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Degenerate Lagrangean systems (*)

by

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ABSTRACT. — The algorithm by which Lagrangeans whose Hessian has constant but not maximal rank can define consistent equations of motion for a system of finitely many degrees of freedom is studied in a modern geometrical framework. It is shown how such a Lagrangean may (but need not) lead to a (locally) well defined manifold $\tilde{M}$ of states of the classical system, the real functions on which form the algebra of observables. If this is the case then the Lagrangean also induces canonically a symplectic structure on $\tilde{M}$ and thus a Poisson bracket on the set observables. This is proved in the case where no constraints of higher than second order appear.

RÉSUMÉ. — L'algorithme, par lequel le lagrangien, dont le hessien a un rang constant mais pas maximal, peut définir des équations de mouvement consistantes pour un système avec un nombre fini de degrés de liberté, est étudié dans un cadre moderne géométrique. On montre comment un tel lagrangien peut (mais pas nécessairement) déterminer (localement) une variété $\tilde{M}$ des états du système classique, dont les fonctions réelles forment l'algèbre des observables. En ce cas le lagrangien induit aussi une structure symplectique canonique sur $\tilde{M}$ et ainsi des crochets de Poisson sur l'ensemble des observables. Ceci est démontré dans le cas ou n'apparaissent pas des contraintes d'un ordre plus que deux.

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1. INTRODUCTION

In classical physics the set of (pure) states of a system with n degrees of freedom is regarded as an n-dimensional differential manifold $\tilde{M}$. If the state of such a system is known at one time t_0 it can be computed for any other time t by means of a given set of equations of motion. The real (C^∞ –) functions on $\tilde{M}$ or dynamical variables, which form a real algebra $\tilde{\mathcal{F}}$, can then be called the « observables » of the classical system, any complete set of which, i. e., any n independent functions on $\tilde{M}$, describes the system completely. (Cf. Souriau [13], where $\tilde{M}$ is called spece of motions).

Although this is unnecessary from the classical point of view one desires, as a first step towards quantization, to give the set $\tilde{\mathcal{F}}$ of observables a Lie algebra structure by defining a Poisson bracket on it. This, however, is possible in a natural way whenever $\tilde{M}$ is a symplectic manifold (see, for example, Abraham [1] and, for more general cases, Hermann [7]). Of course, the quantization of the classical system is by no means completed if such a canonical formalism is set up, but, hopefully, it will become possible to give a straightforward and reasonably unique prescription for constructing irreducible Hilbert space representations of the Lie algebra $\tilde{\mathcal{F}}$, perhaps in the spirit of the direct method of Souriau [13] or by first selecting a certain physically relevant subalgebra of $\tilde{\mathcal{F}}$ and then finding its irreducible representations (cf. Hermann [6] and [7]).

The knowledge of the classical system alone, that is of the state space $\tilde{M}$ and the equations of motion, does not give any clue as to what symplectic form $\tilde{\omega}$ should be given on $\tilde{M}$, except in cases of extreme symmetry if $\tilde{\omega}$ is to be invariant under all symmetries of the system. In fact, the only known way of restricting the arbitrariness of $\tilde{\omega}$ is to require that the equations of motion as well as the canonical formalism (i. e. $\tilde{\omega}$) follow from a variational principle applied to a Lagrangean. This assumption excludes certain physical systems like non-conservative ones and those with non-holonomous constraints (cf., for example, Havas [4]) without, on the other hand, fixing $\tilde{\omega}$ unambiguously because there may exist different Lagrangeans which lead to the same equations of motion but different symplectic forms (cf. Currie and Saletan [2]).

In spite of these drawbacks Hamilton's principle for a Lagrangean seems to remain the most direct method to introduce a canonical formalism, a fact which may justify further study also of the so-called degenerate Lagrangean systems where the Hessian of L is not a regular matrix.

Although no systems with finitely many degrees of freedom and any physical importance seem to be known a full understanding of this situation is no doubt necessary for a rigorous discussion of the canonical formalism in field theories where degeneracy of the Lagrangean is the rule.

After Dirac [3] initiated a general study of these systems they have been treated by various people in different forms. (For a historic survey see Shanmugadhasan [12]). The theory, however, still has unsatisfactory aspects, mathematically as well as physically. As Dirac [3] already noticed degenerate Lagrangeans used in Hamilton's principle may or may not lead to proper equations of motion. To arrive at a reasonable rigorous theory that retains as much generality as possible it must therefore be postulated that such equations of motion are well defined by the given Lagrangean. Mathematically, the theory is not very attractive because various forms and matrices defined by means of the Lagrangean must necessarily change their rank on the manifold in order to lead to consistent equations of motion. On the other hand, some regularity conditions are necessary to make the formalism work, but their choice seems to remain rather arbitrary.

In this paper the question is studied whether and how a degenerate Lagrangean, provided it defines proper equations of motion, also induces a symplectic structure on the state space $\tilde{M}$ and thus defines a Poisson bracket on the set of actual observables. This question appears not to have been settled in other publications, because of the difficulty to agree on what functions on the phase space (= cotangent bundle of the configuration space) should be called observables. (Cf. Kundt [9], also for further references, and the end of section 2*a* of this paper). The definition adopted here that an observable is a real function on the state space, defined invariantly as the set of all physical motions (cf. [13]), seems the simplest and most natural, especially from the point of view of later quantization. The treatment is geometrical and, since carried out in the homogeneous formalism which treats the time on an equal basis with the space coordinates, applicable also to relativistic systems. One would hope therefore that it should illuminate at least some of the simpler difficulties encountered in the development of a rigorous invariant canonical formalism for degenerate Lagrangean field theories.

Unfortunately a complete proof of the conjecture that every degenerate Lagrangean which defines a proper physical motion also leads to a symplectic structure on M has not been achieved. The statement is proved only in the case the Lagrangean leads to constraints of at most second order but, very likely, it holds in general. Moreover, all considerations in this paper are purely local. Global existence of the constraint submanifolds would be difficult to prove under such general assumptions though the topology of $\tilde{M}$ is probably essential for the further steps in the quantization program. (Cf. Kundt [9] and Souriau [13]).

The next section reviews the homogeneous formalism for the non-degenerate case in the modern geometrical language of Abraham [1] and Souriau [14]. In section 3 the constraint submanifolds of a degenerate system are explicitly constructed while section 4 deals with the existence of a symplectic structure on the thus obtained physical state space.

2. HOMOGENEOUS SECOND ORDER EQUATIONS AND LAGRANGEANS

a) Second order equations.

In many simple cases the set of states of a physical system can be regarded as the tangent bundle TQ of an n-dimensional configuration manifold Q (with local coordinates $q^1, \ldots, q^n$, say) and the motion as a curve $\alpha: \mathbb{R} \to Q$ such that its lift $\alpha': \mathbb{R} \to TQ$ is an integral curve of a certain given vector field $X: TQ \to TTQ$. That is

$$\alpha'' = X \circ \alpha' \tag{2.1}$$

where α'' denotes the lift of α' onto the tangent bundle TTQ of TQ and X is a second order equation, i. e., satisfies

$$T\tau_Q \circ X = id_{TQ}. \tag{2.2}$$

(cf. Abraham [1] or Lang [10], T denotes the tangent functor, τ_Q the projection map of the tangent bundle TQ). In terms of a bundle coordinate system $\{(q^k, v^k), k = 1, \ldots, n\}$ of TQ condition (2.2) means that X has the form $X = v^k \dfrac{\partial}{\partial q^k} + \xi^k(q, v) \dfrac{\partial}{\partial v^k}$. The curve parameter t is here regarded as the physical time. The set $\tilde{M}$ of all possible physical motions or states is then the set of all curves in TQ tangent to the given vectorfield X, and since there is exactly one curve starting at anyone point of TQ the set $\tilde{M}$ can be identified with TQ.

In order to treat systems where X is time dependent and relativistic ones (and even general relativistic systems where no global time need be defined at all) in the same framework we will instead use the homogeneous formalism and treat the time as one of the coordinates of the configuration space Q. The latter is then an $(n+1)$-manifold with local coordinates $\{q^\alpha, \alpha = 0, \dots, n\}$ and a physical motion becomes a curve in Q regarded as a point set, i. e. an equivalence class (under parameter transformations $s \to \hat{s} = \hat{s}(s)$) of maps $\alpha: \mathbb{R} \to Q$. Each element α of such an equivalence class satisfies again (2.1) for a vector field X on TQ. But now X is not the same for differently parametrized representatives of the motion. The latter is therefore no longer infinitesimally described by a given vectorfield but rather by a twodimensional differential system $\mathscr{E}$ on TQ.

It is not difficult to see that $\mathscr{E}$ consists of all vectorfields of the form

$$a(v + \dot{\xi}) + b\dot{v} \tag{2.3}$$

where

$$v = v^\alpha \partial_\alpha = v^\alpha \frac{\partial}{\partial q^\alpha}, \qquad \dot{v} = v^\alpha \partial_{\dot{\alpha}} = v^\alpha \frac{\partial}{\partial v^\alpha}, \qquad \dot{\xi} = \xi^\alpha(q, v)\partial_{\dot{\alpha}}$$

a certain fixed vertical vectorfield, and a and b arbitrary functions on TQ. A twodimensional differential system $\mathscr{E}$ of this form will often be referred to simply as a second order equation since it is the natural analogon of the latter in the homogeneous case. As is straightforward to check, $\mathscr{E}$ is involutive and therefore defines a foliation on TQ (cf., for example, Palais [*11*] and Hermann [*5*] or also Hinds [*8*]). If this foliation satisfies certain global conditions (as stated in [*11*], chapter 1) the set of leaves of $\mathscr{E}$, $\tilde{M} = TQ/\mathscr{E}$, carries a natural manifold structure. Moreover, there is a canonical projection map $\pi: TQ \to \tilde{M}$ whose tangent π_* annihilates precisely the subspace $\mathscr{E}_x$ of the tangent space of TQ at each $x \in TQ$, i. e., $\mathscr{E} = \ker \pi_*$. $\tilde{M}$ and π with these properties always exist locally. Though global existence is by no means guaranteed in general cases we will assume it for notational convenience.

In the following $\mathscr{E}$ will always denote a twodimensional differential system of the form (2.3) with respect to a bundle coordinate system of TQ, but will sometimes only be defined on a certain submanifold M of TQ. After Souriau [*14*] M could be called the *evolution space*, while $\tilde{M} = M/\mathscr{E}$ is nothing but the set of possible physical motions or states and hence called motion space or *state space*.

Any presymplectic form ω on M (i. e. a closed 2-form with constant rank) now defines another involutive differential system $\ker \omega$, by

$$\ker_x \omega = \{ X \in M_x / X \lrcorner\, \omega = 0 \} \ (^1)$$

(1) Differential geometric symbols and conventions are those of STERNBERG [*15*].

and the corresponding quotient manifold $\overline{\mathrm{M}} = \mathrm{M}/\ker \omega$ is equipped with a natural symplectic form $\overline{\omega}$ defined by

$$\langle \overline{\mathrm{X}} \wedge \overline{\mathrm{Y}}/\overline{\omega} \rangle |_{\overline{x}} = \langle \mathrm{X} \wedge \mathrm{Y}/\omega \rangle |_{x} \qquad \text{for all} \qquad \overline{\mathrm{X}}, \overline{\mathrm{Y}} \in \overline{\mathrm{M}}_{\overline{x}} \tag{2.4}$$

where x is any point in $\pi^{-1}(\overline{x})$ and $\mathrm{X}, \mathrm{Y} \in \mathrm{M}_x$ any two vectors satisfying $\pi_* \mathrm{X} = \overline{\mathrm{X}}$ and $\pi_* \mathrm{Y} = \overline{\mathrm{Y}}$. (See, eg, Hermann [7]). It follows immediately that $\omega = \pi^* \overline{\omega}$.

Thus, if the evolution space $(\mathrm{M}, \mathscr{E})$ is given a presymplectic structure such that $\mathscr{E} = \ker \omega$ then the state space $\tilde{\mathrm{M}}$ is symplectic and a Poisson bracket on the set $\tilde{\mathscr{F}}$ of dynamical variables can be introduced in the usual way. This is the situation for the ordinary non-degenerate Lagrangean case as reviewed in the next subsection.

If $\mathscr{E} \subset \ker \omega$ is a non-trivial subsystem the relation (2.4) (with bars replaced by tildes) still defines uniquely a closed 2-form $\tilde{\omega}$ on $\tilde{\mathrm{M}} = \mathrm{M}/\mathscr{E}$ which is however only presymplectic. Hermann [7] has shown recently how in this case a Poisson bracket can be defined not on the whole set of dynamical variables but on the subset $\tilde{\mathscr{F}}_{\tilde{\omega}}$ of functions on $\tilde{\mathrm{M}}$ that are invariant under $\ker \tilde{\omega}$, i. e. $\tilde{\mathscr{F}}_{\tilde{\omega}} = f \in \tilde{\mathscr{F}}/\mathrm{X}(f) = 0$ for all $\mathrm{X} \in \ker \tilde{\omega} \}$.

Remark. — It seems likely that $\tilde{\mathscr{F}}_{\tilde{\omega}}$ coincides with what other authors define as the set of observables (cf. Kundt [9] for references). The remaining functions on $\tilde{\mathrm{M}}$ are then called gauge variables.

b) Homogeneous Lagrangeans.

Let $\mathrm{L} : \mathrm{TQ} \to \mathbb{R}$ be a Lagrangean, homogeneous of the first degree in the velocities, i. e., satisfying $\mathrm{L}(q, \lambda v) = \lambda \mathrm{L}(q, v)$ for all $0 \neq \lambda \in \mathbb{R}$ or, equivalently,

$$\mathrm{L} = v^\alpha \partial_{\dot{\alpha}} \mathrm{L}. \tag{2.5}$$

The Lagrange equations,

$$v^\alpha = \frac{dq^\alpha}{ds}, \qquad \frac{d}{ds}\left(\frac{\partial \mathrm{L}}{\partial v^\alpha}\right) = \frac{\partial \mathrm{L}}{\partial q^\alpha}, \tag{2.6}$$

which determine the physical motion $\alpha' : s \to (q(s), v(s)) \in \mathrm{TQ}$ can be restated in terms of the tangent vector $\alpha''(s)$ of $\alpha'(s)$ and the presymplectic form

$$\omega_{\mathrm{L}} = \frac{1}{2} \omega_{\alpha\beta} dq^\alpha \wedge dq^\beta + \sigma_{\alpha\beta} dq^\alpha \wedge dv^\beta$$

where

$$\omega_{\alpha\beta} = \partial_{\dot{\alpha}\beta} \mathrm{L} - \partial_{\alpha\dot{\beta}} \mathrm{L} \qquad \text{and} \qquad \sigma_{\alpha\beta} = \partial_{\dot{\alpha}\dot{\beta}} \mathrm{L}$$

(ω_L is obtained invariantly as pullback of the canonical symplectic form $\omega_0 = dq^\alpha \wedge dp_\alpha$ on the cotangent bundle T*Q by means of the fibre derivative $\mathscr{L}$: TQ $\to$ T*Q: $(q^\alpha, v^\alpha) \to (q^\alpha, p_\alpha = \partial_{\dot\alpha}L)$, cf. Abraham [1] or Sternberg [15]). Equation (2.6) is equivalent to

$$v^\alpha = \frac{dq^\alpha}{ds}, \qquad \frac{dv^\alpha}{ds}\sigma_{\alpha\beta} = \partial_\beta L - v^\alpha \partial_{\alpha\dot\beta} L = v^\alpha \omega_{\alpha\beta} \tag{2.7}$$

On the other hand, a general vector field $X = \eta^\alpha \partial_\alpha + \xi^\alpha \partial_{\dot\alpha}$ on TQ lies in ker ω_L if and only if $X \lrcorner\, \omega_L = 0$ or, explicitly,

$$\eta^\alpha \sigma_{\alpha\beta} = 0 \tag{2.8}$$

and

$$\xi^\alpha \sigma_{\alpha\beta} = \eta^\alpha \omega_{\alpha\beta}. \tag{2.9}$$

If, in particular, X has the form (2.3) these equations become

$$v^\alpha \sigma_{\alpha\beta} = 0 \qquad \text{and} \qquad a\xi^\alpha \sigma_{\alpha\beta} + bv^\alpha \sigma_{\alpha\beta} = av^\alpha \omega_{\alpha\beta}. \tag{2.10}$$

Due to the homogeneity of L the first of (2.10) is trivially satisfied and the second equivalent to (2.7) for $\xi^\alpha = dv^\alpha/ds$. This proves that for any second order equation $\mathscr{E}$ on TQ all integral curves of $\mathscr{E}$ satisfy Lagrange's equations if and only if

$$\mathscr{E} \subset \ker \omega_L. \tag{2.11}$$

Note that this result is independent of any further assumptions on L apart from the homogeneity in v. Moreover, it also holds if $\mathscr{E}$ is not defined on the whole of TQ but only on a submanifold.

In the ordinary case, however, the Hessian $\sigma_{\alpha\beta} = \partial_{\dot\alpha\dot\beta}L$ of the Lagrangean has rank n, then (2.8) implies $\eta^\alpha = av^\alpha$ for an arbitrary function a on TQ and the general solution of (2.9) is $\xi^\alpha = a\overset{*}{\xi}{}^\alpha + bv^\alpha$ for an arbitrary function b and any fixed particular solution $\overset{*}{\xi}{}^\alpha$. Therefore ker ω_L is a twodimensional differential system of the form (2.3), i. e., a second order equation, and the integral curves of ker ω_L are precisely those satisfying Lagrange's equations, that is the physical motions. Thus, $\mathscr{E} = \ker \omega_L$ in this case and the state space $\tilde{M} = TQ/\mathscr{E}$ is symplectic.

c) Symmetries.

To end this preliminary section a few remarks about symmetries are added, mainly because they shed some light on the relation between evolution and state space and the usual phase space formulation. Confining the discussion to the case where $\mathscr{E} = \ker \omega$ on the evolution space M we define with Souriau [14] a dynamical group of M as a Lie group G whose action on M leaves ω invariant, i. e., ψ: $G \times M \to M$: $(a, x) \to \psi_a x$ with

$$\psi_a^* \omega = \omega \qquad \text{for all} \qquad a \in G. \tag{2.12}$$

The infinitesimal form of (2.12) is $\mathfrak{L}_{\bar{A}} \omega = 0$ for all $A \in \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of G and $\bar{A}$ the Killing vector field to A. It follows that ψ leaves ker ω invariant and hence induces an action ψ on the quotient manifold $\tilde{M} = M/\mathscr{E}$ which again leaves the symplectic structure invariant. Moreover, the Killing vectorfields $\bar{A}$ and $\tilde{A}$ on M and $\tilde{M}$ respectively are π-related, i. e. $\tilde{A} = \pi_* \bar{A}$.

Suppose that for example time translations form a dynamical group (this is the case when L does not explicitly depend on t), then $\mathfrak{L}_{\partial_t} \omega = 0$ and hence

$$\mathfrak{L}_{\tilde{X}} \tilde{\omega} = 0 \qquad \text{where} \qquad \tilde{X} = -\pi_* \partial_t. \tag{2.13}$$

Equation (2.13), however, is equivalent to $d(\tilde{X} \lrcorner \tilde{\omega}) = 0$ and therefore to the local existence of a function $\tilde{H}$ on $\tilde{M}$ satisfying $\tilde{X} \lrcorner \tilde{\omega} = d\tilde{H}$, called a Hamiltonian. One can now again consider the time development of the given physical state, i. e. a point in $\tilde{M}$, as an integral curve of the vector field $\tilde{X}$ and thus recovers the usual formalism on the phase space. Moreover, the connection between the Hamiltonian $\tilde{H}$ and time translations is here rather evident.

3. CONSTRAINT ALGORITHM FOR DEGENERATE LAGRANGEAN SYSTEMS

a) First order constraints.

Let L: TQ $\to \mathbb{R}$ be again a Lagrangean, homogeneous of first degree in v, but assume now that the rank r of the Hessian $\sigma_{\alpha\beta} = \partial_{\dot{\alpha}\dot{\beta}} L$ is constant in the open set of $TQ \equiv M^0$ under consideration, but smaller than $n = \dim Q - 1$. (All following considerations will only apply to such an open set of M^0 though this will no longer be mentioned.) A physical motion of such a degenerate system is still given by an equivalence class of curves α: $\mathbb{R} \to Q$ whose lifts α' into TQ satisfy Lagrange's equations or, according to section 2*b*, are integral curves of a differential system $\mathscr{E}$ which has the form (2.3) and satisfies also (2.11).

In this degenerate case, however, these two conditions are no longer in general compatible. For, assume that the vectors $\dot{w}_A = w_A^\alpha \partial_{\dot{\alpha}}$ for $A \in I = \{1, 2, \ldots, R = n - r\}$ span together with $\dot{v}$ the kernel of the symmetric bilinear form $\sigma = \sigma_{\alpha\beta} dv^\alpha \otimes dv^\beta$ on the tangent space V_x to the fibre at each point $x \in TQ$. Let moreover $\{\dot{u}_a, a = 1, \ldots, r\}$ complete this vector system to a basis of V_x and let $\{\overset{*}{v} = v_\alpha dv^\alpha, \dot{w}^A = w_\alpha^A dv^\alpha, \dot{u}^a = u_\alpha^a dv^\alpha\}$ be *a* dual basis of V_x^*. Then $\sigma = g_{ab} \dot{u}^a \otimes \dot{u}^b$ (2) with a regular matrix g_{ab}. If now a vector field X has the general form of a second order equation, i. e.,

$$X = a(v + b\dot{v} + \xi^A \dot{w}_A + \xi^a \dot{u}_a) \tag{3.1}$$

then $X \lrcorner\, \omega_L = a(v^\alpha \omega_{\alpha\beta} - \xi^a g_{ab} u_\beta^b) dq^\beta$. This expression vanishes if and only if

$$\xi^a g_{ab} \dot{u}^b = \xi^a g_{ab} u_\alpha^b dv^\alpha = v^\alpha a_{\alpha\beta} dv^\beta = v^\alpha \omega_{\alpha\beta}(v^\beta \overset{*}{v} + u_b^\beta \dot{u}^b + w_A^\beta \dot{w}^A)$$

i. e., if and only if

$$\tilde{\varphi}_A^1 = v^\alpha \omega_{\alpha\beta} w_A^\beta = 0. \tag{3.2}$$

Therefore a solution of the form (3.1) to

$$X \lrcorner\, \omega_L = 0 \tag{3.3}$$

exists only on the submanifold M^1 of TQ defined as the point set on which the first order constraints $\tilde{\varphi}_A^1$ vanish for all $A \in I$. There, on the other hand, the coefficients ξ^A remain undetermined. In fact, defining

$$\dot{\xi}_0 = v^\alpha \omega_{\alpha\beta} u_a^\beta g^{ab} \dot{u}_b$$

(where g^{ab} is the inverse of g_{ab}) we see that

$$X = a\left(v + b\dot{v} + \dot{\xi}_0 + \sum_{A \in I} \xi^A \dot{w}_A\right) \tag{3.4}$$

for arbitrary functions *a*, *b*, ξ^A on TQ satisfies

$$X \lrcorner\, \omega_L = a\tilde{\varphi}_A^1 (w_\alpha^A dq^\alpha) \tag{3.5}$$

and thus solves (3.3) precisely on M^1.

So far the set of vector fields X of the form (3.4) on M^1 does not yet describe a classical system because the functions ξ^A are still arbitrary. However, since the motion is only defined on the submanifold M^1 it must remain in M^1 which means that X must be tangent to M^1, i. e.

$$X \lrcorner\, d\tilde{\varphi}_B^1 = X(\tilde{\varphi}_B^1) = 0 \qquad \text{for all} \qquad B \in I \qquad \text{on} \qquad M^1 \tag{3.6}$$

(2) Whenever there is no doubt about the range of the indices the summation convention is applied.

or, more explicitly,

$$X_0(\tilde{\varphi}^1_B) + b\dot{v}(\tilde{\varphi}^1_B) + \xi^A \dot{w}_A(\tilde{\varphi}^1_B) = 0 \tag{3.7}$$

where

$$X_0 = v + \dot{\xi}_0. \tag{3.8}$$

It turns out that

$$\dot{v}\tilde{\varphi}^1_B = 0 \qquad \text{on} \qquad M^1. \tag{3.9}$$

To see this note, that by definition $\sigma_{\alpha\beta} w^\beta_A = 0$ identically in TQ. Hence

$$0 = \partial_{\dot{\gamma}}(w^\beta_B \partial_{\dot{\beta}\dot{\alpha}} L) = \partial_{\dot{\gamma}} w^\beta_B \sigma_{\beta\alpha} + w^\beta_B \partial_{\dot{\beta}\dot{\alpha}\dot{\gamma}} L. \tag{3.10}$$

With the help of $v^\gamma \partial_{\dot{\gamma}\dot{\alpha}\dot{\beta}} L = -\sigma_{\alpha\beta}$ this leads to $v^\gamma \partial_{\dot{\gamma}} w^\beta_B \sigma_{\beta\alpha} = \sigma_{\alpha\beta} w^\beta_B = 0$ whence

$$v^\gamma \partial_{\dot{\gamma}} w^\beta_B = c_B v^\beta + c^A_B w^\beta_A \tag{3.11}$$

for certain functions c_B and c^A_B. Using this result and $v^\gamma \partial_{\dot{\gamma}} \omega_{\alpha\beta} = 0$, which follows from (2.5), one obtains

$$\dot{v}(\tilde{\varphi}^1_B) = v^\alpha \omega_{\alpha\beta} w^\beta_B + v^\alpha \omega_{\alpha\beta}(c_B v^\beta + c^A_B w^\beta_A) = \tilde{\varphi}^1_B + c^A_B \tilde{\varphi}^1_A,$$

which vanishes on M^1. Since

$$w^\gamma_A \partial_{\dot{\gamma}} \omega_{\alpha\beta} = w^\gamma_A \partial_\beta \sigma_{\alpha\gamma} - w^\gamma_A \partial_\alpha \sigma_{\beta\gamma} = -\sigma_{\alpha\gamma} \partial_\beta w^\gamma_A + \sigma_{\beta\gamma} \partial_\alpha w^\gamma_A$$

and

$$w^\alpha_A \partial_{\dot{\alpha}} w^\beta_B = d_{AB} v^\beta + d^C_{AB} w^\beta_C,$$

which follows from (3.10) in the same way as (3.11), it is straightforward to deduce that

$$\dot{w}_A(\tilde{\varphi}^1_B) = w^\alpha_A \omega_{\alpha\beta} w^\beta_B + d^C_{AB} \tilde{\varphi}^1_C \tag{3.12}$$

for certain functions d^C_{AB} on M^0. This shows that $\dot{w}_A(\tilde{\varphi}^1_B)$ is actually antisymmetric, at least on M^1.

Equations (3.7) now reduce on M^1 to

$$X_0(\tilde{\varphi}^1_B) + \xi^A \dot{w}_A(\tilde{\varphi}^1_B) = 0 \qquad \text{for all} \qquad B \in I \tag{3.13}$$

and represent R conditions on the still arbitrary functions ξ^A. It may be that they determine ξ^A uniquely, namely when $\dot{w}_A(\tilde{\varphi}^1_B)$ is a regular $R \times R$-matrix everywhere on M^1. Then the vector field X is determined on M^1 precisely as much as necessary to describe a physical motion. Thus $(M^1, \mathscr{E})$ where $\mathscr{E} = \{X/X = a(v + b\dot{v} + \dot{\xi}_0 + \xi^A \dot{w}_A),\ a.\ b, \text{ arbitrary}\}$ can be regarded as the evolution space of the system. The reduction process is then already completed. In general, however,

$$\text{rk } \dot{w}_A(\tilde{\varphi}^1_B) = R_1 < R$$

(we assume it is constant on all of M^1). Then if the functions $C^B{}_{\tilde{B}}$,

$$\tilde{B} \in \tilde{I}_1 = I \backslash I_1$$

span the right kernel of $\dot{w}_A(\tilde{\varphi}^1_B)$ (i. e. $\dot{w}_A(\tilde{\varphi}^1_B)C^B{}_{\tilde{B}} = 0$ and

$$\dot{w}_A(\tilde{\varphi}^1_B)C^B = 0 \Rightarrow C^B = d^{\tilde{B}}C^B{}_{\tilde{B}}$$

for some functions $d^{\tilde{B}}$) (3.13) can be solved for ξ^A if and only if

$$\tilde{\varphi}^2_{\tilde{B}} = X_0(\tilde{\varphi}^1_B)C^B{}_{\tilde{B}} = 0. \tag{3.14}$$

These are new restrictions of def X, the domain of definition of X.

b) Basic assumptions.

It is now clear how in principle this process has to be repeated. Before this can be done in detail, however, and the consistency of the algorithm proved it is necessary to observe that not every degenerate Lagrangean need describe a physical system. Dirac remarked in his first paper on degenerate Lagrangean systems [*3*] already that the consistency conditions like (3.6) may result in actual contradictions. In the homogeneous formalism the corresponding phenomenon would normally be that equations like (3.2) and (3.14) do not define smooth submanifolds but restrict def X to a part of TQ where L is singular. Moreover, this is not the only way how a degenerate Lagrangean can fail to give rise to a second order equation and hence to equations of motion of a classical physical system. It can happen, for example, that there are not enough conditions to determine all the arbitrary functions ξ^A in the general form (3.4) for X because too many of the φ_A are trivial or dependent of others. This case is illustrated by the simple example in which $\sigma_{\alpha\beta}$ is degenerate and $\omega_{\alpha\beta}$ vanishes identically. Then all $\tilde{\varphi}^1_A$ in (3.2) are trivial and no constraint conditions are obtained at all. More generally, noting that $\omega_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ transform like ordinary tensors under configuration space coordinate transformations, suppose that by a transformation $q^\alpha \to \hat{q}^\alpha(q^\beta)$ it can be achieved that simultaneously $\sigma_{\alpha\beta} = 0 = \omega_{\alpha\beta}$ for all $\beta = r + 1, \ldots, n$. The last $n + 1 - r$ components of ξ^α then remain completely indeterminate, and it is clear why: the configuration space was simply chosen unnecessarily large, suggesting many more degrees of freedom than the system actually has. There seems no point in not immediately discarding these supernumerary dimensions.

Our basic assumption before setting up the constraint algorithm and investigating whether the state space of the system carries a natural symplectic structure or not will therefore be that an evolution space M with a second order equation $\mathscr{E}$ actually exists, i. e., that none of the two mentioned difficulties arise in the process of the reduction. Moreover, it will be assumed that whenever a matrix like $\dot{w}_A(\tilde{\varphi}^1_B)$ is defined on a certain submanifold it has constant rank on this submanifold. Without this assumption one would be led to countless case distinctions which would in fact amount to different physical systems described by the same Lagrangean in different regions of the original configuration space.

An immediate consequence of these assumptions is that the whole configuration space is actually needed for the eventual description of the system, i. e., the evolution space M is a subbundle of $M^0 = TQ$ having as its base space still the whole of Q, or infinitesimally, the subspace M_x of M^0_x at each point $x \in M$ is projected onto $Q_{T_Q x}$ by T_{Qx}. From this property it is straightforward to deduce that, if

$$M = \{ x \in TQ / \varphi_{\bar{A}}(x) = 0 \quad \text{for all} \quad \bar{A} \in \bar{I} \},$$

for a certain index set $\bar{I}$, say, then

$$\operatorname{rk} \begin{pmatrix} \partial_{\dot{\alpha}} \varphi_A \\ \partial_\alpha \varphi_{\bar{A}} \end{pmatrix} = \operatorname{rk}(\partial_{\dot{\alpha}} \varphi_{\bar{A}}) = \dim M^0 - \dim M.$$

c) Construction of the evolution space.

According to the last paragraph it is not obvious that the functions $\tilde{\varphi}^1_A$ defined in (3.2) are independent. From the basic assumption that the formalism works, however, it will eventually follow that they in fact are. Since their explicit form is to a high degree arbitrary it is essential to concentrate only on the invariant geometrical properties of the submanifolds defined, in order to avoid physically irrelevant complications in the calculations. As to the notation, indexed or otherwise specified capital letters like e. g. A_k will always range (and sum if occurring twice) over an index set I_k with R_k elements unless indicated otherwise.

Suppose now that $\varphi^1_{\bar{A}_1}$ are any $\bar{R}_1$ independent functions defining M^1 by means of $\varphi^1_{\bar{A}_1}(x) = 0$ for $\bar{A}_1 \in \bar{I}_1$, then $\bar{R}_1 = R - S_1$ for $S_1 \geq 0$. A certain subspace of $\ker \sigma \subset V_x$ will be tangent to M^1_x (at least the vector $\dot{v}$ according to (3.9)); let $\{ \dot{w}_{A_1} / A_1 \in I_1 \subset \bar{I}_1 \}$ span its complement. Then $\dot{w}_{A_1}(\varphi^1_{\bar{B}_1})$ has rank R_1 which is manifestly an invariant number and thus the same as the rank of the antisymmetric matrix $\dot{w}_A(\tilde{\varphi}^1_B)$ in (3.13). Hence R_1 is even. It is clear geometrically that by making a suitable transformation among the $\varphi^1_{\bar{B}_1}$ it can be achieved that $\dot{w}_{A_1}(\varphi^1_{B_1}) = \delta_{A_1 B_1}$ and $\dot{w}_{A_1}(\varphi^1_{\bar{B}_1}) = 0$ for $\bar{B}_1 \in \bar{I}_1 \setminus I_1$ (of course, all these relations generally hold on M^1 only).

The conditions (3.6) that X be tangent to M^1 now split into

$$X \lrcorner\, d\varphi^1_{A_1} = a(X_0(\varphi^1_{A_1}) + \xi^{A_1}) = 0, \qquad A_1 \in I_1,$$

determining R_1 of the R arbitrary functions ξ^A, and, yielding new constraints, $\varphi^2_{\bar{A}_1} = X_1(\varphi^1_{\bar{A}_1}) = X_0(\varphi^1_{\bar{A}_1}) = 0$ for $\bar{A}_1 \in \bar{I}_1 \setminus I_1$ where

$$X_1 = X_0 + \xi^{A_1}\dot{w}_{A_1}.$$

Thus the vector field X is now defined on

$$M^2 = \{ x \in M^1 / \varphi^2_{\bar{A}_1}(x) = 0 \qquad \text{for all} \qquad \bar{A}_1 \in \bar{I}_1 \setminus I_1 \} \tag{3.15}$$

where it is given by

$$X = a\Big(X_1 + b\dot{v} + \sum_{\tilde{A}_1 \in \tilde{I}_1} \xi^{\tilde{A}_1}\dot{w}_{\tilde{A}_1}\Big)$$

for arbitrary a, b, $\xi^{\tilde{A}_1}$ with $\tilde{A}_1 \in \tilde{I}_1 = I \setminus I_1$.

The next step consists in choosing any $\bar{R}_2 = \bar{R}_1 - R_1 - S_2$ $(S_2 \geq 0)$ suitable functions that are independent of the $\varphi^1_{A_1}$ and each other and characterize M^2 as the $\varphi^1_{\bar{A}_1}$ did in (3.15) and then demanding that

$$X \lrcorner\, d\varphi^2_{\bar{A}_2} = 0 \qquad \text{on M}.$$

This process can be continued. It must end after a finite number m of steps, such that on $M = M^m$ the vector field X is of the form (2.3) with ξ completely determined. This M is then clearly the evolution space of the system. We are going to prove inductively that it can be characterized in the following way.

THEOREM 1. — If L is a homogeneous Lagrangean over an $(n+1)$-dimensional configuration space Q whose Hessian $\partial_{\dot{\alpha}\dot{\beta}}L$ has constant rank $n - R < n$ on TQ and if L defines a classical system in the sense of subsection 2 then the evolution space M is a subbundle of TQ which can be described as follows: There exist

$$\bar{R} = \sum_{k=1}^{m} kR_k$$

independent functions $\{ \varphi^k_{A_l} / k \leq l = 1, \ldots, m, A_l \in I_l \}$ on TQ and

$$R = \sum_{k=1}^{m} R_k$$

vertical vector fields $\{ \dot{w}_{A_k}/k = 1, \ldots, m, A_k \in I_k \}$ spanning together with $\dot{v}$ the kernel of the Hessian in V_x for all $x \in TQ$ such that if a sequence of submanifolds $TQ = M^0 \supset M^1 \supset \ldots \supset M^m = M$ is defined by

$$M^k = \{ x \in M^{k-1} / \varphi^k_{A_l}(x) = 0 \qquad \text{for all} \qquad l \geq k, A_l \in I_l \}$$

then

$$\dot{w}_{A_j} \lrcorner\, d\varphi^k_{B_l} = \delta_{jl}\delta_{kl}\delta_{A_jB_l} \qquad \text{on} \qquad M^k. \tag{3.17}$$

Moreover, there exists a vector field $\hat{X}$ of the form $\hat{X} = v^\alpha \partial_\alpha + \xi^\alpha \dot{\partial}_\alpha$ on TQ which satisfies $\hat{X} \lrcorner\, \omega_L = 0$ on M^1 such that

$$\varphi^k_{A_l} = \hat{X}(\varphi^{k-1}_{A_l}) \qquad \text{for} \qquad 2 \leq k \leq l, A_l \in I_l \tag{3.18}$$

and

$$\varphi^1_{A_l} = \sum_{k=1}^{n} \sum_{B_k \in I_k} v^\alpha \omega_{\alpha\beta} w^\beta_{B_k} \cdot C^{B_k}{}_{A_l} \qquad \text{for} \qquad 1 \leq l \leq m, A_l \in I_l$$

where C^B_A is an $(R \times R)$-matrix of functions, nonsingular on M^1. The second order equation on M is then given by all vector fields Y on M of the form $Y = a(\hat{X} + b\dot{v})$ for arbitrary functions a and b. In particular, $\hat{X}$ is tangent to all M^l, $\dot{w}_{A_k}$ tangent to M^l for $k > l$, but transversal for $k \leq l$.

Proof. — To show first that $\dot{v}$ is tangent to all M^k for $0 \leq k \leq m$ suppose as induction hypothesis that by the method of the last paragraph M^k has already been defined as the submanifold of M^{k-1} on which all

$$\overline{R}_k = \overline{R}_{k-1} - R_{k-1} - S_k \qquad (S_k \geq 0) \tag{3.19}$$

independent functions $\varphi^k_{\overline{A}_k}$ vanish and that the general solution X of $X \lrcorner\, \omega_L = 0$ tangent to M^l for $l = 1, \ldots, k-1$ is given on M^k by

$$X = a\left(X_{k-1} + b\dot{v} + \sum_{A \in \tilde{I}_{k-1}} \xi^A \dot{w}_A \right)$$

for arbitrary $\xi^{\tilde{A}_{k-1}}$ where

$$\tilde{I}_{k-1} = I \Big\backslash \bigcup_{l=1}^{k-1} I_l.$$

Thus $\dot{v}$ and all $\dot{w}_{\tilde{A}_{l-1}}$ are tangent to M^{k-1} and among the latter $\dot{w}_{A_k}$ for $A_k \in I_k \subset \tilde{I}_{k-1}$ are assumed to be transversal to M^k. The conditions that X be also tangent to M^k then amount to

$$X_{k-1}(\varphi^k_{\overline{B}_k}) + b\dot{v}(\varphi^k_{\overline{B}_k}) + \sum_{A \in \tilde{I}_k} \xi^A \dot{w}_A(\varphi^k_{\overline{B}_k}) = 0 \qquad \text{on} \qquad M^k.$$

Now by the induction hypothesis the corresponding equation on M^{k-1},

$$X_{k-2}(\varphi^{k-1}_{\overline{B}_{k-1}}) + \sum_{A \in \tilde{I}_{k-1}} \xi^A \dot{w}_A(\varphi^{k-1}_{\overline{B}_{k-1}}) = 0$$

(since $\dot{v}$ is tangent to M^{k-1}), had been used to determine M^k; namely the R_{k-1} vectors $\dot{w}_{A_{k-1}}$ for $A_{k-1} \in I_{k-1} \subset \tilde{I}_{k-2}$ were transversal to M^{k-1}, hence $\mathrm{rk}\,(\dot{w}_{A_{k-1}}(\varphi^{k-1}_{\overline{B}_{k-1}})) = R_{k-1}$. The functions defining M^k are therefore of the general form

$$\varphi^k_{A_k} = X_{k-2}(\varphi^{k-1}_{\overline{B}_{k-1}})C^{\overline{B}_{k-1}}_{A_k} \tag{3.20}$$

where $C^{\overline{B}_{k-1}}_{A_k}$ are functions such that on M^{k-1}

$$\dot{w}_A(\varphi^{k-1}_{\overline{B}_{k-1}})C^{\overline{B}_{k-1}}_{A_k} = 0 \qquad \text{for all} \qquad A \in \tilde{I}_{k-1}. \tag{3.21}$$

Thus, on M^{k-1},

$$\begin{aligned}\dot{v}(\varphi^k_{A_k}) &= \dot{v}[X_{k-2}(\varphi^{k-1}_{\overline{B}_{k-1}})C^{\overline{B}_{k-1}}_{A_k}] \\ &= -\dot{v}\Big[\sum_{D \in \tilde{I}_{k-1}} \xi^D \dot{w}_D(\varphi^{k-1}_{\overline{B}_{k-1}})C^{\overline{B}_{k-1}}_{A_k}\Big] = -\sum_{D \in \tilde{I}_{k-1}} \{ v(\xi^D)\dot{w}_D(\varphi^{k-1}_{\overline{B}_{k-1}})C^{\overline{B}_{k-1}}_{A_k} \\ &\quad + \xi^A \dot{v}[\dot{w}_A(\varphi^{k-1}_{\overline{B}_{k-1}})C^{\overline{B}_{k-1}}_{A_k}] \}.\end{aligned}$$

Due to (3.21) and since $\dot{v}$ is by induction hypothesis tangent to M^{k-1} it follows that both terms vanish; therefore $\dot{v}$ is also tangent to M^k.

Next we prove that there are as many independent functions needed to define M^k as stated in the theorem. This is the case if and only if $S_k = 0$ for $1 \leq k \leq m$ and follows from the basic assumptions that the reduction terminates and defines an evolution space $(M, \mathscr{E})$. For, by construction, $\overline{R}_1 + S_1 = R$ and $\overline{R}_k + S_k = \overline{R}_{k-1} - R_{k-1}$ for $2 \leq k \leq m$. Summing over $k = 1, \ldots, m$ yields

$$\sum_{k=1}^{m} S_k = R_m - \overline{R}_m \leq 0 \tag{3.22}$$

by (3.19). But since the S_k are by construction non-negative integers (3.22) implies that they all vanish. Thus, by (3.19),

$$\overline{R}_k = R - \sum_{l=1}^{k-1} R_l$$

which immediately leads to

$$\overline{\mathrm{R}} = \sum_{k=1}^{m} \overline{\mathrm{R}}_k = \sum_{k=1}^{m} k\mathrm{R}_k .$$

It remains to be shown that the R vector fields $\dot{w}_{\mathrm{A}}$ and the $\overline{\mathrm{R}}$ functions φ_{A} can be chosen and labeled such that the relations (3.17) and (3.18) hold. For $k = 1$ this is already proved. For the general case note that the actual form of $\varphi^k_{\overline{\mathrm{A}}_k}$ is only important as far as it affects the definition of M^k. Hence the set of functions

$$\tilde{\varphi}^k_{\mathrm{A}_k} = \mathrm{C}^{\overline{\mathrm{B}}_k}_{\mathrm{A}_k} \varphi^k_{\overline{\mathrm{B}}_k} + \sum_{l=1}^{k-1} \mathrm{D}^{\overline{\mathrm{B}}_l}_{\overline{\mathrm{A}}_k} \varphi^l_{\overline{\mathrm{B}}_l} \tag{3.23}$$

for arbitrary matrices $\mathrm{C}^{\overline{\mathrm{B}}_k}_{\overline{\mathrm{A}}_k}$ and $\mathrm{D}^{\overline{\mathrm{B}}_l}_{\overline{\mathrm{A}}_k}$ of functions will be equivalent to $\{ \varphi^k_{\overline{\mathrm{A}}_k} \}$ provided only that the matrix $\mathrm{C}^{\overline{\mathrm{B}}_k}_{\overline{\mathrm{A}}_k}$ is non-singular everywhere on M^k. Due to the selection of the $\dot{w}_{\mathrm{A}}$ we have already $\dot{w}_{\mathrm{A}_k}(\varphi^l_{\overline{\mathrm{B}}_l}) = 0$ for $k > l$ and $\mathrm{rk}\,[\dot{w}_{\mathrm{A}_k}(\varphi^k_{\overline{\mathrm{B}}_k})] = \mathrm{R}_k$. By means of a transformation of the type (3.23) with $\mathrm{D} = 0$ it can therefore be achieved that the matrix $\dot{w}_{\mathrm{A}_k}(\varphi_{\overline{\mathrm{B}}_k})$ assumes the form $(1_{\mathrm{R}_k \times \mathrm{R}_k} 0_{\mathrm{R}_k \times (\overline{\mathrm{R}}_k - \mathrm{R}_k)})$. This leaves the possibility that $\dot{w}_{\mathrm{A}_k}(\varphi^l_{\overline{\mathrm{B}}_l}) \neq 0$ for $k < l$. But it is easily seen that adding a linear combination of $\{ \varphi^k_{\overline{\mathrm{B}}_l}, k < l \}$ to $\varphi^l_{\overline{\mathrm{B}}_l}$ eliminates these terms. This proves (3.17). Moreover, from (3.20) and (3.21) it is clear that (3.18) also holds provided only that the $\mathrm{R}_k + \mathrm{R}_{k+1} + \ldots + \mathrm{R}_m = \overline{\mathrm{R}}_k$ functions $\varphi^k_{\overline{\mathrm{A}}_k}$ are suitably relabeled. ▮

4. EXISTENCE OF A SYMPLECTIC STRUCTURE ON THE STATE SPACE

The manifold M constructed in the last section is endowed with a differential system $\mathscr{E}$ of the form (2.3) and is therefore the proper evolution space of the system described by the degenerate Lagrangean. Moreover, there exists a natural presymplectic form on M, namely $\omega = \iota^* \omega_{\mathrm{L}}$ where $\iota : \mathrm{M} \to \mathrm{TQ}$ is the inclusion map. It is easy to check formally that $\mathscr{E} \subset \ker \omega$ as remarked in section 2. For, suppose $\overline{\mathrm{X}} \in \mathscr{E}$ is a vector field on M, then

$$\overline{\mathrm{X}} \lrcorner\, \omega = \overline{\mathrm{X}} \lrcorner\, \iota^* \omega_{\mathrm{L}} = \iota^*(\iota_* \overline{\mathrm{X}} \lrcorner\, \omega_{\mathrm{L}}) = 0 \tag{4.1}$$

because $\mathscr{E}$ has been constructed such as to satisfy $\iota_* \overline{\mathrm{X}} \lrcorner\, \omega_{\mathrm{L}} = 0$ for all $\overline{\mathrm{X}} \in \mathscr{E}$.

Equation (4.1) also shows that a vector $Y \in M_x$ lies in ker ω if and only if its image

$$Y = \iota_* \bar{Y} \tag{4.2}$$

satisfies $\iota^*(Y \lrcorner \omega_L) = 0$ which is equivalent to

$$Y \lrcorner \omega_L = \sum_{\bar{A} \in \bar{I}} c^{\bar{A}} d\varphi_{\bar{A}} \quad \text{on} \quad M \tag{4.3}$$

for certain functions $c^{\bar{A}}$ on M. But Y, being of the form (4.2), also satisfies

$$Y \lrcorner d\varphi_{\bar{A}} = 0 \quad \text{on M for all} \quad \bar{A} \in \bar{I}. \tag{4.4}$$

Thus, if M is considered as a subset of TQ and $\bar{Y}$ identified with Y, then ker ω consists at any point $x \in M$ of the vectors Y satisfying (4.3) and (4.4).

Suppose for the moment that the coefficients $c^{\bar{A}}$ are fixed, then (4.3) can be solved for Y if and only if $Z \lrcorner \Sigma c^{\bar{A}} d\varphi_{\bar{A}} = 0$ for all $Z \in \ker \omega_L$. To construct first a basis for ker ω_L let $Z = \eta^\alpha \partial_\alpha + \zeta^\alpha \dot{\partial}_\alpha$. Then $Z \lrcorner \omega_L = 0$ if and only if

$$\eta^\alpha \sigma_{\alpha\beta} = 0 \tag{4.5}$$

and

$$\zeta^\alpha \sigma_{\alpha\beta} = \eta^\alpha \omega_{\alpha\beta}. \tag{4.6}$$

Equation (4.5) implies that $\eta^\alpha = av^\alpha + b^A w^\alpha_A$ and (4.6) becomes

$$\zeta^\alpha \sigma_{\alpha\beta} = av^\alpha \omega_{\alpha\beta} + b^A w^\alpha_A \omega_{\alpha\beta}.$$

This equation in turn has a solution if and only if the right hand side contracted with any vector of the kernel of $\sigma_{\alpha\beta}$ vanishes, i. e., if and only if

$$av_\alpha \omega_{\alpha\beta} v^\beta + b^A w^\alpha_A \omega_{\alpha\beta} v^\beta = - b^A \tilde{\varphi}^1_A = 0$$

and

$$av^\alpha \omega_{\alpha\beta} w^\beta_B + b^A w^\alpha_A \omega_{\alpha\beta} w^\beta_B = a\tilde{\varphi}^1_B + b^A \dot{w}_A(\tilde{\varphi}^1_B) - b^A d^C_{AB} \tilde{\varphi}^1_C = 0 \tag{4.7}$$

for all $B \in I$. (Here (3.2) and (3.12) have been used.) The first condition is satisfied identically on M^1 (and hence on M) and places non condition on the so far arbitrary factors b^A. Since $\dot{w}_A$ were chosen to be tangent to M^1 if $A = \tilde{A}_1 \in \tilde{I}_1 = I \backslash I_1$ equation (4.7) implies only that $b^{A_1} = 0$ if $A_1 \in I_1$ but all the other $R - R_1 + 1$ coefficients a and $b^{\tilde{A}_1}$ remain arbitrary. It follows that although ker ω_L may have any dimension $\geq R + 1$ at a general point of TQ its dimension for $x \in M$ is necessarily $2R - R_1 + 2$ and as a spanning system of vectors one may take for example

$$\{ \dot{v}, \dot{w}_A, \hat{X}, Z_{\tilde{A}_1} / A \in I, \tilde{A}_1 \in \tilde{I}_1 \}$$

where $\hat{X}$ is as in Theorem 1 and

$$Z_{\tilde{A}_1} = w^{\alpha}_{\tilde{A}_1}\partial_\alpha + \zeta^{\alpha}_{\tilde{A}_1}\partial_{\dot{\alpha}} \tag{4.8}$$

with $\zeta^{\alpha}_{\tilde{A}_1}$ satisfying

$$\zeta^{\alpha}_{\tilde{A}_1}\sigma_{\alpha\beta} = w^{\alpha}_{\tilde{A}_1}\omega_{\alpha\beta}. \tag{4.9}$$

These latter equations do not, however, fix $\zeta^{\alpha}_{\tilde{A}_1}$ completely, but only up to a transformation $Z_{\tilde{A}_1} \to \hat{Z}_{\tilde{A}_1} = Z_{\tilde{A}_1} + a_{\tilde{A}_1}\dot{v} + b^{B}_{A_1}\dot{w}_B$ with arbitrary $a_{\tilde{A}_1}$ and $b^{B}_{\tilde{A}_1}$ (if it is assumed that the $\dot{w}_B$ are fixed according to Theorem 1). Thus $Z_{\tilde{A}_1}$ can be required to satisfy

$$Z_{\tilde{A}_1} \lrcorner\, d\varphi^k_{B_k} = 0 \qquad \text{for all} \qquad k = 1, \ldots, m, \qquad B_k \in I_k \tag{4.10}$$

(Suppose they do not, then

$$\hat{Z}_{\tilde{A}_1} \lrcorner\, d\varphi^k_{B_k} = Z_{\tilde{A}_1} \lrcorner\, d\varphi^k_{B_k} + \sum_{l=1}^{m} b^{C_l}_{\tilde{A}_1}\dot{w}_{C_l}\varphi^k_{B_k} = Z_{\tilde{A}_1} \lrcorner\, d\varphi^k_{B_k} + b^{B_k}_{\tilde{A}_1} = 0$$

according to (3.17) if $b^{B_k}_{\tilde{A}_1}$ is chosen equal to $- Z_{\tilde{A}_1} \lrcorner\, d\varphi^k_{B_k}$). The conditions (4.8), (4.9) and (4.10) then fix $Z_{\tilde{A}_1} \in \ker \omega_L$ up to an irrelevant term proportional to $\dot{v}$. These vector fields $Z_{\tilde{A}_1}$ (which are defined on M^1 only, but can be thought of as extended in an arbitrary smooth fashion to the whole of TQ, where, however, $Z_{\tilde{A}_1} \lrcorner\, \omega_L$ and $Z_{\tilde{A}_1} \lrcorner\, d\varphi^k_{B_k}$ no longer vanish in general) satisfy on M

$$Z_{A_j} \lrcorner\, d\varphi^k_{B_l} = \delta_{jl}\delta_{k+1,l}\delta_{A_jB_l} \tag{4.11}$$

for all $A_j \in I_j$, $B_l \in I_l$ with $1 \leqslant k \leqslant l \leqslant m$ and $2 \leqslant j \leqslant m$.

Proof. — For $l = k$ equation (4.11) agrees with (4.10). If $l > k$ note that the equations $\sigma_{\alpha\beta}\eta^\beta = \partial_{\dot{\alpha}}\varphi^k_{B_l}$ can be solved for η^β on M at least in view of (3.17). Let $\eta^\beta_{kB_l}$ be such a solution, which extended to TQ satisfies

$$\sigma_{\alpha\beta}\eta^\beta_{kB_l} = \partial_{\dot{\alpha}}\varphi^k_{B_l} - w^{A}_{\alpha}\rho^{\overline{C}}_{AB_l}\varphi_{\overline{C}} \tag{4.12}$$

if $\dot{w}_A(\varphi^k_{B_l}) = \rho^{\overline{C}}_{AB_l}\varphi_{\overline{C}}$ for certain functions $\rho^{\overline{C}}_{AB_l}$ and $l > k$ (cf. section 3*c*). Then, always on M,

$$\begin{aligned}
&Z_{A_j} \lrcorner\, d\varphi^k_{B_l} \\
&= w^{\alpha}_{A_j}\partial_\alpha\varphi^k_{B_l} + \zeta^{\alpha}_{A_j}\partial_{\dot{\alpha}}\varphi^k_{B_l} \\
&= w^{\alpha}_{A_j}\partial_\alpha\varphi^k_{B_l} + \zeta^{\alpha}_{A_j}\sigma_{\alpha\gamma}\eta^\gamma_{kB_l} \\
&= w^{\alpha}_{A_j}(\partial_\alpha\varphi^k_{B_l} + \omega_{\alpha\gamma}\eta^\gamma_{kB_l}) \qquad \text{(by (4.9))} \\
&= w^{\alpha}_{A_j}\partial_{\dot{\alpha}}[v^\beta\partial_\beta\varphi^k_{B_l} + v^\beta\omega_{\beta\gamma}\eta^\gamma_{kB_l}] - w^{\alpha}_{A_j}v^\beta\partial_{\dot{\alpha}}(\partial_\beta\varphi^k_{B_l} + \omega_{\beta\gamma}\eta^\gamma_{kB_l}).
\end{aligned} \tag{4.13}$$

In view of (3.5), where ξ_0^β is replaced by $\hat{\xi}^\beta$ as in $\hat{X} = v + \hat{\xi}$ of Theorem 1, the first term becomes on M

$$
\begin{aligned}
& w^\alpha_{A_j}\partial_{\dot\alpha}(v^\beta\partial_\beta\varphi^k_{B_l} + \hat{\xi}^\beta\sigma_{\beta\gamma}\eta^\gamma_{kB_l} + \tilde{\varphi}^1_C w^C_\gamma\eta^\gamma_{kB_l}) \\
&\qquad = \dot{w}_{A_j}(v^\beta\partial_\beta\varphi^k_{B_l} + \hat{\xi}^\beta\partial_{\dot\beta}\varphi^k_{B_l} - \hat{\xi}^\beta w^A_\beta\rho^{\bar{C}}_{AB_l}\varphi_{\bar{C}} + \tilde{\varphi}^1_C w^C_\gamma\eta^\gamma_{kB_l}) \\
&\qquad = \dot{w}_{A_j}(\varphi^{k+1}_{B_l}) - \hat{\xi}^\beta w^A_\beta\rho^C_{AB_l}\dot{w}_{A_j}(\varphi_{\bar{C}}). \qquad (4.14)
\end{aligned}
$$

Since (due to (4.12))

$$
\begin{aligned}
w^\alpha_{A_j}v^\beta\partial_{\dot\alpha}\omega_{\beta\gamma}\eta^\gamma_{kB_l} &= w^\alpha_{A_j}v^\beta\partial_\gamma\sigma_{\alpha\beta}\eta^\gamma_{kB_l} - w^\alpha_{A_j}v^\beta\partial_\beta\sigma_{\alpha\gamma}\eta^\gamma_{kB_l} \\
&= -\, w^\alpha_{A_j}v^\beta\partial_\beta(\sigma_{\alpha\gamma}\eta^\gamma_{kB_l}) + w^\alpha_{A_j}v^\beta\sigma_{\alpha\gamma}\partial_\beta\eta^\gamma_{kB_l} \\
&= -\, w^\alpha_{A_j}v^\beta\partial_{\dot\alpha\beta}\varphi^k_{B_l} + (w^\alpha_{A_j}w^D_\alpha)\rho^{\bar{C}}_{DB_l}v^\beta\partial_\beta\varphi_{\bar{C}}
\end{aligned}
$$

on M^k and

$$
\begin{aligned}
& v^\beta\omega_{\beta\gamma}w^\alpha_{A_j}\partial_{\dot\alpha}\eta^\gamma_{kB_l} \\
&= \hat{\xi}^\beta\sigma_{\beta\gamma}\partial_{\dot\alpha}\eta^\gamma_{kB_l}w^\alpha_{A_j} \\
&= \hat{\xi}^\beta w^\alpha_{A_j}\partial_{\dot\alpha}(\sigma_{\beta\gamma}\eta^\gamma_{kB_l}) - \hat{\xi}^\beta\partial_{\dot\alpha}\sigma_{\beta\gamma}\eta^\gamma_{kB_l}w^\alpha_{A_j} \\
&= \hat{\xi}^\beta w^\alpha_{A_j}\partial_{\dot\alpha\dot\beta}\varphi^k_{B_l} - (\hat{\xi}^\beta w^D_\beta)\rho^{\bar{C}}_{DB_l}\dot{w}_{A_j}(\varphi_C) - \hat{\xi}^\beta\partial_{\dot\beta}(\sigma_{\alpha\gamma}\eta^\gamma_{kB_l})w^\alpha_{A_j} + \hat{\xi}^\beta\partial_{\dot\beta}\eta^\gamma_{kB_l}\sigma_{\gamma\alpha}w^\alpha_{A_j} \\
&= -\,(\hat{\xi}^\beta w^D_\beta)\rho^{\bar{C}}_{DB_l}\dot{w}_{A_j}(\varphi_{\bar{C}}) + (w^\alpha_{A_j}w^D_\alpha)\rho^{\bar{C}}_{DB_l}\hat{\xi}^\beta\partial_{\dot\beta}\varphi_{\bar{C}}
\end{aligned}
$$

on M the second term of (4.13) becomes

$$
-\,\rho^{\bar{C}}_{A_jB_l}X(\varphi_{\bar{C}}) + \hat{\xi}^\beta w^D_\beta\rho^{\bar{C}}_{DB_l}\dot{w}_{A_j}(\varphi_{\bar{C}}) \qquad \text{on} \qquad M.
$$

Here the first term vanishes on M and the second cancels with the second term of (4.14). Then (4.11) follows from (3.17).

This result shows, in particular, that not only $\dot{w}_A$ but also Z_{A_j} with $2 \le j \le m$ are transversal to M.

In order to write down the general solution Y of (4.3) introduce also the vectors $Y^k_{A_l}$ for $k, l = 1, \ldots, m,\ l \ge k + 2$ which are solutions (on M) of

$$
Y^k_{A_l} \lrcorner\ \omega_L = d\varphi^k_{A_l} \qquad (4.15)
$$

fixed up to terms proportional to $\dot{v}$ and $\hat{X}$ by the conditions

$$
Y^k_{A_l} \lrcorner\, d\varphi^i_{B_j} = 0 \quad \text{for all} \quad i, j = 1, \ldots, m \quad \text{and} \quad i \le j \le i+1. \quad (4.16)
$$

(That (4.15) has a solution for precisely the indicated values of k and l and that the conditions (4.16) can be imposed and fix $Y^k_{A_l}$ follows from similar arguments as were used for the construction of Z_{A_k}).

The necessary and sufficient conditions for a solution Y of (4.3) to exist now become

$$
\sum_{k=1}^{m}\sum_{l=k}^{m} c^{B_l}_k\dot{w}_{A_i} \lrcorner\ d\varphi^k_{B_l} = c^{A_i}_i = 0 \qquad \text{for} \qquad i = 1, \ldots, m
$$

and

$$\sum_{k=1}^{m}\sum_{l=k}^{m} c_k^{B_l} Z_{A_j} \lrcorner\, d\varphi_{B_l}^k = c_{j-1}^{A_j} = 0 \qquad \text{for} \qquad j = 2, \ldots, m.$$

If they are satisfied the general solution Y has the form

$$Y = \sum_{k=1}^{m-2}\sum_{l=k+2}^{m} c_k^{A_l} Y_{A_l}^k + a\hat{X} + b\dot{v} + a^A \dot{w}_A + b^{\tilde{A}_1} Z_{\tilde{A}_1}.$$

To lie in ker ω Y must also satisfy (4.4), i. e.

$$0 = \sum_{k=1}^{m-2}\sum_{l=k+2}^{m} c_k^{A_l} Y_{A_l}^k \lrcorner\, d\varphi_{B_j}^i + a^A \dot{w}_A \lrcorner\, d\varphi_{B_j}^i + b^{\tilde{A}_1} Z_{\tilde{A}_1} \lrcorner\, d\varphi_{B_j}^i$$

for all $i = 1, \ldots, m$, $j \geq i$. This yields, in particular, for $i = j$, in view of (4.10) and (4.16),

$$\sum_{k=1}^{m} a^{A_k} \dot{w}_{A_k}(\varphi_{B_i}^i) = a^{B_i} = 0$$

for all $i = 1, \ldots, m$, similarly for $j = i + 1$, $i = 1, \ldots, m - 1$

$$\sum_{k=2}^{m} b^{A_k} Z_{A_k}(\varphi_{B_{i+1}}^i) = b^{B_{i+1}} = 0$$

and, finally,

$$\sum_{k=1}^{m-2}\sum_{l=k+2}^{m} c_k^{A_l}(Y_{A_l}^k \lrcorner\, d\varphi_{B_j}^i) = 0 \tag{4.17}$$

for

$$i = 1, \ldots, m - 2, \qquad j \geq i + 2 = 3, \ldots, m.$$

In the particular case where $m \leq 2$ this last set of equations is empty and it follows already that $Y = a\hat{X} + b\dot{v}$, hence contained in $\mathscr{E}$. This leads, according to section 2*b* to.

THEOREM 2. — A degenerate Lagrangean that describes a classical system and leads to constraints of at most second order induces a symplectic structure on the state space of the system. That is, there are no « gauge variables », cf. the remark in section 2*a*.

If $m > 2$ the equations (4.17) impose restrictions on the

$$S = \sum_{k=3}^{m} (k-2)R_k = \overline{R} - (2R - R_1)$$

remaining arbitrary functions $c_k^{A_l}$ depending on the rank of the (S × S)-matrix $Y_{A_l}^k \lrcorner\, d\varphi_{B_j}^i$, which since by (4.15) it can be written in the form $\langle Y_{A_l}^k \wedge Y_{B_j}^i / \omega \rangle$ turns out to be antisymmetric. If this rank is S then all $c_k^{A_l}$ must vanish and Y has precisely the form $aX + b\dot{v}$, hence $\ker \omega = \mathscr{E}$. In particular, in this case S and hence $\overline{R}$ must be even, which makes it easy to conceive situations where $\mathrm{rk}\,(Y_{A_l}^k \lrcorner\, d\varphi_{B_j}^i) < S$ (e. g. $m = 3$, $R_1 = R_2 = 0$, $R_3 = R = 1$). However, no simple example of this type could be found. It seems indeed more likely that the basic assumption that the construction of (M, $\mathscr{E}$) works implies that R_k is even for odd k, whence it would follow the

$$\overline{R} = \sum_{k=1}^{m} kR_k$$

is even.

These remarks suggest that the conclusion of Theorem 2 holds in general for « physical » Lagrangeans leading to constraints of arbitrary order. The proof, however, in the present framework, would presumably depend on the explicit structure of ω_L and $\{ \varphi_{\overline{A}} \}$—somewhat as the proof of (4.11) did—rather than on general geometrical arguments.

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