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Section A :
Physique théorique.

On lattice spin systems

par

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ABSTRACT. — We start by describing the C*-algebra of the model, whose extremal symmetric states are built and classified into factor types. In order to introduce a ferromagnetic behaviour we specify interactions to be of a « generalized Ising » or a « generalized Heisenberg » type, involving many-body interactions with infinite range. From a boundedness hypothesis, we show that the formal hamiltonians induce an automorphism group on the C*-algebra. We exhibit the extremal symmetric states, invariant under this group, and finally we search for the states which induce positive hamiltonians.

1. INTRODUCTION

Scrutinizing lattice spin systems is appealing in several respects; firstly they appear to be among the simplest models of infinite systems to be built; secondly much progress has recently been made in the mathematical questions involved in this problem. Finally, there is some hope that these discrete models can help in more elaborated ones.

We focus our attention on the C*-algebra of the model. In the second section, we build and describe its main properties (locality, asymptotic abelianness with respect to the permutation group, simplicity). In the third section, we exhibit the extremal symmetric states (*i. e.* extre-

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mal among the symmetric ones) of our algebra with respect to translations and we calculate their entropy. The so-called « symmetric states » are the states invariant under the group of permutations of a finite number of points in the lattice. These states play an important rôle because of their homogeneous character; their physical meaning is that all the ions in the lattice are in the same physical state.

The symmetric states are invariant under the translation group of the lattice; all extremal symmetric states are ergodic with respect to translations (see [12]).

The fourth section is devoted to « dynamics »; we define a formal hamiltonian H in the C*-algebra of the system; we search for the conditions under which this hamiltonian induces a one-parameter group of automorphisms of the algebra. The invariant states under this group are the so-called « stationary states » ω, which in turn induce an infinitesimal generator H_ω. The study of H_ω, specially its boundedness from below, helps to select states of physical interest.

The hamiltonians usually found in the litterature turn out to be the H_ω induced by the Fock representation (*i. e.* corresponding to the state with all the spins pointing in the same direction).

In the fifth and sixth sections, after describing the generalized Ising and generalized Heisenberg hamiltonians, we show that the conditions under which these hamiltonians induce a group of automorphisms of the algebra are weaker than those proposed by D. Ruelle [12] (with his notations, that is $\sum_{0 \in X} N(X) \| \phi(X) \| < +\infty$, instead of $\sum_{0 \in X} e^{N(X)} \| \phi(X) \| < +\infty$); then we look at the extremal symmetric states which induce positive hamiltonians.

2. THE FERROMAGNET C*-ALGEBRA

The ferromagnet C*-algebra has already been described in [1], [10], [11]. For completeness, we briefly build this algebra and we give its main properties, following a slightly more general approach.

2.1. Building the local *-algebra.

Let E be any set, and let K be a real Hilbert space, whose scalar product will be denoted by s. $A(K \times E)$ will be the free complex algebra built upon the alphabet $K \times E$. Let us recall that this algebra is generated through finite formal sums (with complex coefficients) of finite formal products of elements belonging to $K \times E$. Any element in $A(K \times E)$ is written in the following form:

$$\sum_{j=1}^{p} \alpha_j(\psi_{j_1}, x_{j_1}) \ldots (\psi_{j_n}, x_{j_n})$$

with n and p two positive integers, α_j a complex number, $x_{j_k} \in E$, $\psi_{j_k} \in K$, for $j = 1, 2, \ldots, p$ and $k = 1, 2, \ldots, n$. Let $\mathfrak{J}(K \times E, s)$ be the two-sided ideal generated by the following elements:

$$(\alpha\psi + \beta\varphi, x) - \alpha(\psi, x) - \beta(\varphi, x)$$
$$(\psi, x)(\varphi, y) + (2\delta_{xy} - 1)(\varphi, y)(\psi, x) - 2\delta_{xy}s(\psi, \varphi)I$$

with α, β real numbers, φ and $\psi \in K$, x and $y \in E$, δ_{xy} the usual Kronecker symbol and I the identity of the free algebra. We shall denote by $\mathcal{F}(K \times E, s)$ the quotient-algebra $A(K \times E)/\mathfrak{J}(K \times E, s)$. The canonical image of any $(\psi, x) \in A(K \times E)$ will be denoted $B_x(\psi)$ and the identity is still denoted by I. There is only one involution in $\mathcal{F}(K \times E, s)$ such that all $B_x(\psi)$ are hermitian. So $\mathcal{F}(K \times E, s)$ is turned into a *-algebra.

2.2. Properties.

2.2.1. *If* $x \neq y$, $[B_x(\psi), B_y(\varphi)]_- = 0$ *for any* ψ *and* $\varphi \in K$. *Moreover* $[B_x(\psi), B_x(\varphi)]_+ = 2s(\psi, \varphi)I$, *for any* $x \in E$, ψ and $\varphi \in K$.

This property is straightforward, from the definition of $\mathfrak{J}(K \times E, s)$.

2.2.2. *For any* $M \subset E$, $\mathcal{F}(K \times M, s) \subset \mathcal{F}(K \times E, s)$.

This property is derived from the following relations :

$$A(K \times M, s) \subset A(K \times E, s),\ \mathfrak{J}(K \times E, s) \cap A(K \times M) = \mathfrak{J}(K \times M, s).$$

2.2.3. *For any set* $\{x_1, \ldots, x_n\}$, $\mathcal{F}(K \times \{x_1, \ldots, x_n\}, s)$ *is isomorphic with* $\bigotimes_{i=1}^{n} \mathcal{A}_i$, *with* $\mathcal{A}_i = \mathcal{A}(K, s)$ *(the Clifford algebra on* (K, s) *[2]).*

This is a corollary of 2.2.1.

$\mathcal{A}(K, s)$ being postliminar, a unique C*-norm (*i. e.* $\| a^*a \| = \| a \|^2$ for any $a \in \mathcal{A}(K, s)$) can be found on it. Consequently $\mathcal{F}(K \times \{x_1, \ldots, x_n\}, s)$ can also be equipped with a unique C*-norm. Now let F(E) be the set of finite subsets of E. It is easily seen that $\mathcal{F}(K \times E, s)$ is the inductive limit of the directed upward set $\{ \mathcal{F}(K \times M, s) \}_{M \in F(E)}$. This follows from 2.2.2, together with the properties of F(E):

α) for any N and $M \in F(E)$, $M \cup N \in F(E)$,

β) $E = \bigcup_{M \in F(E)} M$.

Consequently, a unique C*-norm also can be found on $\mathcal{F}(K \times E, s)$.

The algebra obtained from $\mathcal{F}(K \times E, s)$ through completion with respect to this C*-norm (we write it $\overline{\mathcal{F}(K \times E, s)}$) is a C*-algebra. From the above, we easily derive the following property:

2.2.4. *$\overline{\mathcal{F}(K \times E, s)}$ is the inductive limit of* C*-*algebras* $\overline{\mathcal{F}(K \times M, s)}$ $M \in F(E)$. *Moreover if the dimension of* K *is even or infinite then* $\overline{\mathcal{F}(K \times E, s)}$ *is simple.*

The second part of this proposition is easily deduced from the first part, adding that, for any $x \in E$, $\overline{\mathcal{A}(K \times \{ x \}, s)}$ is simple if the dimension of K is even or infinite.

From 2.2.1, it follows:

2.2.5. *For any* N *and* $M \subset E$, *such that* $M \cap N = \phi$,

$$[\overline{\mathcal{F}(K \times M, s)}, \overline{\mathcal{F}(K \times N, s)}]_- = 0.$$

2.3. Spatial automorphisms.

Let $\mathfrak{T}(E)$ be the permutation group of E (*i. e.* the group of one to one mappings of E onto itself). The following proposition is straightforward:

2.3.1. *For any* $p \in \mathfrak{T}(E)$, *the mapping* $B_x(\psi) \to B_{p(x)}(\psi)$ *can be extended to a unique automorphism* ζ_p *of* $\overline{\mathcal{F}(K \times E, z)}$. *The mapping* $p \in \mathfrak{T}(E) \to \zeta_p$ *is a one-to-one homomorphism.*

The group $\{ \zeta_p \mid p \in \mathfrak{T}(E) \}$ is called the group of spatial automorphisms.

2.3.2. *If* E *is an infinite set,* $\overline{\mathcal{F}(K \times E, s)}$ *is asymptotically abelian with respect to the spatial automorphisms group* (*i. e. for any a and b* $\in \overline{\mathcal{F}(K \times E, s)}$, $\inf_{p \in \mathfrak{T}(E)} \{ [a, \zeta_p(b)]_- \} = 0$).

This is a direct consequence of 2.2.5.

3. EXTREMAL SYMMETRIC STATES

From now on, attention will be specifically paid to a lattice whose points are occupied by ions, having spin 1/2, and whose motion is neglected. Consequently $E = Z^{\nu}$ (ν is the dimension of the lattice) and K is two-dimensional. The translation-group of the lattice is included in $\mathcal{T}(E)$. We restrict the theoretical study of states on $\overline{\mathcal{F}(K \times Z^{\nu}, s)}$ to invariant ones under the spatial group; nevertheless, to get mathematical properties on these latter states, we shall restrict our interest to the invariant states under the group $\mathcal{T}_0(E)$ of permutations of finite subsets of E.

An invariant state under the group $\mathcal{T}_0(E)$ will be called a « symmetric state » as in [*3*].

3.1. States on $\mathcal{A}(K, s)$.

Let (e_1, e_2) be a basis for K. Then $\mathcal{A}(K, s)$ is generated by $\{ B(e_1), B(e_2) \}$ which satisfy:

$$[B(e_i), B(e_j)]_+ = 2\delta_{ij}I.$$

It follows that any linear form on $\mathcal{A}(K, s)$ is characterized by its values on 1, $B(e_1)$, $B(e_2)$ and $B(e_1)B(e_2)$. As far as states ω are concerned, we know that $\omega(1) = 1$, $\omega(B(e_1)) = \chi_1$, $\omega(B(e_2)) = \chi_2$ are real numbers, while $\omega(B(e_1)B(e_2)) = i\chi_3$, a pure imaginary number. From the inequality $|\omega(x)| \leqslant \| x \|$, it follows: $|\chi_i| \leqslant 1$, for $i = 1, 2, 3$.

3.1.1. *ω is a state if and only if* $\sum_{i=1}^{3} \chi_i^2 \leqslant 1$.

Any positive element in $\mathcal{A}(K, s)$ can be written as y^2 with y self-adjoint, so that:

$$y = \alpha_0 1 + \alpha_1 B(e_1) + \alpha_2 B(e_2) + i\alpha_3 B(e_1)B(e_2)$$

with α_i real numbers.

Therefore:

$$y^2 = (\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)I + 2\alpha_0\alpha_1 B(e_1) + 2\alpha_0\alpha_2 B(e_2) - 2i\alpha_0\alpha_3 B(e_1)B(e_2).$$

An easy computation shows that $\omega(y^2) \geqslant 0$ if and only if

$$\sum_{i=1}^{3} \alpha_i^2 - \left(\sum_{i=1}^{3} \alpha_i\chi_i\right)^2 \geqslant 0$$

and the conclusion follows from the inequality:

$$\left(\sum_{i=1}^{3} \alpha_i \chi_i\right)^2 \leqslant \left(\sum_{i=1}^{3} \alpha_i^2\right)\left(\sum_{i=1}^{3} \chi_i^2\right).$$

Let π be the irreducible representation of $\mathcal{A}$ in $\mathbb{C}^2$ defined by:

$$\pi(\mathrm{B}(e_1)) = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pi(\mathrm{B}(e_2)) = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \pi(\mathrm{B}(e_1)\mathrm{B}(e_2)) = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

We shall denote for brevity, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by $\uparrow$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by $\downarrow$.

3.1.2. *Every state on $\mathcal{A}(\mathrm{K}, s)$ is associated either with the representation $\pi \oplus \pi$ ($a \uparrow + b \downarrow \oplus c \uparrow$ being the cyclic vector, where a is a complex number, b and c positive non zero numbers satisfying $|a|^2 + b^2 + c^2 = 1$), or with the representation π ($a \uparrow + b \downarrow$ being the cyclic vector, where a is a complex number, b a positive number satisfying $|a|^2 + b^2 = 1$).*

Such a state will be denoted by $\rho_{a,b,c}$ where a is a complex number, b and c positive numbers, satisfying the conditions: $|a|^2 + b^2 + c^2 = 1$, $b \neq 0$ if $c \neq 0$.

Let $\chi_1 = \rho(\mathrm{B}(e_1)) = 2\mathfrak{R}ab$,

$\chi_2 = \rho(\mathrm{B}(e_2)) = 2\mathfrak{I}ab$ and $i\chi_3 = \rho(\mathrm{B}(e_1)\mathrm{B}(e_2)) = i(|a|^2 - b^2 + c^2)$.

The result follows from 3.1.1, adding $|a|^2 + b^2 + c^2 = 1$ if and only if $\sum_{i=1}^{3} \chi_i^2 \leqslant 1$; this is easily proved.

3.1.3. *Among the states $\rho_{a,b,c}$, the pure states are the stats $\rho_{a,b,0}$.*

These pure states are associated with the irreducible representation π with $a \uparrow + b \downarrow$ as cyclic vector. Reciprocally, if either $a \neq 0$ or $b \neq 0$ together with $c \neq 0$, $\rho_{a,b,c}$ is associated with $\pi \oplus \pi$, that is a reducible representation. Finally let us notice that the states $\rho_{0,0,c}$ cannot be distinguished from the pure states $\rho_{c,0,0}$.

3.2. Extremal symmetric states on $\overline{\mathcal{F}(\mathrm{K} \times \mathrm{E}, s)}$.

We now exhibit the states ω on $\overline{\mathcal{F}(\mathrm{K} \times \mathrm{E}, s)}$, which are invariant under $\mathfrak{T}_0(\mathrm{E})$ and moreover, extremal among states invariant under $\mathfrak{T}_0(\mathrm{E})$. From now on, we suppose consequently that the conditions on a, b and c, stated in 3.1.2 are always satisfied. Our task is made much easier by Størmer's analysis which shows that extremal symetric states are product states, so we can state:

3.2.1. THEOREM. — *Any extremal symmetric state on* $\overline{\mathcal{F}(\mathrm{K} \times \mathrm{E}, s)}$ *is a product state* $\omega_{a,b,c} = \bigotimes_{i \in \mathrm{E}} \omega_i$, *with, for any* $i \in \mathrm{E}$, $\omega_i = \rho_{a,b,c}$. *Let us denote by* $\mathcal{H}_{a,b,c}$ *the Hilbert space generated by* $\bigotimes_{i \in \mathrm{E}} \mathrm{W}_i$, *where*

$$\mathrm{W}_i = a \uparrow + b \downarrow \oplus c \uparrow$$

for any i, but a finite number; the state $\omega_{a,b,c}$ *is associated with the representation* $\pi_{a,b,c} = \bigotimes_{i \in \mathrm{E}} \eta_i$, *where* $\eta_i = \pi \oplus \pi$ *(see* 3.1.2*) holding in* $\mathcal{H}_{a,b,c}$, *and with the cyclic vector* $\Omega_{a,b,c} = \bigotimes_{i \in \mathrm{E}} \mathrm{V}_i$, *where* $\mathrm{V}_i = a \uparrow + b \downarrow \oplus c \uparrow$ *for any* $i \in \mathrm{E}$. *Moreover, the entropy of* $\omega_{a,b,c}$ *is equal to ;*

$$-(\lambda_1 \log \lambda_1 + \lambda_2 \log \lambda_2)$$

where

$$\lambda_1 = \frac{1}{2}(1 + \sqrt{1 - 4b^2c^2}) \qquad \text{and} \qquad \lambda_2 = \frac{1}{2}(1 - \sqrt{1 - 4b^2c^2}).$$

This can be deduced from the fact that we necessarily deal with a product state of a given state ω_0 on $\mathcal{A}(\mathrm{K}, s)$ ([*3*], theorem 2.7). The existence of $\mathcal{H}_{a,b,c}$ is known from [*8*]. The second part of the theorem is straightforward.

Let us remark that the entropy goes from 0 (for pure states) to log 2 (for the central state).

3.2.2. *Any extremal symmetric state on* $\overline{\mathcal{F}(\mathrm{K} \times \mathrm{Z}^\nu, s)}$ *is primary.*

It is quite straightforward that the condition expressed in ([*4*], theorem 2.5) is satisfied by any $\omega = \bigotimes_{i \in \mathrm{E}} \omega_i$ since on one hand $\overline{\mathcal{F}(\mathrm{K} \times \mathrm{E}, s)}$ is asymptotically abelian (2.3.2) and on the other, ω is a product state.

In order to perform the classification of states into different factor-types, we need these two lemmas.

3.2.3. *Following the notations introduced in* 3.2.1, *the vector* $\bigotimes_{i \in \mathrm{E}} \mathrm{V}_i$ *is separating for the representation* $\pi_{a,b,c}$, *when b and* $c \neq 0$.

Since V_i is separating for the algebra $\mathcal{A}_i$, it is cyclic for its commutant, thus the tensor product $\bigotimes_{i \in \mathrm{E}} \mathrm{V}_i$ is cyclic for the tensor product of these commutants, $\bigotimes_{i\in E} \mathcal{A}_i'$, and for the commutant $(\bigotimes_{i\in E} \mathcal{A}_i)'$ also, since $\bigotimes_{i\in E} \mathcal{A}_i' \subset (\bigotimes_{i\in E} \mathcal{A}_i)'$; consequently $\Omega_{a,b,c}$ is separating for the product $\bigotimes_{i\in E} \mathcal{A}_i$.

Since $\omega_{a,b,c}$ is invariant under the automorphism group induced by $\mathcal{T}_0$, there exists a unique unitary representation U of $\mathcal{T}_0$ into $\mathcal{H}_{a,b,c}$, such that:

$\pi_{a,b,c}(\zeta_p(\gamma)) = \mathrm{U}(p)\pi_{a,b,c}(\gamma)\mathrm{U}(p)^*$ where $\mathrm{U}(p) \bigotimes_{i\in E} \mathrm{W}_i = \bigotimes_{i\in E} \mathrm{W}_{p(i)}$([5], 2.12.11).

3.2.4. *The cyclic vector $\Omega_{a,b,c}$ is the unique vector ψ in $\mathcal{H}_{a,b,c}$, up to a scalar, which satisfies :*

$$\mathrm{U}(p)\psi = \lambda(p)\psi, \qquad \lambda(p) \text{ a complex number} \tag{1}$$

Take $\psi \in \mathcal{H}_{a,b,c}$ ($\|\psi\| = 1$), satisfying (1); since U is unitary, $|\lambda(p)| = 1$, and obviously

$$|(\psi \mid \mathrm{U}(p)\psi)| = 1; \tag{2}$$

from cyclicity of $\Omega_{a,b,c}$, a sequence x_n can be found in $\overline{\mathcal{F}(\mathrm{K} \times \mathrm{E}, s)}$ such that $\psi_n = \pi(x_n)\Omega_{a,b,c}$ converges towards ψ. By (2) and this last remark, it is easily seen that:

$|(\psi_n \mid \mathrm{U}(p)\psi_n)| = |\omega(x_n^*\zeta_p(x_n))|$ converges towards 1 uniformly with respect to p; it follows from the strongly clustering property of ω (3.2.1 and [*3*]) that

$$\inf_{p\in\mathcal{T}_0} \{ \omega(x_n^*\zeta_p(x_n)) - \omega(x_n^*)\omega(x_n) \} = 0.$$

Then, $\omega(x_n^*)\omega(x_n)$ being equal to $|\omega(x_n)|^2$, we have:

$$\lim_{n\to\infty} |\omega(x_n)| = \lim_{n\to\infty} |(\Omega_{a,b,c} \mid \psi_n)| = |(\Omega_{a,b,c} \mid \psi)| = 1.$$

Since Schwartz inequality here turns to be an equality, ψ is equal to $\Omega_{a,b,c}$, up to a scalar.

3.2.5. THEOREM. — *Following the notations of* 3.2.1, *the classification of the extremal symmetric states is obtained :*

α) when $c = 0$, ω is a pure state (with type I_∞),

β) when $a = 0$ and $b = c$, ω is the central state (with type II_1),

γ) in all the other cases, ω has the type III.

When $c = 0$, $\omega_{a,b,c}$ is an infinite product of pure states (3.1.3), thus a pure state also [*6*]. When $a = 0$ and $b = c$, see for instance ([*3*], corollary 2.4. (3)). From both the lemmas 3.1.3 and 3.1.4, the cyclic vector is separating for the representation $\pi_{a,b,c}$ and it is the unique vector invariant, up to a scalar, under the unitary operators $U(p)$, $p \in \mathfrak{S}_0$. Since the state $\omega_{a,b,c}$ is primary (3.2.2), the conditions indicated by Størmer ([7], th. 2.4.) are precisely satisfied, which imply that $\omega_{a,b,c}$ has the type III.

Since the extremal symmetric states are tensorial infinite products of copies of a state (3.2.1), these states are invariant through the group of automorphisms induced by $\mathfrak{S}$; consequently they are invariant through the translation group of Z^ν.

Moreover these states are strongly clustering for this group and consequently, extremal (ergodic states).

4. INTRODUCTION TO DYNAMICS

4.1. Group of automorphisms induced by a formal hamiltonian.

Let H_n be a sequence of hermitian elements belonging to the local algebra $\mathcal{F}(K \times Z^\nu, s)$, which does not necessarily converge in $\overline{\mathcal{F}(K \times Z^\nu, s)}$. Nevertheless, the sequence (*) $(ad^p H_n(\gamma))_{n \in N}$ is supposed to converge in $\overline{\mathcal{F}(K \times Z^\nu, s)}$ for any γ in $\mathcal{F}(K \times Z^\nu, s)$ and any integer p. H will be the « formal » element $\lim\limits_{n\to\infty} H_n$ defined by: $ad^p H(\gamma) = \lim\limits_{n\to\infty} ad^p H_n(\gamma)$ for all γ's in $\mathcal{F}(K \times Z^\nu, s)$. Consequently adH is generally an unbounded derivation on $\mathcal{F}(K \times Z^\nu, s)$, and so cannot be extended to the whole algebra $\overline{\mathcal{F}(K \times Z^\nu, s)}$.

4.1.1. *If, for all γ's $\in \mathcal{F}(K \times Z^\nu, s)$, there exists a neighbourhood of* O *in* R *such that the limit:* $\lim\limits_{n\to\infty} e^{it\, adH_n}(\gamma) = \tau_t(\gamma)$ *is uniform in t, then τ_t is a strongly continuous group of automorphisms of $\overline{\mathcal{F}(K \times Z^\nu, s)}$ induced by the derivation ad*H.

Proof : See [9].

Remark. — Whenever H is an hermitian element in $\overline{\mathcal{F}(K \times Z^\nu, s)}$, it is easily verified that $\exp\{itH\}$ belongs to $\overline{\mathcal{F}(K \times Z^\nu, s)}$, for any t in R, and

$$\tau_t(\gamma) = \exp\{it\, adH\}(\gamma) = \exp\{itH\}\, \gamma \exp\{-itH\}.$$

If H is a formal hamiltonian, $\exp\{itH\}$ does not exist, despite the possible existence of τ_t.

(*) $ad^p H_n$ is defined through the induction formula: $ad^p H_n = adH_n \circ ad^{p-1} H_n$, and $adH_n(\gamma) = H_n\gamma - \gamma H_n$.

4.2. Stationary states.

In this section we assume that the conditions of 4.1.1 are verified. Thus, the abelian group of « time » automorphisms is obtained. The states invariant under this group will be called stationary states.

4.2.1. *A state ω is stationary, if and only if $\omega \circ ad\mathrm{H} = 0$, on the local algebra $\mathcal{F}(\mathrm{K} \times \mathrm{Z}^\nu, s)$.*

Proof : From the continuity character of ω, one obtains for all $\gamma \in \mathcal{F}(\mathrm{K} \times \mathrm{Z}^\nu, s)$, in a neighbourhood of 0,

$$\omega(\tau_t(\gamma)) - \omega(\gamma) = \sum_{p=1}^{\infty} \frac{(it)^p}{p} \omega(ad^p\mathrm{H}(\gamma)).$$

The expansion on the left-hand side is an analytic function of t, so that, this function will be vanishing everywhere, if and only if $\omega \circ ad\mathrm{H} = 0$.

For any state ω on $\overline{\mathcal{F}(\mathrm{K} \times \mathrm{Z}^\nu, s)}$, we denote by $\mathcal{H}_\omega$, π_ω and Ω_ω respectively the Hilbert space, the representation and the cyclic vector, associated with ω through the Gelfand-Segal theorem ([5], 2.4.4.)

4.2.2. *If ω is a stationary state on $\overline{\mathcal{F}(\mathrm{K} \times \mathrm{Z}^\nu, s)}$, then a unique self-adjoint operator H_ω (which is generally unbounded) is defined through:*

$$\mathrm{H}_\omega(\pi_\omega(\gamma)\Omega_\omega) = \pi_\omega(ad\mathrm{H}(\gamma))\Omega_\omega$$

for all γ's $\in \mathcal{F}(\mathrm{K} \times \mathrm{Z}^\nu, s)$. This operator H_ω verifies, for all $\gamma \in \overline{\mathcal{F}(\mathrm{K} \times \mathrm{Z}^\nu, s)}$:

$$\pi_\omega(\tau_t(\gamma)) = \exp\{it\mathrm{H}_\omega\}\, \pi_\omega(\gamma) \exp\{-it\mathrm{H}_\omega\}.$$

Proof: By ([5], 2.12.11), there exists a continuous unitary representation U of R such that:

$$\mathrm{U}(t)(\pi_\omega(\gamma)\Omega_\omega) = \pi_\omega(\tau_t(\gamma))\Omega_\omega$$

and

$$\pi_\omega(\tau_t(\gamma)) = \mathrm{U}(t)\pi_\omega(\gamma)\mathrm{U}(t)^*.$$

Stone's theorem implies the existence of a self-adjoint operator H_ω on $\mathcal{H}_\omega$, verifying:

$$\mathrm{U}(t) = \exp\{it\mathrm{H}_\omega\}$$

$$\mathrm{H}_\omega = -i \lim_{t \to 0} \frac{\mathrm{U}(t) - 1}{t}.$$

The elements $\pi_\omega(\gamma)\Omega_\omega$, with $\gamma \in \mathcal{F}(K \times Z^\nu, s)$, are in the domain of H_ω and one has:

$$H_\omega(\pi_\omega(\gamma)\Omega_\omega) = -i \lim_{t\to 0} \frac{\pi_\omega(\tau_t(\gamma) - \gamma)\Omega_\omega}{t} = \pi_\omega(ad H(\gamma))\Omega_\omega.$$

The hamiltonians usually written in the litterature are, in the framework of this paper, hamiltonians induced by a Fock state. These latter are a convenient tool as guide for the choice of the hamiltonians in the abstract algebra.

5. GENERALIZED ISING MODEL

5.1. Formal hamiltonian.

Let:

$$B_i(e_1) = u_i^1,\ B_i(e_2) = u_i^2 \text{ and } - iB_i(e_1)B_i(e_2) = u_i^3.$$

Introducing:

$$u_i^+ = \frac{u_i^1 + iu_i^2}{2}, \qquad u_i^- = \frac{u_i^1 - iu_i^2}{2}$$

we get the following relations:

$$[u_i^+, u_i^-]_+ = 1, \qquad [u_i^+, u_i^-]_- = u_i^3, \qquad [u_i^3, u_i^+]_+ = [u_i^3, u_i^-]_+ = 0 \quad (5)$$

and $[u_i^\beta, u_j^\rho]_- = 0$ for any $i, j \neq i$, and any $\beta, \rho = 3, +, -$.

The following relations will be of constant use:

$$[u_i^+, u_i^3]_- = 2u^+u_i^3 = -2u_i^+, \qquad [u_i^-, u_i^3]_- = 2u_i^-u_i^3 = 2u_i^-. \quad (6)$$

The « generalized Ising model » will be defined to as the model with a formal hamiltonian written as:

$$H_I = \sum_{r=1}^{\infty} \sum_{i_k \in Z^\nu} g_{i_1 \dots i_{2r}} u_{i_1}^3 u_{i_2}^3 \dots u_{i_{2r}}^3 \quad (7)$$

the real coefficients $g_{i_1 \dots i_{2r}}$ can be taken as completely symmetrical functions of their indices. The translation-invariance of the interaction turns H_i to be a formal hamiltonian since the sums

$$\sum_{i_k \in Z^\nu} | g_{i_1 \ldots i_{2r}} |$$

cannot converge. This hamiltonian is the limit, in the sence precised above (4.1), of local hamiltonians H_i^n, obtained by taking the interacting points in an open ball whose center is at the origin, whose radius is n.

Let us note that for any r, the hamiltonian

$$H_{(r)} = \sum_{i_k \in Z^\nu} g_{i_1 \ldots i_{2r}} u_{i_1}^3 u_{i_2}^3 \ldots u_{i_{2r}}^3$$

describes the $2r$-body interactions. We shall impose the convergence of all the sums:

$$S_r = \sum_{\widehat{i_1}} | g_{i_1 \ldots i_{2r}} |$$

the hatted index means that we do not sum with respect to i_1.

By translation-invariance, S_r cannot depend on the choice of i_1.

To get a ferromagnetic behaviour, that is the spins tending to align themselves along some given direction, we shall impose the same sign for the whole set of coefficients; it will be later verified that, if we take them as negative numbers, the hamiltonian will be a positive operator in the Fock representations.

5.1.1. *If the sum* $\sum_{k=1}^{\infty} k S_k < +\infty$, *the expansion* (3) *is convergent.*

Taking firstly:

$$ad H_{(1)}(u_k^\pm) = \sum_{i,j} g_{ij}[u_i^3 u_j^3, u_k^\pm]_- = -2 \sum_{i,j} g_{ij}(\delta_{ik} u_k^\pm u_k^3 u_j^3 + \delta_{jk} u_k^\pm u_k^3 u_i^3)$$

we see:

$$\| ad H_{(1)}(u_k^\pm) \| \leqslant \left(2 \sum_j | g_{kj} | + \sum_j | g_{ik} |\right) = 4 S_1.$$

This formula can easily be generalized in the following form:

$$\| ad H_{(r)}(u_k^\pm) \| \leqslant 4 r S_r.$$

If follows:

$$\|ad\mathrm{H}_{\mathrm{I}}(u_k^{\pm})\| \leqslant \sum_{r=1}^{\infty} \|ad\mathrm{H}_{(r)}(u_k^{\pm})\| = 4\sum_{r=1}^{\infty} r\mathrm{S}_r = \mathrm{S}.$$

Suppose now that:

$$\|ad^n\mathrm{H}_{\mathrm{I}}(u_k^{\pm})\| \leqslant \mathrm{S}^n,$$

from:

$$\|ad\mathrm{H}_r(ad^n\mathrm{H}_{\mathrm{I}}(u_k^{\pm}))\| \leqslant 4r\mathrm{S}_r\mathrm{S}^n, \qquad r = 1, 2, \ldots$$

we get by induction:

$$\|ad^{n+1}\mathrm{H}_{\mathrm{I}}(u_k^{\pm})\| \leqslant \sum_{r=1}^{\infty} \|ad\mathrm{H}_{(r)}(ad^n\mathrm{H}_{\mathrm{I}}(u_k^{\pm}))\| \leqslant \mathrm{S}^{n+1}.$$

and we can conclude that the expansion (3):

$$\sum_{n=1}^{\infty} \frac{(it)^n}{n\,!} ad^n\mathrm{H}_{\mathrm{I}}(u_k^{\pm})$$

is absolutely convergent whatever t may be in the complex plane.

5.2. Stationary states.

5.2.1. *The states $\omega_{a,b,c}$ with $a \neq 0$, are not stationary if both b and c are not zero.*

By 4.2.1, it is sufficient to prove that $\omega \circ ad\mathrm{H}$ is not everywhere zero. This is verified, when $\omega \circ ad\mathrm{H}$ is applied to $u_k^+ u_r^3$ since we then get:

$$2\bar{a}b\Big\{2 \times 1\Big[\Big(\sum_{j\neq r} g_{kj}\Big)(|a|^2 - b^2 + c^2)^2 + g_{kr}\Big] + 2 \times 2\Big[\Big(\sum_{j\neq r} g_{kjmn}\Big)(|a|^2 - b^2 + c^2)^4 + g_{krmn}\Big] + 2 \times 3[\ldots]\ldots\Big\}$$

which is a non zero number when $ab \neq 0$.

5.2.2. *The states $\omega_{0,b,c}$ and $\omega_{1,0,0}$ are stationary.*

We must prove that $\omega \circ ad\mathrm{H}_{\mathrm{I}} = 0$ (4.2.1). We firstly verify that if ω is one of these states, then $\omega(u_i^3\gamma_i) = \omega(\gamma_i u_i^3)$, $\gamma_i = u_i^{\pm}$, u_i^3, 1. Since we are dealing with product states, we see that:

$$\omega\big(ad(u_{i_1}^3 \cdot \ldots \cdot u_{i_p}^3)\gamma_{i_1} \cdot \ldots \cdot \gamma_{i_p}\big) = 0$$

which proves the proposition.

$\omega_{1,0,0}$ is usually called « Foch state » and $\omega_{0,1,0}$ « anti-Foch state » (they are pure states).

5.3. States with positive hamiltonians.

In section 4.2, we saw that every stationary state ω induces a hamiltonian $H_{I,\omega}$. Here, we search for states whose induced hamiltonian is positive.

5.3.1. « *Fock and anti-Fock states* » *induce positive hamiltonians. The central state induces a vanishing hamiltonian.*

To derive positivity of the hamiltonian induced by the « Fock state », we must show that:

$$\omega_{1,0,0}(\gamma^* ad H_I(\gamma)) \geqslant 0 \tag{8}$$

for every $\gamma \in \overline{\mathcal{F}(K \times E, s)}$. Let: $\gamma_i = d_i 1 + e_i u_i^-$ with d_i and e_i arbitrary complex numbers verifying $|d_i|^2 + |e_i|^2 = 1$. We easily see that:

$$\omega_{1,0,0}(\gamma_i^*[u_i^3, \gamma_i]_-) = -2|e_i|^2 = x_i.$$

The two-body part $H_{(1)}$ of H_I verifies (8) for every $\gamma \in \overline{\mathcal{F}(K \times E, s)}$, if it verifies it for $\gamma_i\gamma_j$ for any $i, j \in Z^\nu$. It follows from:

$$[u_i^3u_j^3, \gamma_i\gamma_j]_- = \gamma_i[u_i^3u_j^3, \gamma_j]_- + [u_i^3, \gamma_i]_- \cdot [u_j^3, \gamma_j]_- + [u_i^3, \gamma_i]_-\gamma_j u_j^3,$$

that:

$$\omega_{1,0,0}(\gamma_i^*\gamma_j^*[u_i^3u_j^3, \gamma_i\gamma_j]_-) = x_i + x_j + x_i x_j.$$

An elementary discussion shows that if

$$-2 \leqslant x_i \leqslant 0 \quad \text{and} \quad -2 \leqslant x_j \leqslant 0, \quad \text{then} \quad -2 \leqslant x_i + x_j + x_i x_j \leqslant 0.$$

The positivity of $H_{(1)}$ follows from the negative character of the interaction coefficients. The positivity is thus easily extended by induction separately to any part of the hamiltonian, using:

$$[u^3_{i_1}u^3_{i_2}\dots u^3_{i_k}, \gamma_{i_1}\gamma_{i_2}\dots\gamma_{i_k}]_- = \gamma_{i_1}u^3_{i_1}[u^3_{i_2}u^3_{i_3}\dots u^3_{i_k}, \gamma_{i_2}\gamma_{i_3}\dots\gamma_{i_k}]_-$$
$$+ [u^3_{i_1}, \gamma_{i_1}][u^3_{i_2}u^3_{i_3}\dots u^3_{i_k}, \gamma_{i_2}\gamma_{i_3}\dots\gamma_{i_k}]_- + [u^3_{i_1}, \gamma_{i_1}]_-\gamma_{i_2}u^3_{i_2}\gamma_{i_3}u^3_{i_3}\dots\gamma_{i_k}u^3_{i_k}.$$

The case of the « anti-Fock representation », where the cyclic vector is obtained when all spins are « down », is quite analogous. As to the central state $\omega_{0,1/\sqrt{2},1/\sqrt{2}}$, from $\omega_{0,1/\sqrt{2},1/\sqrt{2}}(u_i^\pm) = \omega_{0,1/\sqrt{2},1/\sqrt{2}}(u_i^3) = 0$ one sees easily that the corresponding hamiltonian vanishes everywhere.

5.3.2. *The states $\omega_{0,b,c}$ with $b \neq c$, b and $c \neq 0$, induce hamiltonians with unbounded expectation values (without upper or lower bound).*

Take $v_{ij} = du_i^+ + eu_j^-$ with $b^2\,|\,d\,|^2 + c^2\,|\,e\,|^2 = 1$ and calculate:

$\omega_{0,b,c}(v_{ij}^* ad\mathrm{H}_\mathrm{I}(v_{ij})) = \mathrm{C}(b^2\,|\,d\,|^2 - c^2\,|\,e\,|^2)(b^2 - c^2)$, with C some constant.

This term is obviously positive or negative, depending on a suitable choice of d and e. In a first step, let us show that when truncated hamiltonians H_I^n are considered, $ad\mathrm{H}_\mathrm{I}^n$ is not bounded from below. This can be easily seen by taking correspondingly a sequence of $2m$ points in Z^ν (i_1, j_1), (i_2, j_2), $\dots$, (i_m, j_m), such that the distance between the set $\{ i_k, j_k \}$ and the set $\{ i_r, j_r \}$ $(k \neq r)$, be greater than n (i_k and j_k can for instance be taken as nearest neighbours). Thus the following formula holds:

$$\omega_{0,b,c}(v_{i_1j_1}^*\dots v_{i_mj_m}^* ad\mathrm{H}_\mathrm{I}^n(v_{i_1j_1}\dots v_{i_mj_m})) = m\mathrm{C}(b^2\,|\,d\,|^2 - c^2\,|\,e\,|^2)(b^2 - c^2). \quad (9)$$

Since the vector $\prod_{k=1}^{m}\pi_{0,b,c}(v_{i_kj_k})\Omega_{0,b,c}$ is normalized to 1, and since (9) grows linearly with m, the proposition is shown, for truncated hamiltonians. The result will be extended to $ad\mathrm{H}_i$ by noting that:

$$\left|\, \omega_{0,b,c}\Big(\prod_{k=1}^{m} v_{i_kj_k}^*(ad\mathrm{H}_\mathrm{I} - ad\mathrm{H}_\mathrm{I}^n)\Big(\prod_{k=1}^{m} v_{i_kj_k}\Big)\Big) \,\right| \leqslant m\mathrm{R}_n$$

where R_n is the remaining term $4\sum_{r=n+1}^{\infty} r\mathrm{S}_r$ of the expansion $\mathrm{S} = 4\sum_{r=1}^{\infty} r\mathrm{S}_r$. This bound clearly goes to zero (for fixed m), when n goes to infinity, so that the result is obtained.

Remark. — A magnetic field $h\sum_k u_k^3$ being introduced, one can remark firstly that the invariant states are still the same, secondly that the state with positive hamiltonian is one of the Fock states, depending on the sign of h.

6. HEISENBERG MODEL

6.1. Formal hamiltonian.

Let: $\vec{u}_i\vec{u}_j = \sum_{\alpha=1}^{3} u_i^\alpha u_j^\alpha$. The Heisenberg hamiltonian will be written in the following form:

$$\mathrm{H_H} = \sum_{r=1}^{\infty}\sum_{i_k\in Z^\nu}{}' g_{(i_1,i_2)\dots(i_{2r-1},i_{2r})}(\vec{u}_{i_1}\vec{u}_{i_2})\dots(\vec{u}_{i_{2r-1}}\vec{u}_{i_{2r}}) \tag{10}$$

The real coefficients $g'_{(i_1,\, i_2)\cdots(i_{2r-1},\, i_{2r})}$ are chosen symmetric with respect to the exchange of two pairs (i_{2k-1},i_{2k}), $(i_{2j-1},\, i_{2j})$ for any k and j, and also in the exchange $(i_{2k},\, i_{2k-1}) \to (i_{2k-1},\, i_{2k})$. In contrast to the Ising case, the indices can now be repeated, and reduction of the corresponding terms into others of smaller degree is necessary, through the relations:

$$[\vec{u}_i\vec{u}_j,\, \vec{u}_j\vec{u}_k]_+ = 2\vec{u}_i\vec{u}_k$$
$$(\vec{u}_i\vec{u}_j)^2 = 3 - 2\vec{u}_i\vec{u}_j$$

combined with the symmetry properties of the coefficients. Once all the repetitions have been suppressed, we have to deal with a hamiltonian such as (10) but the points $i_1 \dots i_{2r}$ are all different. Then we suppose that the corresponding coefficients are all negative. The Hamiltonian is « formal », since translation invariance is imposed. We suppose again the convergence of the sums:

$$\mathrm{S}_r' = \sum_{\widehat{i_1}} | g'_{(i_1,i_2)\dots(i_{2r-1},i_{2r})} |$$

6.1.1. *If the sum* $\sum_{k=1}^{\infty} k3^k\mathrm{S}_k' < \infty$, *the Dyson expansion is convergent.*

In order to make more apparent the role of spin values, we write:

$$f_{(i_1,i_2)\dots i_{2r-1},i_{2r})} = (4)^r g'_{(i_1,.)\dots(.,i_{2r})}.$$

For instance, we get:

$$H_{(1)} = \sum_{i,j,\alpha} f_{ij} s_i^\alpha s_j^\alpha \qquad \text{with} \qquad s_i^\alpha = \frac{1}{2} u_i^\alpha \quad (\alpha = 1, 2, 3).$$

From:

$$adH_{(1)}(s_k^\gamma) = \sum_{i,j,\alpha} f_{ij}[s_i^\alpha s_j^\alpha, s_k^\gamma] = \sum_{i,j,\alpha} f_{ij}(\delta_{ik}\varepsilon_{\alpha\beta\gamma} s_i^\alpha s_k^\gamma + \delta_{ik}\varepsilon_{\alpha\beta\gamma} s_k^\gamma s_j^\alpha)$$

it follows, for fixed k and γ:

$$adH_{(1)}(s_k^\gamma) = \sum_{i,\alpha} f_{ik}\varepsilon_{\alpha\beta\gamma} s_i^\alpha s_k^\gamma + \sum_{j,\alpha} f_{kj}\varepsilon_{\alpha\beta\gamma} s_k^\gamma s_j^\alpha .$$

For fixed i, or fixed j, there are two terms in the sum over α, so that:

$$\| adH_{(1)}(s_k^\gamma) \| \leqslant \frac{1}{2} \sum_i | f_{ik} | + \frac{1}{2} \sum_j | f_{kj} | = \frac{\sigma}{2} \quad \text{with} \quad \sigma = 2 \sum_j | f_{ij} | \tag{11}$$

Suppose now that:

$$\| ad^n H_{(1)}(s_k^\gamma) \| \leqslant \frac{n ! \, \sigma^n}{2} .$$

If this inequality holds also for 1, 2, ..., $n - 1$, then:

$$\| ad^n H_{(1)}(s_k^\alpha s_j^\beta) \| \leqslant \sum_{p=0}^{n} C_n^p \| ad^p H_{(1)}(s_k^\alpha) \| \, \| ad^{n-p} H_{(1)}(s_j^\beta) \|$$

$$\leqslant \sum_{p=0}^{n} C_n^p \frac{\sigma^p p !}{2} \frac{\sigma^{n-p}(n-p) !}{2} .$$

So we get:

$$\| ad^n H_{(1)}(s_k^\alpha s_j^\beta) \| \leqslant \frac{\sigma^n}{4} \sum_{p=0}^{n} C_n^p p ! \, (n-p) ! = \frac{(n+1) ! \, \sigma^n}{4} .$$

Finally:

$$\| ad^{n+1} H_{(1)}(s_k^\gamma) \| \leqslant 2 \Big(\sum_j | f_{ik} | + \sum_j | f_{kj} | \Big) \| ad^n H_{(1)}(s_k^\alpha s_j^\beta) \| \leqslant \frac{(n+1) ! \, \sigma^{n+1}}{2}$$

More generally:

$$\| adH_{(r)}(s_j^\alpha) \| \leqslant r \left(\frac{3}{4}\right)^{r-1} \sigma_r \quad \text{with} \quad \sigma_r = \sum_{\widehat{i_1}} | f_{(i_1,.)\dots(.,i_{2r})} | = 4^r S_r'$$

since $\| \vec{s}_j \vec{s}_i \| \leqslant \frac{3}{4}$, and it follows:

$$\left\| ad\mathrm{H_H}(s_j^\alpha) \right\| = \sum_{r=1}^{\infty} \left\| ad\mathrm{H}_{(r)}(s_j^\alpha) \right\| \leqslant \sum_{r=1}^{\infty} r \left(\frac{3}{4}\right)^{r-1} \sigma_r = \frac{\sigma}{2}$$

where σ is now some finite number, as in (11); as before, from:

$$\left\| ad^k\mathrm{H_H}(s_i^\gamma) \right\| \leqslant \frac{k\,!\,\sigma^k}{2} \qquad k = 1, 2, \ldots, n$$

we conclude analogously:

$$\left\| ad^{n+1}\mathrm{H_H}(s_k^\gamma) \right\| \leqslant \frac{(n+1)\,!\,\sigma^{n+1}}{2}.$$

Thus the Dyson expansion is absolutely convergent if $|\, t\,| < \frac{1}{\sigma}$, and we are now in position to apply 4.1.1.

6.2. Extremal stationary states.

6.2.1. *Any extremal symmetric state is stationary.*

Let $\omega_{a,b,c}$ be such a state; we must show that $\omega_{a,b,c} \circ ad\mathrm{H_H} = 0$ (4.2.1). We firstly verify that:

$$\omega_{a,b,c}(\vec{u}_{i_1}\vec{u}_{j_1}\gamma_{i_1}\gamma_{j_1}) = \omega_{a,b,c}(\gamma_{i_1}\gamma_{j_1}\vec{u}_{i_1}\vec{u}_{j_1})$$

where γ_{i_1} (respectively γ_{j_1}) take the values $u_{i_1}^3$, $u_{i_1}^\pm$, 1 (respectively $u_{j_1}^3$, $u_{j_1}^\pm$, 1). Since we are dealing with product states, the last equality implies:

$$\omega_{a,b,c}(ad((\vec{u}_{i_1}\vec{u}_{j_1})(\vec{u}_{i_2}\vec{u}_{j_2}) \ldots (\vec{u}_{i_k}\vec{u}_{j_k}))(\gamma_{i_1}\gamma_{j_1} \ldots \gamma_{i_k}\gamma_{j_k})) = 0.$$

The proposition is proved.

6.3. States with positive hamiltonians.

We firstly consider the case:

α) $a = 0$

Take $v_{ij} = du_i^+ + eu_j^-$ with $b^2 \,|\, d\,|^2 + c^2 \,|\, e\,|^2 = 1$ as in the Ising case, and calculate:

$$\omega_{0,b,c}(v_{ij}^* ad\mathrm{H_H}(v_{ij})) = \mathrm{C}'(b^2 \,|\, d\,|^2 - c^2 \,|\, e\,|^2)(b^2 - c^2)$$

where C′ is some constant. This term is still obviously positive or negative, depending on a suitable choice of d and e, and the analysis can be pursued quite analogously to the Ising case. We can state:

6.3.1. *The states $\omega_{0,b,c}$, with $b \neq c$, b and $c \neq 0$, induce hamiltonians with unbounded expectation values (without upper and lower bound).*

Now we turn to the Fock cases and the central one, to derive in a first step the positivity, when only the two-body interactions are taken into account. To derive positivity of the hamiltonian induced by the « Fock state », we must show that:

$$\omega_{1,0,0}(\gamma_{ij}^{*} ad \mathrm{H}_{(1)}(\gamma_{ij})) \geqslant 0$$

for any

$$\gamma_{ij} = d_{ij}1 + e_i u_i^- + e_j u_j^- + h_{ij} u_i^- u_j^- .$$

This is shown through the result :

$$\omega_{1,0,0}(\gamma_{ij}^{*}[\vec{u}_i \vec{u}_j, \gamma_{ij}]) = -2 \mid e_i - e_j \mid^2 = x_{ij}.$$

The four-body part of $\mathrm{H_H}$ does not always induce correspondingly a positive hamiltonian since from:

$$[(\vec{u}_i\vec{u}_j)(\vec{u}_k\vec{u}_r), \gamma_{ij}\gamma_{kr}]_- = \gamma_{ij}(\vec{u}_i\vec{u}_j)[\vec{u}_k\vec{u}_r, \gamma_{kr}]_- + [\vec{u}_i\vec{u}_j, \gamma_{ij}]_- \cdot [\vec{u}_k\vec{u}_r, \gamma_{kr}]_- + [\vec{u}_i\vec{u}_j, \gamma_{ij}]\gamma_{kr}\vec{u}_k\vec{u}_r$$

it follows that:

$$\omega_{1,0,0}(\gamma_{ij}^{*}\gamma_{kr}^{*}[(\vec{u}_i\vec{u}_j)(\vec{u}_k\vec{u}_r), \gamma_{ij}\gamma_{kr}]) = x_{ij} + x_{kr} + x_{ij}x_{kr}$$

and an elementary discussion shows that when:

$$-4 \leqslant x_{ij} \leqslant 0 \quad \text{and} \quad -4 \leqslant x_{kr} \leqslant 0, \quad \text{then} \quad -4 \leqslant x_{ij} + x_{kr} + x_{ij}x_{kr} \leqslant 8.$$

But nevertheless, if we consider all together the two-body and four-body interactions, if the first ones are sufficiently dominant, then a positive hamiltonian is induced through $\mathrm{H}_{(1)} + \mathrm{H}_{(2)}$, more precisely if the following condition is satisfied:

$$4 \mid g'_{ij} \mid > 8 \sum_{k,r} \mid g'_{(ij)(kr)} \mid .$$

A simple graphical analysis shows, that quite generally, the latter inequality must be generalized into the following one:

$$4|g'_{ij}| > 8\sum_{k,r}|g'_{(ij)(kr)}| + (3^4-1)\sum_{k,r,m,n,l,p}|g'_{(ij)(kr)(mn)(l,p)}|$$
$$+(3^{2n}-1)\sum_{i_1\dots i_{2n-1}}|g'_{(ij)(i_1j_1)\dots(i_{2n-1}j_{2n-1})}| + 8\sum_{k,r,m,n} g'_{(ij)(kr)(mn)}$$
$$+(3^4-1)\sum_{k,r,m,n,l,p,s,q}|g'_{(ij)(kr)(m,n)(l,p)(s,q)}| + \dots (3^{2n}-1)\sum_{i_1\dots 2n}|g'_{(ij)\dots(i_{2n}j_{2n})}| \quad (12)$$

The summations are extended over all the indices, but i and j are missing.

The « anti-Fock » case is quite analogous.

In the central case, the corresponding hamiltonian vanishes everywhere, since the result in fact does not effectively depend on the hamiltonian.

6.3.2. *If the condition* (12) *is satisfied, the « Fock and anti-Fock » states induce positive hamiltonians. The central state induces a vanishing hamiltonian.*

β) $a \neq 0$

6.3.3. *The stationary states inducing positive hamiltonians are $\omega_{a,b,0}$ where $|a| = b$.*

From the commutation relation:

$$[\vec{u}_i\vec{u}_j,\ u_i^+]_- = 2u_i^+u_j^3 - 2u_i^3u_j^+$$

it follows that

$$\omega_{a,b,c}((u_i^+)^*[\vec{u}_i\vec{u}_j,\ u_i^+]_+) = 2b^2(-b^2+c^2).$$

More generally:

$$[\vec{u}_i\vec{u}_j + (\vec{u}_i\vec{u}_j)(\vec{u}_k\vec{u}_r) + \dots,\ u_i^+]_- = 2(u_i^+u_j^3 - u_i^3u_j^+)(1 + \vec{u}_k\vec{u}_r + (\vec{u}_k\vec{u}_r)(\vec{u}_m\vec{u}_n) + \dots)$$

Using the product law of states we obtain:

$$\omega_{a,b,c}(u_i^{+*}[\vec{u}_i\vec{u}_j + \dots,\ u_i^+]) = 2b^2(-b^2+c^2)(1 + \{(|a|^2-b^2+c^2)^2 + 4|a|^2b^2)\}$$
$$+ \{(|a|^2-b^2+c^2)^2 + 4|a|^2b^2\}^2 + \dots).$$

In order to have positivity, we must verify:

$$-b^2+c^2 \leqslant 0 \quad \text{or} \quad b=0, \quad \text{that is} \quad b=c=0 \quad \text{or} \quad b \geqslant c.$$

Quite analogously, a more restrictive condition will be obtained through:

$$[\vec{u}_i\vec{u}_j+(\vec{u}_i\vec{u}_j)(\vec{u}_k\vec{u}_r)+\ldots,\ u_i^+u_j^3]_- = (2u_i^+ - 2u_i^3u_j^+u_j^3 - 2u_i^+u_i^-u_j^+)(1+\vec{u}_k\vec{u}_r+\ldots).$$

That is:

$$b^2(|\,a\,|^2 - b^2 + c^2)(1 + \{(|\,a\,|^2 - b^2 + c^2)^2 + 4\,|\,a\,|^2b^2\} + \ldots) \leqslant 0\,;$$

it follows that either $b = 0$ and consequently $c = 0$, or $b^2 \geqslant |\,a\,|^2 + c^2$.

Inversely, from:

$$[\vec{u}_i\vec{u}_j+(\vec{u}_i\vec{u}_j)(\vec{u}_k\vec{u}_r)+\ldots,\ u_i^-u_j^3]_- = (-2u_i^- + 2u_i^3u_j^-u_j^3 + 2u_i^-u_i^+u_j^-)(1+\vec{u}_k\vec{u}_r+\ldots)$$

we obtain:

$$-(|\,a\,|^2+c^2)(|\,a\,|^2 - b^2 + c^2)(1 + \{(|\,a\,|^2 - b^2 + c^2)^2 + 4\,|\,a\,|^2b^2\} + \ldots) \leqslant 0.$$

That is, since the case where $a = c = 0$ is already examined:

$$b^2 \leqslant |\,a\,|^2 + c^2.$$

Consequently $b^2 = |\,a\,|^2 + c^2$ if b and c differ from 0. Clearly since we are now looking for states different from the Fock states and the central one, we may suppose that this last condition is verified; from a further calculation involving the most general term belonging to an algebra at a given point i, it follows, with $\gamma_i = \alpha 1 + \beta u_i^+ + \gamma u_i^- + \delta u_i^3$

$$\begin{aligned}\omega_{a,b,c}(\gamma_i^*[\vec{u}_i\vec{u}_j + (\vec{u}_i\vec{u}_j)(\vec{u}_k\vec{u}_r) + \ldots,\ \gamma_i]_-) &= (1 + 4\,|\,a\,|^2b^2 + (4\,|\,a\,|^2b^2)^2 + \ldots)\\ &\cdot(-2\beta\delta b^3a - 2\,|\,\beta\,|^2b^2\,|\,a\,|^2 + 2\bar{\beta}\gamma b^2a^2 + 2\bar{\gamma}\delta b^3\bar{a} + 2\bar{\gamma}\beta(\bar{a}b)^2\\ &-|\,\gamma\,|^2\,|\,a\,|^2b^2 - 4\bar{\delta}\beta b^3\bar{a} + 4\bar{\delta}\gamma b^3a - 4\,|\,\delta\,|^2\,|\,a\,|^2b^2),\end{aligned}$$

when the relation $|\,a\,|^2 - b^2 + c^2 = 0$ is taken into account.

It is possible to suppose a to be a real positive number, by multiplying β, γ, δ by some phases, so that, if we take $\beta = -\gamma = -\delta$, we get the condition:

$$-12b^2a^2\gamma^2 + 12b^3a\gamma^2 = -12\gamma^2b^2a(a - b) \leqslant 0,$$

so that necessarily $a > b$.

Since $b^2 = a^2 + c^2 \geqslant a^2$, it follows $c = 0$ and finally $|\,a\,|^2 = b^2$. So we get only pure states; moreover, there is not any privileged direction, and this is an expected result; it is now quite easy to verify positivity, which results, just as in the case $\alpha)$ from the dominance condition of two-body interactions.

Remark. — A magnetic field $h \sum_k u_k^3$ being introduced, one can remark firstly that the invariant states are $\omega_{0,b,c}$, secondly the state with positive hamiltonian is one of the Fock states, depending on the sign of h.

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