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Quantum theory of scalar field in de Sitter space-time

by

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ABSTRACT. — The quantum theory of scalar field is constructed in the de Sitter spherical world. The field equation in a Riemannian space-time is chosen in the form $\square\varphi + \frac{1}{6}R\varphi + \left(\frac{mc}{\hbar}\right)^2 \varphi = 0$ owing to its conformal invariance for $m = 0$. In the de Sitter world the conserved quantities are obtained which correspond to isometric and conformal transformations. The Fock representations with the cyclic vector which is invariant under isometries are shown to form an one-parametric family. They are inequivalent for different values of the parameter. However, its single value is picked out by the requirement for motion to be quasiclassis for large values of square of space momentum. Then the basis vectors of the Fock representation can be interpreted as the states with definite number of particles. For $m = 0$ this result can also be obtained from the condition of conformal invariance. It is proved that the above requirement for motion to be quasiclassic cannot be satisfied at all in the theory with equation $\square\varphi + \left(\frac{mc}{\hbar}\right)^2 \varphi = 0$.

RÉSUMÉ. — La théorie quantique d'un champ scalaire libre est construite dans le monde sphérique de de Sitter. L'équation de champ dans l'espace-temps riemannien est choisie comme $\square\varphi + \frac{1}{6}R\varphi + \left(\frac{mc}{\hbar}\right)^2 \varphi = 0$ tenant compte de son invariance conforme pour $m = 0$.

Dans le cas de de Sitter, on a obtenu des quantités conservées qui correspondent aux transformations isométriques et conformes. On montre que les représentations de Fock avec un vecteur cyclique qui est invariant par rapport au groupe d'isométries forment une famille à un seul paramètre et sont non équivalentes pour des valeurs différentes de ce paramètre. Cependant, en exigeant que le mouvement soit quasi classique pour de grandes valeurs du carré de l'impulsion spatial, on choisit une seule valeur du paramètre pour laquelle les vecteurs de base de l'espace de Fock sont interprétés comme des états avec un nombre défini de particules. Pour $m = 0$, on obtient ce résultat aussi de la condition de l'invariance conforme. On a établi que dans la théorie avec équation $\Box\varphi + \left(\frac{mc}{\hbar}\right)^2 \varphi = 0$ il n'est pas possible de satisfaire l'exigence que le mouvement soit quasi classique.

1. INTRODUCTION

In an earlier paper [1] we constructed the quantum field theory in the two-dimensional de Sitter space-time. As we have known, Thirring carried out an analogous work [2]. The results obtained in [1] will be extended here to the four-dimensional de Sitter space-time.

Interest to the de Sitter space-time increased considerably during the last years in connection with investigations of elementary particles symmetries [3, 4]. From our point of view the following circumstance is also not of small importance. In the quantum field theory space-time relations are set usually by the Minkowsky geometry and, consequently, there is no possibility for a satisfactory account of gravitation. It seems therefore desirable to adapt the quantum field theory machinery to the general case of a pseudo-Riemannian space-time. As the latter appears in the problem globally it is not possible to confine oneself to consideration of its local metric properties. The de Sitter space-time is a remarkable example in this respect for it differs from the Minkowsky one not only by curvature but also by topology.

First of all the question arises as to how the Fock-Klein-Gordon equation is to be written in the general case of space-time with a nonvanishing curvature. Replacement alone of partial derivatives by covariant ones ∇_α gives

$$\Box\varphi + \left(\frac{mc}{\hbar}\right)^2 \varphi = 0, \qquad (1.1)$$

where

$$\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\,g^{\alpha\beta}\frac{\partial}{\partial x^\beta}\right).$$

Most authors consider this equation. However, the equation of scalar field with zero mass must be conformal invariant while equation $\Box\varphi = 0$ does not satisfy this requirement by any means. The conformal invariant equation is

$$\Box\varphi + \frac{n-2}{4(n-1)}R\varphi = 0, \tag{1.2}$$

where R is the scalar curvature of the space-time and n is its dimensionality. Penrose [5] considered just such an equation for $n = 4$. One may speak about conformal invariance of eq. (1.2) in view of the identity

$$\left(\Box + \frac{n-2}{4(n-1)}R\right)\varphi = \Omega^{\frac{n+2}{2}}\left(\tilde{\Box} + \frac{n-2}{4(n-1)}\tilde{R}\right)\Omega^{\frac{2-n}{2}}\varphi,$$

the quantities marked by $\sim$ being defined through the metric tensor

$$\tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}.$$

So we come to the equation

$$\Box\varphi + \frac{n-2}{4(n-1)}R\varphi + \left(\frac{mc}{\hbar}\right)^2\varphi = 0. \tag{1.3}$$

We disagree in the choice of the scalar field equation with Nachtmann [6] who developed Thirring's results considering eq. (1.1) in the four dimensional de Sitter space-time.

We note, that eq. (1.3) corresponds to the classical one $g^{\alpha\beta}p_\alpha p_\beta = m^2c^2$ so that the operator of the square of momentum is

$$P^2 = -\hbar^2\left(\Box + \frac{n-2}{4(n-1)}R\right). \tag{1.4}$$

In the Heisenberg picture the field operator obeys eq. (1.3) chosen by us. To fix a certain Heisenberg picture one must choose a space-like hypersurface Σ such that the Cauchy data on Σ define uniquely a solution of eq. (1.3) in the whole space-time. We shall consider a real field and so the field operator must obey the following commutation relations on Σ (see for example [7]) :

$$[\varphi(M_1), \varphi(M_2)] = 0, \quad [\varphi_\alpha(M_1)d\sigma^\alpha(M_1), \varphi_\beta(M_2)d\sigma^\beta(M_2)] = 0 \quad (1.5$$

$$\int_\Sigma f(M)[\varphi(M_1), \varphi_\alpha(M)]d\sigma^\alpha(M) = i\hbar f(M_1),$$

where M, M_1, $M_2 \in \Sigma$, $\varphi_\alpha = \dfrac{\partial \varphi}{\partial x^\alpha}$, $d\sigma^\alpha$ is the vector element of area of Σ and $f(M)$ is an arbitrary function.

The following step in the canonical quantization method is to be a choice of representation of commutation relations (1.5). It would seem quite easy to do this by considering a state vector as a wave functional $\Psi[\varphi(M)]$ the argument $\varphi(M)$ being a function on Σ, by dealing with the field operator $\varphi(M)$ as with an operator of multiplication of Ψ by its argument and by equating the operator $\varphi_\alpha(M)d\sigma^\alpha(M)$ to $-i\hbar \dfrac{\delta}{\delta\varphi(M)} d\sigma(M)$. However, one encounters here the difficulties of functional integration because the probability of a field configuration is given by the functional integral $\int |\Psi|^2 \delta\varphi$. Besides, on this way one does not obtain a corpuscular interpretation of the quantum field theory even in the case of the flat spacetime. It is known that in the latter case the Fock representation and the second quantization method enable one to avoid these difficulties. Using the method suggested in [*8*, *9*] one can construct different Fock representations in the case of curved spacetime, too. In essence every Fock representation is characterized completely by the quasivacuum state vector, otherwise by the cyclic vector of representation of the algebra generated by operators $\varphi(M)$, $\varphi_\alpha(M)d\sigma^\alpha(M)$, $M \in \Sigma$. In the general case we do not known a principle which would enable to prefer one of the quasivacua and so to single out the true vacuum. If the space-time admits however an isometric group, then there is a class of quasivacua which are invariant with respect to the group. For the Minkowsky space-time this class consists of the single element which is just the vacuum state. One can assert the same about any static space-time. The corresponding Fock representation then gives the corpuscular interpretation of quantized field.

The principle purpose of this paper consists in defining the vacuum state and in attaining the corpuscular interpretation of the quantum field theory in the de Sitter space-time. Although the de Sitter space-time is a space of constant curvature and consequently admits the isometry group with maximal number of parameters it turns out that the requirement of invariance with respect to the group alone is not sufficient: it picks over an one-parametric family of invariant quasivacuum states. In paper [*1*] the correspondence principle was used in order to choose the vacuum among them: under some conditions particle motion must be quasiclassic and defined by the geodesic equations. It has turned out that this principle is inapplicable to eq. (1.1) if $n > 2$, but gives a good result for eq. (1.3).

For us this fact is another argument in favour of eq. (1.3). As in the two-dimensional case the correspondence principle together with the principle of invariance has enabled us to define the vacuum and the creation and annihilation operators in the de Sitter space-time of the real dimensionality $n = 4$.

As a consequence of compactness of the hypersurface Σ in the de Sitter space-time the set of linear independent particle creation operators is denumerable. This circumstance facilitates essentially the consideration of the problems related to the functional integration because the correct definition of integral over a denumerable set of variables is well known. Fortunately the de Sitter space-time in this respect differs from the Minkowsky space-time, where one is to deal with continual integration. In the latter case one uses the trick of enclosing the system into a box and enlarging the dimensions of the box to infinity after calculations having been performed. In view of compactness of the box the set of degrees of freedom becomes denumerable but the price for this is the lost of the isometric invariance. The latter arises only in the limit of infinite dimensions. From this point of view the de Sitter space-time may be considered as an invariant box. The de Sitter space-time turns into the Minkowsky space-time and the de Sitter group turns into the Poincaré group in the limit of infinite radius. So one may consider the field theory in the de Sitter space-time as a calculation method for the Minkowsky space-time where the continuum of degrees of freedom is replaced by a denumerable set. In contrast to the usual box-method the invariance of the theory is maintained till passing to the limit of infinite dimensions.

2. VARIATIONAL PRINCIPLE

One obtains eq. (1.3) by variation with respect to φ of the action integral

$$A = \int L dv,$$

$dv = \sqrt{(-1)^{n-1} g}\, dx^0 dx^1 \ldots dx^{n-1}$ being the volume element,

$$L = \frac{1}{2} g^{\alpha\beta} \varphi_\alpha \varphi_\beta - \frac{1}{2}\left[\left(\frac{mc}{\hbar}\right)^2 + \frac{n-2}{4(n-1)} R\right]\varphi^2. \tag{2.1}$$

The scalar curvature is $R = g^{\gamma\beta} R_{\gamma\beta}$, where

$$R_{\gamma\beta} = R^{\nu}_{\gamma,\nu\beta},$$

$$R^{\nu}_{\gamma,\alpha\beta} = \frac{\partial \Gamma^{\nu}_{\gamma\alpha}}{\partial x^\beta} - \frac{\partial \Gamma^{\nu}_{\gamma\beta}}{\partial x^\alpha} + \Gamma^{\mu}_{\gamma\alpha}\Gamma^{\nu}_{\mu\beta} - \Gamma^{\mu}_{\gamma\beta}\Gamma^{\nu}_{\mu\alpha}.$$

Following Gilbert [*10*] the variation

$$\delta A = \frac{1}{2}\int T_{\alpha\beta}\delta g^{\alpha\beta} dv$$

gives the (metric) energy-momentum tensor $T_{\alpha\beta}$. Obviously

$$\delta A = \frac{1}{2}\int T^{(\mathrm{can})}_{\alpha\beta}\delta g^{\alpha\beta} dv - \frac{n-2}{8(n-1)}\int \varphi^2 \delta R dv,$$

where $T^{(\mathrm{can})}_{\alpha\beta}$ is the canonical energy momentum tensor:

$$T^{(\mathrm{can})}_{\alpha\beta} = \frac{1}{2}(\varphi_\alpha\varphi_\beta + \varphi_\beta\varphi_\alpha) - L g_{\alpha\beta}. \tag{2.2}$$

To find δR we notice that

$$\delta R^{\nu}_{\gamma,\alpha\beta} = \nabla_\beta \delta\Gamma^{\nu}_{\gamma\alpha} - \nabla_\alpha \delta\Gamma^{\nu}_{\gamma\beta},$$

$$\delta\Gamma^{\nu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(\nabla_\beta \delta g_{\mu\alpha} + \nabla_\alpha \delta g_{\mu\beta} - \nabla_\mu \delta g_{\alpha\beta}).$$

Therefore

$$\delta R = \left\{ R_{\alpha\beta} + \frac{1}{2}(\nabla_\alpha\nabla_\beta + \nabla_\beta\nabla_\alpha) - g_{\alpha\beta}\Box \right\} \delta g^{\alpha\beta}.$$

The identity

$$\nabla_\alpha(A\nabla_\beta B^{\mu\nu}) - \nabla_\beta(B^{\mu\nu}\nabla_\alpha A) = A\nabla_\alpha\nabla_\beta B^{\mu\nu} - B^{\mu\nu}\nabla_\alpha\nabla_\beta A$$

being valid for any scalar A and any tensor $B^{\mu\nu}$ allows to prove the equality

$$\int \varphi^2 \delta R dv = \int \delta g^{\alpha\beta} \left\{ R_{\alpha\beta} + \frac{1}{2}(\nabla_\alpha\nabla_\beta + \nabla_\beta\nabla_\alpha) - g_{\alpha\beta}\Box \right\} \varphi^2 dv$$

provided that $\delta g^{\alpha\beta} = 0$, $\nabla_\gamma \delta g^{\alpha\beta} = 0$ on the boundary of the integration region. From where we find the energy-momentum tensor

$$T_{\alpha\beta} = T^{(\mathrm{can})}_{\alpha\beta} - \frac{n-2}{4(n-1)}\{ R_{\alpha\beta} + \nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box \}\varphi^2. \tag{2.3}$$

This tensor has the following properties:

$$T_{\alpha\beta} = T_{\beta\alpha}, \qquad T^{\alpha}_{\alpha} = \left(\frac{mc\varphi}{\hbar}\right)^2, \qquad \nabla_\alpha T^{\alpha\beta} = 0. \tag{2.4}$$

Therefore the integral

$$M = \int_{\Sigma} \zeta^{\alpha} T_{\alpha\beta} d\sigma^{\beta} \tag{2.5}$$

does not depend on the choice of Σ (is conserved) if this hypersurface is analogous to the one on which commutation relations (1.5) are defined and ζ^{α} is a Killing's vector field i. e. $\nabla_{\alpha}\zeta_{\beta} + \nabla_{\beta}\zeta_{\alpha} = 0$. If $m = 0$, this integral is also conserved when ζ^{α} is a conformal Killing's vector i. e.

$$\nabla_{\alpha}\zeta_{\beta} + \nabla_{\beta}\zeta_{\alpha} = 2fg_{\alpha\beta} \tag{2.6}$$

f being a scalar function.

Integral (2.5) can be considerably simplified. It can be shown [11] that owing to the generalized Killing's equation (2.6)

$$\nabla_{\gamma}\nabla_{\beta}\zeta_{\alpha} = \zeta_{\nu}R^{\nu}_{\gamma,\alpha\beta} + g_{\gamma\alpha}\frac{\partial f}{\partial x^{\beta}} + g_{\alpha\beta}\frac{\partial f}{\partial x^{\gamma}} - g_{\beta\gamma}\frac{\partial f}{\partial x^{\alpha}},$$

whence

$$\zeta^{\mu}R_{\mu\alpha} = \Box\zeta_{\alpha} + (n-2)\frac{\partial f}{\partial x^{\alpha}}.$$

Consequently

$$\zeta^{\alpha}(R_{\alpha\beta} + \nabla_{\alpha}\nabla_{\beta} - g_{\alpha\beta}\Box)\varphi^{2} = (n-1)\left(\varphi^{2}\frac{\partial f}{\partial x^{\beta}} - f\frac{\partial\varphi^{2}}{\partial x^{\beta}}\right)\nabla^{\alpha}S_{\alpha\beta},$$

where

$$S_{\alpha\beta} = \zeta_{\alpha}\nabla_{\beta}\varphi^{2} - \zeta_{\beta}\nabla_{\alpha}\varphi^{2} + \varphi^{2}(\nabla_{\alpha}\zeta_{\beta} - fg_{\alpha\beta}).$$

Since $S_{\alpha\beta} + S_{\beta\alpha} = 0$,

$$M = \int_{\Sigma} \zeta^{\alpha} T^{(\mathrm{can})}_{\alpha\beta} d\sigma^{\beta} + \frac{n-2}{4}\int_{\Sigma}\left(f\frac{\partial\varphi^{2}}{\partial x^{\beta}} - \varphi^{2}\frac{\partial f}{\partial x^{\beta}}\right)d\sigma^{\beta} \tag{2.7}$$

by the Stockes' theorem. For Killing's vector $f = 0$ and only the integral of the canonical energy-momentum tensor remains.

We note finally that the integral

$$(\varphi, \psi) = i\int_{\Sigma}\left(\varphi^{+}\frac{\partial\psi}{\partial x^{\beta}} - \frac{\partial\varphi^{+}}{\partial x^{\beta}}\psi\right)d\sigma^{\beta} \tag{2.8}$$

does not depend on Σ provided φ and ψ satisfy eq. (1.3) and φ^{+} is Hermitean conjugate to φ.

3. SOLUTION OF FIELD EQUATION

We will dwell on the de Sitter space-time of the 1st type which can be represented as a sphere (a hyperboloid of one sheet) in the $(n + 1)$-dimensional Minkowsky space

$$\eta_{AB}X^A X^B = (X^0)^2 - (X^1)^2 - \dots - (X^n)^2 = -r^2. \qquad (3.1)$$

Therefore the isometry group of the de Sitter space-time is isomorphic to the homogeneous Lorentz group of the embedding Minkowsky space.

In the de Sitter space-time

$$R_{\gamma\mu,\alpha\beta} = g_{\mu\nu}R^{\nu}_{\gamma,\alpha\beta} = \frac{1}{r^2}\{g_{\mu\alpha}g_{\gamma\beta} - g_{\gamma\alpha}g_{\mu\beta}\},$$

$$R_{\gamma\beta} = g^{\mu\alpha}R_{\gamma\mu,\alpha\beta} = \frac{n-1}{r^2}g_{\gamma\beta}, \qquad R = \frac{n(n-1)}{r^2}.$$

and so eq. (1.3) can be written as

$$\Box\varphi + \frac{n(n-2)}{4r^2}\varphi + \left(\frac{mc}{\hbar}\right)^2\varphi = 0. \qquad (3.2)$$

It is convenient to introduce the coordinates $\theta, \xi^1, \dots, \xi^{n-1}$ (*) :

$$X^0 = r \operatorname{tg}\theta, \qquad X^a = \frac{r}{\cos\theta}k_a(\xi^1, \dots, \xi^{n-1}), \qquad a = 1, 2, \dots, n,$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$\xi^1, \dots, \xi^{n-1}$ being coordinates on the sphere $k_1^2 + \dots + k_n^2 = 1$. If one denotes

$$(dk_1)^2 + \dots + (dk_n)^2 = \omega_{ij}(\xi^1, \dots, \xi^{n-1})d\xi^i d\xi^j,$$

where $\omega_{ij} = \frac{\partial k_a}{\partial \xi^i}\frac{\partial k_a}{\partial \xi^j}$, the interval of the de Sitter space-time is written in the form

$$ds^2 = \frac{r^2}{\cos^2\theta}\{d\theta^2 - \omega_{ij}(\xi^1, \dots, \xi^{n-1})d\xi^i d\xi^j\}$$

(*) We agree the capital Latin indices A, B, ... to take values from 0 to n, the small ones from the beginning of the alphabet $a, b, \dots, h$ to take values from 1 to n, the rest small Latin indices $i, j, \dots$ to take values from 1 to $n - 1$. As beforenow the Greek indices take values from 0 to $n - 1$.

and eq. (3.1) as

$$\cos^n\theta \frac{\partial}{\partial\theta}\left(\cos^{2-n}\theta \frac{\partial\varphi}{\partial\theta}\right) - \cos^2\theta\Delta\varphi + \left[\frac{n(n-2)}{4} + \mathfrak{m}^2\right]\varphi = 0, \quad (3.3)$$

where

$$\Delta = \frac{1}{\sqrt{\omega}}\frac{\partial}{\partial\xi^i}\left(\sqrt{\omega}\,\omega^{ij}\frac{\partial}{\partial\xi^j}\right)$$

is the Laplace operator on the sphere $k_a k_a = 1$ and $\mathfrak{m} = \frac{mc}{\hbar} r$ is a dimensionless parameter.

Eq. (3.3) can be solved by separation of variables. Putting

$$\varphi = \mathrm{T}(\theta)\Xi(\xi^1, \ldots, \xi^{n-1}),$$

one obtains

$$(\Delta + \kappa^2)\Xi = 0,$$

$$\cos^n\theta \frac{d}{d\theta}\left(\cos^{2-n}\theta \frac{d\mathrm{T}}{d\theta}\right) + \left[\kappa^2\cos^2\theta + \frac{n(n-2)}{4} + \mathfrak{m}^2\right]\mathrm{T} = 0.$$

It is well-known that the functions Ξ which are regular on the sphere $k_a k_a = 1$ can be expressed through the harmonic polynomials of k_a

$$\Xi = c_{a_1\ldots a_s} k_{a_1} \ldots k_{a_s}$$

s being the degree of the polynomial.

In the embedding euclidean space the coefficients $c_{a_1\ldots a_s}$ form a symmetric tensor with zero trace for any pair of indices : $c_{aaa_3\ldots a_s} = 0$. They are subjected to no limitation when $s < 2$. The eigenvalues κ^2 are equal to

$$\kappa^2 = s(s + n - 2).$$

The substitution

$$\mathrm{T}(\theta) = \cos^{\frac{n-2}{2}}\theta u(\theta)$$

results in the equation

$$\frac{d^2u}{d\theta^2} + \left(p^2 + \frac{\mathfrak{m}^2}{\cos^2\theta}\right)u = 0, \quad (3.4)$$

where $p = s + \frac{n-2}{2}$.

The physical meaning of quantum number p can be explained as follows. The square of space momentum is equal to $\frac{r^2}{\cos^2\theta}\omega_{ij}p^ip^j$ on the sphere $\theta = \text{const}$ and in conformity with (1.4) the operator

$$-\frac{\hbar^2}{r^2}\cos^2\theta\left[\Delta - \frac{(n-1)(n-3)}{4}\right] \tag{3.5}$$

corresponds to it.

The eigenvalues of the latter are

$$\left(p^2 - \frac{1}{4}\right)\frac{\hbar^2}{r^2}\cos^2\theta. \tag{3.6}$$

We pass now to eq. (3.4). A pair of its linear independent solution is

$$u_p^{\pm}(\theta) = \frac{2^{\mu}}{p\,!}\sqrt{\Gamma(p+\mu)\Gamma(p-\mu+1)}\cos^{\mu}\theta e^{\pm i(p+\mu)\theta}\mathrm{F}(p+\mu, \mu\,; p+1\,; -e^{\pm 2i\theta}) \tag{3.7a}$$

or otherwise

$$u_p^{\pm}(\theta) = \frac{1}{p\,!}\sqrt{\Gamma(p+\mu)\Gamma(p-\mu+1)}e^{\pm ip\theta}\mathrm{F}\left(\mu, 1-\mu\,; p+1\,; \frac{1\pm i\,\mathrm{tg}\,\theta}{2}\right) \tag{3.7b}$$

where $\mu = \frac{1}{2}(1 - \sqrt{1-4\mathfrak{m}^2})$, F is the hypergeometric function.

We will list the following properties of these functions:

1. $(u_p^+)^* = u_p^-$.
2. $u_p^+(\theta) = u_p^-(-\theta)$.
3. $u_p^- \dfrac{du_p^+}{d\theta} - u_p^+\dfrac{du_p^-}{d\theta} = 2i$.
4. $\dfrac{du_p^+}{d\theta} = pu_p^+\,\mathrm{tg}\,\theta + i\dfrac{\sqrt{p(p+1)+\mathfrak{m}^2}}{\cos\theta}u_{p+1}^+$.
5. $\sqrt{p(p+1)+\mathfrak{m}^2}u_{p+1}^+ - \sqrt{p(p-1)+\mathfrak{m}^2}u_{p-1}^+ = 2ip\sin\theta u_p^+$.
6. $u_{p+1}^+u_p^- + u_p^+u_{p+1}^- = \dfrac{2\cos\theta}{\sqrt{p(p+1)+\mathfrak{m}^2}}$.
7. $0 < u_p^+u_p^- < \infty$, if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

8. $u_p^+(0) = \dfrac{1}{\sqrt{\gamma_p}}$, where $\gamma_p = \dfrac{2\Gamma\left(\frac{p+\mu+1}{2}\right)\Gamma\left(\frac{p-\mu+2}{2}\right)}{\Gamma\left(\frac{p+\mu}{2}\right)\Gamma\left(\frac{p-\mu+1}{2}\right)}$.

9. $u_p^+ e^{-ip\theta}$ can be expanded into a Fourier series of positive frequency exponentials.

10. For $m^2 = 0$

$$u_p^{\pm}(\theta) = \frac{1}{\sqrt{p}}\, e^{\pm ip\theta}. \tag{3.8}$$

The simplicity of the last expression is an additional argument in credit of eq. (1.3).

Finally we give the following approximate expression

$$u_p^{\pm}(\theta) = \frac{1}{\sqrt{p}}\, e^{\pm ip\theta}\left(1 \pm \frac{i\mathfrak{m}^2}{2p}\operatorname{tg}\theta - \frac{\mathfrak{m}^2}{4p^2\cos^2\theta} - \frac{\mathfrak{m}^4\operatorname{tg}^2\theta}{8p^2}\cdots\right), \tag{3.9}$$

the dots denoting terms of the order p^{-3} and still higher. Further consideration of the n-dimensional case is not of special interest and we shall satisfy ourselves with the case of $n = 4$.

With the above considerations we can solve the Cauchy problem for eq. (3.3). Let be given

$$\varphi\,|_{\theta=0} = \frac{1}{\pi r\sqrt{2}}\sum_{s=0}^{\infty}\sqrt{2^s(s+1)}\,q_{a_1\ldots a_s}k_{a_1}\ldots k_{a_s}, \tag{3.10}$$

$$\left.\frac{\partial\varphi}{\partial\theta}\right|_{\theta=0} = \frac{1}{\pi r\sqrt{2}}\sum_{s=0}^{\infty}\sqrt{2^s(s+1)}\,p_{a_1\ldots a_s}k_{a_1}\ldots k_{a_s}, \tag{3.11}$$

where q and p are symmetric tensors with zero trace for any pair of indices. Then

$$\varphi = \frac{\cos\theta}{\pi r\sqrt{2}}\sum_{s=0}^{\infty}\sqrt{2^s(s+1)}\,u_{a_1\ldots a_s}k_{a_1}\ldots k_{a_s}, \tag{3.12}$$

where

$$u_{a_1\ldots a_s} = \frac{\sqrt{\gamma_{s+1}}}{2}(u_{s+1}^+ + u_{s+1}^-)q_{a_1\ldots a_s} + \frac{i}{2\sqrt{\gamma_{s+1}}}(u_{s+1}^- - u_{s+1}^+)p_{a_1\ldots a_s}.$$

4. FIELD COMMUTATOR

We will deduce commutation relations between q and p from (1.5). As a hypersurface Σ it is possible to choose the sphere θ = const. Generally

$$\varphi_\alpha d\sigma^\alpha = \sqrt{-g}\begin{vmatrix} \varphi^0 & \varphi^1 & \varphi^2 & \varphi^3 \\ d_1x^0 & d_1x^1 & d_1x^2 & d_1x^3 \\ d_2x^0 & d_2x^1 & d_2x^2 & d_2x^3 \\ d_3x^0 & d_3x^1 & d_3x^2 & d_3x^3 \end{vmatrix}, \qquad \varphi^\alpha = g^{\alpha\beta}\varphi_\beta$$

so that on the sphere θ = const

$$\varphi_\alpha d\sigma^\alpha = \frac{r^2}{\cos^2\theta}\varphi_\theta d\sigma \tag{4.1}$$

where $\varphi_\theta = \dfrac{\partial\varphi}{\partial\theta}$, $d\sigma = \sqrt{\omega}d\xi^1 d\xi^2 d\xi^3$.

Assuming $\theta = 0$ and denoting

$$\varphi(f) = \int \varphi(0, \xi)f(\xi)d\sigma, \qquad \varphi_\theta(f) = \int \varphi_\theta(0, \xi)f(\xi)d\sigma, \tag{4.2}$$

one obtains the commutation relations from (1.5)

$$\begin{gathered}[\varphi(f), \varphi(g)] = 0, \qquad [\varphi_\theta(f), \varphi_\theta(g)] = 0, \\ r^2[\varphi(g), \varphi_\theta(f)] = i\hbar\int f(\xi)g(\xi)d\sigma\end{gathered} \tag{4.3}$$

Further, for any pair of harmonic polynomials

$$P_{(s)} = P_{a_1\dots a_s}k_{a_1}\dots k_{a_s}, \qquad Q_{(t)} = Q_{b_1\dots b_t}k_{b_1}\dots k_{b_t}$$

one has

$$\int P_{(s)}Q_{(t)}d\sigma = \frac{2\pi^2}{2^2(s+1)}\delta_{ts}P_{a_1\dots a_s}Q_{a_1\dots a_s}. \tag{4.4}$$

Consider a tensor $\delta_{a_1\dots a_s;b_1\dots b_s}$ which results from the product $\delta_{a_1b_1}\dots\delta_{a_sb_s}$ after symmetrization in indices $a_1, \dots, a_s$ and subtraction of trace. Apparently

$$P_{a_1\dots a_s} = \delta_{a_1\dots a_s;b_1\dots b_s}P_{b_1\dots b_s} \tag{4.5}$$

for any symmetric tensor $P_{a_1\dots a_s}$ with zero trace for any pair of indices.

On the basis of (4.4) and (4.5) one concludes that in expansion (3.12)

$$u_{a_1 \dots a_s} = \frac{r\sqrt{2^s(s+1)}}{\sqrt{2\pi}\cos\theta} \int \varphi(\theta, \xi)\delta_{a_1 \dots a_s ; b_1 \dots b_s} k_{b_1} \dots k_{b_s} d\sigma.$$

Assuming in (4.2) that

$$f(\xi) = \delta_{a_1 \dots a_s ; b_1 \dots b_s} k_{a_1} \dots k_{a_s},$$

one finds

$$\varphi(f) = \frac{\sqrt{2\pi}}{r\sqrt{2^s(s+1)}} q_{a_1 \dots a_s}, \qquad \varphi_\theta(f) = \frac{\sqrt{2\pi}}{r\sqrt{2^s(s+1)}} p_{a_1 \dots a_s}.$$

Now from (4.3) it is not difficult to get the commutation relations which were sought for

$$\begin{gathered} [q_{a_1 \dots a_s}, q_{b_1 \dots b_t}] = 0, \qquad [p_{a_1 \dots a_s}, p_{b_1 \dots b_t}] = 0, \\ [p_{a_1 \dots a_s}, q_{b_1 \dots b_t}] = -i\hbar\delta_{st}\delta_{a_1 \dots a_s ; b_1 \dots b_s}. \end{gathered} \tag{4.6}$$

Using (4.6) one can get the commutator

$$D = \frac{i}{\hbar}[\varphi(\theta_1, \xi_1), \varphi(\theta_2, \eta)].$$

Explicit commutation gives

$$D = \frac{\cos\theta_1 \cos\theta_2}{\pi r^2} \sum_{s=0}^{\infty} 2^s(s+1)\Delta_{s+1} k_{a_1}(\xi) \dots k_{a_s}(\xi)\delta_{a_1 \dots a_s ; b_1 \dots b_s} k_{b_1}(\eta) \dots k_{b_s}(\eta)$$

where

$$\Delta_s = i \begin{vmatrix} u_s^-(\theta_1) & u_s^-(\theta_2) \\ u_s^+(\theta_1) & u_s^+(\theta_2) \end{vmatrix}. \tag{4.7}$$

It can be proved that for any vectors x_a and y_a

$$2^s x_{a_1} \dots x_{a_s} \delta_{a_1 \dots a_s ; b_1 \dots b_s} y_{b_1} \dots y_{b_s} = x^s y^s C_s^1(\cos\gamma) \tag{4.8}$$

where

$$x = \sqrt{x_a x_a}, \qquad y = \sqrt{y_a y_a}, \qquad \cos\gamma = \frac{x_a y_a}{xy},$$

C_s^1 is the Gegenbauer polynomial, namely

$$C_s^1(\cos\gamma) = \frac{\sin(s+1)\gamma}{\sin\gamma}.$$

Assuming

$$k_a(\xi) = \frac{x_a}{x}, \qquad k_a(\eta) = \frac{y_a}{y}, \qquad \cos\gamma = k_a(\xi)k_a(\eta)$$

one gets

$$\mathrm{D} = \frac{\cos\theta_1 \cos\theta_2}{\pi r^2 \sin\gamma} \sum_{s=1}^{\infty} s\Delta_s \sin s\gamma. \tag{4.9}$$

Further, it can be shown [1] that

$$\Delta_s = 2\int_0^{\theta_1-\theta_2} \mathrm{P}_{-\mu}(\mathrm{G}) \cos s\gamma d\gamma, \tag{4.10}$$

$\mathrm{P}_{-\mu}$ being the Legendre function:

$$\mathrm{P}_{-\mu}(\mathrm{G}) = \mathrm{F}\left(\mu, 1-\mu\,;1\,;\frac{1-\mathrm{G}}{2}\right)$$

and its argument being equal to

$$\mathrm{G} = \frac{\cos\gamma - \sin\theta_1 \sin\theta_2}{\cos\theta_1 \cos\theta_2}.$$

For this it is sufficient to prove that the integral (4.10) as a function of θ_1 satisfies the same differential equation and the same initial conditions as the determinant (4.7), namely

$$\frac{\partial^2\Delta_s}{\partial\theta_1^2} + \left(s^2 + \frac{\mathfrak{m}^2}{\cos^2\theta_1}\right)\Delta_s = 0, \qquad \Delta_s\Big|_{\theta_1=\theta_2} = 0, \qquad \frac{\partial\Delta_s}{\partial\theta_1}\Big|_{\theta_1=\theta_2} = 2.$$

In differentiating the integral (4.10) with respect to θ_1 one is to use the equalities

$$(\mathrm{G}^2-1)\frac{d^2\mathrm{P}_{-\mu}}{d\mathrm{G}^2} + 2\mathrm{G}\frac{d\mathrm{P}_{-\mu}}{d\mathrm{G}} + \mathfrak{m}^2\mathrm{P}_{-\mu} = 0,$$

$$\frac{\partial^2\mathrm{G}}{\partial\theta_1^2} - \frac{\partial^2\mathrm{G}}{\partial\gamma^2} = \frac{2\mathrm{G}}{\cos^2\theta_1}, \qquad \left(\frac{\partial\mathrm{G}}{\partial\theta_1}\right)^2 - \left(\frac{\partial\mathrm{G}}{\partial\gamma}\right)^2 = \frac{\mathrm{G}^2-1}{\cos^2\theta_1},$$

$$\left(\frac{\partial\mathrm{G}}{\partial\theta_1} + \frac{\partial\mathrm{G}}{\partial\gamma}\right)_{\gamma=\theta_1-\theta_2} = 0.$$

It follows from (4.10) that the trigonometric series

$$\mathrm{Q} = \frac{1}{2\pi r^2}\Delta_0 + \frac{1}{\pi r^2}\sum_{s=0}^{\infty}\Delta_s \cos s\gamma$$

is the Fourier series of the function

$$Q = \varepsilon(\theta_1 - \theta_2)\frac{1 + \varepsilon(G - 1)}{2r^2} P_{-\mu}(G),$$

where $\varepsilon(x)$ is the sign of x. Since

$$\frac{\partial}{\partial G} = - \frac{\cos\theta_1 \cos\theta_2}{\sin\gamma}\frac{\partial}{\partial\gamma},$$

the sum of series (4.9) is

$$D = \frac{\partial Q}{\partial G} = \frac{\varepsilon(\theta_1 - \theta_2)}{r^2}\left[\delta(G - 1) + \frac{1 + \varepsilon(G - 1)}{2}\frac{dP_{-\mu}(G)}{dG}\right].$$

This is the relation between the commutator in the four-dimensional space-time D and that in the two dimensional space-time which is just $\frac{1}{2}$Q as it has been shown in [1].

Geometric meaning of invariant G is the following: if the geodetic distance between (θ_1, ξ) and (θ_2, η) is $r\Gamma$ then $G = \mathrm{Ch}\Gamma$. The conditions $G = 1$ and $G < 1$ define respectively the light cone and its exterior. The conditions $G > 1$ and $\theta_1 > \theta_2$ mean that the point (θ_1, ξ) is « in the future » with respect to the point (θ_2, η).

5. CONSERVED QUANTITIES

If the space-time admits a continuous group of conformal transformations (i. e. the vector field ζ_α existe such that $\nabla_\alpha\zeta_\beta + \nabla_\beta\zeta_\alpha = 2f\, g_{\alpha\beta}$) and φ is a solution of eq. (1.2) then $\psi = \frac{i}{\hbar}Z\varphi$ is also a solution of the same equation, Z being the operator

$$Z = -i\hbar\left(\zeta^\alpha\frac{\partial}{\partial x^\alpha} + \frac{n-2}{2}f\right).$$

If $f = 0$ (and the conformal transformation turns into the isometric one) this last assertion is equally true for eq. (1.3).

For the de Sitter space-time the general form of Z can be obtained from the corresponding operator in the embedding Minkowsky space-time.

In the latter the general form of the conformal Killing's vector is [11]

$$\zeta^A = C^{AB}X_B + D^A + (C, X)X^A - \frac{1}{2}(X, X)C^A + DX^A \qquad (5.1)$$

where $C^{AB} = - C^{BA}$, D^A, C^A, D, are constants and $(C, X) = C^B X_B$. Therefore, the general form of the conformal Killing's vector in the de Sitter space-time is

$$\zeta^A = C^{AB}X_B + (C, X)X^A + r^2 C^A. \qquad (5.2)$$

In fact, the vector ζ is to be tangent to sphere (3.1). This means that $\zeta^A X_A = 0$ whence $D = 0$, $D^A = \frac{1}{2} r^2 C^A$ and consequently equality (5.2).

Further since for a vector defined by (5.1) we have

$$\frac{\partial \zeta_A}{\partial X_B} + \frac{\partial \zeta_B}{\partial X_A} = 2[D + CX]\eta_{AB},$$

then the dilatation coefficient f of conformal transformation (5.2) is

$$f = (C, X) = \frac{r}{\cos\theta}[C^0 \sin\theta - C^a k_a].$$

So we have found the general form of Z in the de Sitter space-time. Its decomposition into linear independent parts is

$$Z = \frac{1}{2} C^{AB} \underset{(AB)}{Z} + r C^A \underset{(A)}{Z}. \qquad (5.3)$$

The operator $\underset{(AB)}{Z} = - \underset{(BA)}{Z}$ corresponds to embedding space-time rotation in the plane (AB)

$$\frac{i}{\hbar} \underset{(AB)}{Z} = X_B \frac{\partial}{\partial X_A} - X_A \frac{\partial}{\partial X_B}.$$

The operatorz $\underset{(A)}{Z}$ define nonisometric conformal transformations. Passing to the coordinates r, θ, ξ one obtains

$$\begin{aligned}
\frac{i}{\hbar} \underset{(ab)}{Z} &= \left(k_a \frac{\partial k_b}{\partial \xi^i} - k_b \frac{\partial k_a}{\partial \xi^i}\right)\omega^{ij} \frac{\partial}{\partial \xi^j} \\
\frac{i}{\hbar} \underset{(a0)}{Z} &= k_a \cos\theta \frac{\partial}{\partial \theta} + \sin\theta \frac{\partial k_a}{\partial \xi^i} \omega^{ij} \frac{\partial}{\partial \xi^j}, \\
\frac{i}{\hbar} \underset{(a)}{Z} &= - k_a \sin\theta \frac{\partial}{\partial \theta} + \cos\theta \frac{\partial k_a}{\partial \xi^i} \omega^{ij} \frac{\partial}{\partial \xi^j} - \frac{n-2}{2} \frac{k_a}{\cos\theta} \\
\frac{i}{\hbar} \underset{(0)}{Z} &= \frac{\partial}{\partial \theta} + \frac{n-2}{2} \operatorname{tg}\theta.
\end{aligned} \qquad (5.4)$$

The components of the vector ζ^{α} in the coordinates θ, ξ can be easily determined from (5.4). We substitute this vector into (2.6) and choose the sphere $\theta = \text{const}$ as Σ. By analogy with (5.3) we have

$$\mathrm{M} = \frac{1}{2}\mathrm{C}^{\mathrm{AB}}\underset{(\mathrm{AB})}{\mathrm{M}} + r\mathrm{C}^{\mathrm{A}}\underset{(\mathrm{A})}{\mathrm{M}}.$$

Further we will consider again $n = 4$.

The calculation of $\underset{(\mathrm{A})}{\mathrm{M}}$ and $\underset{(\mathrm{AB})}{\mathrm{M}}$ reduces to taking integrals of the form (4.4) and

$$\int k_a \mathrm{P}_{(s)}\mathrm{Q}_{(t)}d\sigma = \frac{\pi^2}{2^s(s+2)}\delta_{t,s+1}\mathrm{P}_{a_1\ldots a_s}\mathrm{Q}_{a_1\ldots a_s a} + \frac{\pi^2}{2^t(t+2)}\delta_{s,t+1}\mathrm{P}_{a a_1\ldots a_t}\mathrm{Q}_{a_1\ldots a_t}.$$

Using the combinations

$$\underset{(a)}{\mathrm{K}} = \underset{(a0)}{\mathrm{M}}\cos\theta - \underset{(a,}{\mathrm{M}}\sin\theta, \qquad \underset{(a)}{\mathrm{L}} = \underset{(a0)}{\mathrm{M}}\sin\theta + \underset{(a)}{\mathrm{M}}\cos\theta$$

which are more convenient for calculation of $\underset{(a)}{\mathrm{M}}$ and $\underset{(a0)}{\mathrm{M}}$. One obtains as result of integration

$$\underset{(a)}{\mathrm{K}} = \frac{1}{\sqrt{2}}\sum_{s=0}^{\infty}\sqrt{\frac{s+1}{s+2}} \times \left\{\dot{u}_{a_1\ldots a_s}\dot{u}_{a_1\ldots a_s a} + \left[(s+1)(s+2) + \frac{\mathrm{m}^2}{\cos^2\theta}\right]u_{a_1\ldots a_s}u_{a_1\ldots a_s a}\right\}$$

$$\underset{(a)}{\mathrm{L}} = \frac{1}{\sqrt{2}}\sum_{s=0}^{\infty}\{(s+2)\dot{u}_{a_1\ldots a_s}u_{a_1\ldots a_s a} - (s+1)u_{a_1\ldots a_s}\dot{u}_{a_1\ldots a_s a}\}$$

$$\underset{(0)}{\mathrm{M}} = \frac{1}{2}\sum_{s=0}^{\infty}\left\{\dot{u}_{a_1\ldots a_s}\dot{u}_{a_1\ldots a_s} + \left[(s+1)^2 + \frac{\mathrm{m}^2}{\cos^2\theta}\right]u_{a_1\ldots a_s}u_{a_1\ldots a_s}\right\}$$

$$\underset{(ab)}{\mathrm{M}} = \sum_{s=0}^{\infty}(s+1)\{\dot{u}_{a_1\ldots a_s a}u_{a_1\ldots a_s b} - \dot{u}_{a_1\ldots a_s b}u_{a_1\ldots a_s a}\}.$$

The dot over u signifies the differentiation with respect to θ.

The integrals $\underset{(AB)}{M}$ do not depend on θ and are

$$\underset{(ab)}{M} = \sum_{s=0}^{\infty} (s+1)\{p_{a_1\ldots a_s a} q_{a_1\ldots a_s b} - q_{a_1\ldots a_s b} q_{a_1\ldots a_s a}\}$$

$$\underset{(a0)}{M} = \frac{1}{\sqrt{2}} \sum_{s=0}^{\infty} \sqrt{\frac{s+1}{s+2}} \times \{p_{a_1\ldots a_s} p_{a_1\ldots a_s a} + [(s+1)(s+2) + m^2] q_{a_1\ldots a_s} q_{a_1\ldots a_s a}\} \tag{5.5}$$

If $m = 0$ the integrals $\underset{(A)}{M}$ do not depend on θ as well and are

$$\underset{(0)}{M} = \frac{1}{2} \sum_{s=0}^{\infty} \{p_{a_1\ldots a_s} p_{a_1\ldots a_s} + (s+1)^2 q_{a_1\ldots a_s} q_{a_1\ldots a_s}\}$$

$$\underset{(a)}{M} = \frac{1}{\sqrt{2}} \sum_{s=0}^{\infty} \{(s+2) p_{a_1\ldots a_s} q_{a_1\ldots a_s} - (s+1) q_{a_1\ldots a_s} p_{a_1\ldots a_s a}\} \tag{5.6}$$

The operators $\underset{(AB)}{Z}$ define the structure of the isometric group and together with $\underset{(A)}{Z}$ define the structure of the conformal transformation group

$$\frac{i}{\hbar} [\underset{(AB)}{Z}, \underset{(CD)}{Z}] = \eta_{AC} \underset{(BD)}{Z} + \eta_{BD} \underset{(AC)}{Z} - \eta_{AD} \underset{(BC)}{Z} - \eta_{BC} \underset{(AD)}{Z},$$

$$\frac{i}{\hbar} [\underset{(A)}{Z}, \underset{(B)}{Z}] = \underset{(AB)}{Z}, \qquad \frac{i}{\hbar} [\underset{(A)}{Z}, \underset{(BC)}{Z}] = \eta_{AC} \underset{(B)}{Z} - \eta_{AB} \underset{(C)}{Z}.$$

The conserved quantities satisfy the same commutation relations, namely: for any m

$$\frac{i}{\hbar} [\underset{(AB)}{M}, \underset{(CD)}{M}] = \eta_{AC} \underset{(BD)}{M} + \eta_{BD} \underset{(AC)}{M} - \eta_{AD} \underset{(BC)}{M} - \eta_{BC} \underset{(AD)}{M}$$

and for $m = 0$

$$\frac{i}{\hbar} [\underset{(A)}{M}, \underset{(B)}{M}] = \underset{(AB)}{M}, \qquad \frac{i}{\hbar} [\underset{(A)}{M}, \underset{(BC)}{M}] = \eta_{AC} \underset{(B)}{M} - \eta_{AB} \underset{(C)}{M}.$$

6. INVARIANT QUASIVACUM STATES

According to [8, 9] the general form of the quasivacuum state is defined by eq. $z_{a_1\ldots a_s} | 0 \rangle = 0$ where

$$z_{a_1\ldots a_s} = \frac{i}{\sqrt{2\hbar}} \left\{ p_{a_1\ldots a_s} - \sum_{t=0}^{\infty} S_{a_1\ldots a_s; b_1\ldots b_t} q_{b_1\ldots b_t} \right\} \tag{6.1}$$

The linear transformation $S = R + iQ$ has the following properties

$$S_{a_1 \ldots a_s; b_1 \ldots b_t} = S_{b_1 \ldots b_t; a_1 \ldots a_s}$$
$$S_{a_1 \ldots a_s; b_1 \ldots b_t} = S_{(a_1 \ldots a_s); b_1 \ldots b_t} = S_{a_1 \ldots a_s; (b_1 \ldots b_t)}$$
$$S_{aaa_3 \ldots a_s; b_1 \ldots b_t} = 0 \qquad S_{a_1 \ldots a_s; bbb_3 \ldots b_t} = 0$$

and at last

$$\sum_{s=0}^{\infty} \sum_{t=0}^{\infty} Q_{a_1 \ldots a_s; b_1 \ldots b_t} q_{a_1 \ldots a_s} q_{b_1 \ldots b_t} > 0 \tag{6.2}$$

if not all $q's$ vanish. It is natural to call the operators $z_{a_1 \ldots a_s}$ and the hermitean conjugate operators $z^+_{a_1 \ldots a_s}$ quasiparticle annihilation and creation operators respectively. The operator of the quasiparticle number is

$$N = \sum_{s=0}^{\infty} \tilde{z}^+_{a_1 \ldots a_s} z_{a_1 \ldots a_s} \tag{6.3}$$

where

$$z^+_{a_1 \ldots a_s} = \sum_{t=0}^{\infty} \tilde{Q}_{a_1 \ldots a_s; b_1 \ldots b_t} z^+_{b_1 \ldots b_t}$$

the linear transformation $\tilde{Q}$ being the inverse of Q.

An arbitrary state can be represented by a Fock functional $|\,\rangle = \Phi^+ \,|\, 0 \rangle$, Φ^+ being a power series in the operators $z^+_{a_1 \ldots a_s}$. The state vector norm $\langle\,|\,\rangle = \langle 0 \,|\, \Phi\Phi^+ \,|\, 0 \rangle$ is defined from the condition $\langle 0 \,|\, 0 \rangle = 1$.

Among all quasivacua there are such which are invariant with respect to the de Sitter space-time isometric group. One can simply show that the invariance under time reflection $\theta \to -\theta$ takes place if

$$R_{a_1 \ldots a_s; b_1 \ldots b_t} = 0.$$

However, we confine ourselves to weaker condition of invariance under continuous isometries, what means

$$\underset{(AB)}{M} \,|\, 0 \rangle = \underset{(AB)}{\mu} \,|\, 0 \rangle, \tag{6.4}$$

$\underset{(AB)}{\mu}$ are constants, they will turn out to be zero.

To use this condition one should express q and p through z and z^+:

$$q_{a_1\dots a_s} = \sqrt{\frac{\hbar}{2}}(\tilde{z}_{a_1\dots a_s} + \tilde{z}^+_{a_1\dots a_s}) \tag{6.5}$$

$$p_{a_1\dots a_s} = \sqrt{\frac{\hbar}{2}}\sum_{t=0}^{\infty}(S^*_{a_1\dots a_s;b_1\dots b_t}\tilde{z}_{b_1\dots b_t} + S_{a_1\dots a_s;b_1\dots b_t}\tilde{z}^+_{b_1\dots b_t})$$

Substituting this expressions into (5.5) we first obtain

$$\underset{(ab)}{M}|0\rangle = \underset{(ab)}{\mu}|0\rangle + \hbar\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(s+1)\tilde{z}^+_{a_1\dots a_s[b}S_{a]a_1\dots a_s;b_1\dots b_t}z^+_{b_1\dots b_t}|0\rangle$$

where

$$\underset{(ab)}{\mu} = \hbar\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(s+1)\tilde{Q}_{b_1\dots b_t;a_1\dots a_s[b}R_{a]a_1\dots a_s;b_1\dots b_t}.$$

So the condition of invariance under space rotations gives

$$\begin{aligned} s\delta_{b(a_s}S_{a_1\dots a_{s-1})a;b_1\dots b_t} - s\delta_{a(a_s}S_{a_1\dots a_{s-1})b;b_1\dots b_t} \\ = t\delta_{a(b_t}S_{b_1\dots b_{t-1})b;a_1\dots a_s} - t\delta_{b(b_t}S_{b_1\dots b_{t-1})a;a_1\dots a_s} \end{aligned} \tag{6.6}$$

These equations can be written more simply if one introduces the polylinear forms

$$S_{st}(x, y) = S_{a_1\dots a_s;b_1\dots b_t}x_{a_1}\cdots x_{a_s}y_{b_1}\cdots y_{b_t}$$

They are harmonic polynomials in x of degree s and in y of degree t. Besides, $S_{st}(x, y) = S_{ts}(y, x)$. Instead of (6.6) one has the equivalent equations

$$\left(x_b\frac{\partial}{\partial x_a} - x_a\frac{\partial}{\partial x_b} + y_b\frac{\partial}{\partial y_a} - y_a\frac{\partial}{\partial y_b}\right)S_{st}(x, y) = 0. \tag{6.7}$$

We will prove at first that $S_{st}(x, y) = 0$ if $s \neq t$. In fact, the operator $x_b\frac{\partial}{\partial x_a} - x_a\frac{\partial}{\partial x_b}$ as applied to (6.7) gives

$$\frac{1}{2}\left(x_b\frac{\partial}{\partial x_a} - x_a\frac{\partial}{\partial x_b}\right)\left(y_b\frac{\partial}{\partial y_a} - y_a\frac{\partial}{\partial y_b}\right)S_{st} = s(s+2)S_{st}$$

while $y_b \frac{\partial}{\partial y_a} - y_a \frac{\partial}{\partial y_b}$ gives

$$\frac{1}{2}\left(y_b \frac{\partial}{\partial y_a} - y_a \frac{\partial}{\partial y_b}\right)\left(x_b \frac{\partial}{\partial x_a} - x_a \frac{\partial}{\partial x_b}\right) S_{st} = t(t+2)S_{st}.$$

Consequently $S_{st}(x, y) = 0$ if $s \neq t$.

Further, from eq. (6.7) it follows that S_{st} depends only on invariant combinations $x_a x_a$, $y_a y_a$, $x_a y_a$. Therefore the form $S_{st}(x, y)$ is proportional to (4.8) for $s = t$.

Thus, we have proved that

$$S_{a_1 \ldots a_s; b_1 \ldots b_t} = (R_s + iQ_s)\delta_{st}\delta_{a_1 \ldots a_s; b_1 \ldots b_s} \tag{6.8}$$

where R_s and Q_s are real numbers. Owing to (6.2) $Q_s > 0$ for any s.

Substitution of (6.8) into (6.1) gives

$$z_{a_1 \ldots a_s} = \frac{i}{\sqrt{2\hbar}}\left[p_{a_1 \ldots a_s} - (R_s + iQ_s)q_{a_1 \ldots a_s}\right] \tag{6.9}$$

whence one finds

$$\begin{aligned} q_{a_1 \ldots a_s} &= Q_s^{-1}\sqrt{\frac{\hbar}{2}}\left(z_{a_1 \ldots a_s} + z^+_{a_1 \ldots a_s}\right) \\ p_{a_1 \ldots a_s} &= \sqrt{\frac{\hbar}{2}}\left(\frac{R_s - iQ_s}{Q_s} z_{a_1 \ldots a_s} + \frac{R_s + iQ_s}{Q_s} z^+_{a_1 \ldots a_s}\right) \end{aligned} \tag{6.10}$$

Now $\underset{(ab)}{M}$ expressed through z and z^+ takes on a far simpler form. Indeed, according to (5.5) and (6.10)

$$\underset{(ab)}{M} = i\hbar \sum_{s=0}^{\infty} \frac{s+1}{Q_{s+1}}\left(z^+_{a_1 \ldots a_s a} z_{a_1 \ldots a_s b} - z^+_{a_1 \ldots a_s} z_{a_1 \ldots a_s a}\right). \tag{6.11}$$

We pass to the quantities $\underset{(a0)}{M}$ and have

$$\underset{(a0)}{M}|0\rangle = \frac{\hbar}{2^{3/2}} \times \sum_{s=0}^{\infty} \sqrt{\frac{s+1}{s+2}} \frac{(R_s + iQ_s)(R_{s+1} + iQ_{s+1}) + (s+1)(s+2) + m^2}{Q_s Q_{s+1}} z^+_{a_1 \ldots a_s} z^+_{a_1 \ldots a_s a}|0\rangle. \tag{6.12}$$

The numbers $\underset{(AB)}{\mu}$ which enter (6.3) are equal to zero as is seen from (6.11) and (6.12). Owing to the invariance condition

$$(R_s + iQ_s)(R_{s+1} + iQ_{s+1}) + (s+1)(s+2) + m^2 = 0. \qquad (6.13)$$

Then

$$\underset{(a0)}{M} = \frac{\hbar}{\sqrt{2}} \sum_{s=0}^{\infty} \sqrt{\frac{s+1}{s+2}} \left(\frac{Q_s - iR_s}{Q_s} z^+_{a_1 \ldots a_s} z_{a_1 \ldots a_s a} + \frac{Q_{s+1} - iR_{s+1}}{Q_{s+1}} z^+_{a_1 \ldots a_s a} z_{a_1 \ldots a_s} \right). \qquad (6.14)$$

To solve the reccurrent relations (6.13) we notice that the numbers γ_{s+1}, through which $u^{\pm}_{s+1}(0)$ are expressed (see § 3, prop. 8) satisfy the relation $\gamma_{s+1}\gamma_{s+2} = (s+1)(s+2) + m^2$. Substitution into (6.13)

$$R_s + iQ_s = i\gamma_{s+1} \frac{1 - \lambda_s}{1 + \lambda_s}$$

gives $\lambda_s + \lambda_{s+1} = 0$, whence

$$\gamma_s = (-1)^s \lambda.$$

Since

$$Q_s = \gamma_{s+1} \frac{1 - |\lambda|^2}{|1 + (-1)^s \lambda|^2} > 0$$

then $|\lambda| < 1$. This is the single limitation on λ given by the isometry group. If the invariance under time reflection $\theta \to -\theta$ is taken into account then as was already pointed out, R_s is to be zero, i. e. $\lambda = \lambda^*$ and the space reflections give no additional limitations. We shall not require for the present λ to be real.

The numbers involved in (6.14) are equal to

$$\frac{Q_s - iR_s}{Q_s} = \frac{[1 - (-1)^s \lambda][1 + (-1)^s \lambda^*]}{1 - |\lambda|^2}.$$

Going over from the operators $z_{a_1 \ldots a_s}$ to

$$c_{a_1 \ldots a_s} = \frac{1 + (-1)^s \lambda}{\sqrt{\gamma_{s+1}(1 - |\lambda|^2)}} z_{\alpha_1 \ldots a_s}$$

one obtains finally

$$\underset{(ab)}{\mathrm{M}} = i\hbar \sum_{s=0}^{\infty} (s+1)\left(c^{+}_{a_1\ldots a_s a} c_{a_1\ldots a_s b} - c^{+}_{a_1\ldots a_s b} c_{a_1\ldots a_s a}\right)$$

$$\underset{(a0)}{\mathrm{M}} = \frac{\hbar}{\sqrt{2}} \sum_{s=0}^{\infty} \sqrt{\frac{s+1}{s+2}} \sqrt{(s+1)(s+2) + \mathfrak{m}^2} \times \left(c^{+}_{a_1\ldots a_s} c_{a_1\ldots a_s a} + c^{+}_{a a_1\ldots a_s} c_{a_1\ldots a_s}\right). \tag{6.16}$$

The operators c obey the commutation relations

$$[c_{a_1\ldots a_s}, c_{b_1\ldots b_t}] = 0, \qquad [c^{+}_{a_1\ldots a_s}, c^{+}_{b_1\ldots b_t}] = 0, \qquad [c_{a_1\ldots a_s}, c^{+}_{b_1\ldots b_t}] = \delta_{st}\delta_{a_1\ldots a_s; b_1\ldots b_s} \tag{6.17}$$

as it follows from the expression

$$c_{a_1\ldots a_s} = \frac{[1-(-1)^s\lambda]q_{a_1\ldots a_s} + i[1+(-1)^s\lambda]p_{a_1\ldots a_s}}{\sqrt{2\hbar\gamma_{s+1}(1-|\lambda|^2)}}. \tag{6.18}$$

The quasiparticle number operator is

$$\mathrm{N} = \sum_{s=0}^{\infty} c^{+}_{a_1\ldots a_s} c_{a_1\ldots a_s}. \tag{6.19}$$

We will show that two Fock spaces constructed on invariant cyclic vectors with different values of λ have no common state vectors. Really, from expressions (6.18) and their inverse expressions

$$q_{a_1\ldots a_s} = \sqrt{\frac{\hbar}{2\gamma_{s+1}}} \left\{ \frac{1+(-1)^s\lambda}{\sqrt{1-|\lambda|^2}} c^{+}_{a_1\ldots a_s} + \frac{1+(-1)^s\lambda^*}{\sqrt{1-|\lambda|^2}} c_{a_1\ldots a_s} \right\}$$

$$p_{a_1\ldots a_s} = i\sqrt{\frac{\hbar\gamma_{s+1}}{2}} \left\{ \frac{1-(-1)^s\lambda}{\sqrt{1-|\lambda|^2}} c^{+}_{a_1\ldots a_s} - \frac{1-(-1)^s\lambda^*}{\sqrt{1-|\lambda|^2}} c_{a_1\ldots a_s} \right\} \tag{6.20}$$

it follows that

$$c_{a_1\ldots a_s}(\lambda_2) = \frac{(1-\lambda_1^*\lambda_2)c_{a_1\ldots a_s}(\lambda_1) + (-1)^s(\lambda_1-\lambda_2)c^{+}_{a_1\ldots a_s}(\lambda_1)}{\sqrt{1-|\lambda_1|^2}\sqrt{1-|\lambda_2|^2}} \tag{6.21}$$

for different values of λ. This transformation is similar to those which were introduced by N. N. Bogolubov in his microscopic theory of superfluidity [12]. It follows from (6.21) that the vector $|0\rangle_{\lambda_2}$ is proportional to $\Phi^+ |0\rangle_{\lambda_1}$ where

$$\Phi^+ = \exp\left\{\frac{\lambda_2 - \lambda_1}{2(1 - \lambda_1^* \lambda_2)} \sum_{s=0}^{\infty} (-1)^s c^+_{a_1 \ldots a_s}(\lambda_1) c^+_{a_1 \ldots a_s}(\lambda_2)\right\}.$$

Our assertion is proved if it turns out that

$$_{\lambda_1}\langle 0 | \Phi \Phi^+ | 0 \rangle_{\lambda_1} = \infty. \tag{6.22}$$

To evaluate this norm we choose an orthonormal basis

$$\mathrm{P}^{(\sigma)}_{a_1 \ldots a_s}, \quad \sigma = 1, \ldots, (s+1)^2$$

in the space of symmetric tensors $\mathrm{P}_{a_1 \ldots a_s}$ with zero trace for any pair of indices. By definition

$$\mathrm{P}^{(\sigma)}_{a_1 \ldots a_s} \mathrm{P}^{(\rho)*}_{a_1 \ldots a_s} = \delta_{\sigma\rho}.$$

Expanding $\mathrm{C}_{a_1 \ldots a_s}$ in this basis

$$c_{a_1 \ldots a_s} = \sum_{\sigma=1}^{(s+1)^2} \mathrm{P}^{(\sigma)}_{a_1 \ldots a_s} c_{s\sigma} \quad , \quad c_{s\sigma} = \mathrm{P}^{(\sigma)*}_{a_1 \ldots a_s} c_{a_1 \ldots a_s}$$

we find

$$[c_{s\sigma}, c_{t\tau}] = 0, \qquad [c^+_{s\sigma}, c^+_{t\tau}] = 0, \qquad [c_{s\sigma}, c^+_{t\tau}] = \delta_{st}\delta_{\sigma\tau},$$

$$c^+_{a_1 \ldots a_s} c^+_{a_1 \ldots a_s} = \sum_{\sigma=1}^{(s+1)^2} c^+_{s\sigma} c^+_{s\sigma}.$$

Consequently

$$_{\lambda_1}\langle 0 | \Phi \Phi^+ | 0 \rangle_{\lambda_1} = \prod_{s=0}^{\infty} \prod_{\sigma=1}^{(s+1)^2} {}_{\lambda_1}\langle 0 | \Phi_{s\sigma} \Phi^+_{s\sigma} | 0 \rangle_{\lambda_1}$$

where

$$\Phi^+_{s\sigma} = \exp\left\{\frac{(-1^s)\Lambda}{2} c^+_{s\sigma}(\lambda_1) c^+_{s\sigma}(\lambda_1)\right\}, \qquad \Lambda = \frac{\lambda_2 - \lambda_1}{1 - \lambda_1^* \lambda_2}.$$

It is easy to see that

$$_{\lambda_1}\langle 0 | \Phi_{s\sigma} \Phi^+_{s\sigma} | 0 \rangle_{\lambda_1} = \sum_{\kappa=0}^{\infty} \frac{|\Lambda|^{2\kappa}(2\kappa)!}{2^{2\kappa}(\kappa!)^2} = \frac{1}{\sqrt{1 - |\Lambda|^2}}$$

the summation performed here may be justified owing to

$$1 > 1 - |\Lambda|^2 = \frac{(1 - |\lambda_1|^2)(1 - |\lambda_2|^2)}{(1 - \lambda_2^*\lambda_1)(1 - \lambda_2\lambda_1^*)} > 0$$

but for the same reason one obtains (6.22)

$$_{\lambda_1}\langle 0 | \Phi\Phi^+ | 0 \rangle_{\lambda_1} = \prod_{s=0}^{\infty} \left(1 - |\Lambda|^2\right)^{-\frac{(s+1)^2}{2}} = \infty.$$

7. TRANSITION TO SECOND QUANTIZATION

When $m = 0$ the unique state vector is picked out among the invariant quasivacua which is also invariant under conformal transformations. Indeed, from (5.6) and (6.20) one obtains

$$\underset{(a)}{M} | 0 \rangle = \frac{i\sqrt{2}\hbar\lambda}{1 - |\lambda|^2} \sum_{s=0}^{\infty} \sqrt{(s+1)(s+2)}(-1)^{s+1} c^+_{a_1\ldots a_s} c^+_{a_1\ldots a_s a} | 0 \rangle.$$

So the requirement of conformal invariance gives $\lambda = 0$ and the state $| 0 \rangle$ for $\lambda = 0$ and $m = 0$ is the true vacuum. The conserved quantities for this case are

$$\begin{aligned}
\underset{(ab)}{M} &= i\hbar \sum_{s=0}^{\infty} (s+1)\left(c^+_{a_1\ldots a_s a} c_{a_1\ldots a_s b} - c^+_{a_1\ldots a_s b} c_{a_1\ldots a_s a}\right) \\
\underset{(a0)}{M} &= \frac{\hbar}{\sqrt{2}} \sum_{s=0}^{\infty} (s+1)\left(c^+_{a_1\ldots a_s} c_{a_1\ldots a_s a} + c^+_{a_1\ldots a_s a} c_{a_1\ldots a_s}\right) \\
\underset{(a)}{M} &= \frac{i\hbar}{\sqrt{2}} \sum_{s=0}^{\infty} (s+1)\left(c^+_{a_1\ldots a_s} c_{a_1\ldots a_s a} - c^+_{a_1\ldots a_s a} c_{a_1\ldots a_s}\right) \\
\underset{(0)}{M} &= \frac{\hbar}{2} \sum_{s=0}^{\infty} (s+1)\left(c^+_{a_1\ldots a_s} c_{a_1\ldots a_s} + c_{a_1\ldots a_s} c^+_{a_1\ldots a_s}\right)
\end{aligned} \qquad (7.1)$$

The relation between the operators q, p, c is also essentially simplified in this case:

$$c_{a_1\ldots a_s} = \frac{(s+1)q_{a_1\ldots a_s} + ip_{a_1\ldots a_s}}{\sqrt{2\hbar(s+1)}}$$

$$q_{a_1 \ldots a_s} = \sqrt{\frac{\hbar}{2(s+1)}}\,(\overset{+}{c}_{a_1 \ldots a_s} + c_{a_1 \ldots a_s})$$

$$p_{a_1 \ldots a_s} = i\sqrt{\frac{\hbar(s+1)}{2}}\,(\overset{+}{c}_{a_1 \ldots a_s} - c_{a_1 \ldots a_s}).$$

Using these formulae, one can write the field operator φ as

$$\varphi = \sqrt{\hbar}(\varphi^- + \varphi^+) \tag{7.2}$$

where

$$\varphi^- = \frac{\cos\theta}{2\pi r}\sum_{s=0}^{\infty} 2^{\frac{s}{2}} e^{-i(s+1)\theta} c_{a_1 \ldots a_s} k_{a_1} \ldots k_{a_s}$$

and φ^+ is the hermitean conjugate of φ^-. Through the operator φ^- the particle number operator N and conserved quantities (7.1) are represented as (2.8) namely

$$\begin{gathered} \mathrm{N} = \sum_{s=0}^{\infty} \overset{+}{c}_{a_1 \ldots a_s} c_{a_1 \ldots a_s} = (\varphi^-, \varphi^+), \\ \underset{(AB)}{\mathrm{M}} = -(\varphi^-, \underset{(AB)}{\mathrm{Z}}\varphi^-), \quad \underset{(a)}{\mathrm{M}} = -(\varphi^-, \underset{(a)}{\mathrm{Z}}\varphi^-), \quad :\underset{(0)}{\mathrm{M}}: = -(\varphi^-, \underset{(0)}{\mathrm{Z}}\varphi^-). \end{gathered} \tag{7.3}$$

The colons signify as usual the normal product. So proceeding from the canonical method we come to the method of second quantization.

However, the operators N and $\underset{(AB)}{\mathrm{M}}$ (in contrast to $\underset{(A)}{\mathrm{M}}$) can be written in the form (7.3) not only for $m = 0, \lambda = 0$ but for $m^2 \geqslant 0 \mid \lambda \mid < 1$. Indeed, using (6.20) one can represent the field operator as (7.2) in the general case. Of course, now φ^- is another operator, namely

$$\varphi^- = \frac{\cos\theta}{2\pi r}\sum_{s=0}^{\infty}\sqrt{2^s(s+1)}\,\frac{u_{s+1}^-(\theta) + (-1)^s \lambda u_{s+1}^+(\theta)}{\sqrt{1-|\lambda|^2}}\, c_{a_1 \ldots a_s} k_{a_1} \ldots k_{a_s}. \tag{7.4}$$

Then it is not difficult to verify the correctness of our assertion.

The connection between the canonical method and the method of second quantization can be shown by considering the Casimir operators constructed from $\underset{(AB)}{\mathrm{M}}$, Really, since

$$\frac{1}{2}\,\overset{(AB)}{\mathrm{Z}}\,\underset{(AB)}{\mathrm{Z}} = \hbar^2 r^2 \Box, \qquad \text{where} \qquad \overset{(AB)}{\mathrm{Z}} = \eta^{AC}\eta^{BD}\,\underset{(CD)}{\mathrm{Z}}$$

then one can write eq. (3.2) as

$$\left[\frac{1}{2\hbar^2}\underset{(AB)}{Z}\overset{(AB)}{Z}+\frac{n(n-2)}{4}+\mathfrak{m}^2\right]\varphi=0$$

Similarly one has the identity

$$\frac{1}{2\hbar^2}\underset{(AB)}{M}\overset{(AB)}{M}+\frac{n(n-2)}{4}N+\mathfrak{m}^2N=\frac{1}{2\hbar^2}:\underset{(AB)}{M}\overset{AB}{M}:$$

This correspondence shows that the operator

$$\mathfrak{M}^2=-\frac{1}{2\hbar^2}\underset{(AB)}{M}\overset{(AB)}{M}-\frac{n(n-2)}{4}N$$

is to be called operator of the square of field mass in units of $\frac{cr}{\hbar}$. It is easy to show, that

$$\mathfrak{M}^2\,|\,0\,\rangle=0 \qquad \mathfrak{M}^2c^+_{a_1\ldots a_s}\,|\,0\,\rangle=\mathfrak{m}^2c^+_{a_1\ldots a_s}\,|\,0\,\rangle$$

Further,

$$\frac{1}{2}\underset{(ab)}{Z}\ \underset{(ab)}{Z}=\hbar^2\Delta. \tag{7.5}$$

Therefore the operator of the square of space momentum (3.5) can be written as

$$\frac{\cos^2\theta}{r^2}\left[\frac{1}{2}\underset{(ab)}{Z}\ \underset{(ab)}{Z}+\frac{(n-1)(n-3)}{4}\hbar^2\right] \tag{7.6}$$

Similarly as (7.5)

$$\frac{1}{2}\underset{(ab)}{M}\ \underset{(ab)}{M}=\frac{1}{2}:\underset{(ab)}{M}\ \underset{(ab)}{M}:+\hbar^2\sum_{s=0}^{\infty}s(s+n-2)c^+_{a_1\ldots a_s}c_{a_1\ldots a_s}$$

and in correspondence with (7.6) the operator

$$\mathfrak{P}^2=\frac{\cos^2\theta}{r^2}\left[\frac{1}{2}\underset{(ab)}{M}\ \underset{(ab)}{M}+\frac{(n-1)(n-3)}{4}\hbar^2N\right]$$

should be called operator of the square of field space momentum at the moment of time θ. It is easy to see that

$$\mathfrak{P}^2\,|\,0\,\rangle=0, \qquad \mathfrak{P}^2c^+_{a_1\ldots a_s}\,|\,0\,\rangle=\left(p^2-\frac{1}{4}\right)\frac{\hbar^2\cos^2\theta}{r^2}\,c^+_{a_1\ldots a_s}\,|\,0\,\rangle$$

where as in (3.7) $p = s + \frac{n-2}{2}$. Of course, we have a right to write these formulae only for $n = 4$, but their validity can be proved for arbitrary $n \geqslant 2$.

We do not consider in detail the remaining Casimir operators but we note that for $n = 4$ the second Casimir operator $\frac{1}{2}\eta_{AB}L^A L^B$ is constructed out of the operators

$$L^A = \varepsilon^{ABCDE} \underset{(BC)}{M} \underset{(DE)}{M}$$

having the following properties

$$L^A \,|\, 0 \rangle = 0, \qquad L^A c^+_{a_1 \ldots a_s} \,|\, 0 \rangle.$$

Equally we do not dwell on the Casimir operators of conformal group.

Now our main purpose is to prove that if $\lambda = 0$ the state $|\,0\,\rangle$ is the true vacuum for $m^2 > 0$ too. We have known that on the one hand this is the case for $m = 0$ and arbitrary r and, on the other hand, for $m^2 \geqslant 0$ and $r = \infty$ when the de Sitter space-time is converted into the Minkowsky space-time. However, we may not do the same assertion for $m^2 > 0$ and $0 < r < \infty$ since in our preceding considerations the constant λ was limited by the only condition $|\,\lambda\,| < 1$ (and by stronger condition $-1 < \lambda < 1$ if time reflection $\theta \to -\theta$ was taken into account). In other respects λ might be an arbitrary function of m^2 and r. For that reason we will consider the method of second quantization in detail and try to obtain conclusive arguments in favour of our assertion that, if $\lambda = 0$ the state $|\,0\,\rangle$ is the true vacuum for $m^2 > 0$ too.

8. THE VACUUM

A classic free particle moves in space-time along geodesics, i. e. its equations of motion are

$$g^{\alpha\beta} p_\alpha p_\beta = m^2 c^2$$

$$\frac{dx^0}{2g^{0\alpha}p_\alpha} = \ldots = \frac{dx^{n-1}}{2g^{n\alpha}p_\alpha} = \frac{-\,dp_0}{\frac{\partial g^{\alpha\beta}}{\partial x^0}\, p_\alpha p_\beta} = \ldots = \frac{-\,dp_{n-1}}{\frac{\partial g^{\alpha\beta}}{\partial x^{n-1}}\, p_\alpha p_\beta}. \tag{8.1}$$

The corresponding quantum motion is described by the wave function φ^- satisfying eq. (1.3). As in the flat space-time not any solution of eq. (1.3) is a wave function. In the space of all solutions wave functions form a subspace of maximal dimension on which integral (2.8) is positive definite for $\psi = \varphi^-$, $\varphi^+ = (\varphi^-)^*$. Deliberately this subspace does not contain real solutions for their scalar squares (2.8) are zero. Any complex solution of (3.3) can be represented as (3.12) where

$$u_{a_1 \ldots a_s} = \frac{1}{\sqrt{2}} \{ u_{s+1}^-(\theta) \mathrm{P}_{a_1 \ldots a_s} + u_{s+\alpha}^+(\theta) \mathrm{Q}_{a_1 \ldots a_s} \} \tag{8.2}$$

and P, Q are some symmetric tensors with zero trace for any pair of indices. Scalar square (2.8) is equal to

$$(\varphi, \varphi) = \sum_{s=0}^{\infty} (\mathrm{P}^*_{a_1 \ldots a_s} \mathrm{P}_{a_1 \ldots a_s} - \mathrm{Q}^*_{a_1 \ldots a_s} \mathrm{Q}_{a_1 \ldots a_s}). \tag{8.3}$$

The desired subspace of solutions is defined first of all by the condition that

$$\mathrm{Q}_{a_1 \ldots a_s} = \sum_{t=0}^{\infty} \Lambda_{a_1 \ldots a_s ; b_1 \ldots b_t} \mathrm{P}_{1 \ldots b_t} \tag{8.4}$$

and after substitution of (8.4) into (8.3) the quadratic form of P is to be positive definite.

Certainly the condition of positive definitness alone is not sufficient to pick out uniquely the subspace. We demand the subspace (8.4) to be invariant with respect to the isometry group of the de Sitter space-time. This means that if φ^- belongs to subspace (8.4) then $\underset{(AB)}{Z}\varphi^-$ does as well. It is not difficult to show, that the space rotations leads to eq. (6.6) for Λ, whence

$$\Lambda_{a_1 \ldots a_s ; b_1 \ldots b_t} = \lambda_s \delta_{st} \delta_{a_1 \ldots a_s ; b_1 \ldots b_s}$$

λ_s being some complex numbers. (8.3) is positive definite if $| \lambda_s | < 1$. Consideration of rotations in the planes $(a0)$ gives $\lambda_s = (-1)^s \lambda$. Putting

$$\mathrm{P}_{a_1 \ldots a_s} = \frac{c_{a_1 \ldots a_s}}{\sqrt{1 - |\lambda|^2}}, \quad \mathrm{Q}_{a_1 \ldots a_s} = \frac{(-1)^s \lambda c_{a_1 \ldots a_s}}{\sqrt{1 - |\lambda|^2}}$$

One obtains the subspace of solutions (7.4). Naturally one has the same arbitrariness in the choice of λ and again for $m = 0$ the condition of conformal invariance gives $\lambda = 0$.

Having used all invariance conditions we turn to the connection between (1.3) and (8.1). If one represents φ as

$$\varphi = \sqrt{\rho}e^{i\frac{\sigma}{\hbar}}$$

then from eq. (1.3) the two classic equations follow in the limit $\hbar \to 0$: the Hamilton-Jacobi equation

$$g^{\alpha\beta} \frac{\partial \sigma}{\partial x^\alpha} \frac{\partial \sigma}{\partial x^\beta} = m^2 c^2 \tag{8.5}$$

and the equation of continuity

$$g^{\alpha\beta} \nabla_\alpha \left(\rho \frac{\partial \sigma}{\partial x^\beta} \right) = 0. \tag{8.6}$$

Geodesics are characteristics of eq. (8.5). The condition $\frac{\partial \sigma}{\partial x^0} < 0$ corresponds to motion of a particle « into the future ». For the Sitter space-time one has

$$\frac{\partial \sigma}{\partial \theta} + \sqrt{\frac{m^2 c^2 r^2}{\cos^2 \theta} + \omega^{ij} \frac{\partial \sigma}{\partial \xi^i} \frac{\partial \sigma}{\partial \xi^j}} = 0$$

$$\cos^2 \theta \frac{\partial}{\partial \theta} \left(\frac{\rho}{\cos^2 \theta} \frac{\partial \sigma}{\partial \theta} \right) - \frac{1}{\sqrt{\omega}} \frac{\partial}{\partial \xi^i} \left(\rho \sqrt{\omega}\, \omega^{ij} \frac{\partial \sigma}{\partial \xi^j} \right) = 0.$$

These equations can be solved by separation of variables:

$$\sigma = \sigma_0(\theta) + \tilde{\sigma}(\xi), \qquad \rho = \rho_0(\theta)\tilde{\rho}(\xi).$$

Assuming

$$\rho_0 \frac{d\sigma_0}{d\theta} = - \mathrm{A} \cos^2 \theta \tag{8.7}$$

$$\varphi^{ij} \frac{\partial \tilde{\sigma}}{\partial \xi^i} \frac{\partial \tilde{\sigma}}{\partial \xi^j} = \kappa^2 \tag{8.8}$$

where A and κ^2 are constants, we obtain

$$\frac{d\sigma_0}{d\theta} + \sqrt{\frac{m^2 c^2 r^2}{\cos^2 \theta} + \kappa^2} = 0 \tag{8.9}$$

$$\frac{1}{\sqrt{\omega}} \frac{\partial}{\partial \xi^i} \left(\tilde{\rho} \sqrt{\omega}\, \omega^{ij} \frac{\partial \tilde{\sigma}}{\partial \xi^j} \right) = 0. \tag{8.10}$$

From (8.7) (8.9) we find

$$\rho_0 = \frac{A \cos^2 \theta}{\sqrt{\frac{m^2c^2r^2}{\cos^2 \theta} + \kappa^2}} \tag{8.11}$$

$$\sigma_0 = \frac{mcr}{2} \ln \frac{\sqrt{m^2c^2r^2 + \kappa^2 \cos^2 \theta} - mcr \sin \theta}{\sqrt{m^2c^2r^2 + \kappa^2 \cos^2 \theta} + mcr \sin \theta} + \frac{\kappa}{2i} \ln \frac{\sqrt{m^2c^2r^2 + \kappa^2 \cos^2 \theta} - i\kappa \sin \theta}{\sqrt{m^2c^2r^2 + \kappa^2 \cos^2 \theta} + i\kappa \sin \theta}.$$

Particularly for $m = 0$

$$\rho_0 = \frac{A \cos^2 \theta}{\kappa}, \qquad \sigma_0 = -\kappa\theta.$$

Now let us consider a separate summand in (7.4):

$$\frac{\cos \theta}{2\pi r} \sqrt{2^s(s+1)} \frac{\bar{u}_{s+1} + (-1)^s \lambda \overset{+}{u}_{s+1}(\theta)}{\sqrt{1 - |\lambda|^2}} c_{a_1 \dots a_s} k_{a_1} \dots k_{a_s}.$$

It is an eigenfunction of the operator of the square of space momentum (3.5). We shall be interested in its time dependence

$$v_{s+1}(\theta) = \cos \theta \frac{\bar{u}_{s+1}(\theta) + (-1)^s \lambda \overset{+}{u}_{s+1}(\theta)}{\sqrt{1 - |\lambda|^2}} \tag{8.12}$$

because the remaining factor does not depend on m but for $m = 0$ the definition of vacuum state does not give rise to doubt. For the same reason we do not need to consider eq. (8.8) (8.10). If $m = 0$ the function

$$v_{s+1}\big|_{m=0} = \frac{\cos \theta}{\sqrt{s+1}} \frac{e^{-i(s+1)\theta} + (-1)^s \lambda e^{i(s+1)\theta}}{\sqrt{1 - |\lambda|^2}}. \tag{8.13}$$

is evidently of quasiclassic form exactly and describes the motion of a particle « into the future » only when $\lambda = 0$ and in this case $\kappa = \hbar(s + 1)$. So this condition for $m = 0$ gives the same result as the conformal invariance condition. We try to proceed in the same way when $m^2 > 0$.

We demand the function (8.12) to be of quasiclassic form and to describe the motion of a particle « into the future » at least for large values of s. We rewrite (8.12) as

$$v_{s+1}(\theta) = \sqrt{\rho_0} e^{i\frac{\sigma_0}{\hbar}} \tag{8.14}$$

where, obviously

$$\rho_0 = \frac{\cos^2\theta}{1-|\lambda|^2}\left| u^-_{s+1}(\theta) + (-1)^s \lambda u^+_{s+1}(\theta)\right|^2$$

$$\sigma_0 = \frac{\hbar}{2i}\ln\frac{u^-_{s+1}(\theta) + (-1)^s\lambda u^+_{s+1}(\theta)}{u^+_{s+1}(\theta) + (-1)^s\lambda^* u^-_{s+1}(\theta)}.$$

From this the identity follows

$$\rho_0 \frac{d\sigma_0}{d\theta} = -\hbar\cos^2\theta.$$

Comparing it with (8.7) we find $A = \hbar$. Further,

$$\left(\frac{d\sigma_0}{d\theta}\right)^2 = \frac{\hbar^2}{|u^-_{s+1}(\theta)|^4}\frac{(1-|\lambda|^2)^2}{(1+|\lambda|^2+\lambda e^{i\mu}+\lambda^* e^{-i\mu})^2}$$

where

$$e^{-i\mu} = \frac{u^-_{s+1}(\theta)}{|u^-_{s+1}(\theta)|}.$$

Now we use approximate expression (3.9) and up to higher orders in $\frac{1}{s}$ we obtain

$$\frac{\hbar^2}{|u^-_{s+1}(\theta)|^4} = \hbar(s+1)^2 + \frac{m^2c^2r^2}{\cos^2\theta},$$

i. e.

$$\frac{d\sigma_0}{d\theta} = -\sqrt{\hbar^2(s+1)^2 + \frac{m^2c^2r^2}{\cos^2\theta}}\,\frac{1-|\lambda|^2}{1+|\lambda|^2+\lambda e^{i\mu}+\lambda^* e^{-i\mu}}.$$

Since μ depends essentially on θ this expression may coincide with (8.9) only if $\lambda = 0$ and then $\kappa = \hbar(s+1)$ irrespectively of m. Thus, we obtain that the wave function of a particle is (7.4) for $\lambda = 0$ i. e.

$$\varphi^- = \frac{\cos\theta}{2\pi r}\sum_{s=0}^{\infty}\sqrt{2^s(s+1)}u^-_{s+1}(\theta)c_{a_1\ldots a_s}k_{a_1}\ldots k_{a_s} \qquad (8.15)$$

Subjecting $c_{a_1\ldots a_s}$ to commutation relations (6.17) we return to the second quantized theory, but now we know that $\lambda = 0$ irrespectively of mass.

We would like to make two remarks in conclusion. It is not difficult to obtain the results analogous to (8.15) for any $n \geqslant 2$ too. Since in the de Sitter space-time eq. (1.1) is obtained from (1.3) by replacing m^2 by

$m^2 - \frac{n(n-2)}{4}\frac{\hbar^2 r^2}{c^2}$ then (1.3) describes in quasiclassic approximation the motion of a particle with effective mass $\sqrt{m^2 - \frac{n(n-2)}{4}\frac{\hbar^2 r^2}{c^2}}$ rather than m. It may be assumed therefore that (1.1) describes the field with selfaction rather than the free field.

Substituting $\lambda = 0$, σ_0 and ρ_0 from (8.11) and $A = \hbar$, $\kappa = \hbar p$ we obtain one more approximate expression for the function $u_p^-(\theta)$ valid for large values of p:

$$u_p^-(\theta) = \left(\frac{\mathrm{m}^2}{\cos^2\theta} + p^2\right)^{-\frac{1}{4}}$$

$$\times \exp\left\{- ip \operatorname{arc\,tg} \frac{p\sin\theta}{\sqrt{\mathrm{m}^2 + p^2\cos^2\theta}} - i\mathrm{m} \ln \frac{\sqrt{\mathrm{m}^2 + p^2\cos^2\theta} + \mathrm{m}\sin\theta}{\sqrt{\mathrm{m}^2 + p^2\cos^2\theta}}\right\}$$

It is convenient to use this expression in the vicinity of $r = \infty$ when one passes in the limit to the flat space-time. Assuming $\operatorname{tg}\theta = \frac{tc}{r}$, $p = kr$ one finds

$$\lim_{r\to\infty} \sqrt{r}u_p^-(\theta) = \left[\frac{m^2c^2}{\hbar^2} + \kappa^2\right]^{-\frac{1}{4}} \exp\left\{- i\frac{tc}{\hbar}\sqrt{m^2c^2 + \hbar^2k^2}\right\}.$$

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