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Section A :
Physique théorique.

Generalization of the Kaluza-Klein theory for an arbitrary non-abelian gauge group

by

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RÉSUMÉ. — L'isomorphisme entre les théories d'électromagnétisme d'Utiyama et de Kaluza et Klein est généralisé pour un groupe non-abelien semi-simple de jauge. Les équations de champ de jauge non-abelien sont dérivées d'un principe variationnel géométrique. L'interaction entre le champ gravitationnel et le champ généralisé de Yang et Mills est donnée sous une forme explicite. Les équations de mouvement d'une particule dans le champ extérieur sont obtenus à partir de l'équation des géodésiques dans l'espace fibré principal.

SUMMARY. — The isomorphism between the Utiyama and the Kaluza-Klein theories of electromagnetism is generalized to the case of a non-abelian semi-simple gauge group. The equations of the non-abelian gauge field are derived from a geometrical least action principle. The interaction between the gravitational and the generalized Yang-Mills field is given explicitly. The equations of motion of a particle in an external field are obtained from the geodetic equation in the principal fibre bundle space.

1. THE EQUIVALENCE OF THE UTIYAMA AND KALUZA-KLEIN APPROACHES TO THE THEORY OF THE ELECTROMAGNETIC FIELD

Utiyama has put forward a theory of invariant interaction defining the quantities on which the lagrangian invariant with respect to a given gauge group must depend [5]. In the case of electromagnetic field this theory can be regarded as a theory of connections in a principal fibre bundle over a Riemannian manifold V_4 with a structure group isomorphic with a one-dimensional sphere S^1. In this case there exists an isomorphism between the theory of Utiyama and the theory of Kaluza and Klein that treats the electromagnetic and gravitational field in a unified way, as some special metric geometry in a five-dimensional differential manifold. The existence of such an isomorphism was shown recently by Trautman and Tulczyjew [*6*], [*7*].

The possibility of introducing a metric in the principal fibre bundle, which in this case is also a five-dimensional manifold, is inherent in the very structure of a bundle with connection. This special metric is defined by the following requirements:

a) The horizontal subspaces of the tangent space to the bundle must be orthogonal in this metric to the vertical subspaces;

b) The projection of the metric onto the horizontal space must be isomorphic with the Riemannian metric of the base manifold;

c) The vertical part of the metric must be isomorphic to some metric of the space tangent to the fibre, *i. e.* to some metric on the Lie algebra of the structure group.

The only arbitrariness present in the definition of the metric is contained in *c)*; indeed, there are many different metrics on the group. For the Kaluza-Klein theory the group metric is the most trivial, defined by the notion of the natural length in R^1. This choice is equivalent to the « strong cylindricity condition » of the Kaluza-Klein theory (cf. Einstein [*2*]).

Utiyama's theory does not define the lagrangian; it only dictates its dependence on geometric quantities, *i. e.* on the curvature form of the bundle. However the explicit form of this dependence is not defined. In order to generalize the equivalence of the Utiyama and Kaluza-Klein approaches for non-abelian gauge groups, we shall assume that the lagrangian is equal to the scalar density made of the square root of the determinant of the Riemannian metric tensor multiplied by the scalar Riemannian curvature, R, of the bundle manifold. There still remains the problem of choosing a metric on the group space. We shall assume the metric group invariant, and for the case of a semi-simple Lie group we shall take the Killing tensor constructed from the structure constants of the group:

$$g_{ab} = C^c_{ad} C^d_{cb}$$

It is easy to see, that for a one-dimensional abelian gauge group (*i. e.* in the theory of gravitation and electromagnetism) the Riemann scalar of the bundle, constructed with the metric tensor defined above, is equal to the sum of the Riemann scalar of the base manifold and the lagrangian of the electromagnetic field, and that the determinant of the five-dimensional metric tensor is equal to the determinant of the metric tensor of the base. However, it is by no means obvious that for any non-abelian gauge group the Riemannian scalar curvature of the bundle will be the sum of the Riemann scalar of the base and a lagrangian of the Yang-Mills type. Our first aim is to prove this statement.

2. GENERALIZATION OF THE KALUZA-KLEIN THEORY FOR A NON-ABELIAN GAUGE GROUP

Following the program given above, we construct a principal fibre bundle over some Riemannian space-time manifold V_4 with metric tensor g_{ij}, the structure group being some finite-dimensional Lie group G. We assume that there is a connection in the bundle, given by a Lie algebra valued 1-form A on the bundle manifold. The curvature form of this connection, being the covariant differential of the connection form, DA, is interpreted as a generalized Yang-Mills field tensor. We also assume an invariant metric on the structure group, choosing the Killing form in the case of a semi-simple group.

In the following, we restrict our considerations to a neighbourhood of some point of the bundle; it is well known that such a neighbourhood can be regarded as a direct product of some neighbourhood in the base and a neighbourhood in the group space. This suggests choosing special coordinate system natural to the problem. Let the indices $i, j, k, \ldots$ take values from 1 to 4, the indices $a, b, c, \ldots$ – from 4 to M + 4, M = dim G, and the indices $\alpha, \beta, \gamma, \ldots$ – from 1 to M + 4. The natural choice is taking the first 4 of them as the coordinates in the base neighbourhood, and the remaining M as the coordinates in the neighbourhood of the group. When this is done, the differential of the projection onto the base, dp, has the coordinates $dp^i_j = \delta^i_j$ and $dp^i_a = 0$. Similarly, the isomorphism σ of the Lie algebra of the structure group onto the vertical subspace of the tangent space to the bundle has the coordinates

$$\sigma^a_b = \delta^a_b, \qquad \sigma^i_b = 0.$$

The connection form A has coordinates A^a_j, and the curvature form in our coordinate frame is

$$F^a_{ij} = \frac{1}{2}(\partial_i A^a_j - \partial_j A^a_i) + C^a_{bc} A^b_i A^c_j \tag{1}$$

C^a_{bc} being the structure constants of the group G. Other components of the curvature form vanish in our coordinate frame.

The coordinate system in the base neighbourhood can be chosen at will, whereas in the group neighbourhood, in order to simplify the calculations, we choose locally geodesic coordinates. That is, we require the covariant derivative of A^a_i in the group space to vanish; as the connection coefficients in the group space are equal to the structure constants, this means that

$$\partial_a A^b_j + C^b_{ac} A = 0 \tag{2}$$

or

$$\partial_a A^b_j = C^b_{ca} A^c_j \tag{3}$$

because of the antisymmetry of C^a_{bc} in the two lower indices.

Let the metric of the base be g_{ij} and the invariant metric on the group be g_{ab}. Here invariance means that

$$R_a g = g = L_a g \quad , \quad a \in G \tag{4}$$

where R_a and L_a are the right and left translations respectively. Without lack of generality we can reduce (4) to the condition

$$\partial_a g_{bc} = 0 \tag{5}$$

Local triviality of the bundle gives

$$\partial_a g_{ij} = 0 \qquad\qquad \partial_j g_{ab} = 0 \tag{6}$$

Let us now find the explicit form of the bundle metric satisfying the requirements *a), b), c)*. Expressed in coordinates these requirements are

$$dp^i_\alpha dp^j_\beta g^{\alpha\beta} = g^{ij}, \qquad A^a_\alpha A^b_\beta g^{\alpha\beta} = g^{ab} \tag{7}$$

Analogous equations can be written for the covariant tensor:

$$d\tau^\alpha_i d\tau^\beta_j g_{\alpha\beta} = g_{ij}, \qquad \sigma^\alpha_a \sigma^\beta_b g^{\alpha\beta} = g_{ab} \tag{8}$$

where $d\tau^\alpha_i$ is the differential of the lift and σ^α_a is the isomorphism of the

Lie algebra onto the vertical subspace tangent to the fibre. By virtue of the general relations in principal fibre bundles

$$\mathrm{A}^a_\alpha \sigma^\alpha_b = \delta^a_b, \qquad dp^i_\alpha d\tau^\alpha_j = \delta^i_j,$$
$$\mathrm{A}^a_\alpha d\tau^\alpha_j = 0, \qquad dp^i_\alpha \sigma^\alpha_a = 0. \tag{9}$$

one can easily deduce that the only solution for $g_{\alpha\beta}$ and $g^{\alpha\beta}$ satisfying conditions (7) and (8) is:

$$g_{\alpha\beta} = dp^i_\alpha dp^j_\beta g_{ij} + \mathrm{A}^a_\alpha \mathrm{A}^b_\beta g_{ab} \tag{10}$$
$$g^{\alpha\beta} = d\tau^\alpha_i d\tau^\beta_j g^{ij} + \sigma^\alpha_a \sigma^\beta_b g^{ab} \tag{11}$$

In our special coordinates these formulae take the form:

$$g_{\alpha\beta} = \left(\begin{array}{c|c} g_{ij} + g_{ab}\mathrm{A}^a_i \mathrm{A}^b_i & g_{ab}\mathrm{A}^a_i \\ \hline g_{ab}\mathrm{A}^b_j & g_{ab} \end{array}\right) \tag{12}$$

and

$$g^{\alpha\beta} = \left(\begin{array}{c|c} g^{ij} & -g^{ij}\mathrm{A}^b_j \\ \hline -g_{ij}\mathrm{A}^a_i & g^{ab} + g^{ij}\mathrm{A}^a_i \mathrm{A}^b_j \end{array}\right) \tag{13}$$

A simple computation shows that (13) is the inverse of (12) and that the determinant of (12) is equal to the determinant of g_{ij}. With these preparations we can calculate the Christoffel symbols for the bundle manifold. The calculations are done in the coordinate system defined above. It is useful to remark, that the field of vectors $\mathrm{A}^\alpha_a = g^{\alpha\beta} g_{ab} \mathrm{A}^b_\beta$ is a Killing field with respect to our metric. This can be easily seen in our coordinate system. First, one can easily see that the only non vanishing components of A^α_a are $\mathrm{A}^b_a = \delta^b_a$. Now, since the $\partial_c \delta^b_a$ vanish as well as $\partial_a g_{bc}$ one easily obtains

$$\mathrm{A}^\alpha_a \partial_\alpha g_{\beta\gamma} + g_{\alpha\gamma}\partial_\beta \mathrm{A}^\alpha_a + g_{\beta\alpha}\partial_\gamma \mathrm{A}^\alpha_a = 0 \tag{14}$$

After some calculations we obtain the formulae for the connection coefficients in our coordinate system:

$$\Gamma^a_{bc} = 0, \qquad \Gamma^j_{bc} = 0, \qquad \Gamma^i_{bj} = g^{ik} g_{ab} \mathrm{F}^a_{jk},$$
$$\Gamma^a_{bj} = g^{ik} g_{bc} \mathrm{A}^a_k \mathrm{F}^c_{ij} + \mathrm{C}^a_{cb} \mathrm{A}^c_j,$$
$$\Gamma^a_{jk} = \frac{1}{2}(\partial_j \mathrm{A}^a_k + \partial_k \mathrm{A}^a_j) - \left\{ {m \atop jk} \right\} \mathrm{A}^a_m + g^{im} g_{bc} \mathrm{A}^a_m \mathrm{A}^b_k f^c_{ij} + g^{im} g_{bc} \mathrm{A}^a_m \mathrm{A}^b_j f^c_{ik},$$

$$\Gamma^i_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + g^{im} g_{bc} A^b_k f^c_{jm} + g^{im} g_{bc} A^b_j f^c_{km} \tag{15}$$

where we use the notation

$$f^a_{ij} = \frac{1}{2} (\partial_i A^a_j - \partial_j A^a_i) \tag{16}$$

Not all connection coefficients are explicitly covariant with respect to the gauge group; however, if the fully covariant structure constants $C_{abc} = g_{ad} C^d_{bc}$ are antisymmetric in all three indices, we can replace f^a_{ij} by the covariant quantities F^a_{ij} in the last two of connection coefficients without affecting them. In the case when g_{ab} is the Killing metric the full antisymmetry of C_{abc} takes place by virtue of the Jacobi identities.

Taking the lagrangian, by analogy with the Kaluza-Klein theory, equal to $\sqrt{-g}.R$, by well known standard procedure we obtain the Euler-Lagrange equations in the form:

$$\left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) \delta g^{\alpha\beta} = 0 \tag{17}$$

The variations $\delta g^{\alpha\beta}$ shall not be completely arbitrary; we assume that they do not affect the particular structure of the fibre bundle. In other words, the variations $\delta g^{\alpha\beta}$ must not affect the form of the metric (13). Within these conditions the variations of g^{ij} and of A^a_i are arbitrary.

Before computing the explicit form of (17) it is useful to indentify the parts of Euler-Lagrange equations that arise from independent variations of A^a_j and g^{ij}. Replacing $g_{\alpha\beta}$ and $g^{\alpha\beta}$ by their explicit form, we obtain two systems of equations:

$$A^a_k R_{ab} - R_{bk} = 0 \tag{18}$$

corresponding to the variations of A^a_k, and

$$R_{ab} A^a_j A^b_k - 2 R_{bk} A^b_j + R_{jk} - \frac{1}{2} g_{jk} R = 0 \tag{19}$$

corresponding to the variations of g^{jk}.

Using (18) we can reduce equation (19) to the simpler form

$$R_{jk} - \frac{1}{2} g_{jk} R - A^a_j R_{ak} = 0 \tag{20}$$

Equations (18) are interpreted as the field equations, whereas (20) can be regarded as the definition of the energy-momentum tensor for the combined Yang-Mills type and gravitational fields.

After performing some computations we obtain the following expressions for the Ricci tensor:

$$R_{ac} = g^{im}g^{kn}g_{ab}g_{cd}F^b_{mn}F^d_{ik} \tag{21}$$

$$R_{bj} = g^{im}g_{ab}\nabla_i F^a_{jm} - g^{il}g^{km}g_{ab}g_{cd}A^c_j F^a_{kl}F^d_{im} - 2g^{ik}g_{bc}C^c_{ad}A^a_k F^d_{ij} \tag{22}$$

$$\begin{aligned} R_{jk} = {} & K_{jk} + g^{im}g_{ab}A^a_j\nabla_i F^b_{km} + g^{im}g_{ab}A^a_k\nabla_i F^b_{jm} + g^{im}g_{ab}F^a_{ij}F^b_{km} \\ & + g^{im}g_{ab}F^a_{ik}F^b_{jm} - 2g^{im}g_{ab}C^a_{cd}A^c_i A^d_k F^b_{jm} - 2g^{im}g_{ab}C^a_{cd}A^c_i A^d_j F^b_{km} \\ & + C^d_{ac}C^c_{db}A^a_j A^b_k - g^{im}g^{ln}g_{ab}g_{cd}A^a_j A^c_k F^b_{lm}F^d_{in} \\ & + g^{im}g^{ln}g_{ab}g_{cd}A^a_j A^c_m F^b_{kn}F^d_{il} + g^{im}g^{ln}g_{ab}g_{cd}A^a_k A^c_m F^b_{jn}F^d_{il}. \end{aligned} \tag{23}$$

Here ∇_i means covariant differentiation with respect to the Christoffel symbols of the base and K_{jk} the Ricci tensor of the base. Putting these expressions into (18) we obtain

$$g^{ik}g_{ab}\nabla_i F^a_{jk} = 2g^{ik}g_{ab}C^a_{cd}A^c_k F^d_{ij}$$

or, by virtue of the non-degeneracy of the group metric tensor

$$g^{ik}\nabla_i F^a_{jk} = 2g^{ik}C^a_{bc}A^b_k F^c_{ij} \tag{25}$$

For flat space-time these equations are identical with the well known Yang-Mills equations for a non-abelian gauge field. Equation (25) generalizes them to the case of curved space-time, and also provides the form of the interaction between the gravitational and the generalized Yang-Mills field.

To obtain an explicit form of (20) and calculate the energy-momentum tensor we must compute also the Riemannian scalar R. The result is

$$R = K + g^{ik}g^{im}g_{ab}F^a_{ik}F^b_{jm} + g^{jk}C^c_{ad}C^d_{cb}A^a_j A^b_k \tag{26}$$

where K means the Riemann scalar of the base. There is always a lack of invariance due to the last term in (26); nevertheless we can see that this term vanishes by the following considerations: Since

$$C^c_{ad}A^a_j = \partial_d A^c_j \tag{27}$$

we have

$$\begin{aligned} -g^{jk}C^c_{ad}C^d_{cb}A^a_j A^b_k &= g^{jk}(\partial_d A^c_j)(\partial_c A^d_k) \\ &= g^{ik}\partial_d(A^c_j\partial_c A^d_k) - g^{ik}A^c_j\partial^2_{cd}A^d_k = g^{ik}\partial_d(A^c_j\partial_c A^d_k) \end{aligned} \tag{27}$$

because, with our assumptions concerning the full antisymmetry of C_{abc}, we have $C^d_{bd} = 0$, and then $\partial_d A^d_j = 0$.

Now

$$g^{jk}\partial_d(A^c_j\partial_c A^d_k) = \partial_d(g^{jk}A^c_j\partial_c A^d_k) = \partial_d(g^{jk}C^d_{ac}A^c_jA^a_k) = 0 \tag{28}$$

because

$$g^{jk}C^d_{ac}A^c_jA^a_k = 0$$

The final expression for the Lagrangian is:

$$L = \sqrt{-g}R = \sqrt{-g}K + \sqrt{-g}g^{ik}g^{im}g_{ab}F^a_{ik}F^b_{jm} \tag{29}$$

From the eq. (20) we can easily evaluate the energy momentum tensor by using the relation

$$\frac{8\pi\kappa}{c^4}T_{rs} = K_{rs} - \frac{1}{2}g_{rs}K \tag{30}$$

where K_{rs} is the Ricci tensor of the base and κ is the gravitational constant.

From (20) we obtain:

$$\frac{8\pi\kappa}{c^4}T_{rs} = 2g^{im}g_{ab}F^a_{ri}F^b_{sm} + \frac{1}{2}g_{rs}g^{jk}g^{im}g_{ab}F^a_{ik}F^b_{jm} \tag{31}$$

This expression is in agreement with the well known case of electromagnetism, where F^a_{jk} reduces to f_{jk} and g_{ab} to 1. The covariant divergence of T_{rs} vanishes from its definition or, if computed from the right-hand side of (31) by virtue of the field equations (25). Having at our disposal the equations (25) and (31) we can solve — at least theoretically — the problem of the interaction of a field of the Yang-Mills type with the gravitational field. Practically this can be done only with very simplifying assumptions, *e. g.* some special symmetries of the space-time and the field, or by means of perturbation calculus.

3. THE MOTION OF PARTICLES IN A FIELD OF THE YANG-MILLS TYPE AND THE GRAVITATIONAL FIELD

In analogy with the Jordan-Thirry variant of the Kaluza-Klein theory of gravitation and electromagnetism we assume, that the trajectory of a point particle is a geodesic in our M + 4 dimensional bundle manifold. Then, inserting into the equation of a geodesic the expressions for the connection coefficients (15) we obtain the equations of motion in the form:

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} + 2g^{im}g_{ab}A^a_k F^b_{jm}\frac{dx^j}{ds}\frac{dx^k}{ds} + 2g^{im}g_{ab}F^b_{jm}\frac{dx^j}{ds}\frac{dx^a}{ds} = 0 \quad (32)$$

In our coordinates the components A^a_b of the bundle connection are equal to δ^a_b, hence we can transform (32) to the form

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} + 2g^{im}g_{ab}F^b_{jm}\left(A^a_\alpha \frac{dx^\alpha}{ds}\right)\frac{dx^j}{ds} = 0 \quad (33)$$

The scalar product $A^a_\alpha \dfrac{dx^\alpha}{ds}$ is constant since in any metric space the geodesics are always at the constant angle to the Killing field. Introducing the notion of the generalized charge Q^a we obtain

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} + g^{im}g_{ab}Q^aF^b_{jm}\frac{dx^j}{ds} = 0 \quad (34)$$

where $Q^a = 2A^a_\alpha \dfrac{dx^\alpha}{ds}$. It is worthful to remark, that Q^a really does correspond to the quantity analogous to the charge divided by the mass of the particle.

The equations (34) are not complete, during the motion the charge-vector rotates in the group space. This can be seen by differentiating its definition; then the way of rotating depends explicitly on the external field.

In the case of one-dimensional abelian gauge group with g_{ab} reduced to 1 we obtain the classical Lorentz equation of an electron moving in an external field. In the non-abelian case there is an interesting new aspect to this equation of motion. Whereas in the case of the electromagnetism the necessary and sufficient conditions for the particle to move along a geodesic in space-time was either the vanishing of the charge, or the vanishing of the field, here we obtain yet another possibility: the vanishing of the group scalar product of the « charge-vector » and the field: $g_{ab}Q^aF^b_{ij} = 0$, in spite of non vanishing of both Q^a and F^b_{ij}. It is difficult to see, however, if this circumstance has any physical meaning at all.

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REFERENCES

[1] Th. KALUZA, Zum Unitätsproblem der Physik. *Berl. Berichte*, 1921, p. 966.
[2] A. EINSTEIN, Sitz. Preuss. Akad. Wiss., phys. math., Kl 23-25, 26-30, 1927. Zu Kaluzas Theorie des Zusammenhängs von Gravitation und Elektrizität.
[3] C. N. YANG and R. MILLS, Conservation of Isotopic Spin and Isotopic Gauge invariance. *Phys. Rev.*, t. **6**, 1954, p. 191.
[4] A. LICHNEROWICZ, *Théories relativistes de la Gravitation et de l'Électromagnétisme.* Masson et Cie, éd., Paris, 1955.
[5] R. UTIYAMA, Invariant theory of interactions. *Phys. Rev.*, t. **101**, 1956, p. 1597.
[6] A. TRAUTMAN, Lectures on Fibre Bundles, King's College, London, 1967 (mimographed).
[7] W. TULCZYJEW, private communication.
[8] H. LOOS, The range of Gauge Fields. *Nucl. Phys.*, t. **72**, 1965, p. 677.

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