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C. M. EDWARDS

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C*-algebras of central group extensions I

par

C. M. EDWARDS
The Queen's College, Oxford, United Kingdom

ABSTRACT. — The C*-group algebra of a central group extension of a compact abelian group by a locally compact group is studied. It is shown that this C*-algebra is a direct sum of closed *-ideals each isomorphic to a « twisted » C*-group algebra. Applications to the direct sum of abelian groups and to the C*-algebra of local observables of a quantum system with one degree of freedom are considered.

RÉSUMÉ. — *Les C*-algèbres des extensions centrales des groupes* I. — On étudie la C*-algèbre d'une extension centrale d'un groupe abélien compact par un groupe localement compact. On démontre que cette C*-algèbre constitue une somme directe d'idéaux bilatères fermés dont chacun est isomorphe à une C*-algèbre « gauche » d'un groupe. On étudie des applications à la somme directe de groupes abéliens et à la C*-algèbre d'observables locaux d'un système quantique à une dimension.

§ 1. INTRODUCTION

Among the algebras associated with a locally compact group G the C*-group algebra C*(G) is of importance since its properties determine the structure of the set of unitary representations of G. When G is abelian C*(G) may be identified with the algebra $C_0(G^\wedge)$ of continuous functions taking arbitrarily small values outside compact subsets of the dual $G^\wedge$ of G. However, in general, little is known of its structure. In this paper the structure of C*(G) is studied in the particular case in which G is the central group extension of a compact abelian group by a locally compact group. In order that use can be made of the equivalence of weak measurability and weak continuity for unitary representations it is assumed that all the groups considered are separable although many of the results are true in the wider situation.

In the final section two examples are discussed. In the first our results are applied to the direct product of abelian groups. In the second we use our results to make certain remarks about the C*-algebra of observables of a quantum-mechanical system with n-degrees of freedom.

Loupias and Miracle-Sole [*16*] have studied the C*-algebra of a particular central group extension of the group T of complex numbers of unit modulus by the vector group R^{2n} and some of their results are consequences of the results of this paper.

§ 2. DEFINITIONS AND NOTATION

Let G be a separable locally compact group with unit element e; let m be a left invariant Haar measure on G and let δ be the modular function on G. Let A be a separable compact abelian group with unit element 0 and let ν be the normalised Haar measure on A. Let $A^{\wedge}$ be the dual of A. Let $Z^2(G, A)$ denote the group of Borel 2-co-cycles from G to A (for the properties of Borel structures on groups see [*14*]). An element f of $Z^2(G, A)$ is a Borel mapping from $G \times G$ to A such that for all x, y, z in G,

$$f(x, y) + f(xy, z) = f(x, yz) + f(y, z) \tag{2.1}$$

$$f(x, e) = f(e, x) = 0 \tag{2.2}$$

For each pair (a, x), (b, y) of elements of $A \times G$, let

$$(a, x)(b, y) = (a + b + f(x, y), xy). \tag{2.3}$$

Then with respect to this multiplication $A \times G$ is a separable locally compact group G^f with Borel structure identical to that of $A \times G$. Moreover $\nu \times m$ is a left-invariant Haar measure on G^f and the modular function Δ is defined for each element (a, x) of G^f by $\Delta(a, x) = \delta(x)$ [*15*]. G^f is said to be the central group extension of A by G corresponding to f (Central group extensions may be equivalently defined in terms of exact sequences).

Let T be the compact abelian group of complex numbers of unit modulus. Elements of $Z^2(G, T)$ are said to be multipliers on G. For each element α of $A^\wedge$ and each element f of $Z^2(G, A)$, $\alpha \circ f$ is clearly an element of $Z^2(G, T)$. A multiplier ω on G is said to be trivial if there exists a Borel mapping ρ from G to T such that $\rho(e) = 1$ and for each pair x, y of elements of G,

$$\omega(x, y) = \rho(x)\rho(y)\rho(xy)^{-1} \tag{2.4}$$

Let $L_1(G)$ be the Banach space of complex-valued, measurable, absolutely-integrable functions on G where two functions which differ only on m-null sets are regarded as identical; let $\| \ \|_1$ denote the norm on $L_1(G)$. Let $L_2(G)$ be the Hilbert space of complex-valued, measurable, absolutely square-integrable functions on G, where again functions which differ only on m-null sets are regarded as identical; let $\| \ \|_2$ denote the norm on $L_2(G)$ (for the properties of $L_1(G)$, $L_2(G)$, see [*12*]).

For each multiplier ω on G and each pair ψ_1, ψ_2 of elements of $L_1(G)$ let $\psi_1 \,\omega\, \psi_2$, ψ_1^ω be functions defined for each element x of G by

$$(\psi_1 \,\omega\, \psi_2)(x) = \int_G \psi_1(y)\psi_2(y^{-1}x)\omega(y, y^{-1}x)dm(y) \tag{2.5}$$

$$\psi_1^\omega(x) = \psi_1(x^{-1})\omega(x, x^{-1})\delta(x^{-1}). \tag{2.6}$$

Then with respect to this multiplication and involution $L_1(G)$ is a Banach *-algebra $L_1(G, \omega)$, the twisted group algebra over G corresponding to ω [*8*].

Let $L_1(G^f)$ be the group algebra of G^f; let the norm, multiplication and involution in $L_1(G^f)$ be denoted $\| \ \|_1$, $*$, $\sim$ respectively. For each element Ψ of $L_1(G^f)$ and each element α of $A^\wedge$, let $\alpha(\Psi)$ be the function defined for each element x of G by

$$\alpha(\Psi)(x) = \int_A \Psi(a, x)\alpha(a)dv(a). \tag{2.7}$$

Then α is a norm non-increasing *-homomorphism onto $L_1(G, \alpha \circ f)$. For each element ψ of $L_1(G, \alpha \circ f)$, let $\bar{\alpha} \otimes \psi$ be the function defined for each element (a, x) of G^f by

$$(\bar{\alpha} \otimes \psi)(a, x) = \overline{\alpha(a)}\psi(x). \tag{2.8}$$

Then the mapping $\psi \mapsto \bar{\alpha} \otimes \psi$ is an isometric *-isomorphism into $L_1(G^f)$. Moreover, $\alpha(\bar{\alpha} \otimes \psi) = \psi$ and the set $\{\bar{\alpha} \otimes \alpha;\ \alpha \in A^\wedge\}$ is a family of mutually disjoint projections onto closed two-sided ideals $L_1(G^f, \alpha)$ in

$L_1(G^f)$. $L_1(G^f, \alpha)$ is isometrically *-isomorphic to $L_1(G, \alpha \circ f)$ and

$$L_1(G^f) = \bigoplus_{\alpha \in A^\wedge} L_1(G^f, \alpha). \tag{2.9}$$

A similar construction shows that $L_2(G^f)$ may be written as the direct sum of closed mutually orthogonal subspaces $L_2(G^f, \alpha)$ where $L_2(G^f, \alpha)$ is isometric to $L_2(G)$.

A projective representation of G consists of:

(*i*) A weakly Borel mapping π from G to the group of unitary operators on some separable Hilbert space $\mathfrak{h}$ such that $\pi(e) = 1$ the identity operator (A mapping F from G to the algebra $B(\mathfrak{h})$ of bounded linear operators on $\mathfrak{h}$ is said to be weakly (strongly) Borel when for each pair ξ, η of elements of $\mathfrak{h}$ the mapping $x \to (F(x)\xi, \eta)$ $(x \to \| F(x)\xi \|)$ is Borel).

(*ii*) A mapping ω from $G \times G$ to T such that for each pair x, y of elements of G,

$$\pi(x)\pi(y) = \omega(x, y)\pi(xy). \tag{2.10}$$

It follows that ω is a multiplier on G. π is said to be a projective representation of G on $\mathfrak{h}$ with multiplier ω. Moreover to every multiplier on G there corresponds a particular projective representation π_ω of G on $L_2(G)$ defined for each pair x, y of elements of G and each element ϕ of $L_2(G)$ by

$$(\pi_\omega(x)\phi)(y) = \omega(x, x^{-1}y)\phi(x^{-1}y). \tag{2.11}$$

Let π_1, π_2 be projective representations of G on $\mathfrak{h}_1$, $\mathfrak{h}_2$ respectively both with multiplier ω and let $R(\pi_1, \pi_2)$ denote the set of operators U from $\mathfrak{h}_1$ to $\mathfrak{h}_2$ such that for each element x of G, $U\pi_1(x) = \pi_2(x)U$. When there exists a unitary operator in $R(\pi_1, \pi_2)$, π_1, π_2 are said to be unitarily equivalent. A projective representation π is said to be irreducible if $R(\pi, \pi)$ consists only of the zero operator and multiples of the identity.

Let $\mathfrak{U}$ be a C*-algebra; $\mathfrak{U}$ is said to be liminal if for every irreducible representation π of $\mathfrak{U}$ and every element S of $\mathfrak{U}$, $\pi(S)$ is compact; $\mathfrak{U}$ is said to be post-liminal if every non-null quotient algebra of $\mathfrak{U}$ has a non-null liminal closed two-sided ideal; $\mathfrak{U}$ is said to be of Type I if for every representation π of $\mathfrak{U}$, the weak closure $\pi(\mathfrak{U})^-$ of $\pi(\mathfrak{U})$ is a von Neumann algebra of Type I (for the definitions and properties of C*-algebras and von Neumann algebras see [*6*] [*7*]). $\mathfrak{U}$ is of Type I if and only if it is post-liminal [*10*] [*19*]. The dual $\mathfrak{U}^\wedge$ of the separable C*-algebra $\mathfrak{U}$ is the set of unitary equivalence classes of irreducible representations of $\mathfrak{U}$. $\mathfrak{U}^\wedge$ is a Borel space [*14*]. $\mathfrak{U}$ is said to have a smooth dual if there exists a countable family of Borel subsets of $\mathfrak{U}^\wedge$ which separate points in $\mathfrak{U}$. $\mathfrak{U}$ is of Type I if and only if $\mathfrak{U}^\wedge$ is smooth [*9*] [*10*].

Let $C^*(G)$ be the C^*-group algebra of G (for the definitions and properties of $C^*(G)$ see [*6*]). Then G is said to be liminal, post-liminal, or of Type I according as $C^*(G)$ is liminal, post-liminal or of Type I.

§ 3. THE C*-ALGEBRA C*(G, ω)

Let ω be a multiplier on G and let $L_1(G, \omega)$ be the corresponding twisted group algebra over G.

LEMMA 3.1. — $L_1(G, \omega)$ possesses an approximate identity.

Proof. — Let G^ω be the central group extension of T by G defined by ω and let $L_1(G^\omega)$ be the group algebra of G. Then it follows from 20.27 of [*12*] that $L_1(G^\omega)$ possesses an approximate identity $\{ \Xi_i : i \in \Lambda \}$. The dual $T^\wedge$ of T is isomorphic to the additive group of integers and so it follows that the mapping $1 : \Psi \mapsto 1(\Psi)$ defined by

$$1(\Psi) = \int_T t\Psi(t, x)d\lambda(t) \tag{3.1}$$

where λ is the normalised Haar measure on T and Ψ is an element of $L_1(G^\omega)$, is a norm non-increasing *-homomorphism onto $L_1(G, \omega)$. A simple calculation shows that $\{ 1(\Xi_i) : i \in \Lambda \}$ is an approximate identity for $L_1(G, \omega)$.

LEMMA 3.2. — $L_1(G, \omega)$ possesses a faithful *-representation.

Proof. — Let ψ be an element of $L_1(G)$, let ϕ be an element of $L_2(G)$ and let $\psi \,\omega\, \phi$ be the function defined for each element x of G by

$$(\psi \,\omega\, \phi)(x) = \int_G \psi(y)\phi(y^{-1}x)\omega(y, y^{-1}x)dm(y). \tag{3.2}$$

Then, for each element ϕ' of $L_2(G)$,

$$\left| \int_G \overline{\phi'(x)}(\psi \,\omega\, \phi)(x)dm(x) \right| \leqslant \int_G\int_G |\phi'(x)|\,|\phi(y^{-1}x)|\,|\psi(y)|\,dm(y)dm(x) = \|\phi'\|_2 \|\phi\|_2 \|\psi\|_1 \tag{3.3}$$

from which it follows that $\psi\,\omega\,\phi$ is an element of $L_2(G)$. Let $\pi(\psi)\phi = \psi\,\omega\,\phi$. Then $\pi(\psi)$ is a bounded linear operator on $L_2(G)$. Simple calculations show that π is an essential *-representation of $L_1(G, \omega)$ on $L_2(G)$.

Let ψ be an element of $L_1(G, \omega)$ such that for each element ϕ of $L_2(G)$, $\psi\,\omega\,\phi = 0$. In particular for each function ϕ which is continuous and of compact support $\psi\,\omega\,\phi = 0$ and so making the change of variable $y \to xy^{-1}$ in (3.2) it follows that for each element y of G,

$$\psi(xy^{-1})\omega(xy^{-1}, y)\delta(y^{-1}) = 0.$$

Choosing $y = e$ it follows that $\psi(x) = 0$, m-almost everywhere. Hence $\psi = 0$ and so π is a faithful representation of $L_1(G, \omega)$.

Let $\mathfrak{R}(G, \omega)$ be the set of essential *-representations of $L_1(G, \omega)$ on Hilbert space and let $\mathfrak{R}'(G, \omega)$ be the subset consisting if irreducible representations. It follows from Lemma 3.2 that the mapping $\|\ .\ \|$ defined for each element ψ of $L_1(G, \omega)$ by

$$\|\psi\| = \sup_{\pi \in \mathfrak{R}(G,\omega)} \|\pi(\psi)\| \tag{3.4}$$

is a norm on $L_1(G, \omega)$. Also Lemma 3.1 and Propn. 2.7.4 of [*6*] show that

$$\|\psi\| = \sup_{\pi \in \mathfrak{R}'(G,\omega)} \|\pi(\psi)\| \tag{3.5}$$

and that the completion of $L_1(G, \omega)$ with respect to this norm is a C*-algebra $C^*(G, \omega)$. $C^*(G, \omega)$ is said to be the twisted C*-group algebra over G corresponding to ω.

§ 4. REPRESENTATIONS OF $L_1(G^f)$

Let A be a separable compact abelian group and let G^f be the central group extension of A by G corresponding to the element f of $Z^2(G, A)$. Let $A^\wedge$ be the dual of A. Let $\mathfrak{R}(G^f)$ be the set of essential *-representations of $L_1(G^f)$ and let $\mathfrak{R}'(G^f)$ be the subset of $\mathfrak{R}(G^f)$ consisting of irreducible representations. Let $\mathfrak{P}(G^f)$ be the set of strongly continuous unitary representations of G^f and let $\mathfrak{P}'(G^f)$ be the subset of $\mathfrak{P}(G^f)$ consisting of irreducible representations. For each element α of $A^\wedge$ let $\mathfrak{R}(G, \alpha \circ f)$ be the set of essential *-representations of $L_1(G, \alpha \circ f)$ and let $\mathfrak{R}'(G, \alpha \circ f)$ be the subset of $\mathfrak{R}(G, \alpha \circ f)$ consisting of irreducible representations. Let $\mathfrak{P}(G, \alpha \circ f)$ be the set of projective representations of G with multiplier $\alpha \circ f$

and let $\mathfrak{P}'(G, \alpha \circ f)$ be the subset of $\mathfrak{P}(G, \alpha \circ f)$ consisting of irreducible representations.

Let Π be an element of $\mathfrak{R}(G^f)$ and let α be an element of $A^\wedge$. Let $\alpha^\sim(\Pi)$ denote the mapping defined for each element ψ of $L_1(G, \alpha \circ f)$ by

$$\alpha^\sim(\Pi)(\psi) = \pi(\bar{\alpha} \otimes \psi). \tag{4.1}$$

Let π be an element of $\mathfrak{R}(G, \alpha \circ f)$. Let $\alpha'(\pi)$ be the mapping defined for each element Ψ of $L_1(G^f)$ by

$$\alpha'(\pi)(\Psi) = \pi(\alpha(\Psi)). \tag{4.2}$$

The proofs of the following results are given in [8]. Our first result describes the properties of the mappings $\alpha^\sim$, α'.

THEOREM 4.1. — (*i*) $\alpha^\sim$ maps $\mathfrak{R}(G^f)$ onto $\mathfrak{R}(G, \alpha \circ f)$ and $\mathfrak{R}'(G^f)$ onto $\mathfrak{R}'(G, \alpha \circ f)$.

(*ii*) α' maps $\mathfrak{R}(G, \alpha \circ f)$ one to one onto a subset $\mathfrak{R}(G^f, \alpha)$ of $\mathfrak{R}(G^f)$ and $\mathfrak{R}'(G, \alpha \circ f)$ one to one onto $\mathfrak{R}'(G^f, \alpha) = \mathfrak{R}'(G^f) \cap \mathfrak{R}(G^f, \alpha)$.

(*iii*) $\alpha^\sim\alpha'$ is the identity on $\mathfrak{R}(G, \alpha \circ f)$ and $\alpha'\alpha^\sim$ is the identity on $\mathfrak{R}(G^f, \alpha)$.

(*iv*) For each distinct pair α, β of elements of $A^\wedge$, $\alpha^\sim\beta' = 0$ on $\mathfrak{R}(G, \beta \circ f)$ and so $\mathfrak{R}(G^f, \alpha) \cap \mathfrak{R}(G^f, \beta) = \Phi$.

(*v*) Every element Π of $\mathfrak{R}(G^f)$ has a unique decomposition into elements of $\mathfrak{R}(G^f, \alpha)$, $\alpha \in A^\wedge$ of the form

$$\Pi = \bigoplus_{\alpha \in A^\wedge} \alpha'\alpha^\sim\Pi \tag{4.3}$$

and so every element of $\mathfrak{R}'(G^f)$ is an element of $\mathfrak{R}'(G^f, \alpha)$ for some α. Hence

$$\mathfrak{R}'(G^f) = \bigcup_{\alpha \in A^\wedge} \mathfrak{R}'(G^f, \alpha)$$

where $\mathfrak{R}'(G^f, \alpha) \cap \mathfrak{R}'(G^f, \beta) = \Phi$, $\alpha \neq \beta$.

(*vi*) The mappings $\alpha^\sim$, α' preserve unitary equivalence.

Let Π be an element of $\mathfrak{P}(G^f)$ and for each element α of $A^\wedge$ let $\alpha^\sim(\Pi)$ be the mapping defined for each element x of G by

$$\alpha^\sim(\Pi)(x) = \Pi(0, x). \tag{4.4}$$

Let π be an element of $\mathfrak{P}(G, \alpha \circ f)$ and let $\alpha'(\pi)$ be the mapping defined for each element (a, x) of G^f by

$$\alpha'(\pi)(a, x) = \alpha(a)\pi(x). \tag{4.5}$$

THEOREM 4.2. — Theorem 4.1 holds when $\mathfrak{R}$ is replaced by $\mathfrak{P}$.

The classical Gelfand-Naimark theorem (Propn. 6.28 of [17]) may be stated in the following way. There exists a one to one map i from $\mathfrak{R}(G^f)$ onto $\mathfrak{P}(G^f)$ which maps $\mathfrak{R}'(G^f)$ onto $\mathfrak{P}'(G^f)$ and preserves unitary equivalence. This mapping is defined for each element Π of $\mathfrak{R}(G^f)$, each element Ψ of $L_1(G^f)$ and each pair ξ, η of elements of the Hilbert space $\mathfrak{h}$ on which Π is defined by

$$(\Pi(\Psi)\xi, \eta) = \int_{G^f} \Psi(a, x)((i\Pi)(a, x)\xi, \eta)dv(a)dm(x). \tag{4.6}$$

Making use of this result and those of Theorems 4.1-2, the following theorem is easily proved.

THEOREM 4.3. — (*i*) There exists a one to one mapping i_α from $\mathfrak{R}(G, \alpha \circ f)$ onto $\mathfrak{P}(G, \alpha \circ f)$ which maps $\mathfrak{R}'(G, \alpha \circ f)$ onto $\mathfrak{P}'(G, \alpha \circ f)$ and preserves unitary equivalence. Then, for each element π of $\mathfrak{R}(G, \alpha \circ f)$, each element ψ of $L_1(G, \alpha \circ f)$ and each pair ξ, η of elements of the Hilbert space $\mathfrak{h}$ on which π is defined,

$$(\pi(\psi)\xi, \eta) = \int_{G} \psi(x)((i_\alpha(\pi))(x)\xi, \eta)dm(x). \tag{4.7}$$

(*ii*) $i_\alpha \alpha^\sim = \alpha^\sim i$ on $\mathfrak{R}(G^f)$ and $\mathfrak{R}'(G^f)$.

Since no confusion arises, in what follows we will simply denote $i\Pi$ and $i_\alpha \pi$ by the symbols Π, π.

§ 5. THE STRUCTURE OF $C^*(G^f)$

$C^*(G^f)$ is the completion of $L_1(G^f)$ with respect to the norm defined for each element Ψ of $L_1(G^f)$ by

$$\| \Psi \| = \sup_{\Pi \in \mathfrak{R}'(G^f)} \| \Pi(\Psi) \| = \sup_{\Pi \in \mathfrak{R}'(G^f)} \| \Pi(\Psi) \|. \tag{5.1}$$

The results of [8] may be extended from $L_1(G^f)$ to $C^*(G^f)$ in the following way.

LEMMA 5.1. — Let $\psi \mapsto \bar{\alpha} \otimes \psi$ be the isometric *-isomorphism from $L_1(G, \alpha \circ f)$ into $L_1(G^f)$ defined by (2.8). This mapping has a unique extension to an isometric *-isomorphism from $C^*(G, \alpha \circ f)$ into $C^*(G^f)$.

Proof. — Let ψ be an element of $C^*(G, \psi \circ f)$ and let $\{ \psi_n \}$ be a sequence of elements of $L_1(G, \alpha \circ f)$ converging to ψ. Then, for each pair m, n of integers

$$\begin{aligned} \| \bar{\alpha} \otimes \psi_n - \bar{\alpha} \otimes \psi_m \| &= \sup_{\Pi \in \mathfrak{R}'(G^f)} \| \Pi(\bar{\alpha} \otimes \psi_n) - \Pi(\bar{\alpha} \otimes \psi_m) \| \\ &= \sup_{\Pi \in \mathfrak{R}'(G^f)} \| \alpha^{\sim}(\Pi)(\psi_n - \psi_m) \| \\ &= \sup_{\pi \in \mathfrak{R}'(G, \alpha \circ f)} \| \pi(\psi_n - \psi_m) \| \\ &= \| \psi_n - \psi_m \|. \end{aligned} \tag{5.2}$$

Hence there exists an element $\alpha \otimes \psi$ of $C^*(G^f)$ such that $\bar{\alpha} \otimes \psi_n \to \bar{\alpha} \otimes \psi$. Simple limit arguments show that the mapping $\psi \mapsto \alpha \otimes \psi$ is a *-homomorphism into $C^*(G^f)$ and (5.2) shows the mapping to be an isometry.

LEMMA 5.2. — Let $\alpha : \Psi \to \alpha(\Psi)$ be the norm non-increasing *-homomorphism from $L_1(G^f)$ onto $L_1(G, \alpha \circ f)$ defined by (2.7). Then α has a unique extension to a norm non-increasing *-homomorphism from $C^*(G^f)$ onto $C^*(G, \alpha \circ f)$.

Proof. — Let Ψ be an element of $C^*(G^f)$ and let $\{ \Psi_n \}$ be a sequence of elements of $L_1(G^f)$ converging to Ψ. Then, for each pair m, n of integers,

$$\begin{aligned} \| \alpha(\Psi_n) - \alpha(\Psi_m) \| &= \sup_{\pi \in \mathfrak{R}'(G, \alpha \circ f)} \| \pi(\alpha(\Psi_n)) - \pi(\alpha(\Psi_m)) \| \\ &= \sup_{\pi \in \mathfrak{R}'(G, \alpha \circ f)} \| \alpha'(\pi)(\Psi_n - \Psi_m) \| \\ &= \sup_{\Pi \in \mathfrak{R}'(G^f, \alpha)} \| \Pi(\Psi_n - \Psi_m) \| \end{aligned}$$

by Theorem 4.1 (*ii*),

$$\begin{aligned} &\leqslant \sup_{\Pi \in \mathfrak{R}'(G^f)} \| \Pi(\Psi_n - \Psi_m) \| \\ &= \| \Psi_n - \Psi_m \|. \end{aligned} \tag{5.3}$$

Hence there exists an element $\alpha(\Psi)$ of $C^*(G, \alpha \circ f)$ such that $\alpha(\Psi_n) \to \alpha(\Psi)$. Simple calculations show that α is a *-homomorphism from $C^*(G^f)$ to $C^*(G, \alpha \circ f)$ and (5.3) shows that α is a norm non-increasing. Let ψ be an arbitrary element of $C^*(G, \alpha \circ f)$. Then, clearly $\bar{\alpha} \otimes \psi$ is an element of $C^*(G^f)$ such that $\alpha(\bar{\alpha} \otimes \psi) = \psi$. Hence α maps onto $C^*(G, \alpha \circ f)$.

THEOREM 5.3. — $C^*(G^f)$ is the direct sum over $A^\wedge$ of closed two-sided ideals $C^*(G^f, \alpha)$ where $C^*(G^f, \alpha)$ is isometrically *-isomorphic to $C^*(G, \alpha \circ f)$.

Proof. — $L_1(G^f)$ is the direct sum over $A^\wedge$ of closed two-sided ideals $L_1(G^f, \alpha)$ where $L_1(G^f, \alpha)$ is isometrically *-isomorphic to $L_1(G, \alpha \circ f)$.

It follows from Lemmas 3.1-2 that $L_1(G^f, \alpha)$ is a Banach *-algebra possessing an approximate identity and a faithful *-representation. Therefore, using Theorem 4.1 (*ii*) and Propn. 2.7.4 of [6] its completion with respect to the norm $\| \,.\, \|_\alpha$ defined for each element Ψ of $L_1(G^f, \alpha)$ by

$$\| \Psi \|_\alpha = \sup_{\Pi \in \mathfrak{R}'(G^f, \alpha)} \| \Pi(\Psi) \|$$

is a C*-algebra $C^*(G^f, \alpha)$. We will show that $C^*(G^f, \alpha)$ is identical to the image J_α of $C^*(G, \alpha \circ f)$ under the mapping $\psi \mapsto \alpha \otimes \psi$ defined in Lemma 5.1.

First notice that Theorem 4.1 (*v*) shows that for each element Ψ of $L_1(G^f, \alpha)$,

$$\| \Psi \|_\alpha = \sup_{\Pi \in \mathfrak{R}'(G^f, \alpha)} \| \Pi(\Psi) \| = \sup_{\Pi \in \mathfrak{R}'(G^f)} \| \Pi(\Psi) \| = \| \Psi \|.$$

Hence the norms on $C^*(G^f, \alpha)$ and J_α are identical. Since $L_1(G, \alpha \circ f)$ is contained in $C^*(G, \alpha \circ f)$, $L_1(G^f, \alpha)$ is contained in J_α and so it remains to show that J_α is contained in $C^*(G^f, \alpha)$. Let Ψ be an element of J_α. Then there exists a sequence $\{ \psi_n \}$ of elements of $L_1(G, \alpha \circ f)$ such that $\bar{\alpha} \otimes \psi_n \rightarrow \Psi$. It follows that Ψ lies in the completion of $L_1(G^f, \alpha)$ which is of course $C^*(G^f, \alpha)$. Hence $C^*(G^f, \alpha) = J_\alpha$.

It remains to prove that the family $\{ C^*(G^f, \alpha), \alpha \in A^\wedge \}$ have the required properties. Since $C^*(G^f, \alpha)$ is the isometric image of $C^*(G, \alpha \circ f)$, an easy calculation shows that $C^*(G^f, \alpha)$ is closed in $C^*(G^f)$. Further, for each pair Ψ_1, Ψ_2 of elements of $L_1(G^f)$,

$$\bar{\alpha} \otimes \alpha(\Psi_1 * \Psi_2) = \bar{\alpha} \otimes \alpha(\Psi_1) * \Psi_2 = \Psi_1 * \bar{\alpha} \otimes \alpha(\Psi_2) \tag{5.4}$$

and simple limit arguments show that the same result holds in $C^*(G^f)$. It follows that $C^*(G^f, \alpha)$ is a two-sided ideal in $C^*(G^f)$. Also, for each pair α, β of elements of $A^\wedge$ and each element ψ of $L_1(G, \alpha \circ f)$,

$$\begin{aligned} (\bar{\alpha} \otimes \beta)\psi &= 0 \quad \text{if} \quad \alpha \neq \beta \\ &= \psi \quad \text{if} \quad \alpha = \beta \end{aligned} \tag{5.5}$$

from which it follows that

$$(\bar{\alpha} \otimes \alpha)(\bar{\beta} \otimes \beta) = \delta_{\alpha\beta}(\bar{\alpha} \otimes \alpha) \tag{5.6}$$

and

$$\sum_{\alpha \in A^\wedge} \bar{\alpha} \otimes \alpha = 1 \tag{5.7}$$

on $L_1(G^f)$. Simple limit arguments show that the same applies on $C^*(G^f)$. This completes the proof of the theorem.

Propn. 2.7.4 of [6] shows that there is a one-to-one correspondence between essential *-representations of a Banach *-algebra possessing a faithful *-representation and an approximate identity and essential *-representations of its C*-completion. Moreover this correspondence preserves irreducibility and unitary equivalence. It follows that we may regard $\mathfrak{R}(G^f)$, $\mathfrak{R}'(G^f)$, $\mathfrak{R}(G, \alpha \circ f)$, $\mathfrak{R}'(G, \alpha \circ f)$ as consisting of representations of the corresponding C*-algebras.

Theorem 5.3 allows a study of the properties of $C^*(G^f)$ to be made by considering the properties of the closed two-sided ideals $C^*(G^f, \alpha)$. It is therefore necessary to discuss the properties of $C^*(G, \alpha \circ f)$ in some detail.

§ 6. PROPERTIES OF C*(G, ω)

Let ω be an arbitrary multiplier on G and let $C^*(G, \omega)$ be the corresponding twisted C*-group algebra. Certain properties of $C^*(G, \omega)$ are immediate consequences of the corresponding properties of $L_1(G, \omega)$.

THEOREM 6.1. — $C^*(G, \omega)$ possesses an identity if G is discrete.

Proof. — Since G is discrete Theorem 5 of [8] shows that $L_1(G, \omega)$ possesses an identity E. Since $L_1(G, \omega)$ is dense in $C^*(G, \omega)$, E is an identity in $C^*(G, \omega)$.

For each element x of G and each element ψ of $L_1(G, \omega)$ let ${}^{\omega}_{x}\psi$ be the function defined for each element y of G by

$$ {}^{\omega}_{x}\psi(y) = \omega(x, x^{-1}y)\psi(x^{-1}y). \tag{6.1} $$

Then the mapping $\psi \mapsto {}^{\omega}_{x}\psi$ is isometric and linear from $L_1(G, \omega)$ onto itself. Also, for each pair x, y of elements of G and each element of $L_1(G)$,

$$ {}^{\omega}_{x}({}^{\omega}_{y}\psi) = \omega(x, y)\,{}^{\omega}_{xy}\psi. \tag{6.2} $$

Before proving the next theorem the following Lemmas are required.

LEMMA 6.2. — For each element x of G, the mapping $\psi \mapsto {}^{\omega}_{x}\psi$ has a unique extension to an isometry on $C^*(G, \omega)$.

Proof. — Let ψ be an element of $C^*(G, \omega)$ and let $\{ \psi_n \}$ be a sequence of elements of $L_1(G, \omega)$ such that $\psi_n \to \psi$. Then, for each pair m, n of integers.

$$\| {}^{\omega}_{x}(\psi_n) - {}^{\omega}_{x}(\psi_m) \| = \sup_{\pi \in \mathfrak{R}'(G,\omega)} \| \pi({}^{\omega}_{x}(\psi_n) - {}^{\omega}_{x}(\psi_m)) \|. \tag{6.3}$$

Let π be the element of $\mathfrak{P}'(G, \omega)$ corresponding to an element π of $\mathfrak{R}'(G, \omega)$ according to Theorem 4.3. Then it is easily proved that for each element x of G and each element ψ' of $L_1(G, \omega)$,

$$\pi({}^{\omega}_{x}\psi') = \pi(x)\pi(\psi'). \tag{6.4}$$

It follows from (6.3) and (6.4) that

$$\begin{aligned} \| {}^{\omega}_{x}(\psi_n) - {}^{\omega}_{x}(\psi_m) \| &= \sup_{\pi \in \mathfrak{R}'(G,\omega)} \| \pi(x)(\pi(\psi_n) - \pi(\psi_m)) \| \\ &\leqslant \sup_{\pi \in \mathfrak{R}'(G,\omega)} \| \pi(\psi_n) - \pi(\psi_m) \| \end{aligned}$$

since $\pi(x)$ is unitary,

$$= \| \psi_n - \psi_m \|. \tag{6.5}$$

Hence $\{ {}^{\omega}_{x}(\psi_n) \}$ converges to an element ${}^{\omega}_{x}\psi$ of $C^*(G, \omega)$. It now follows easily that the mapping $\psi \mapsto {}^{\omega}_{\alpha}\psi$ is linear from $C^*(G, \omega)$ to itself and is such that for each pair x, y of elements of G, (6.2) holds. Also (6.5) shows that the mapping is norm non-increasing. Then, replacing y by x^{-1} in (6.2) it follows that the mapping has a norm non-increasing inverse. This completes the proof.

LEMMA 6.3. — (*i*) Let ψ, ψ' be elements of $C^*(G, \omega)$ and let x be an element of G. Then,

$${}^{\omega}_{x}(\psi \, \omega \, \psi') = {}^{\omega}_{x}\psi \, \omega \, \psi'. \tag{6.6}$$

(*ii*) Let $\{ \chi_i : i \in \Lambda \}$ be an approximate identity for $L_1(G, \omega)$. Then $\{ \chi_i : i \in \Lambda \}$ is an approximate identity for $C^*(G, \omega)$.

Proof. — Both these results are immediately proved by simple limit arguments applied to elements of $L_1(G, \omega)$.

THEOREM 6.4. — The mapping $\psi \mapsto {}^{\omega}_{x}\psi$ on $C^*(G, \omega)$ maps every closed left-ideal in $C^*(G, \omega)$ into itself.

Proof. — Let $\mathfrak{J}$ be a closed left-ideal in $C^*(G, \omega)$; let ψ be an element of $\mathfrak{J}$ and let x be an element of G. Then, for each element i of Λ,

$$\| {}^{\omega}_{x}(\chi_i) \, \omega \, \psi - {}^{\omega}_{x}\psi \| = \| {}^{\omega}_{x}(\chi_i \, \omega \, \psi - \psi) \|$$

by Lemma 6.3 (i),

$$= \| \chi_i \, \omega \, \psi - \psi \|$$

by Lemma 6.2. Hence $\{ {}^{\omega}_{x}(\chi_i) \, \omega \, \psi : i \in \Lambda \}$ is a sequence of elements of $\mathfrak{J}$ converging to ${}^{\omega}_{x}\psi$. Since $\mathfrak{J}$ is closed, ${}^{\omega}_{x}\psi$ is an element of $\mathfrak{J}$.

THEOREM 6.5. — $C^*(G, \omega)$ is commutative if and only if G is abelian and ω is trivial.

Proof. — Let $C^*(G, \omega)$ be commutative. Then, $L_1(G, \omega)$ is commutative and so for each pair ψ, ψ' of elements of $L_1(G, \omega)$ and m-almost each element x of G,

$$\begin{aligned}\int_G \psi(y)\psi'(y^{-1}x)\omega(y, y^{-1}x)dm(y) &= (\psi \, \omega \, \psi')(x) \\ &= (\psi' \, \omega \, \psi)(x) \\ &= \int_G \psi'(y)\psi(y^{-1}x)\omega(y, y^{-1}x)dm(y) \\ &= \int_G \psi(y)\psi'(xy^{-1})\omega(xy^{-1}, y)\delta(y^{-1})dm(y). \quad (6.7)\end{aligned}$$

In particular (6.7) holds for each continuous function ψ of compact support and so it follows that for each element y of G,

$$\psi'(y^{-1}x)\omega(y, \ y^{-1}x) = \psi'(xy^{-1})\omega(xy^{-1}, \ y)\delta(y^{-1}). \qquad (6.8)$$

Therefore, putting $y = x$, it follows that for m-almost each element x of G, $\delta(x^{-1}) = 1$. But since δ is continuous it follows that G is unimodular. Therefore, for each element y of G and m-almost each element x of G,

$$\psi'(y^{-1}x)\omega(y, \ y^{-1}x) = \psi'(xy^{-1})\omega(xy^{-1}, \ y). \qquad (6.9)$$

Hence,

$$\int_G \psi'(y)\omega(xy^{-1}, y)dm(y) = \int_G \psi'(y)\omega(y, y^{-1}x)dm(y). \qquad (6.10)$$

In particular (6.10) holds for every continuous function ψ' of compact support and so it follows that for each element y of G,

$$\omega(xy^{-1}, y) = \omega(y, y^{-1}x). \qquad (6.11)$$

It follows from (6.9) that for each element y of G and m-almost each element x of G, $\psi'(y^{-1}x) = \psi'(xy^{-1})$. When ψ' is chosen to be continuous and of compact support both sides of this expression are continuous functions of x and so the expression is valid for all x and y in G. It follows that G is abelian. Further (6.9) shows that ω is symmetric, and so it follows from Corollary 3 of Propn. 18.4 of [5] that ω is trivial.

Conversely, let G be abelian and let ω be trivial. Then a simple calculation shows that $L_1(G, \omega)$ is commutative. Since $L_1(G, \omega)$ is dense in $C^*(G, \omega)$ it follows that $C^*(G, \omega)$ is commutative.

§ 7. PROPERTIES OF $C^*(G^f)$

In this section, the results of § 6 are used to study the properties of $C^*(G^f)$ where G^f is the central extension of A by G corresponding to f.

THEOREM 7.1. — $C^*(G^f)$ is commutative if and only if $C^*(G, \alpha \circ f)$ is commutative for each element α of $A^\wedge$.

Proof. — Let $C^*(G^f)$ be commutative. Then, for each element α of $A^\wedge$ the closed two-sided ideal $C^*(G^f, \alpha)$ is commutative. Since $C^*(G^f, \alpha)$ is isometrically *-isomorphic to $C^*(G, \alpha \circ f)$ it follows that $C^*(G, \alpha \circ f)$ is commutative.

Conversely, let $C^*(G, \alpha \circ f)$ be commutative for each element α of $A^\wedge$. Let Ψ, Ψ' be elements of $C^*(G^f)$. Then,

$$\Psi * \Psi' = \sum_{\alpha \in A^\wedge} \sum_{\beta \in A^\wedge} \bar{\alpha} \otimes \alpha(\Psi) * \bar{\beta} \otimes \beta(\Psi')$$
$$= \sum_{\alpha \in A^\wedge} \bar{\alpha} \otimes \alpha(\Psi * \Psi')$$

using (5.6-7),

$$= \sum_{\alpha \in A^\wedge} \bar{\alpha} \otimes \alpha(\Psi) * \bar{\alpha} \otimes \alpha(\Psi')$$
$$= \sum_{\alpha \in A^\wedge} \bar{\alpha} \otimes \alpha(\Psi') * \bar{\alpha} \otimes \alpha(\Psi)$$
$$= \Psi' * \Psi.$$

Hence $C^*(G^f)$ is commutative.

THEOREM 7.2. — $C^*(G^f)$ is liminal if and only if $C^*(G, \alpha \circ f)$ is liminal for each element α of $A^\wedge$.

Proof. — Let $C^*(G^f)$ be liminal. Then, for each element Π of $\mathfrak{R}'(G^f)$ and each element Ψ of $C^*(G^f)$, $\Pi(\Psi)$ is a compact operator. Hence, for each element Π of $\mathfrak{R}'(G^f, \alpha)$ and each element Ψ of $C^*(G^f)$, $\Pi(\Psi)$ is a compact operator. However Theorem 4.1 shows that $\alpha'\alpha^\sim\Pi = \Pi$ and so it follows that $\alpha^\sim(\Pi)(\alpha(\Psi)) = \Pi(\Psi)$. But $\alpha^\sim$ maps $\mathfrak{R}'(G^f, \alpha)$ onto $\mathfrak{R}'(G, \alpha \circ f)$ and α maps $C^*(G^f)$ onto $C^*(G, \alpha \circ f)$. Hence, for each element π of $\mathfrak{R}'(G, \alpha \circ f)$ and each element ψ of $C^*(G, \alpha \circ f)$, $\pi(\psi)$ is a compact operator. Hence $C^*(G, \alpha \circ f)$ is liminal.

Conversely, let $C^*(G, \alpha \circ f)$ be liminal for each element α of $A^\wedge$. Then, for each element π of $\mathfrak{R}'(G, \alpha \circ f)$ and each element ψ of $C^*(G, \alpha \circ f)$, $\pi(\psi)$ is a compact operator. Hence, for each element Π of $\mathfrak{R}'(G^f, \alpha)$ and each element Ψ of $C^*(G^f)$, $\Pi(\Psi)$ is compact. It follows from Theorem 4.1 (*v*) that for each element Π of $\mathfrak{R}'(G^f)$ and each element Ψ of $C^*(G^f)$, $\Pi(\Psi)$ is compact. It therefore follows that $C^*(G^f)$ is liminal.

THEOREM 7.3. — $C^*(G^f)$ is post-liminal if and only if $C^*(G, \alpha \circ f)$ is post-liminal for each element α of $A^\wedge$.

Proof. — Let $C^*(G^f)$ be post-liminal. Since G^f is separable, $C^*(G^f)$ is separable and hence of Type I [*10*]. But $C^*(G, \alpha \circ f)$ is isometrically *-isomorphic to the closed two-sided ideal $C^*(G^f, \alpha)$ of $C^*(G^f)$. Hence it follows from 5.7.4 of [*6*] and [*22*] that $C^*(G^f, \alpha)$ is of Type I and being separable is therefore post-liminal.

Conversely, let $C^*(G, \alpha \circ f)$ be post-liminal. Then $C^*(G, \alpha \circ f)$ is of Type I as above. Let Π be an arbitrary element of $\mathfrak{R}(G^f, \alpha)$. Then Theorem 4.1 (*v*) shows that Π may be written as the direct sum of elements of $\mathfrak{R}(G^f, \alpha)$. But since $\mathfrak{R}(G^f, \alpha)$ is isomorphie to $\mathfrak{R}(G, \alpha \circ f)$ every element of $\mathfrak{R}(G^f, \alpha)$ is of Type I. It follows that Π is of Type I. Therefore $C^*(G^f)$ is post-liminal.

§ 8. EXAMPLES

(*a*) Let G be an abelian group and let $f = 0$. Then, $G^f = A \times G$ and for each element α of $A^\wedge$, $\mathfrak{R}(G, \alpha \circ f) = \mathfrak{R}(G)$. $\mathfrak{R}'(G)$ may be identified with the dual $G^\wedge$ of G and so $C^*(G)$ may be identified with the algebra $C_0(G^\wedge)$ of continuous functions ψ on $G^\wedge$ which take arbitrarily, small values outside compact subsets (see, for example, Chapter 11 of [*18*]).

Since $(A \times G)^\wedge$ is isomorphic to $A^\wedge \times G^\wedge$ the mappings $\psi \mapsto \bar{\alpha} \otimes \psi$

from $L_1(G)$ into $L_1(A \times G)$ and $\Psi \mapsto \alpha(\Psi)$ from $L_1(A \times G)$ onto $L_1(G)$ induce mappings $\psi \mapsto \bar{\alpha} \otimes \psi$, $\Psi \mapsto \alpha(\Psi)$ on dense subspaces of $C_0(G^\wedge)$ and on $C_0(A^\wedge \times G^\wedge)$ respectively, defined by

$$\begin{aligned}(\bar{\alpha} \otimes \psi)(\beta, \gamma) &= 0 && \text{if} \quad \alpha \neq \beta \\ &= \psi(\gamma) && \text{if} \quad \alpha = \beta \end{aligned} \tag{8.1}$$

for each element (β, γ) of $A^\wedge \times G^\wedge$ and

$$\alpha(\Psi)(\gamma) = \Psi(\alpha, \gamma) \tag{8.2}$$

for each element γ of $G^\wedge$. It follows from Lemmas 5.1 and 5.2 that the mapping $\psi \mapsto \bar{\alpha} \otimes \psi$ extends uniquely to an isometric *-isomorphism from $C_0(G^\wedge)$ onto a closed two-sided ideal in $C_0(A^\wedge \times G^\wedge)$, defined by (8.1) and that the mapping $\Psi \mapsto \alpha(\Psi)$ extends uniquely to a norm non-increasing *-homomorphism from $C_0(A^\wedge \times G^\wedge)$ onto $C_0(G^\wedge)$.

The following results, whose proofs by other methods are known, are immediate consequences of the remarks above.

THEOREM 8.1. — Let B be a countable, discrete abelian group and let H be a separable locally compact abelian group.

(*i*) For each element ϕ of $C_0(H)$ and each element b of B, let $\bar{b} \otimes \phi$ be the function on $B \times H$ defined for each element (c, y) of $B \times H$ by,

$$\begin{aligned}(\bar{b} \otimes \phi)(c, y) &= 0 && \text{if} \quad b \neq c \\ &= \phi(y) && \text{if} \quad b = c. \end{aligned} \tag{8.3}$$

Then $\bar{b} \otimes \phi$ is an element of $C_0(B \times H)$ and the mapping $\phi \mapsto \bar{b} \otimes \phi$ is an isometric *-isomorphism onto a closed two-sided ideal in $C_0(B \times H)$.

(*ii*) For each element Φ of $C_0(B \times H)$ and each element b of B, let $b(\Phi)$ be the function on H defined for each element y of H by

$$b(\Phi)(y) = \Phi(b, y). \tag{8.4}$$

Then $b(\Phi)$ is an element of $C_0(H)$ and the mapping $\Phi \mapsto b(\Phi)$ is a norm non-increasing *-homomorphism onto $C_0(H.)$

(*iii*) $C_0(B \times H)$ is the direct sum over B of closed two-sided ideals, each isometrically *-isomorphic to $C_0(H)$.

(*b*) Let $G = R_2$ the direct product of the additive real numbers with itself, let $A = T$ and lef f be defined for each pair $x = (x_1, x_2)$, $y = (y_1, y_2)$ of elements of R_2 by

$$f(x, y) = \exp \frac{1}{2} i(x_1 y_2 - x_2 y_1). \tag{8.5}$$

Let π be a projective representation of R_2 with multiplier f. Then, for each pair x, y of elements of R_2,

$$\pi(x_1, x_2)\pi(y_1, y_2) = \exp\frac{1}{2}i(x_1y_2 - x_2y_1)\pi(x_1 + x_2, y_1 + y_2). \quad (8.6)$$

This is of course the Weyl form of the canonical commutation relations. Von Neumann [*20*] showed that up to unitary equivalence there is a unique irreducible projective representation π of R_2 with multiplier f which may be represented on $L_2(R)$ by

$$\pi(x_1, x_2) = e^{i(x_1Q+x_2P)} \quad (8.7)$$

where Q, P are the self-adjoint operators defined for each element ϕ of $L_2(R)$ and each element s of R by

$$(Q\phi)(s) = s\phi(s), \qquad (P\phi)(s) = id\phi/ds. \quad (8.8)$$

It follows that the norm on $C^*(R_2, f)$ may be defined for each element ψ by $\| \psi \| = \| \pi(\psi) \|$ where

$$\pi(\psi) = \int_{R_2} \psi(x_1, x_2)e^{i(x_1Q+x_2P)}dx_1dx_2. \quad (8.9)$$

Weyl [*21*] suggested that the quantum mechanical observable corresponding to the classical observable, represented by the function ψ' on the phase space R_2 of a one-dimensional system, should be represented by the self-adjoint operator $\pi(\psi)$ defined by (8.9), where ψ is the Fourier transform of ψ'. In this sense one can regard $C^*(R_2, f)$ as the C*-algebra of observables of a one-dimensional quantum-mechanical system in the usual algebraic approach to quantum theory (see for example [*11*]). The same results of course apply for an n-dimensional system when R_{2n} replaces R_2.

The central extension R_2^f of T by R_2 corresponding to f, is a connected, simply connected nilpotent Lie group and as such is post-liminal. It follows from Theorem 7.3 that $C^*(R_2, f)$ is also post-liminal and hence of Type I. There has been some discussion as to the Type of a C*-algebra of observables (see for example [*1*]-[*4*], [*13*]). In the case of a one-dimensional system we have shown the C*-algebra of observables to be of Type I.

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