ term by term, and it

(*) Institute of Theoretical Physics, Warsaw University. Warsaw, Hoza 69, Poland.

was further investigated by many authors (see, e. g., [4], [6], [7], [8], [9] and the literature cited there). The advent of quantum theories has stimulated an interest in the continuous analog of (1.1). Given the « evolution equation » $\frac{d}{dt}\mathrm{E}(t, t_0) = \mathrm{H}(t)\mathrm{E}(t, t_0)$ with initial condition $\mathrm{E}(t_0, t_0) = 1$, determining an operator valued function $\mathrm{E}(t, t_0)$, the solution is conventionally written as the formal exponential series under the sign of the chronological ordering operator:

$$(1.2)\quad \mathrm{E}(t, t_0) = 1 + \int_{t_0}^{t} \mathrm{H}(t_1)dt_1 + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \mathrm{H}(t_2)\mathrm{H}(t_1) + \ldots = \mathrm{T}\left(\exp \int_{t_0}^{t} d\tau \mathrm{H}(\tau)\right)$$

The same solution, however, can be represented as the genuine exponential function, $\mathrm{E}(t, t_0) = \exp \Omega(t, t_0)$, where the exponent $\Omega(t, t_0)$ can be obtained from the following prescription of Schwinger: one divides the interval $[t_0, t]$ into n sub-intervals

$$[t_0, t] = [t_0, t_1] \cup [t_1, t_2] \cup \ldots \cup [t_{n-1}, t_n = t],$$

where

$$t_k = t_0 + k\frac{t - t_0}{n} = t_0 + k\Delta t$$

and constructs an approximant to $\mathrm{E}(t, t_0)$:

$$(1.3)\qquad \mathrm{E}_n(t, t_0) = e^{\mathrm{H}(t_n)\Delta t} e^{\mathrm{H}(t_{n-1})\Delta t} \ldots e^{\mathrm{H}(t_1)\Delta t}$$

This with the help of (1.1) can be written as:

$$(1.4)\qquad \mathrm{E}_n(t, t_0) = e^{\mathrm{H}(t_n)\Delta t \,\#\, \mathrm{H}(t_{n-1})\Delta t \,\#\, \ldots + \mathrm{H}(t_1)\Delta t} \ (*)$$

In agreement with Schwinger's prescription, $\mathrm{E}(t, t_0)$ is the limit of (1.4) for $n \to \infty$. This indicates that $\Omega(t, t_0)$ is the limit of

$$\mathrm{H}(t_n)\Delta t \,\#\, \ldots \,\#\, \mathrm{H}(t_1)\Delta t.$$

The last expression has structure similar to that of an integral sum; thus, $\Omega(t, t_0)$ can be interpreted as the result of an « integration » in the sense of the non-abelian operation $\#$; symbolically

$$(1.5)\qquad \Omega(t, t_0) = \mathop{\#\!\!\!\#}_{t_0}^{t} d\tau \mathrm{H}(\tau).$$

(*) The composition $x \# y$ is associative, so that one does not need any parentheses in the exponent.

This representation elucidates why $\Omega(t, t_0)$ is called the « continuous BCH exponent ». An interesting study of (1.5) has been published by W. Magnus [6], who seeks $\Omega(t, t_0)$ in the form of an infinite series of multiple commutators:

$$(1.6) \quad \Omega(t, t_0) = \int_{t_0}^{t} dt_1 \mathrm{H}(t_1) + \frac{1}{2} \int_{t_0}^{t} dt_2 \int_{t_0}^{t_1} dt_1 [\mathrm{H}(t_2), \mathrm{H}(t_1)] + \dots$$

and provides a recursive method for evaluation of the series (1.6) term by term.

Formulæ (1.1) and (1.6) are of great interest for mathematical « technology » of quantum theories. However, up to now, they have found only limited applications. One of reasons appears to be of technical nature. Although the BCH formulæ (1.1) and (1.6) have been known for some time and could be evaluated term by term, no explicit expression for the n-th term of (1.1) and (1.6) has been available. The traditional iterative methods when applied to computing higher terms of (1.1) and (1.6) led to a mess of multiple commutators, so that even the possibility of using computers has been mentioned. This, of course, restricted the area of practical applications to the first few terms of the iterative solutions. However, since the time of Hausdorff, the « art of exponentiation » has considerably progressed. The aim of the present paper is to give a survey of the algebraic methods which make the BCH formulæ practically manageable. In part I we present the three principal sources of information about the classical « discrete » BCH formula: 1) the differential algorithm; 2) Dynkin's explicit expression, and 3) functional equations. In part II we outline recent results concerning the continuous BCH-exponent (briefly reported in paper by I. Bialynicki-Birula and the present authors [10]) including some number-theoretical aspects. We then interpret the results derived in terms of new natural ordering operations; some basic facts concerning the algebra spanned by these operations are established. In our review we concentrate on combinatorial problems, leaving the topological questions open. The validity of our limiting transitions is assured only in a very weak topology of the free algebra; the convergence of our formulæ in stronger topologies and their validity in quotient algebras is to be separately investigated. For clarity, we decided to emphasize intuitive ideas rather than strictly formal constructions. Comments of more formal nature are printed small type or shifted to footnotes.

Although our principal aim is to present the general theory, the form of our review is dictated by applications. These can be found in many areas of mathematics and mathematical physics as, for example, statistical physics (Liouville, Bloch, Boltzmann and master equations), Lie group theory (the effective construction of group elements from generators), and all problems which can be reduced to systems of linear, ordinary differential equations whith variable coefficients. Some particularly important applications can be expected in quantum field theory (in the theory of S-matrix). The most immediate consequences of the continuous BCH formula for non-relativistic quantum mechanics which have already been explored in [*10*] lead to an explicit expression for phase shifts in any order of the perturbation theory. An excellent review of various applications of the BCH-formula in physical theories is contained in Wilcox [*9*]; we hope that many problems discussed there can be advantageously approached by the use of the new techniques presented below.

I. — Classical BCH-formula

2. Differential identities

According to the Friedrichs criterion [*5*] the BCH exponent $x \# y$ in (1.1) is a Lie element, i. e., a linear combination of x, y and their multiple commutators. More detailed information concerning the structure of $x \# y$ is traditionally derived from differential identities. In order to establish them some simple combinatorial concepts are needed.

Multiple commutators. — Let X be a finite set of symbols x_i called « operator variables » and $\mathbb{C}$ be the complex field. We construct the free algebra $\mathscr{A}$ as the set of all formal power series a of the form

$$a = \alpha^0 + \alpha^k x_k + \alpha^{k_1 k_2} x_{k_1} x_{k_2} + \dots \tag{2.1}$$
$$(\alpha^{k_1 \dots k_n} \in \mathbb{C};\quad \text{summation convention applies})$$

equipped with the obvious definitions of addition and multiplication of the series. Any number $z \in \mathbb{C}$ is identified with $z + 0x_k + 0x_{k_1}x_{k_2} + \dots \in \mathscr{A}$. In particular, the number 1 is the unity of $\mathscr{A}$, and each operator variable $x \in \mathrm{X}$ is identified with $0 + 1x + 0 + \dots \in \mathscr{A}$. The natural topology in $\mathscr{A}$ is the weakest topology in which all mappings

$$\mathscr{A} \ni a \to \alpha^{k_1 \dots k_n} \in \mathbb{C}$$

are continuous. (In this topology our infinite summation and contour integration formulæ are convergent). For any pair $a, b \in \mathscr{A}$ the commutator symbol $[a, b]$ denotes $ab - ba$. A series a is called a *Lie element* if it is constructed out of the symbols x_i by addition, multiplication by numbers and commutation. We shall now introduce a generalized commutation operation $\{\ \}$ assigning to each $a \in \mathscr{A}$ a Lie element $\{a\}$ defined by:

$$
\begin{aligned}
&\{1\} = 0,\ \{x_k\} = x_k \\
&\{x_{k_1}x_{k_2} \ldots x_{k_n}\} = [x_{k_1}, \{x_{k_2} \ldots x_{k_n}\}] \\
&\{a\} = \alpha^k \{x_k\} + \alpha^{k_1k_2}\{x_{k_1}x_{k_2}\} + \ldots
\end{aligned}
\tag{2.2}
$$

Of course, $\{ax^n\} \overset{\text{df}}{=} \{a\underbrace{x \ldots x}_{n \text{ times}}\} = 0$, for $x \in \mathrm{X}$, $a \in \mathscr{A}$ and $n > 1$. In what follows it is also convenient to use a more general multiple commutation operation $\{\ ,\ \} : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ defined (for any $b \in \mathscr{A}$) by

$$
\begin{aligned}
&\{1, b\} = b,\ \{x_k, b\} = [x_k, b] \\
&\{x_{k_1} \ldots x_{k_n}, b\} = [x_{k_1}, \{x_{k_2} \ldots x_{k_n}, b\}] \\
&\{a, b\} = \alpha^0\{1, b\} + \alpha^k\{x_k, b\} + \alpha^{k_1k_2}\{x_{k_1}x_{k_2}, b\} + \ldots
\end{aligned}
\tag{2.3}
$$

The obvious relations between the operation $\{\ \}$ and $\{\ ,\ \}$ are:

$$
\begin{aligned}
\{ab\} &= \{a, \{b\}\} \\
\{ab, c\} &= \{a, \{b, c\}\}
\end{aligned}
\qquad \text{for all} \qquad a, b, c \in \mathscr{A}
\tag{2.4}
$$

Multiple commutators of the particular form

$$\{x^n, a\} \overset{\text{df}}{=} \{\underbrace{x \ldots x}_{n \text{ times}}, y\}\ (x \in \mathrm{X},\ y \in \mathscr{A})$$

are of special importance. If

$$f(x) = f_0 + f_1x + f_2x^2 + \ldots\ (f_n \in \mathbb{C})$$

is a formal power series in $x \in \mathrm{X}$, then according to our definitions:

$$\{f(x), a\} = \sum_{n=0}^{\infty} f_n\{x^n, a\}; \qquad a \in \mathscr{A} \tag{2.5}$$

New, one easily sees that if $e^{\lambda x} = \sum_{n=0}^{\infty} \frac{\lambda^n x^n}{n!} (\lambda \in \mathbb{C})$ then

$$e^{\lambda x}ae^{-\lambda x} = \{e^{\lambda x}, a\} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\{x^n, a\} \tag{2.6}$$

The proof of (2.6) follows by noticing that

$$(d/d\lambda)^n e^{\lambda x}ae^{-\lambda x} = e^{\lambda x}\{x^n, a\}e^{-\lambda x}$$

and by developing the left hand member of (2.6) into the formal Taylor series around $\lambda = 0$. On the other hand, direct series multiplication yields

$$e^{\lambda x} a e^{-\lambda x} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k} a x^k$$

This leads to the useful identity

$$\{ x^n, a \} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k} a x^k \tag{2.7}$$

Generalized commutation formulæ. — Expressions of the form $\{ f(x), a \}$ with $\mathscr{A} \in a$, $f(x) = f_0 + f_1 x + f_2 x^2 + \ldots$, $(x \in X, f_k \in \mathbb{C})$ can be conveniently represented with the use of the multi-commutation operation $\{ \, , \}$. Consider first the special case

$$f = \frac{1}{z} + \frac{1}{z^2} x + \frac{1}{z^3} x^2 + \ldots \overset{\text{df}}{=} \frac{1}{z - x}, \qquad (0 \neq z \in \mathbb{C}, \ x \in X).$$

We shall show that:

$$\frac{1}{z - x} a = \sum_{n=0}^{\infty} \{ x^n, a \} \frac{1}{(z - x)^{n+1}} \tag{2.8 a}$$

$$a \frac{1}{z - x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(z - x)^{n+1}} \{ x^n, a \} \tag{2.8 b}$$

where $1/(z - x)^{n+1}$ is the abbreviation for the corresponding formal power series in x. Indeed:

$$\begin{aligned} (2.9) \quad & (z - x) \sum_{n=0}^{\infty} \{ x^n, a \} \frac{1}{(z - x)^{n+1}} \\ &= \sum_{n=0}^{\infty} \{ x^n, a \} (z - x) \frac{1}{(z - x)^{n+1}} - \sum_{n=0}^{\infty} [x, \{ x^n, a \}] \frac{1}{(z - x)^{n+1}} \\ &= \sum_{n=0}^{\infty} \{ x^n, a \} \frac{1}{(z - x)^n} - \sum_{n=0}^{\infty} \{ x^{n+1}, a \} \frac{1}{(z - x)^{n+1}} \\ &= \{ x^0, a \} \frac{1}{(z - x)^0} = a \end{aligned}$$

proving (2.8 *a*). The proof of (2.8 *b*) is parallel to (2.9).

The formulæ (2.8 *a-b*) are convenient as starting points in the proof of more general identities. Indeed, let $f(z) = f_0 + f_1 z + f_2 z^2 + \dots$ be an analytic function (at $z = 0$) and let $f^{(n)}(z) = \dfrac{d^n}{dz^n} f(z)$. Then the corresponding formal power series $f(x)$ of the operator variable $x \in X$ can be represented as a contour integral over a contour around $z = 0$ (*)

$$f(x) = \frac{1}{2\pi i} \oint_{z=0} \frac{f(z)}{z - x} dz \tag{2.10}$$

and similarly $f^{(n)}(x)$ can be represented in the form

$$f^{(n)}(x) = \frac{n!}{2\pi i} \oint_{z=0} \frac{f(z)}{z - x} dz \tag{2.11}$$

Both integrals are convergent in the natural topology of the free algebra $\mathscr{A}$. We can now multiply both sides of (2.8 *a-b*) by $\dfrac{1}{2\pi i} f(z)$ and carry out the contour integration around $z = 0$. Beacuse of (2.10-11) this leads to

$$f(x)a = \sum_{n=0}^{\infty} \frac{1}{n!} \{ x^n, a \} f^{(n)}(x) \tag{2.12 a}$$

$$af(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} f^{(n)}(x) \{ x^n, a \} \tag{2.12 b}$$

Of course, (2.12 *a-b*) can be equivalently rewritten as

$$[f(x), a] = \sum_{n=1}^{\infty} \frac{1}{n!} \{ x^n, a \} f^{(n)}(x) \tag{2.13 a}$$

$$[a, f(x)] = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} f^{(n)}(x) \{ x^n, a \} \tag{2.13 b}$$

The formulæ (2.13 *a-b*) generalize the known relation $[p, f(q)] = -i\hbar f'(q)$ which holds for the quantum mechanical operators of momentum and

---

(*) Such a representation is possible because each generator x of the free algebra $\mathscr{A}$ belongs to the radical of $\mathscr{A}$ and henceforth has the « spectrum » concentrated at $z = 0$. However, in the case of any concrete operator realization of x, the contours in (2.10-11) must be selected so that they contain the whole spectrum of the operator x. The same remark concerns our further integral formulæ.

position. Further on, we shall use (2.12 *a-b*) to develope a technique of differentiation which plays an essential role in the theory of BCH exponents.

Polarization derivatives. — Let $x \in \mathrm{X}$ be an arbitrary free variable and let $y \in \mathscr{A}$. Following Hausdorff we introduce the polarization derivative $y \dfrac{\partial}{\partial x} : \mathscr{A} \to \mathscr{A}$, defined as follows:

1) $y \dfrac{\partial}{\partial x}$ is a derivation of the algebra $\mathscr{A}$, i. e.

$$
\begin{aligned}
(2.14) \qquad & y \frac{\partial}{\partial x}(a + b) = \left(y \frac{\partial}{\partial x} a \right) + \left(y \frac{\partial}{\partial x} b \right) \\
& y \frac{\partial}{\partial x}(ab) = \left(y \frac{\partial}{\partial x} a \right) b + a \left(y \frac{\partial}{\partial x} b \right)
\end{aligned}
$$

2) it acts on the generators $x_1, \ldots, x_k = x, \ldots$ of $\mathscr{A}$ according to

$$
(2.15) \qquad y \frac{\partial}{\partial x} x_j = \begin{cases} y & \text{if} \quad x_j = x \ (j = k) \\ 0 & \text{if} \quad x_j \neq x \ (j \neq k) \end{cases}
$$

3) $y \dfrac{\partial}{\partial x}$ is continuous in the natural topology of $\mathscr{A}$.

Of course, conditions 1) and 2) determine uniquely the action of $y \dfrac{\partial}{\partial x}$ on any polynomial in the variables $x_1, x_2, \ldots$. Condition 3) means only that each infinite series of the form (2.1) has to be differentiated term by term. Conditions 1), 2) and 3) determine $y \dfrac{\partial}{\partial x}$ as an algebraic operation. An alternative definition, which uses a limiting transition, is also possible. In fact, one easily sees that for an arbitrary series

$$
a(x_1, x_2, \ldots, x_k = x, \ldots) \in \mathscr{A}
$$

one has

$$
(2.16) \quad y \frac{\partial}{\partial x} a = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [a(\ldots, x + \varepsilon y, \ldots) + - a(\ldots, x, \ldots)]; \qquad 0 \neq \varepsilon \in \mathbb{C}
$$

Thus the concept of the polarization derivative is similar to that of the directional derivative. In what follows we shall determine the action of $y \dfrac{\partial}{\partial x}$ on the formal power series $f(x) = f_0 + f_1 x + f_2 x^2 + \ldots$ in a single variable x. For this purpose we shall first find $y\frac{\partial}{\partial x}1/(z-x)$, $z \in \mathbb{C}$. Because

$$0 = y\frac{\partial}{\partial x}1 = y\frac{\partial}{\partial x}(z-x)\frac{1}{(z-x)} = (z-x)y\frac{\partial}{\partial x}\frac{1}{z-x} - y\frac{1}{z-x},$$

hence:

$$y\frac{\partial}{\partial x}\frac{1}{z-x} = \frac{1}{z-x}y\frac{1}{z-x} \tag{2.17}$$

and from (2.8 *a-b*)

$$y\frac{\partial}{\partial x}\frac{1}{z-x} = \sum_{n=0}^{\infty}\{x^n, y\}\frac{1}{(z-x)^{n+2}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(z-x)^{n+2}}\{x^n, y\} \tag{2.18}$$

Because of (2.10-11) these relations multiplied by $\frac{1}{2\pi i}f(z)$ and integrated around the contour containing $z = 0$ lead to the identities (*):

$$y\frac{\partial}{\partial x}f(x) = \frac{1}{2\pi i}\oint_{z=0} f(z)\frac{1}{z-x}y\frac{1}{z-x}dz \tag{2.19}$$

$$\begin{aligned} y\frac{\partial}{\partial x}f(x) &= \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\{x^n, y\}f^{(n+1)}(x) \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}f^{(n+1)}(x)\{x^n, y\} \end{aligned} \tag{2.20}$$

Now, (2.20) specialized to $f(x) = e^x$ yields:

$$\left(y\frac{\partial}{\partial x}\right)e^x = \left\{\frac{e^x-1}{x}, y\right\}e^x = e^x\left\{\frac{e^{-x}-1}{-x}, y\right\} \tag{2.21}$$

The formula admits a convenient integral representation (see [9]):

$$\left(y\frac{\partial}{\partial x}\right)e^x = \int_0^1 d\tau\,\{e^{\tau x}, y\}\,e^x = \int_0^1 d\tau e^{\tau x}ye^{(1-\tau)x} \tag{2.22}$$

The parametric derivatives and differential algorithm. — The calculus of polarization derivatives leads to simple rules of differentiation of functions, which depend on operators containing a parameter. Let $F(\lambda)$ be a diffe-

(*) On the level of the free algebra $\mathscr{A}$, (2.19) and (2.20) are equivalent. In some specializations of $\mathscr{A}$ to specific algebras one of them may be more convenient than the other.

rentiable operator function of $\lambda \in \mathbb{R}$ and let $f(\mathrm{F}) = f_0 + f_1\mathrm{F} + f_2\mathrm{F}^2 + \ldots$, $f_k \in \mathbb{C}$, be a formal power series in the variable F. We shall assume that $f(\mathrm{F}(\lambda))$ is a differentiable function of λ and that $\frac{d}{d\lambda} f(\mathrm{F}(\lambda))$ can be computed by term by term differentiation of the series. In that case, the structure of $\frac{d}{d\lambda} f(\mathrm{F}(\lambda))$ can be described by notocing that $\frac{d}{d\lambda}$ acts on the series in F as the polarization derivative $\dot{\mathrm{F}}\frac{\partial}{\partial \mathrm{F}}\left(\dot{\mathrm{F}} = \frac{d\mathrm{F}}{d\lambda}\right)$:

$$(2.23) \quad \frac{d}{d\lambda} f(\mathrm{F}(\lambda)) = \left(\dot{\mathrm{F}}\frac{\partial}{\partial \mathrm{F}}\right) f(\mathrm{F}) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\{\,\mathrm{F}^n, \dot{\mathrm{F}}\,\}\, f^{(n+1)}(\mathrm{F})$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} f^{(n+1)}(\mathrm{F})\,\{\,\mathrm{F}^n, \dot{\mathrm{F}}\,\}$$

This formula is the generalization of the theorem about the differentiation of composite functions to the domain of the operator valued functions. A particular case is:

$$(2.24) \qquad \frac{d}{d\lambda} e^{\mathrm{F}} = \left\{\frac{e^{\mathrm{F}}-1}{\mathrm{F}}, \dot{\mathrm{F}}\right\} e^{\mathrm{F}} = e^{\mathrm{F}}\left\{\frac{e^{-\mathrm{F}}-1}{-\mathrm{F}}, \dot{\mathrm{F}}\right\}$$

The last identity is the traditional source of the differential algorithms for the BCH exponent $x \# y$. In order to investigate the dependence of this exponent on x set $\mathrm{F}(\lambda) = (\lambda x) \# y$, i. e., $e^{\mathrm{F}(\lambda)} = e^{\lambda x} e^{y}$. Then from (2.24)

$$(2.25) \qquad x = \left(\frac{d}{d\lambda} e^{\lambda x} e^{y}\right) e^{-y} e^{-\lambda x} = \left(\frac{d}{d\lambda} e^{\mathrm{F}}\right) e^{-\mathrm{F}} = \left\{\frac{e^{\mathrm{F}}-1}{\mathrm{F}}, \dot{\mathrm{F}}\right\}$$

Therefore by the second of the general rules (2.4):

$$(2.26) \quad \left\{\frac{\mathrm{F}}{e^{\mathrm{F}}-1}, x\right\} = \left\{\frac{\mathrm{F}}{e^{\mathrm{F}}-1}, \left\{\frac{e^{\mathrm{F}}-1}{\mathrm{F}}, \dot{\mathrm{F}}\right\}\right\} = \{\,1, \dot{\mathrm{F}}\,\} = \dot{\mathrm{F}}$$

But this means that $\frac{d}{d\lambda}$ when acting on any power series of our particular $\mathrm{F} = (\lambda x) \# y$ can be replaced by $\left\{\frac{\mathrm{F}}{e^{\mathrm{F}}-1}, x\right\}\frac{\partial}{\partial \mathrm{F}}$. It follows by induction that:

$$(2.27) \qquad \frac{d^n}{d\lambda^n} f(\mathrm{F}(\lambda)) = \left(\left\{\frac{\mathrm{F}}{e^{\mathrm{F}}-1}, x\right\}\frac{\partial}{\partial \mathrm{F}}\right)^n f(\mathrm{F})$$

Specialising (2.27) for $\lambda = 0$ we must refer all operations on the right hand side to $F = y$, so that:

$$\frac{d^n}{d\lambda^n} f(F(\lambda))\bigg|_{\lambda=0} = \left(\left\{\frac{y}{e^y - 1}, x\right\}\frac{\partial}{\partial y}\right)^n f(y) \tag{2.28}$$

No effective knowledge of the structure of $F(\lambda) = (\lambda x) \# y$ is needed to calculate this expression. Inversely, once given the derivatives (2.28) we can easily reconstruct $f(x \# y)$ by applying the Taylor development:

$$\begin{aligned} f(x \# y) = f(F(1)) &= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n}{d\lambda^n} f(F(\lambda))\bigg|_{\lambda=0} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\left\{\frac{y}{e^y - 1}, x\right\}\frac{\partial}{\partial y}\right)^n f(y) \\ &= e^{\left\{\frac{y}{e^y-1}, x\right\}\frac{\partial}{\partial y}} f(y). \end{aligned} \tag{2.29}$$

In particular

$$x \# y = e^{\left\{\frac{y}{e^y-1}, x\right\}\frac{\partial}{\partial y}} y. \tag{2.30}$$

By a parallel argument (working with $\tilde{F}(\lambda) = x \# \lambda y$) one obtains:

$$x \# y = e^{\left\{\frac{-x}{e^{-x}-1}, y\right\}\frac{\partial}{\partial x}} x \tag{2.31}$$

The formulæ (2.30-31) known as the Campbell-Hausdorff series provide the decompositions of $x \# y$ in powers of x and y respectively (*).

(*) It is to be mentioned, however, that the use of the BCH series (2.30) for concrete algebras is somewhat ineffective: one has first to execute all subsequent polarization derivatives in (2.30) and only ofterwards can one substitute some specific operators for x and y. The computation of the polarization derivatives of any order can be simplified by the identity:

$$\left(y\frac{\partial}{\partial x}\right)\{ f(x), z \} = \sum_{k=1}^{\infty} \frac{1}{k!}[\{ x^{k-1}, y \}, \{ f^{(k)}(x), z \}] \qquad (x, y \in X,\ z \in \mathscr{A}) \tag{2.32}$$

which ammounts to:

$$\left(y\frac{\partial}{\partial x}\right)\{ x^n, z \} = \sum_{k=1}^{n} \binom{n}{k}[\{ x^{k-1}, y \}, \{ x^{n-k}, z \}] \tag{2.33}$$

The last identity can be demonstrated by using the integral representation

$$\{ x^n, z \} = \frac{n!}{2\pi i}\oint_{\lambda=0} e^{\lambda x} z e^{-\lambda x}\frac{d\lambda}{\lambda^{n+1}}$$

and (2.22). However, even with the knowledge of (2.32) the formulæ (2.30-31) are still not completely effective.

By executing the operations on the right hand side of (2.30) (or (2.31)) and by decomposing the resulting terms into orders of y (or x) we can also obtain the development of $x \# y$ into the homogeneous Lie elements of various orders in x and y. However, as far as this double decomposition is concerned the most explicit solution was obtained by Dynkin [4].

3. Dynkin's explicit expression

Dynkin's method of determining the explicit form of $x \# y$ starts from direct multiplication of the series which results in:

$$e^{x \# y} = e^x e^y = 1 + Z, \qquad \text{where} \qquad Z = \sum_{\substack{n,m=0 \\ n+m>0}}^{\infty} \frac{x^n}{n!}\frac{y^m}{m!} \tag{3.1}$$

so that

$$x \# y = ln(1 + Z) = Z - \frac{1}{2}Z^2 + \frac{1}{3}Z^3 - \dots \tag{3.2}$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sum_{\substack{n_j,m_j=0 \\ n_j+m_j>0}}^{\infty} \frac{x^{n_1}}{n_1!}\frac{y^{m_1}}{m_1!} \cdots \frac{x^{n_k}}{n_k!}\frac{y^{m_k}}{m_k!}$$

This expression, although explicit, is not yet satisfactory. As known from the Friedrichs criterion, $x \# y$ is a Lie element, i. e., a linear combination of the multiple commutators of x and y. However, the multiple commutator structure is not evident in (3.2). The main problem of the BCH exponent consists precisely in expressing $x \# y$ as a series of the multiple commutators. This problem has been radically simplified by the combinatorial lemmas of Dynkin, which we quote below in a further simplified and generalized form:

LEMMA 1. — Let A be a free algebra generated by a set X. If $a \in \mathscr{A}$ is a Lie element, then for every $b \in \mathscr{A}$:

$$\{ a, b \} = [a, b]. \tag{3.3}$$

Proof : (3.3) holds if $a \in$ X, because from the definition $\{ x_i, b \} = [x_i, b]$. Suppose that (3.3) holds for two Lie elements $a_1, a_2 \in \mathscr{A}$. Then it must hold for $a = [a_1, a_2]$. Indeed:

$$\{ a, b \} = \{ a_1 a_2 - a_2 a_1, b \} = \{ a_1, \{ a_2, b \} \} - \{ a_2, \{ a_1, b \} \}$$
$$= [a_1, [a_2, b]] - [a_2, [a_1, b]] = [[a_1, a_2], b] = [a, b].$$

Thus, the lemma holds for any Lie element $a \in \mathscr{A}$.

LEMMA 2. — If a Lie element $a_n \in \mathscr{A}$ is an homogeneous n-th order polynomial in the generators of the algebra, then $\{ a_n \} = na_n$. In particular $\{ \{ x_{k_1} \dots x_{k_n} \} \} = n \{ x_{k_1} \dots x_{k_n} \}$.

Proof. — The assertion holds for $n = 1$, since $\{ x_k \} = x_k$ for any generator $x_k \in X$. Suppose that the lemma holds for two Lie elements a_n and a_m, of the orders n and m respectively. We shall show that then it holds for $a_{n+m} = [a_n, a_m]$. Indeed:

$$\{ a_{n+m} \} = \{ a_n a_m - a_m a_n \} = \{ a_n, \{ a_m \} \} - \{ a_m, \{ a_n \} \} = m \{ a_n, a_m \} - n \{ a_m, a_n \} = m[a_n, a_m] - n[a_m, a_n] = (n + m)[a_n, a_m] = (n + m)a_{n+m}.$$

Hence, our lemma holds for every integer n.

This lemma suggests the construction of the following operator D which projects the algebra $\mathscr{A}$ onto the subset of its Lie elements:

$$\mathrm{D}a \overset{\mathrm{df}}{=} \alpha^j \{ x_j \} + \frac{1}{2}\alpha^{j_1 j_2} \{ x_{j_1} x_{j_2} \} + \frac{1}{3}\alpha^{j_1 j_2 j_3} \{ x_{j_1} x_{j_2} x_{j_3} \} + \dots \tag{3.4}$$

(a being given by (2.1)). With its help lemma 2 can be equivalently expressed by either of the two statements:

(1) a is a Lie element $\Leftrightarrow$ $\mathrm{D}a = a$.

(2) $\mathrm{D}^2 = \mathrm{D}$.

Now, Dynkin's method of finding $x \# y$ reduces to the following remark: it is already known that $x \# y$ is a Lie element, so that $\mathrm{D}(x \# y) = x \# y$. Thus applying D to both sides of (3.2) one obtains:

$$x \# y = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sum_{\substack{n_j, m_j = 0 \\ n_j + m_j > 0}}^{\infty} \frac{1}{n_1 + m_1 + \dots + n_k + m_k} \frac{\{ x^{n_1} y^{m_1} \dots x^{n_k} y^{m_k} \}}{n_1 ! m_1 ! \dots n_k ! m_k !} \tag{3.5}$$

which represents the sought for expression of $x \# y$ in terms of multiple commutators. The calculation of the few first terms yields:

$$x \# y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}(\{ x^2 y \} + \{ y^2 x \}) + \dots \tag{3.6}$$

in agreement with results obtained by other methods.

Dynkin's solution (3.5) has a distinguished role in the literature: it exhibits the regular structure of the n-th term of $x \# y$ and, moreover, is more directly applicable at the level of specific algebras than the original BCH-formula (2.30). It ought to be noticed, however, that solution (3.5) cannot eliminate the other approaches to the BCH-formula but rather completes them. In fact, expression (3.5), although explicit, is still somewhat overcharged. In order to find the third order term of $x \# y$ which is $\frac{1}{12}\{x^2y\} + \frac{1}{12}\{y^2x\}$ one has first to write down 8 multiple commutators and then to make cancellations using elementary identites like $\{xyx\} = -\{x^2y\}$. In higher orders this combinatorial task increases, making the deduction of the final output of (3.5) troublesome. Thus, the problem of finding more economical expressions is open. It is also possible that in certain cases some other methods such as differential algorithms can lead more directly to closed form results (*).

4. General properties

The methods reviewed in § 2 and 3 determine the « perturbative expansions » of $\Omega(x, y) = x \# y$. It is also worth while to mention some general regularities of $x \# y$ which follow from the defining equation

$$e^x e^y = e^{x \# y} = e^{\Omega(x,y)}.$$

The most obvious of them are:

$$x \# 0 = x = 0 \# x, \tag{4.1 a}$$

$$(x \# y) \# z = x \# (y \# z), \tag{4.1 b}$$

$$x \# y \# (-x) = \{e^x, y\} = e^x y e^{-x}. \tag{4.1 c}$$

These basic properties of the composition $\#$ permit us to prove some secondary ones. Thus, for example, combining (4.1 c) with $y = 0$ and (4.1 a) and making use of (4.1 b) one shows that

$$x \# (-x) = 0. \tag{4.2}$$

Having established (4.2) and using (4.1 a-b), one easily proves the implication:

$$x \# y = 0 \Leftrightarrow x = -y. \tag{4.3}$$

(*) In § 11 we apply the solution of the continuous BCH problem to obtain the integral representation (11.12) of (3.5) which in some cases may be convenient.

From this rule and associatitivity (4.1 *b*) one easily infers that:

$$(4.4) \qquad x \# y = -(-y) \# (-x) \Leftrightarrow \Omega(x, y) = -\Omega(-y, -x).$$

Finally, setting in (4.1 *c*) $y \to y \# x$ and using associatitivity one finds $e^{-x}\Omega(x, y)e^{x} = \Omega(y, x)$. Similarly $e^{y}\Omega(x, y)e^{-y} = \Omega(y, x)$. By combining these identities with (4.4) one obtains:

$$(4.5\,a) \qquad \{ e^{-x}, \Omega(x, y) \} = -\Omega(-x, -y),$$

$$(4.5\,b) \qquad \{ e^{y}, \Omega(x, y) \} = -\Omega(-x, -y).$$

Now, it is worth while to decompose Ω into « even » and « odd » parts:

$$\Omega_{+}(x, y) = \frac{1}{2}[\Omega(x, y) + \Omega(-x, -y)]$$

and

$$\Omega_{-}(x, y) = \frac{1}{2}[\Omega(x, y) - \Omega(-x, -y)].$$

Because of (4.4) we have:

$$(4.6) \qquad \Omega_{+}(x, y) = -\Omega_{+}(y, x), \qquad \Omega_{-}(x, y) = \Omega_{-}(y, x).$$

By splitting (5.5 *a-b*) into even and odd parts one easily obtains:

$$(4.7) \qquad \Omega_{+}(x, y) = \left\{ \operatorname{th}\frac{x}{2}, \Omega_{-}(x, y) \right\} = -\left\{ \operatorname{th}\frac{y}{2}, \Omega_{-}(x, y) \right\}.$$

This result can be equivalently represented as:

$$(4.8\,a) \qquad \Omega_{+} = \frac{1}{2}\left\{ \operatorname{th}\frac{x}{2} - \operatorname{th}\frac{y}{2}, \Omega_{-} \right\},$$

$$(4.8\,b) \qquad 0 = \left\{ \operatorname{th}\frac{x}{2} + \operatorname{th}\frac{y}{2}, \Omega_{-} \right\}.$$

Equation (4.8 *a*) shows that Ω_{-} determines Ω_{+} uniquely, while (4.8 *b*) is a sort of consistency condition restricting the possible structure of Ω_{-}. Given x and y, the conditions (4.8 *a-b*) contain relevant information about the structure of $\Omega = \Omega_{+} + \Omega_{-}$. In fact, in some cases these conditions plus the associativity property of the composition operation $\#$ permit us to deduce the closed form of $\Omega(x, y)$. Thus, e. g., if $\vec{I} = (I_1, I_2, I_3)$ are the generators of the group $0^{+}(3)$, i. e., $[I_a, I_b] = i\epsilon_{abc}I_c$, and

$$x = i\frac{2 \operatorname{arc\,tg} |\vec{x}|}{|\vec{x}|}\vec{x}\vec{I}, \qquad y = i\frac{2 \operatorname{arc\,tg} |\vec{y}|}{|\vec{y}|}\vec{y}\vec{I}$$

($\vec{x}$ and $\vec{y}$ are real vectors) then using (4.8 *a-b*) and the associativity law (4.1 *b*) one easily shows that

$$\Omega(\vec{x}, \vec{y}) = i\frac{2 \operatorname{arc\,tg} |\vec{z}|}{|\vec{z}|}\vec{z}\vec{I},$$

where

$$z = \frac{\vec{x} + \vec{y} - \vec{x} \wedge \vec{y}}{1 - \vec{x}\vec{y}}, \tag{4.9}$$

with $\vec{x} \wedge \vec{y}$ denoting the vector product (see also [8]).

II. — CONTINUOUS FORMULA

5. Preliminaries. The traditional approach

As has already been mentioned in the introduction, the problem of the continuous BCH formula arises in the study of the « evolution operator » $E(t, \tau)$ depending on real parameters t and τ and defined be either of the conditions:

$$\frac{d}{dt}E(t, \tau) = H(t)\,E(t, \tau); \qquad E(\tau, \tau) = 1, \tag{5.1}$$

$$\frac{d}{dt}E(t, \tau) = -\,E(t, \tau)\,H(t); \qquad E(t, t) = 1. \tag{5.1 b}$$

The equivalence of (5.1 *a-b*) is due to their common origin: for differentiable $E(t, \tau)$ (5.1 *a-b*) imply and are implied by the functional equations:

$$E(t, s)\,E(s, \tau) = E(t, \tau), \tag{5.2 a}$$

$$E(s, s) = 1. \tag{5.2 b}$$

Indeed, assuming that the conventional differentiation rules apply to the operator-valued function $E(t, \tau)$ and differentiating (5.2 *a*) with respect to s one obtains:

$$\left(\frac{d}{ds}E(t, s)\right)E(s, \tau) + E(t, s)\left(\frac{d}{ds}E(s, \tau)\right) = 0.$$

Because (5.2 *a-b*) implies $E(s, t)\,E(t, s) = 1$, this is equivalent to

$$-\,E(s, t)\left(\frac{d}{ds}E(t, s)\right) = \left(\frac{d}{ds}E(s, \tau)\right)E(\tau, s).$$

The left hand member of this equations is independent of τ while the right hand member is independent of t. Thus, each member represents an operator function depedning on s only. Denoting this function by $H(s)$ one obtains (5.1 *a*) and (5.1 *b*). The proof that either of (5.1 *a-b*)implies (5.2 *a-b*) is equally simple.

The known solution of either of (5.1 *a-b*) in terms of $H(t)$ is obtained by the transition to the corresponding integral equation and then the application of the iterative procedure. This yields:

$$(5.3)\quad E(t, t_0)$$

$$= 1 + \sum_{n=1}^{\infty} \int_{t_0}^{t} \ldots \int_{t_0}^{t} dt_n \ldots dt_1 H(t_n)\, \theta_{n,n-1} H(t_{n-1}) \ldots \theta_{2,1} H(t_1) \qquad (\text{if} \quad t > t_0)$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \int_{t}^{t_0} \ldots \int_{t}^{t_0} dt_n \ldots dt_1 H(t_1)\, \theta_{2,1} H(t_2) \ldots \theta_{n,n-1} H(t_n) \qquad (\text{if} \quad t < t_0)$$

$$= T \exp\left(\int_{t_0}^{t} d\tau H(\tau) \right) \qquad (\text{if} \quad t > t_0)$$

$$= \overline{T} \exp\left(- \int_{t}^{t_0} d\tau H(\tau) \right) \qquad (\text{if} \quad t < t_0)$$

where $\theta_{k,j} = \theta(t_k - t_j)$, θ being the step function:

$$\theta(\tau) = \begin{cases} 1 & \tau > 0 \\ 0 & \tau < 0, \end{cases}$$

and the symbols T and $\overline{\mathrm{T}}$ stand for the operators of chronological and the anti-chronological orderings respectively.

The problem of the convergence of the series (5.3) is for some operator realisations of $H(t)$ nontrivial. We do not intend to enter into this problem here, restricting ourselves to the purely algebraic investigation of $E(t, t_0)$. This can be formally achieved in two ways: 1) we can assume that the symbols $H(t)$ are certain parameter-dependent elements of a free algebra $\mathscr{A}$ of the form $H(t) = C_1(t)x_1 + C_2(t)x_2 + \ldots$ where $C_i(t)$ are complex valued functions of the real variable t and x_i are the generators of $\mathscr{A}$. Within this assumption, our formulæ will be convergent in the natural topology of $\mathscr{A}$. 2) We can also interpret the symbols $H(t)$ as generating a « continuous analog » of the free algebra. By this we understand the set $\mathscr{A}_{[t_0,t]}$ of all symbolic series A, B, ... of the form:

$$(5.4)\qquad A = A_0 + \int_{t_0}^{t} dt_1 A_1(t_1) H(t_1) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_1 . A_2(t_1, t_2) H(t_2) H(t_1) + \ldots$$

where A_n are the complex valued n variable distributions ($A_0 \in \mathbb{C}$ is a number). For any two symbolic series A, $B \in \mathscr{A}_{[t_0,t]}$ the linear combination $\alpha A + \beta B (\alpha, \beta \in \mathbb{C})$ is defined by $(\alpha A + \beta B)_n = \alpha A_n + \beta B_n$, while the product AB is defined by

$$(AB)_n(t_1 \ldots t_n) = \sum_{k=0}^{n} A_k(t_1 \ldots t_k) B_{n-k}(t_{k+1} \ldots t_n).$$

Since the sets of n-variable distributions are topological spaces, the natural topology can be defined in $\mathscr{A}_{[t_0,t]}$ as the weakest topology in which all mappings $A \to A_n$ are continuous. The resulting topological algebra $\mathscr{A}_{[t_0,t]}$ is an extended version of the Borchers algebra. It can be noticed that for $[t, \tau] \subset [t', \tau']$ the algebra $\mathscr{A}_{[t,\tau]}$ admitts a natural embedding in $\mathscr{A}_{[t',\tau']}$. This fact makes it possible to attribute a definite sense to the processes of differentiation of the series (5.3) with respect to t or t_0 and to the equations like (5.1) or (5.2). In what follows, our considerations refer to the algebra $\mathscr{A}_{[t_0,t]}$, although all our results can be reinterpreted immediately in the spirit of the first possibility.

The problem of representing the solution (5.3) as a genuine exponential function, $E(t, t_0) = e^{\Omega(t,t_0)}$, is traditionally approached according to the following method [6] [8] [9]. One substitutes $E = e^{\Omega}$ into (5.1 a) and one uses (2.24). This leads to the relation

$$\left\{ \frac{e^{\Omega} - 1}{\Omega}, \frac{\partial \Omega}{\partial t} \right\} E = HE \Rightarrow H = \left\{ \frac{e^{\Omega} - 1}{\Omega}, \frac{\partial \Omega}{\partial t} \right\} .$$

Therefore, because the second of (2.4),

$$\left\{ \frac{\Omega}{e^{\Omega} - 1}, H \right\} = \left\{ \frac{\Omega}{e^{\Omega} - 1}, \left\{ \frac{e^{\Omega} - 1}{\Omega}, \frac{\partial \Omega}{\partial t} \right\} \right\} = \frac{\partial \Omega}{\partial t}$$

and so:

$$\frac{\partial \Omega}{\partial t} = \left\{ \frac{\Omega}{e^{\Omega} - 1}, H \right\} = \sum_{n=0}^{\infty} \frac{B_n}{n!} \{ \Omega^n, H \}, \tag{5.5}$$

where the B_n, defined by

$$t/(e^t - 1) = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n$$

are Bernoulli numbers ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6, \ldots$). Of course $E(t_0, t_0) = 1$ suggests an initial condition for Ω: $\Omega(t_0, t_0) = 0$. Thus (5.5) can be replaced by the integral equation:

$$\Omega(t, t_0) = \sum_{n=0}^{\infty} \frac{B_n}{n!} \int_{t_0}^{t} d\tau \underbrace{[\Omega(\tau, t_0), [\Omega(\tau, t_0), \ldots [\Omega(\tau, t_0),}_{n \text{ times}} H(\tau)] \ldots]] \tag{5.6}$$

which is the starting point of the following algorithm: one seeks Ω as a formal series composed of terms of various orders in H,

$$\Omega = \underset{1}{\Omega} + \underset{2}{\Omega} + \underset{3}{\Omega} + \ldots.$$

Then by comparing terms of the same order on both sides of (5.6) one successively obtains

$$(5.7)\quad \underset{1}{\Omega} = \int_{t_0}^{t} dt_1\, \mathrm{H}(t_1)$$

$$\underset{2}{\Omega} = \frac{1}{2}\int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_1 [\mathrm{H}(t_2), \mathrm{H}(t_1)]$$

$$\underset{3}{\Omega} = \frac{1}{6}\int_{t_0}^{t} dt_3 \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_2} dt_1 ([\mathrm{H}(t_3), [\mathrm{H}(t_2), \mathrm{H}(t_1)]] + [\mathrm{H}(t_1), [\mathrm{H}(t_2), \mathrm{H}(t_3)]])\ (*)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

This procedure suffices in principle to determine Ω up to any desired order. However, because of the increasing combinatorial difficulties in higher orders it is still inadequate to find a general structure for $\underset{n}{\Omega}$. Until very recent times it was even doubtful whether any compact explicit expression for $\underset{n}{\Omega}$ could be given. The existence of such an expression was demonstrated and its explicit form was found in [*10*] through the application of a straightforward method similar to Dynkin's technique of evaluating the discrete BCH exponent.

6. New method (Heuristics)

We shall first outline the new method in its most primitive version; a more complete formalism will be developed subsequently. Following Dynkin we start from the power series expansion of the logarithm yielding:

$$(6.1)\qquad \Omega = ln\,\mathrm{E} = ln(1 + \mathrm{T}) = \mathrm{T} - \frac{1}{2}\mathrm{T}^2 + \frac{1}{3}\mathrm{T}^3 + \ldots$$

where for $t > t_0$:

$$(6.2)\quad \mathrm{T} = \int_{t_0}^{t} dt_1\, \mathrm{H}(t_1) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_1\, \mathrm{H}(t_2)\,\theta_{2,1}\mathrm{H}(t_1) + \int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} dt_3 dt_2 dt_1\, \mathrm{H}(t_3)\,\theta_{3,2}\mathrm{H}(t_2)\,\theta_{2,1}\mathrm{H}(t_1) + \ldots$$

(*) $\underset{3}{\Omega}$ in this form is given in [*9*] where the calculation is carried out as far as $\underset{4}{\Omega}$.

The main difficulty in determining Ω is in evaluating the powers of the infinite series T and then in grouping the resulting terms in (6.1) according to the orders of H. This task can be simplified by means of the following techniques which covers the case of θ in (6.2) being an arbitrary function instead of the step function. Let us represent T^2 as:

$$(6.3)\quad T^2 = \int_{t_0}^{t}\int_{t_0}^{t} dt_2 dt_1\, H(t_2)\, H(t_1) + \int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} dt_3 dt_2 dt_1\, H(t_3)\, H(t_2)\, \theta_{2,1} H(t_1) + \int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} dt_3 dt_2 dt_1\, H(t_3)\, \theta_{3,2} H(t_2)\, H(t_1) + \dots$$

Comparing (6.3) with (6.2) we notice that all terms for T^2 can be obtained out of the terms for T by dropping out some θ' as. The needed operation of « dropping out θ' as » possesses the formal properties of differentiation. This fact can be described by introducing a symbolic operation $\frac{d}{d\theta}$ which acts on formal series A, B, ... constructed from $H's$ and θ' according to the following rules:

1) $\frac{d}{d\theta} H(t_j) = 0,$ 2) $\frac{d}{d\theta}\theta_{k,l} = 1,$

3) $\frac{d}{d\theta}(A + B) = \frac{d}{d\theta}A + \frac{d}{d\theta}B,$ 4) $\frac{d}{d\theta}(AB) = \left(\frac{d}{d\theta}A\right)B + A\left(\frac{d}{d\theta}B\right).$

With the help of $\frac{d}{d\theta}$ the structure of (6.3) is described by:

$$(6.4)\qquad T^2 = \frac{d}{d\theta}T \text{ (*)}$$

It follows by induction that

$$(6.5)\qquad T^n = \frac{1}{(n-1)!}\frac{d^{n-1}}{d\theta^{n-1}}T.$$

(*) This formula, in facts, is not so strange: by examining the structure of the series (6.2) one notice that T is a particular realization of the symbolic series

$$IH + IHJH + IHJHJH + \dots = IH\frac{1}{1 - JH}.$$

The $\frac{d}{d\theta}$ operation corresponds to the polarization derivative $I\frac{d}{dJ}$ and hence, (6.4) is a special realization of (2.25).

Consequently, (6.1) becomes

$$(6.6)\quad ln(1 + \mathrm{T}) = \mathrm{T} - \frac{1}{2}\frac{d}{d\theta}\mathrm{T} + \frac{1}{3}\frac{1}{2}\frac{d^2}{d\theta^2}\mathrm{T} + \ldots = \left(\frac{e^{-\frac{d}{d\theta}} - 1}{-\frac{d}{d\theta}}\right)\mathrm{T}.$$

Now by writing down T as

$$(6.7)\quad \mathrm{T} = \sum_{n=1}^{\infty} \int_{t_0}^{t} \ldots \int_{t_0}^{t} dt_n \ldots dt_1 \mathrm{H}(t_n) \ldots \mathrm{H}(t_1)\theta_{n,n-1}\theta_{n-1,n-2} \ldots \theta_{2,1},$$

and by introducing the operator $(e^{-d/d\theta} - 1)/\left(-\dfrac{d}{d\theta}\right)$ under the sum and integrals one obtains:

$$(6.8)\quad ln(1 - \mathrm{T}) = \sum_{n=1}^{\infty} \int_{t_0}^{t} \ldots \int_{t_0}^{t} dt_n \ldots dt_1 \mathrm{L}_n(t_n, \ldots, t_1)\mathrm{H}(t_n) \ldots \mathrm{H}(t_1)$$

where

$$(6.9)\qquad \mathrm{L}_n(t_n \ldots t_1) = \left(\frac{e^{-(d/d\theta)} - 1}{-d/d\theta}\right)\theta_{n,n-1} \ldots \theta_{2,1}.$$

Since we can neglect here all terms with $(d/d\theta)^m$ for $m > n$, the kernels L_n can be explicitly computed; e. g.,

$$\mathrm{L}_1 = 1, \qquad \mathrm{L}_2 = \theta_{2,1} - \frac{1}{2}, \qquad \mathrm{L}_3 = \theta_{3,2}\theta_{2,1} - \frac{1}{2}\theta_{2,1} - \frac{1}{2}\theta_{3,2} + \frac{1}{3},$$

$$\mathrm{L}_4 = \theta_{4,3}\theta_{3,2}\theta_{2,1} - \frac{1}{2}\theta_{4,3}\theta_{3,2} - \frac{1}{2}\theta_{4,3}\theta_{2,1} - \frac{1}{2}\theta_{3,2}\theta_{2,1} + \frac{1}{3}\theta_{2,1} + \frac{1}{3}\theta_{3,2} + \frac{1}{3}\theta_{4,3} - \frac{1}{4}, \text{ etc.}$$

By setting θ in (2.5) to be the step function we obtain the required series for Ω ordered according to the « powers » of H. Due to the Friedrichs criterion Ω must be a Lie element. Hence, $\mathrm{D}\Omega = \Omega$, and by applying Dynkin's operator D to (6.8) we can express Ω through multiple commutators of H's. This is, in essence, the new method in its most primitive version.

Although (6.9) allows one to evaluate the kernels L_n, it still involves some combinatorial task of executing multiple differentiations. In order to obtain a simpler technique of determining L_n's through complex contour integrals we shall reformulate our method by using the concept of the resolvent operator.

7. Method of the resolvent operator

The symbolic method outlined above can be generalized by the use of the resolvent operator $R(t, t_0; \lambda) \in \mathscr{A}_{[t_0,t]}(\lambda \in \mathbb{C})$ defined by:

(7.1 *a*) $R(\lambda) = R(t, t_0; \lambda)$

$$\stackrel{\text{df}}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \int_{t_0}^{t} \cdots \int_{t_0}^{t} dt_n \ldots dt_1 H(t_n)(\varepsilon_{n,n-1} + \lambda)H(t_{n-1}) \ldots (\varepsilon_{2,1} + \lambda)H(t_1) \quad (\text{for } t > t_0)$$

(7.1 *b*) $R(\lambda) = R(t, t_0; \lambda)$

$$\stackrel{\text{df}}{=} \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} \int_{t_0}^{t} \cdots \int_{t_0}^{t} dt_n \ldots dt_1 H(t_1)(\varepsilon_{1,2} - \lambda)H(t_2) \ldots (\varepsilon_{n-1,n} - \lambda)H(t_n) \quad (\text{for } t < t_0)$$

where $\varepsilon_{k,l} = \varepsilon(t_k - t_l)$, $\varepsilon(t) = t/|t|$. We shall show that from the combinatorial point of view this operator provides the most economical representation for an arbitrary formal power series $f(t) = f(E - 1)$. An immediate analog of (6.4) is the differential equation:

$$\frac{d}{d\lambda} R(\lambda) = R^2(\lambda), \tag{7.2}$$

which determines the dependence of R on λ. Because $\frac{1}{2}(\varepsilon(t) + 1) = \theta(t)$ therefore, for $t > t_0$ (*):

$$R(1) = \frac{1}{2} \sum_{n=1}^{\infty} \int_{t_0}^{t} \cdots \int_{t_0}^{t} dt_n \ldots dt_1 H(t_n)\theta_{n,n-1}H(t_{n-1}) \ldots \theta_{2,1}H(t_1) = \frac{1}{2}T = \frac{1}{2}(E - 1). \tag{7.3}$$

Thus, $R(\lambda)$ has the form $R(\lambda) = \dfrac{C}{1 - \lambda C}$, where $R(1) = \dfrac{C}{1 - C} = \dfrac{1}{2}T$ and $C = R(1) = \dfrac{1/2T}{1 + \frac{1}{2}T} = \dfrac{E - 1}{E + 1}$. Therefore

$$R(\lambda) = \frac{1/2T}{1 + \frac{1}{2}T} \frac{1}{1 - \lambda \dfrac{1/2T}{1 + \frac{1}{2}T}} = \frac{1/2T}{1 + (1 - \lambda)\frac{1}{2}T} \tag{7.4}$$

(*) Also for $t < t_0$, $R(1) = \frac{1}{2}T = \frac{1}{2}(E - 1)$.

This formula already allows one to find the analytic functions of T through the corresponding contour integrals of $R(\lambda)$.

Before this will be done it is worthwhile to list same useful properties of the resolvent operator. Since $T = E - 1$, (7.4) is equivalent to:

$$R(\lambda) = \frac{E - 1}{2 + (1 - \lambda)(E - 1)} = \frac{E - 1}{(1 + \lambda) + (1 - \lambda)E} \tag{7.5}$$

and since $E = e^{\Omega}$ where Ω is the BCH exponent, (7.5) leads to

$$R(\lambda) = \frac{\operatorname{th}(\Omega/2)}{1 - \lambda \operatorname{th}(\Omega/2)} \tag{7.6}$$

Each of these relations implies that $R(\lambda)$ satisfies the Hilbert finite difference equation:

$$\frac{R(\lambda) - R(\mu)}{\lambda - \mu} = R(\lambda)R(\mu), \tag{7.7}$$

which in the limit $\lambda \to \mu$ leads back to (7.2). The distinguished values of λ are $\lambda = 1, 0, -1$ where

$$R(1) = \frac{1}{2}(E - 1) \tag{7.8 a}$$

$$R(0) = \frac{E - 1}{E + 1} = C \qquad \text{(Cayley's operator)} \tag{7.8 b}$$

$$R(-1) = \frac{1}{2}(E^{-1} - 1). \tag{7.8 c}$$

Note that because of the general representation (7.1) formula (7.8 *b*) yields immediately the Schwinger's representation for the Cayley operator C, while (7.8 *c*) ammounts to the « antichronological » representation of $E^{-1}(t, t_0) = E(t_0, t)$. To close the review of the properties of R we quote the differential equations:

$$\frac{\partial}{\partial t} R = \frac{1}{2}[1 - (1 - \lambda)R]H(t)[1 + (1 + \lambda)R] \tag{7.9}$$

$$\frac{\partial}{\partial t_0} R = \frac{1}{2}[1 - (1 - \lambda)R]H(t_0)[1 + (1 + \lambda)R],$$

which follow easily from (7.5) and (5.1 *a-b*). With $\lambda = 0$, (7.9 *a*) reduces to

$$\frac{dC}{dt} = \frac{1}{2}(1 - C)H(1 + C)$$

which have been traditionally used to derive the Schwinger's algorithm for the Cayley operator.

We now return to the problem of evaluating an arbitrary series

$$f(T) = f_0 + f_1 T + f_2 T^2 + \dots$$

in the operator $T = E - 1$. We first note that from (7.4)

$$(7.10)\quad \frac{1}{z - T} = \frac{1}{z}\frac{1}{1 - \frac{1}{z}T} = \frac{1}{z}\frac{1 - \frac{1}{z}T + \frac{1}{z}T}{1 - \frac{1}{z}T}$$

$$= \frac{1}{z} + \frac{2}{z^2}\frac{\frac{1}{z}T}{1 - \frac{2}{z^2}\frac{1}{z}T} = \frac{1}{z} + \frac{2}{z^2}R\left(1 + \frac{2}{z}\right).$$

It follows that:

$$(7.11)\quad f(T) = \frac{1}{2\pi i}\oint_{z=0} dz\frac{f(z)}{z - T} = f(0) + \frac{1}{2\pi i}\oint_{z=0} dz\frac{2f(z)}{2^2}R\left(1 + \frac{2}{z}\right).$$

Therefore, by using (7.1) for $t > t_0$ one obtains:

$$(7.12)\quad f(E - 1) - f(0)$$
$$= \sum_{n=1}^{\infty}\int_{t_0}^{t}\cdots\int_{t_0}^{t} dt_n \ldots dt_1 F_n(t_n, \ldots, t_1)H(t_n)\ldots H(t_1)$$

where

$$(7.13)\quad F_n(t_n, \ldots, t_1) \stackrel{\mathrm{df}}{=} \frac{1}{2\pi i}\oint_{z=0}\frac{dz}{z^2}\left(\theta_{n,n-1} + \frac{1}{1}\right)\cdots\left(\theta_{2,1} + \frac{1}{z}\right)f(z)$$

This formula represents the sought for generalization of (6.9). Because θ is the step function, (7.13) can be represented more conveniently. Indeed, each of the factors $\theta_{k,k-1} + \frac{1}{z}$ equals either $\frac{1}{z}$ or $1 + \frac{1}{z}$. The number of the factors equal to $1 + \frac{1}{z}$ is obviously:

$$(7.14)\quad \Theta_n \stackrel{\mathrm{df}}{=} \theta_{n,n-1} + \theta_{n-1,n-2} + \ldots + \theta_{2,1},$$

while the remaining $n - 1 - \Theta_n$ factors are equal to $\frac{1}{z}$. Therefore

$$(7.15)\quad F_n(t_n, \ldots, t_1) = \frac{1}{2\pi i}\oint_{z=0}\frac{dz}{z^{n+1}} f(z)(1 + z)^{\Theta_n},$$

depends on $t's$ only through Θ_n. The formulæ (7.12) and (7.15) provide the basic « perturbative » expansion of an arbitrary series $f(\mathrm{T}) = f(\mathrm{E}-1)$ (*).

The structure of Kernels. — According to (7.14) and (7.15) the kernel $\mathrm{F}_n(t_n, \ldots, t_1)$ is a step function of n variables constant in each of the $n!$ sectors defined by inequalities of the type $t_{i_n} > t_{i_{n-1}} > \ldots > t_{i_1}$ ($i_1, i_2, \ldots, i_n$ being a permutation of $1, 2, \ldots, n$): e. g., $t_n > t_{n-1} > \ldots > t_1$ is the *chronological sector,* $t_1 > t_2 > \ldots > t_n$ is the *anti-chronological sector.* Moreover, the value of F_n must be the same for all sectors characterised by the same number $\Theta = \theta(t_n - t_{n-1}) + \ldots + \theta(t_2 - t_1)$ which will be called the *chronological type* of the sector. Of course, for each fixed n, the chronological types of the sectors in the n-dimensional space of the variables $t_1, t_2, \ldots, t_n$ can vary between 0 and $n-1$: the maximal value $\Theta = n - 1$ characterises the chronological sector, while the minimal value $\Theta = 0$ characterises the anti-chronological sector. The number of the sectors of the other chronological types ($0 < \Theta < n - 1$) can be found by an elementary combinatorial argument. However, we shall show in § 8 that the complete information about these numbers is already contained in the analytic formulæ (7.12)-(7.15). Summing up: all the kernels F_n have a common feature: each is a step function constant on the area of all sectors with fixed chronological type; the actual value of this constant is given explicitly by the integral (7.15).

Examples: By specializing to $f(z) = (1 + z)^\alpha = 1 + \binom{\alpha}{1}z + \binom{\alpha}{2}z^2 + \ldots$, where $\alpha \in \mathbb{C}$, we obtain the integral representation (7.12) for $\mathrm{E}^\alpha = (1 + \mathrm{T})^\alpha$ with the kernels:

$$(7.16)\quad \mathrm{F}_n = \Xi_n^\alpha(t_n, \ldots, t_1) = \binom{\Theta_n + \alpha}{n} = \frac{(\Theta_n + \alpha)(\Theta_n + \alpha - 1)\ldots(\Theta_n + \alpha - n + 1)}{n!}$$

The integral representation of the continuous BCH exponent $\Omega = ln\,\mathrm{E}$ is obtained by taking $f(z) = ln(1 + z)$; the corresponding kernels are:

$$(7.17)\quad \mathrm{L}_n(t_n, \ldots, t_1) = \frac{1}{2\pi i}\oint_{z=0} \frac{dz}{z^{n+1}}\, ln(1 + z)(1 + z)^{\Theta_n} = (-1)^{n-1-\Theta_n}\frac{\Theta_n!(n - 1 - \Theta_n)!}{n!} = \frac{(-1)^{n-1-\Theta_n}}{n}\binom{n - 1}{\Theta_n}^{-1}.$$

(*) Similar formulæ can be obtained from the second line of (7.1) when $t < t_0$.

The resulting explicit representation is:

$$(7.18)\quad \Omega(t, t_0) = \sum_{n=1} \int_{t_0}^{t} \dots \int_{t_0}^{t} dt_n \dots dt_1 \frac{(-1)^{n-1-\Theta_n}}{n} \binom{n-1}{\Theta_n}^{-1} \mathrm{H}(t_n) \dots \mathrm{H}(t_1); \qquad t > t_0.$$

The last formula can be also derived more directly. Indeed, (7.6) implies that:

$$(7.19)\qquad \Omega = \int_{-1}^{1} d\lambda \mathrm{R}(t, t_0; \lambda),$$

which leads to

$$\mathrm{L}_n(t_n, \dots, t_1) = \frac{1}{2^n} \int_{-1}^{1} d\lambda(\varepsilon_{n,n-1} + \lambda) \dots (\varepsilon_{2,1} + \lambda)$$

coinciding with (7.17).

Now, because Ω is a Lie element, $\Omega = \mathrm{D}\Omega$ where D is the Dynkin operation (3.4) and therefore Ω can be represented as a linear combination of the multiple commutators:

$$(7.20)\quad \Omega(t, t_0) = \sum_{n=1}^{\infty} \int_{t_0}^{t} \dots \int_{t_0}^{t} dt_n \dots dt_1 \frac{(-1)^{n-1-\Theta_n}}{n^2} \binom{n-1}{\Theta_n}^{-1} \{ \mathrm{H}(t_n) \dots \mathrm{H}(t_1) \}; \qquad t > t_0.$$

The result (7.20) is the continuous conterpart of Dynkin's classical expression (3.5). It can be remarked, however, that the scope of our new method is wider than just the applications to the continuous BCH exponent problem: indeed, within this method we are in possession of the algorithm for $f(\mathrm{E} - 1)$ while the problem of BCH exponent concerns only the specific series $ln[1 + (\mathrm{E} - 1)]$.

8. **Numbers** N(k, l)

The numbers of sectors of given chronological type accounts for certain common structural properties of all functions $f(\mathrm{E} - 1)$. Since:

$$(8.1)\qquad \Theta_n(t_n, \dots, t_1) = n - 1 - \Theta_n(t_1, \dots, t_n),$$

the number of sectors type $\Theta = k$ in n-dimensional space is the same as the number of sectors with $\Theta = l = n - 1 - k$; the common value of these numbers will be denoted by $\mathrm{N}(k, l)$. (Thus, $\mathrm{N}(k, l)$ stands either for the number of sectors of type $\Theta = k$ or for the number of sectors of type $\Theta = l$ in the space of dimension $n = k + l + 1$). Obviously:

$$\mathrm{N}(k, l) = \mathrm{N}(l, k). \tag{8.2}$$

Since the total number of sectors in the n-dimensional space is $n!$, the following identity holds:

$$\sum_{k=0}^{n-1} \mathrm{N}(k, n-1-k) = n! \tag{8.3}$$

The numbers $\mathrm{N}(k, l)$ could be in principle determined by combinatorial considerations. However, complete information concerning them is already contained in formulæ (7.12) and (7.15).

Indeed, specialise (7.12) for $\mathrm{H} =$ constant $=$ number. Then the integration can be carried out, and leads to

$$f(e^{\zeta} - 1) = f(0) + \sum_{n=1}^{\infty} \frac{\zeta^n}{n!} \sum_{k=0}^{n-1} \mathrm{N}(k, n-1-k) \frac{1}{2\pi i} \oint_{z=0} \frac{dz}{z^{n+1}} f(z)(1+z)^k, \tag{8.4}$$

where ζ is an indetermined variable introduced in the place of $\mathrm{H}(t - t_0)$. Since $f(z)$ is arbitrary, this identity is sufficient to determine the numbers $\mathrm{N}(k, l)$. Taking $f(z) = \dfrac{z}{2 + (1-\lambda)z}$ and $\zeta = 2w$, where λ and w are complex variables, we reduce (8.4) to:

$$\frac{1}{\operatorname{cth} w - \lambda} = \sum_{n=1}^{\infty} \frac{w^n}{n!} \sum_{k=0}^{n-1} \mathrm{N}(k, n-1-k)(\lambda - 1)^{n-1-k}(\lambda + 1)^k \tag{8.5}$$

By introducing the new complex variables $u = w(\lambda - 1)$, $v = w(\lambda + 1)$ one arrives at the following identity which means that the $\mathrm{N}(k, l)$'s possess a simple generating function:

$$\Psi(u, v) \overset{\mathrm{df}}{=} \frac{e^{-u} - e^{-v}}{(-ue^{-u}) - (-ve^{-v})} = \sum_{k,l=0}^{\infty} \frac{\mathrm{N}(k, l)}{(k+l+1)!} u^k v^l \left(= \frac{1}{w} \frac{1}{\operatorname{cth} w - \lambda} \right). \tag{8.6}$$

Now, the partial differential equation:

$$\left[(v-1)u\frac{\partial}{\partial u}+(u-1)v\frac{\partial}{\partial v}+(u+v-1)\right]\Psi=-1, \tag{8.7}$$

implies the recurrence rule:

$$\mathrm{N}(k,l)=(l+1)\,\mathrm{N}(k-1,l)+(k+1)\,\mathrm{N}(k,l-1), \tag{8.8}$$

which permits one to construct $\mathrm{N}(k,l)$'s successively in the form of a triangle where each row is determined by the previous rows according to (8.8)

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&4&&1&&& \\ &&1&&11&&11&&1&& \\ &1&&26&&66&&26&&1& \\ 1&&57&&302&&302&&57&&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{array} \tag{8.9}$$

The generating function (4.43) leads also to an explicit expression for $\mathrm{N}(k,l)$:

$$\begin{aligned}\mathrm{N}(k,l)&=\frac{(k+l+1)!}{(2\pi i)^2}\oint_{u=0}\frac{du}{u^{k+1}}\oint_{v=0}\frac{dv}{v^{l+1}}\Psi(u,v)\\ &=\sum_{n=0}^{k}(-1)^m\binom{k+l+2}{m}(k+1-m)^{k+l+1}\end{aligned} \tag{8.10}$$

which is an analytic function of $n=k+l+1$. This result illustrates how our solution of the continuous BCH problem can induce the solutions of some other, apparently unrelated combinatorial problems.

It is of some interest to notice that the numbers $\mathrm{N}(k,l)$ induce the polynomials:

$$\mathscr{P}_{n-1}(\lambda)\overset{\mathrm{df}}{=}\frac{1}{2^n}\sum_{k=0}^{n-1}\mathrm{N}(k,n-1-k)(\lambda-1)^{n-1-k}(\lambda+1)^k, \tag{8.11}$$

which are closely related with the structure of the resolvent operator. In fact, from (8.5)

$$f(w,\lambda)\overset{\mathrm{df}}{=}\frac{1}{\operatorname{cth}\frac{w}{2}-\lambda}=\sum_{n=1}^{\prime}\mathscr{P}_{n-1}(\lambda)\frac{w^n}{n!}. \tag{8.12}$$

Therefore, according to (7.6) the polynomials $\mathscr{P}_n$ provide a development of $R(\lambda)$ in powers of the BCH exponent Ω:

$$R(\lambda) = \sum_{n=1}^{\infty} \mathscr{P}_{n-1}(\lambda) \frac{\Omega^n}{n!} \tag{8.13}$$

The polynomials $\mathscr{P}_n$ can be evaluated easily from an expression similar to the Rodriguez formula. In fact, the partial differential equation

$$2\frac{\partial f}{\partial w} = 1 + \frac{\partial}{\partial \lambda}(\lambda^2 - 1)f, \tag{8.14}$$

leads to the recurrence relation

$$\mathscr{P}_n(\lambda) = \frac{1}{2}\frac{d}{d\lambda}[(\lambda^2 - 1)\mathscr{P}_{n-1}(\lambda)], \qquad n \geqslant 1, \tag{8.15}$$

which, since $\mathscr{P}_0(\lambda) = \frac{1}{2}$, yields

$$\mathscr{P}_n(\lambda) = \frac{1}{2^{n+1}}\left[\frac{d}{d\lambda}(\lambda^2 - 1)\right]^n . 1 \tag{8.16 a}$$

An equivalent representation is:

$$\mathscr{P}_n\,(\mathrm{th}\ v/2) = \frac{1}{2}\mathrm{ch}^2\, v/2\left(-\frac{d}{dv}\right)^n \mathrm{ch}^{-2}\, v/2 \tag{8.16 b}$$

whence the n roots of $\mathscr{P}_n(\lambda) = 0$ are all contained in the real interval $-1 \leqslant \lambda \leqslant 1$ and are symmetrically distributed with respect to $\lambda = 0$.

The polynomials $\mathscr{P}_n(\lambda)$ as given by (8.16) contain all information concerning the numbers $N(k, l)$ (e. g., (8.16) $\Rightarrow$ (8.8)). In particular, a relation among the N's and the Bernoulli numbers B_n can be obtained by setting $\lambda = 0$ in (8.12) and using the well known development $\mathrm{th}\ v/2 = 2\sum_{k=1}^{\infty}(2^{2k} - 1)\frac{B_{2k}}{(2k)!}v^{2k-1}$. By comparing the terms of the same order in w on both sides of (8.12) one obtains:

$$(8.17\,a)\quad B_n = \begin{cases} \dfrac{1}{2^n(2^n - 1)}\left[n! - 4n\displaystyle\sum_{k=1}^{l} N(2k - 1, n - 2k - 1)\right] & \text{for } n = 4l + 2, l \geqslant 0 \\ \dfrac{1}{2^n(2^n - 1)}\left[n! - 4n\displaystyle\sum_{k=0}^{l-1} N(2k, n - 2k - 2)\right] & \text{for } n = 4l, \quad l \geqslant 1 \end{cases}$$

Specializing (8.4) to $f(z) = \frac{1}{2} ln\,(1 + z)$ and using (7.17) with $n \to n + 1$ one also finds that for $n \geqslant 1$:

$$B_n = \sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{n+1}\binom{n}{k}^{-1} N(k, n - 1 - k)\ (*) \tag{8.17 b}$$

(*) This formula contains the numbers $N(k, l)$ with alternating signs. If one wishes to determine effectively B_n's through $N(k, l)$'s derivable from the triangle (8.9) the formulæ (8.17) are more economical.

A more general relation involving N's and B_n's can be obtained by executing the integrals $\int_{-1}^{1} d\lambda(\lambda^2 - 1)\mathscr{P}_n(\lambda)\mathscr{P}_m(\lambda)$. Because of (8.16) and (8.12) the integration by parts leads to the following « normalization condition »:

(8.18) $$\int_{-1}^{1} d\lambda(\lambda^2 - 1)\mathscr{P}_n\mathscr{P}_m = 2(-1)^{n+1}B_{n+m+2}$$

which contains (8.17) as a particular case.

The close realtionship among N's and B_n's explains the following problem: it can appear strange that the numbers B_n so essential in the traditional algorithm (5.6) do not intervene in our explicit solution (7.18). It now becomes clear that the B_n's enter (7.18) in the disguise of the numbers N(k, l) describing the combinatorics of sectors.

9. Consistency with the differential algorithm

An explicit verification of the consistency of formulæ (7.18) and (7.20) with the differential equation (5.6) can be achieved by the use of an integral representation of $\partial\Omega/\partial t$. Indeed, (7.5), (5.1) and (2.17) imply that:

(9.1) $$\frac{\partial R}{\partial t} = 2\frac{1}{(1+\lambda)+(1-\lambda)E}HE\frac{1}{(1+\lambda)+(1-\lambda)E} \quad (*)$$

Now, because of (7.19), (9.1) integrated over λ in the limits -1 and 1 gives

(9.2) $$\frac{\partial\Omega}{\partial t} = 2\int_{-1}^{1} d\lambda\frac{1}{1+\lambda+(1-\lambda)E}HE\frac{1}{1+\lambda+(1-\lambda)E}$$

Introducing $\mu = 2\,\text{arth}\,\lambda$, $E = e^{\Omega}$ one obtains:

$$\frac{\partial\Omega}{\partial t} = \int_{-\infty}^{+\infty} d\mu\frac{1}{1+e^{\Omega+\mu}}H\frac{1}{1+e^{-\Omega-\mu}}$$

Using here the integral representation

(9.3) $$\frac{1}{1+e^{A}} = \int_{-\infty}^{+\infty}\frac{d\sigma}{2\,\text{ch}\,\pi\sigma}e^{(i\sigma-\frac{1}{2})A}$$

for $A = \Omega + \mu$ and $A = -\Omega - \mu$, and then performing the integration over μ and one integration over σ one obtains:

(9.4) $$\frac{\partial\Omega}{\partial t} = \frac{\pi}{2}\int_{-\infty}^{+\infty}\frac{d\sigma}{\text{ch}^2\,\pi\sigma}e^{(i\sigma-\frac{1}{2})\Omega}He^{-(i\sigma-\frac{1}{2})\Omega}$$
$$= \frac{i\pi}{2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{d\rho}{\sin^2\pi\rho}e^{\rho\Omega}He^{-\rho\Omega} = \left\{\frac{i\pi}{2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}-i\infty}\frac{d\rho e^{\rho\Omega}}{\sin^2\pi\rho}, H\right\}.$$

(*) Of course, (9.1) is equivalent to (7.9a).

It is known, however, that

$$\text{(9.5)} \qquad \frac{i\pi}{2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{d\rho e^{\rho\Omega}}{\sin^2 \pi\rho} = \frac{\Omega}{e^{\Omega}-1}.$$

Thus (9.4) is simply an integral version of (5.6).

10. **Permuting operators**

As has already been shown in [*10*] our results (7.12) and (17.15) can be plausibly interpreted in terms of « F-orderings » which generalize the chronological and anti-chronological ordering operators. In the next sections of this paper we present a more complete and systematic study of this aspect and outline some new results concerning the algebra of orderings.

Preliminary remarks. — Consider the formulæ (7.12) and (7.15). Since all kernels F_n are constant inside sectors, it is natural to decompose the integration domains in (7.12) into single sectors and then by the change of variables to reduce all the integrations to those over the chronological sector. This procedure leads to

$$\text{(10.1)} \quad f(E-1) - f(0) = \sum_{n=1}^{\infty} \int_{t_0}^{t} dt_n \int_{t_0}^{t_n} dt_{n-1} \dots \int_{t_0}^{t_2} dt_1 \sum_{\binom{1,\dots,n}{i_1,\dots,i_n}} F_{(i_1,\dots,i_n)} H(t_{i_n}) \dots H(t_{i_1}),$$

where the last summation runs over all permutations

$$\pi = \begin{pmatrix} 1, & \dots, & n \\ i_1, & \dots, & i_n \end{pmatrix} \in S_n$$

and the number $F_{(i_1,\dots,i_n)}$ stands for the value of the kernel F_n in the sector $t_{i_n} > t_{i_{n-1}} > \dots > t_{i_1}$:

$$\text{(10.2 } a) \qquad F_{(i_1,\dots,i_n)} = \frac{1}{2\pi i}\oint_{z=0} \frac{dz}{z^{n+1}} f(z)(1+z)^{\Theta_n},$$

$$\text{(10.2 } b) \quad \Theta_n = \Theta(i_n, \dots, i_1) = \theta(i_n - i_{n-1}) + \dots + \theta(i_2 - i_1) = \Theta(\pi).$$

It is natural now to introduce some conventions: let $\pi = \begin{pmatrix} 1, & \dots, & n \\ i_1, & \dots, & i_n \end{pmatrix}$ be an element of the permutation group S_n and $x_1, \dots, x_n$ any generators of an arbitraty free algebra (either discrete or continuous); we shall use the symbol $\pi(x_n \ldots x_1)$ to denote the permuted product

$$\pi(x_n \ldots x_1) = x_{i_n} \ldots x_{i_1}.$$

More generally, for

$$P_n = \sum_{\pi \in S_n} \alpha(\pi)\pi$$

($\alpha(\pi) \in \mathbb{C}$) an element of the natural group algebra $\mathscr{P}_n$ of the group S_n, the symbol $P_n(x_n \ldots x_1)$ shall denote:

$$P_n(x_n \ldots x_1) = \sum_{\in S_n} \alpha(\pi)\pi(x_n \ldots x_1). \tag{10.3}$$

Within this convention (7.12) can be represented as:

$$f(E-1) = f(0) + \sum_{n=1}^{\infty} \int_{t_0}^{t} dt_n \int_{t_0}^{t_n} dt_{n-1} \ldots \int_{t_0}^{t_2} dt_1 \; P_n[f]H(t_n) \ldots H(t_1), \tag{10.4}$$

where

$$\begin{aligned} P_n[f] &= \sum_{\binom{1,\ldots,n}{i_1,\ldots,i_n}} F_{(i_1,\ldots,i_n)}\begin{pmatrix} 1, & \ldots, & n \\ i_1, & \ldots, & i_n \end{pmatrix} \\ &= \sum_{\pi \in S_n} \frac{1}{2\pi i} \oint_{z=0} \frac{dz}{z^{n+1}} f(z)(1+z)^{\Theta(\pi)} \pi \in \mathscr{P}_n \end{aligned} \tag{10.5}$$

By comparing (10.4) with the well known series for the evolution operator

$$E = 1 + \sum_{n=1}^{\infty} \int_{t_0}^{t} dt_n \int_{t_0}^{t_n} dt_{n-1} \ldots \int_{t_0}^{t_2} dt_1 H(t_n) \ldots H(t_1) \tag{10.6}$$

one is led to the idea that the series (10.4) can be considered as a result of a certain linear operation applied to E, the operation being determined by the sequence of the elements $P_n[f] \in \mathscr{P}_n$.

Permutors. — In order to formalize this idea let us consider the set $\mathscr{P}$ of all infinite sequences

$$P = [P_n], \qquad \text{where} \qquad P_0 \in \mathbb{C},\ P_n \in \mathscr{P}_n \qquad (n > 0). \tag{10.7}$$

Every sequence of this type will be called a *permutor*. For any two permutors P, $Q \in \mathscr{P}$ the linear combination $\lambda P + \mu Q$ ($\lambda, \mu \in \mathbb{C}$) is defined by $(\lambda P + \mu Q)_n = \lambda P_n + \mu Q_n$ and the product PQ is defined by $(PQ)_n = P_n Q_n$. With these definitions the set $\mathscr{P}$ becomes an associative algebra; we shall call it the *algebra of permutors*. The unity of $\mathscr{P}$ is the sequence $[\mathbf{1}_n]$ where $\mathbf{1}_n = \begin{pmatrix} 1 & \dots & n \\ 1 & \dots & n \end{pmatrix} \in \mathscr{P}_n$. Since each $\mathscr{P}_n$ has the topology of a finite dimensional linear space, a natural topology can be defined in $\mathscr{P}$ as the weakest topology in which all mappings $P \to P_n$ are continuous. With this convention $\mathscr{P}$ becomes a linear topological algebra. We shall further assume that each element $P \in \mathscr{P}$ acts in $\mathscr{A}_{[t_0,t]}$ as a linear operator mapping any series of the form

$$A = A_0 + \int_{t_0}^{t} dt_1 A_1(t_1) H(t_1) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_1 A_2(t_2, t_1) H(t_2) H(t_1) + \dots$$

into

$$\text{(10.8)} \quad PA = P_0 A_0 + \int_{t_0}^{t} dt_1 A_1(t_1) P_1 H(t_1) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_1 A_2(t_2, t_1) P_2 H(t_2) H(t_1) + \dots,$$

where $P_n H(t_n) \dots H(t_1)$ is given by (10.3) with $x_i = H(t_i)$. It is evident that each permutor P is completely defined by specyfying its action on E. Therefore, (10.8) defines an isomorphic representation of permutors by linear operators in $\mathscr{A}_{[t_0,t]}$.

Of course, we are not bound to interpret permutors as linear operators in $\mathscr{A}_{[t_0,t]}$. The concept of permutor contains a « universal » prescription for the transformation of any free algebra. Indeed, given a free algebra $\mathscr{A}$ and $P \in \mathscr{P}$ we associate with P the linear operation in $\mathscr{A}$ mapping any product of generators $x_n \dots x_1$ into $Px_n \dots x_1 = P_n x_n \dots x_1$, where the last expression is given by (10.3). The interesting examples of permutors are the bracketing operation $\{\ \}$ and the Dynkin operator D. Denoting the cyclic permutations by:

$$\text{(10.9)} \qquad C_{n,k} = \begin{pmatrix} 1 & \dots & k-1 & k & k+1 & \dots & n \\ k & 1 & \dots & k-1 & k+1 & \dots & n \end{pmatrix}$$

we hawe

$$\text{(10.10 a)} \qquad \{\ \} = [(\mathbf{1}_n - C_{n,n})(\mathbf{1}_n - C_{n,n-1}) \dots (\mathbf{1}_n - C_{n2})],$$

$$\text{(10.10 b)} \qquad D = \left[\frac{1}{n}(\mathbf{1}_n - C_{n,n})(\mathbf{1}_n - C_{n,n-1}) \dots (\mathbf{1}_n - C_{n2})\right].$$

(The zero-order terms of both sequences vanish; the first-order terms are both equal to $\mathbf{1}_1$ = numerical unity).

Within the formalism of permutors we can rewrite (10.4) more compactly as:

$$\text{(10.11)} \qquad P[f]E = f(E - 1)$$

with $P_0[f] = f(0)$ and $P_n[f]$ defined for $n > 0$ by (10.5).

The dependence of $P[f]$ on f is linear and continuous with respect to the natural topology in the set of all formal power series $f(z)$ of an indeterminete z and with respect to the natural topology of the algebra of permutors. In what follows it will be convenient to denote any permutor P such that $PE = F$ by $P(F)$. Thus: $P[f] = P(f(E - 1))$.

Examples. — The permutor $L = P[\ln(1 + z)] = P(\Omega)$ which generates the BCH exponent ($\Omega = LE$) is determined by:

$$\text{(10.12)} \qquad L_0 = 0,\ L_n = \sum_{\pi \in S_n} \frac{(-1)^{n-1-\Theta(\pi)}}{n} \binom{n-1}{\Theta(\pi)}^{-1} \pi.$$

The permutor Ξ^α such that $\Xi^\alpha E = E^\alpha$ ($\alpha \in \mathbb{C}$) is given by

$$\text{(10.13)} \qquad \Xi_0^\alpha = 1, \qquad \Xi_n^\alpha = \sum_{\pi \in S_n} \binom{\Theta(\pi) + \alpha}{n} \pi.$$

Since

$$E^\alpha = e^{\alpha\Omega} = \sum_{n=0}^{\infty} \alpha^n \frac{\Omega^n}{n!},$$

we have the following development

$$\text{(10.14)} \qquad \Xi^\alpha = P(E^\alpha) = L^0 + \alpha L^1 + \alpha^2 L^2 + \dots,$$

where $L^k = P\left(\frac{\Omega^k}{k!}\right)$ are permutors which generate the powers of the BCH exponent according to $L^k E = \frac{\Omega^k}{k!}$. In particular $L_0^0 = 1$, $L_n^0 = 0$ for $n > 0$ and $L^1 = L$. A convenient expression for L_n^k follows from the identity $L^k = \frac{1}{k!}\left(\frac{d}{d\alpha}\right) \Xi^\alpha \Big|_{\alpha=0}$:

$$\text{(10.15)} \qquad L_n^k = \frac{1}{k!} \sum_{\pi \in S_n} \left(\frac{d}{d\alpha}\right)^k \binom{\Theta(\pi) + \alpha}{n} \Bigg|_{\alpha=0} \pi.$$

Because $\binom{\Theta(\pi) + \alpha}{n}$ is a polynomial of the n-th order in α, therefore $L_0^k = L_1^k = \ldots = L_{m-1}^k = 0$, which embodies the fact that the series for Ω^k starts from the k-th order in H. The first non-vanishing term of L^k is $L_k^k = \frac{1}{k!} \sum_{\pi \in S_k} \pi$ (the « symmetrisation » operation in $\mathscr{P}_k$). The other non-trivial $L_k^{k'}s$ can be worked out explicitly from (10.15) in terms of the Stirling numbers (*).

The technique of permutors forms an essential step in the theory of analytic functions of $T = E - 1$. The formula $P[f]E = f(E - 1)$ gives a detailed insight into the algebraic structure of functions of the evolution operator, and moreover, it stimulates some obvious associations with the familar ordering operators of quantum theories, such as the chronological or normal orderings [*10*]. This last aspect deserves a special discussion.

11. **Ordering operators**

The general concept of an « ordering operation » although frequently applied in quantum theories has no been formally defined. Intuitively, however, the concept seems to be clear enough. First, it is natural to assume that an ordering operator should be a linear operator in a free algebra which maps each homogeneous element into an homogeneous element of the same order. Next, the ordering operator ought to « force » the elements of the free algebra to acquire a certain particular structure (the « ordering »). Once this structure is introduced, it should be immaterial whether it is introduced again or not: thus, the ordering operator should be idempotent. In what follows these two conditions will be assumed as the definition of an ordering operator. In that sense, the Dynkin operator D is an ordering operator which « forces » the elements of a free algebra to acquire the Lie element structure. Similarly, the symmetrisation operator $S = \left[\frac{1}{n!} \sum_{\pi \in S_n} \pi\right] \in \mathscr{P}$ is idempotent ($S^2 = S$) and therefore is an ordering operator. It is much less trivial that the permutors $L^k = P\left(\frac{\Omega^k}{k!}\right)$ act also as ordering operators.

(*) The definition and basic properties of these numbers are found in Milne-Thomson: Calculus of Finite Differences. St. Martin's Press, New York, 1960.

To show this, we shall first demonstrate the following multiplication law for the permutors Ξ^α:

$$\Xi^\alpha \Xi^\beta = \Xi^{\alpha\beta}. \tag{11.1}$$

The origin of this law lies in a certain universality property of the permutors P[f] understood as linear operators in $\mathscr{A}_{[t_0,t]}$: each P[f] gives a « universal » prescription for constructing $f(\mathrm{E} - 1)$ out of E, valid for all series $\mathrm{E}(t, t_0)$ independently of t and t_0. Since $\mathrm{E}(t', t)\mathrm{E}(t, t_0) = \mathrm{E}(t', t_0)$, it follows that

$$\mathrm{P}[f]\mathrm{E}(t', t)\mathrm{E}(t, t_0) = f(\mathrm{E}(t', t)\mathrm{E}(t, t_0) - 1). \tag{11.2}$$

By selecting now $t' = t + (t - t_0)$ and specializing $\mathrm{H}(\tau)$ in the interval $[t, t']$ to be the « replica » of $\mathrm{H}(\tau)$ in $[t_0, t]$, i. e., $\mathrm{H}(\tau) = \mathrm{H}(\tau + t - t_0)$ we get $\mathrm{E}(t', t) = \mathrm{E}(t, t_0)$ and therefore (*) (11.2) implies:

$$\mathrm{P}[f]\mathrm{E}^2 = f(\mathrm{E}^2 - 1). \tag{11.3}$$

Similarly one shows that

$$\mathrm{P}[f]\mathrm{E}^n = f(\mathrm{E}^n - 1) \tag{11.4}$$

By specializing to $f(z) = (1 + z)^\alpha$, $\alpha \in \mathbb{C}$, we have

$$\Xi^\alpha \mathrm{E}^n = (\mathrm{E}^n)^\alpha = (\mathrm{E}^\alpha)^n, \tag{11.5}$$

so that

$$\Xi^\alpha g(\mathrm{E}) = g(\mathrm{E}^\alpha) = g(\Xi^\alpha \mathrm{E}) \tag{11.6}$$

where $g(\mathrm{E})$ is any series of the form $g_0 + g_1(\mathrm{E} - 1) + g_2(\mathrm{E} - 1)^2 + \ldots$; $g_k \in \mathbb{C}$.

In particular

$$\Xi^\alpha \Xi^\beta \mathrm{E} = \Xi^\alpha \mathrm{E}^\beta = (\mathrm{E}^\alpha)^\beta = \mathrm{E}^{\alpha\beta} = \Xi^{\alpha\beta}\mathrm{E}, \tag{11.7}$$

which proves (11.1).

(*) We presented the argument in the text for its intuitive clarity. More formalised reasoning can employ the existence of the homomorphism h:

$$\mathscr{A}_{[t_0,t']} \to \mathscr{A}_{[t_0,t]} \ (t' = t + (t - t_0))$$

defined by: $h\mathrm{H}(\tau) = \mathrm{H}(\tau)$ and $h\mathrm{H}(\tau + t - t_0) = \mathrm{H}(\tau)$ for $\tau \in [t_0, t]$. The basic property of this homomorphism is: $h\mathrm{E}(t', t_0) = \mathrm{E}^2(t, t_0)$. As an homomorphism, h can be introduced under the sign of any analytic series: $hf(\mathrm{A}) = f(h\mathrm{A})$ where $\mathrm{A} \in \mathscr{A}_{[t_0,t']}$ with $\mathrm{A}_0 = 0$. Furthermore, since the permutation of the generators of a free algebra and the identification of some of them are commuting operations, therefore h commutes with any permutor $\mathrm{P} \in \mathscr{P}$ understood as a linear operator in the free algebra. Consequently:

$$\mathrm{P}[f]\mathrm{E}^2(t, t_0) = \mathrm{P}[f]h\mathrm{E}(t', t_0) = hf(\mathrm{E}(t', t_0) - 1) = f(h\mathrm{E}(t', t_0) - 1) = f(\mathrm{E}^2(t, t_0) - 1).$$

The law (11.1) has important consequences concerning the permutors L^k. By substituting in (11.1) the developments (10.14) and by comparing the terms of the same order in α on the two sides one obtains

$$(11.8) \qquad L^k L^l = \delta_{kl} L^k.$$

By setting $k = l$ we see that the permutors L^k are all idempotent; thus they define ordering operations. Since $L^k L^l = 0$ for $k \neq l$, we shall say that the orderings L^k's are « mutually exclusive » (*). Now, the summation rule

$$(11.9) \quad \sum_{k=0}^{\infty} L^k = \sum_{k=0}^{\infty} P\left(\frac{\Omega^k}{k!}\right) = P(e^{\Omega}) = P(E) = \text{unity of } P,$$

implies that the sub-spaces « ordered » according to the distinct operators L^k span the entire free algebra.

Il should be observed that the equations (11.1) and (11.8) although easily derived from our general arguments are equivalent to collections of nontrivial combinatorial identities. E. g., (11.1) is the abbreviated form of the sequence of identifies $\Xi_n^\alpha \Xi_n^\beta = \Xi_n^{\alpha\beta}$, which after multiplying the expressions of form (10.13) as the elements of the natural group algebra $\mathscr{P}_n$ lead to the numerical equalities:

$$(11.10) \qquad \sum_{\pi \in S_n} \binom{\Theta(\pi\pi'^{-1}) + \alpha}{n} \binom{\Theta(\pi') + \beta}{n} = \binom{\Theta(\pi) + \alpha\beta}{n}.$$

Using the definition of $\Theta(\pi)$, (10.2 *b*), one can easily verify (11.10) for $n = 0, 1, 2, 3$. However, for higher n's the straightforward proof of (11.10) would be troublesome.

As a particular consequence of (11.8) the permutor $L = L^1$ which generates the BCH exponent is idempotent and therefore represents an ordering operation. This fact is important in the understanding of the algebraic structure of formula (7.18). By the application of $D\Omega = \Omega$ we replaced (7.18) by the equivalent (7.20), the last expression manifestly constructed from multicommutators. Our present considerations, however, imply that the last transition is, in fact, unnecessary. We have shown that L as well as D is an ordering operator producing Lie elements, and therefore, it is reasonable to consider the ordered products

$$LH(t_n) \dots H(t_1)$$

instead of $\{ H(t_n) \dots H(t_1) \}$ as the most natural base for the Lie elements constituting the continuous BCH exponent.

(*) According to this terminology, the symmetrization permutor S and Dynkin's permutor D also induce two « mutually excluding » types of ordering. In fact, one has: $DS = SD = 0$.

It is worthwhile to acknowledge some simple interrelations among the permutors D and L. It can be shown that L maps any free algebra onto the sub-space of all Lie elements. Thus, D and L act as two idempotent operators with the common range, and hence:

$$\text{(11.11)} \qquad \begin{aligned} \mathrm{DL} &= \mathrm{L} \\ \mathrm{LD} &= \mathrm{D} \ (*). \end{aligned}$$

We sould like to observe that our solution for the continous BCH exponent contains as a special case the solution of the classical « discrete » BCH problem. Indeed, specializing $\mathrm{H}(t)$ to the form $\mathrm{H}(t) = \theta(t)x + \theta(-t)y$ with x, y being two generators of a « discrete » free algebra, one easily sees from (5.1 *a*) that because of (5.2 *a*), $\mathrm{E}(1, -1) = e^x e^y = e^{x \# y}$ and therefore (7.20) implies

$$\text{(11.12)} \quad x \# y = \Omega(1, -1) = \sum_{n=1}^{\infty} \int_{-1}^{1} \cdots \int_{-1} dt_n \ldots dt_1 \frac{(-1)^{n-1-\Theta_n}}{n^2} \binom{n-1}{\Theta_n}^{-1} \{ \mathrm{H}(t_n) \ldots \mathrm{H}(t_1) \},$$

where $\mathrm{H}(t) = \theta(t)x + \theta(-t)y$.

Of course, this expression is equivalent to the permutor L acting on the actual « evolution operator »:

$$x \# y = \mathrm{L}(e^x e^y) = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \mathrm{L}(x^n y^m),$$

where the action of L in the discrete free algebra yields:

$$\text{(11.13)} \qquad \mathrm{L}(x^n y^m) = \sum_{\pi \in \mathrm{S}_n} \frac{(-1)^{n-1-\Theta(\pi)}}{n} \binom{n-1}{\Theta(\pi)}^{-1} \pi(\underbrace{x \ldots x}_{n} \underbrace{y \ldots y}_{m})$$

F-ordering algebra. — The formulæ (11.1) and (11.8) determine the general multiplication law for the permutors of the form $\mathrm{P}[f]$. In what follows the set of these permutors will be denoted by $\mathcal{O}$. Any element $\mathrm{P}[f] = \mathrm{P}(f(\mathrm{E}-1)) \in \mathcal{O}$ will be alternatively denoted by $\mathrm{P}(\mathscr{F}(\Omega))$ or simply $\mathrm{P}(\mathscr{F})$ where $\mathscr{F}$ is the power series:

$$\mathscr{F} \doteq \mathscr{F}(z) \overset{\mathrm{df}}{=} f(e^z - 1) = \sum_{n=0}^{\infty} \mathscr{F}_n z^n.$$

(*) By working directly with permutations one can easily verify $\mathrm{L}_n \mathrm{D}_n = \mathrm{D}_n$ for low n's. A strictly combinatoric proof of LD = D appears to be nontrivial. The identity DL = L is obvious from the fact that LE is a Lie element, however a combinatorial proof of DL = D is also nontrivial.

Because any permutor $P(\mathscr{F}) \in \mathscr{O}$ can be represented as a linear combination of L^{k}'s:

$$(11.14) \qquad P(\mathscr{F}) = P(\mathscr{F}(\Omega)) = \sum_{n=0}^{\infty} \mathscr{F}_n P(\Omega^n) = \sum_{n=0}^{\infty} n!\mathscr{F}_n L^n,$$

therefore the multiplication rules (11.8) imply that the product of any two permutors $P(\mathscr{F})$, $P(\mathscr{G}) \in \mathscr{O}$ is again an element of $\mathscr{O}$: $P(\mathscr{F})\,P(\mathscr{G}) = P(\mathscr{K})$, where the series $\mathscr{K}(z)$ is given by: $\mathscr{K}_n = n!\mathscr{F}_n\mathscr{G}_n$. Thus, denoting $\mathscr{K} = \mathscr{F} * \mathscr{G}$ we obtain the following multiplication law:

$$(11.15) \qquad P(\mathscr{F})P(\mathscr{G}) = P(\mathscr{F} * \mathscr{G})$$

where $*$ can be defined as the unique associative and distributive operation in the set of the formal series of the free variable z such that

$$\frac{z^n}{n!} * \frac{z^m}{m!} = \delta_{nm}\frac{z^n}{n!}.$$

A closed form expression for $\mathscr{F} * \mathscr{G}$ can be symbolically written as

$$(11.16) \qquad \mathscr{F} * \mathscr{G} = \left[\mathscr{F}(\zeta)e^{\frac{\overleftarrow{d}}{d\zeta} z \frac{\overrightarrow{d}}{d\zeta}}\mathscr{G}(\zeta)\right]_{\zeta=0}$$

Since $\mathscr{F} * \mathscr{G} = \mathscr{G} * \mathscr{F}$ the set $\mathscr{O}$ is an abelian sub-algebra of the algebra of permutors $\mathscr{P}$. We will call it the *F-ordering algebra* (*). The minimal idempotents of this algebra are the ordering operators L^k and the spectral decomposition of any element $P(\mathscr{F}) \in \mathscr{O}$ into minimal idempotents is given by (11.14). The equivalent decomposition of $P[f]$ is given by:

$$(11.17) \qquad P[f] = \sum_{n=0}^{\infty}\left(\frac{n!}{2\pi i}\right)\oint_{z=0} \frac{dz}{z^{n+1}} f(e^z - 1)L^n.$$

These results provide a complete insight into the mechanism employed to create arbitrary functions of the evolution operator. This can be of significance in quantum field theories. Up to now, these theories have worked mainly with the perturbative expansion of the evolution operator (and its limit, the S-matrix). However, in some cases the perturbative expansions of some other operators which are functions of $E - 1$ (as, e. g. the Cayley operator C or the phase operator Ω) can be of interest. Our

(*) $\mathscr{C}$ is a relatively narrow sub-set of $\mathscr{P}$; e. g., the Dynkin operator D does not belong to $\mathscr{C}$ because it does not commute with L.

analysis of $\mathcal{O}$ shows that the structure of these expansions is not unduly involved: all the functions of $T = E - 1$ can be constructed with the help of the fixed sequence of ordering operators $L^0, L^1, L^2, \ldots$, which, therefore, provide a natural extension of the familly of traditional ordering operators of quantum field theory.

Acknowledgments. — The authors are indebted to Dr S. Woronowicz and Mr. A. Sym for the helpful and interesting discussions. One of us (J. P.) acknowledges a stimulating visit to Nordita, Copenhagen, in 1968, where a part of this work was carried out.

REFERENCES

[*1*] J. E. Campbell, *Proc. London Math. Soc.*, **29**, 1898, 14.
[*2*] H. F. Baker, *Proc. London Math. Soc.*, **34**, 1902, 347 ; **35**, 1903, 333 ; **3**, 1904, 24.
[*3*] F. Hausdorff, *Ber. Verhandl. Saechs. Akad. Wiss. Leipzig, Math. Naturwis. Kl.*, **58**, 1906, 19.
[*4*] E. B. Dynkin, *Doklady Akad. Nauk USSR* (N. S.), **57**, 1947, 323.
[*5*] K. O. Friedrichs, *Commun. Pure Appl. Math.*, **6**, 1953, 1.
[*6*] W. Magnus, *Commun. Pure Appl. Math.*, **7**, 1954, 649.
[*7*] L. Infeld and J. Plebański, *Acta Phys. Polon.*, **14**, 1955, 41.
[*8*] J. Plebanski, On the generators of unitary and pseudo-orthogonal groups, report CIEA, Mexico, 1966.
[*9*] R. M. Wilcox, *J. Math. Phys.*, **8**, 1967, 962.
[*10*] I. Bialynicki-Birula, B. Mielnik and J. Plebański, *Ann. of Phys.*, **51**, 1969, 187.

Manuscrit reçu le 13 octobre 1969.