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Collective Motions of the Relativistic Gravitational Gas

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ABSTRACT. — The relativistic gravitational gas is studied with the help of the linearized Einstein equations coupled to the one-particle Liouville equation. As a consequence, we derive a kinetic equation for the collective motions of a small perturbation of the gas. This equation is a linear integro-differential equation of the same kind as the Vlasov equation.

RÉSUMÉ. — Le gaz gravitationnel relativiste est étudié à l'aide des équations d'Einstein linéarisées, couplées avec l'équation de Liouville à une particule.
En conséquence, nous obtenons une équation cinétique pour les mouvements collectifs d'une petite perturbation du gaz.
Cette équation est une équation intégro-différentielle linéaire du même genre que l'équation de Vlasov.

1. INTRODUCTION

This paper is devoted to the study of the simplest kinetic equation for the gravitating gas, namely the Vlasov equation. In other words, we consider the case of a relativistic gravitating gas interacting only through its

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self-consistent gravitational field (collective field). This kinetic equation is obtained in a phenomenological way by coupling Einstein's equations for the metric tensor and the one-particle Liouville equation. In fact, this phenomenological kinetic equation may be derived from a more complicated chain of equations by assuming (as usual in the electromagnetic case) that correlations of the metric tensor and of matter are negligibly small [*1*].

Although the theory presented in this paper might have interesting applications in astrophysics, our main purpose is to erect methods allowing us to deal with more sophisticated cases where correlations are no longer neglected.

When matter is in a state of sufficiently low density to permit to deal only with the *collective* gravitational field, then our kinetic equation is useful. However, in most stars matter is too dense so that we must deal with a more involved model. Fortunately there exists one « gas » satisfying the requirement of a low density. This « gas » is constituted by the whole universe so that the theory developed below does apply to cosmological problems. In fact, we shall see later that we are led to rather difficult calculations even though they involve linear equations « only ».

Another case where all this may be applied is when dealing with the *early stages* of gravitational collapse of a dense star. Indeed, as the star collapses, correlations become stronger and stronger since matter gets more and more dense.

Finally we shall briefly indicate what modifications of the formalism are needed if we want to include in the theory the emession of gravitational radiation.

Further references on the gravitating gas may be found elsewhere [*2*].

Notations.

In the following, the metric tensor $g_{\mu\nu}$ is of signature $(+ \, - \, - \, -)$ and the velocity of light has the value 1.

Furthermore, η designates the differential form « volume element » in the *various spaces* under consideration.

2. BASIC EQUATIONS AND DEFINITIONS

In this section we indicate the basic notions on which the subsequent results are founded.

Let $\mathcal{V}^4$ be the spacetime manifold where physical phenomena take place.

We assume that $\mathcal{V}^4$ is differentiable and orientable. It is important to realize that the metric tensor of this manifold, *i. e.* the gravitational field, is not known *a priori* : it is determined by the knowledge of the history of the states of the gas under study. There lies the main difficulty of general relativistic statistical mechanics; spacetime has no longer an absolute meaning. If $\mathcal{V}^4$ (and its metric tensor) were given, this generalization of relativistic statistical mechanics [*3*] would be a simple question of algebra. However it is not so and we first have to define phase space (or rather μ-space since we are dealing with kinetic theory). As usual it will be defined as the tangent fibre bundle of $\mathcal{V}^4$, say

$$\mu = \mathcal{T}(\mathcal{V}^4) \tag{1}$$

(where $\mathcal{T}$ denotes the tangent bundle). Since spacetime has no absolute meaning (*i. e.* independent of the state of the gas) it follows that this is equally the case for μ. Note that this definition is by no means new: it has already been used N. A. Chernikov [*2*] and corresponds also to the nonrelativistic definition of phase space [*4*]. In fact, the actual μ-space is rather a sub-bundle of (1). It consists of the bundle constructed with the same basis $\mathcal{V}^4$ but with the following sub-fibre:

$$\mathrm{H}_m^+(x) : \begin{cases} g_{\mu\nu}(x)p^\mu p^\nu = m^2 \\ p^0 > 0 \end{cases} \tag{2}$$

where p^μ belongs to the tangent space at point $x \in \mathcal{V}^4$ and $g_{\mu\nu}(x)$ is the metric tensor of $\mathcal{V}^4$, *i. e.* the gravitational field; m represents the mass of the particles constituting the gas and assumed to be identical. In fact we rather deal with 4-velocities u^μ (*i. e.* with $\mathrm{H}_1^+(x)$ instead of the 4-momenta p^μ), since the constraint (2) may be included in the distribution function.

A typical particle of the gas is assumed to interact with the others only through the collective gravitational field of the gas. In other words, such a typical particle follows a timelike geodesic of the manifold $\mathcal{V}^4$, the metric tensor of which is in turn determined by the motion of the gas :

$$\begin{aligned} u^\mu &= \frac{dx^\mu}{d\tau} \\ \frac{du^\mu}{d\mu} &= - \Gamma_\alpha{}^\mu{}_\beta u^\alpha u^\beta \end{aligned} \tag{3}$$

[The x^μ's are the spacetime coordinates of the generic particle considered in $\mathcal{V}^4$ while the $\Gamma_\alpha{}^\mu{}_\beta$'s are the well-known Christoffel symbols of second kind.] In μ-space the solutions of the differential system (3) form a congruence of trajectories, say $x^A(\tau)$ (where x^A denotes an arbitrary coordinate system in μ-space, which of course is not necessarily (x^μ, u^μ); the index A runs from 1 to 8; τ is the proper time). It is nothing but the geodesic flow. According to the definition of a relativistic Gibbs ensemble [3], we have to introduce a positive measure of total weight one on this family of trajectories. To this end, let us consider a 7-dimensioned manifold Σ embedded in μ and restricted to cut once (and only once) each trajectory of the congruence, but otherwise being arbitrary [5]. Let us denote by $d\Sigma_A$ the differential form with vectorial values « element of surface »:

$$d\Sigma_A = \frac{1}{7!}\sqrt{G}\,\varepsilon_{AB_1\ldots B_7}dx^{B_1} \wedge \ldots \wedge dx^{B_7} \tag{4}$$

where G is the absolute value of the determinant of the metric tensor G^{AB} of μ-space [6] and where

$$\varepsilon_{AB_1\ldots B_7} = \begin{cases} +1 \text{ when } (AB_1 \ldots B_7) \text{ is an even permutation of } (1 \ldots 8) \\ -1 \text{ when } (AB_1 \ldots B_7) \text{ is an odd permutation of } (1 \ldots 8) \\ 0 \text{ otherwise.} \end{cases}$$

The distribution function $\mathcal{N}(x^A)$ is now defined invariantly through the differential form

$$\omega = \mathcal{N}(x^A)i(\dot{x}^A)\eta \equiv \mathcal{N}(x^A)\dot{x}^A d\Sigma_A \tag{5}$$

(where η is the 8-form « element of volume » in μ-space

$$\eta = \sqrt{G}dx^1 \wedge \ldots \wedge dx^8 = dx^A \wedge d\Sigma_A \tag{6}$$

and where $i(\dot{x}^A)\eta$ designates the inner product of the field $\dot{x}^A$ [Eq. (3)] by the form η which represents the number of trajectories which cut

$$d\Sigma = n^A d\Sigma_A$$

(n^A = normal unit to Σ) centered at point x^A). From its definition $\mathcal{N}$ appears to be a scalar. The reasons why $\mathcal{N}(x^A)$ has been defined through Eq. (5) have already been explained in detail elsewhere [2] [3] and it has been shown how $\mathcal{N}(x^A)$ actually leads to a positive measure (although not of the most general kind) on the congruence defined by the solutions of Eq. (3). Accordingly, $\mathcal{N}(x^A)$ is normalized by

$$\int_\Sigma \omega = +1 \tag{7}$$

(for all Σ crossing once and only once each curve of the geodesic flow.) Eq. (7) rests on the assumption of a constant number of particles. Note also that the support of $\mathcal{N}$ in velocity space is restricted to the hyperboloid (2) with $m = 1$.

With usual coordinates (x^μ, u^μ), Eq. (7) may be rewritten as

$$\int_\Sigma \mathcal{N}(x^\mu, u^\mu) u^\mu d\sigma_\mu d_4 u = 1 \qquad \forall \mathrm{S} \text{ spacelike} \subset \mathcal{V}^4 \tag{8}$$

where $\mathrm{U}^4(x)$ is the tangent space to $\mathcal{V}^4$ at point x and Σ is a bundle of which S is the basis.

Note that if we want to take the constraint (2) (with $m = 1$) *explicitly* into account, then in Eq. (8) d_4u has to be replaced by $\sqrt{g} d_3 u / u_0(\boldsymbol{u})$.

From the arbitrary character of Σ occurring in Eq. (7), or equivalently from the conservation of the number of particles, it follows that ω is a *closed form, i. e.*

$$d\omega = 0 \tag{9}$$

which immediately leads to

$$\nabla_{\mathrm{A}} \{ \dot{x}^{\mathrm{A}} \mathcal{N}(x_{\mathrm{B}}) \} = 0 \tag{10}$$

or

$$\partial_{\mathrm{A}} \{ \sqrt{\mathrm{G}} \dot{x}^{\mathrm{A}} \mathcal{N}(x_{\mathrm{B}}) \} = 0. \tag{11}$$

In order to specify our basic equations more precisely, we first prove the general relativistic Liouville theorem:

$$\frac{d}{d\tau} \mathcal{N}(x^{\mathrm{A}}) \equiv \dot{x}^{\mathrm{A}} \partial_{\mathrm{A}} \mathcal{N}(x^{\mathrm{A}}) = 0. \tag{12}$$

Due to Eq. (11), the validity of Eq. (12) will b- insured when

$$\partial_{\mathrm{A}} \{ \sqrt{\mathrm{G}} \dot{x}^{\mathrm{A}} \} = 0 \tag{13}$$

or equivalently the geodesic flow is incompressible as has been shown by Sasaki.

Let us verify this last equation by using the coordinates (x^μ, u^μ) and the differential system (3). Taking into account the fact that [6]

$$\| \mathrm{G}_{\mathrm{AB}} \| = \left| \begin{array}{c|c} \| g_{\mu\nu} \| & 0 \\ \hline 0 & \| g_{\mu\nu} \| \end{array} \right| \tag{14}$$

we get:

$$\partial_\mu (g u^\mu) - \frac{\partial}{\partial u^\mu} (g \Gamma_\alpha{}^\mu{}_\beta u^\alpha u^\beta) = 2 u^\mu \partial_\mu (\log \sqrt{g}) - 2 \Gamma_\nu{}^\nu{}_\alpha u^\alpha = 0 \tag{15}$$

since

$$\partial_\mu (\log \sqrt{g}) = \Gamma_\nu{}^\nu{}_\mu. \tag{16}$$

Eq. (13) is equivalent to

$$d\{\dot{x}^{A}d\Sigma_{A}\} = 0,$$

which could also have been proved by casting Eq. (3) into a Hamiltonian form as Tauber and Weinberg did [2]. This shows that the « effective volume » [7] in μ-space is invariant under the group of motion.

Finally our basic equation (12) will read explicitly

$$u^{\mu}\partial_{\mu}\mathcal{N}(x^{\nu}, u^{\nu}) - \Gamma_{\alpha}{}^{\mu}{}_{\beta}u^{\alpha}u^{\beta}\frac{\partial}{\partial u^{\mu}}\mathcal{N}(x^{\nu}, u^{\nu}) = 0. \tag{17}$$

To Eq. (17) should be added the Einstein equations for the gravitational field

$$S_{\mu\nu} = \chi T_{\mu\nu} \tag{18}$$

(χ = gravitational constant) where

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \lambda g_{\mu\nu} \tag{19}$$

(λ = cosmological constant, $R_{\mu\nu}$ = Ricci tensor, $R \equiv R^{\mu}_{\mu}$ = scalar curvature).

The Einstein's equations (18), (19) are coupled to the Liouville equation (17) through the definition of the momentum-energy tensor $T^{\mu\nu}(x^{\rho})$ [2] [3]:

$$T^{\mu\nu}(x^{\rho}) = m\int \mathcal{N}(x^{\rho}, u^{\rho})u^{\mu}u^{\nu}d_{4}u \tag{20}$$

It is easy to verify that this definition (see Appendix 1), joined to the conservation law

$$\nabla_{\mu}T^{\mu\nu}(x_{\rho}) = 0, \tag{21}$$

is actually consistent with Eq. (17).

Finally it should be noticed that we could consider more complicated fluids allowing electromagnetic or classical nuclear interactions [3], for instance. In such a case Eq. (20) would no longer represent the total momentum-energy tensor of the fluid and we should add the contributions of the electromagnetic field or of the classical nuclear field.

Note also that, since we considered Eq. (3) as equations of motion, we neglected the effect of emission of gravitational radiation. This latter effect may be dealt with, (*a*) by considering the complete equations of motion [8], (*b*) by generalizing our previous definitions so as to take acceleration variables into account [9].

3. DERIVATION OF THE KINETIC EQUATION

The system of basic equations (2.17), (2.18), (2.19) and (2.20) is highly non linear and only particular solutions, or approximate solutions, may be found. We now assume that we know a particular solution of this system, characterized by a distribution function $\mathcal{N}(x^\rho, u^\rho)$ and a metric tensor $g_{\mu\nu}(x^\rho)$. In the sequel, these last quantities will be referred to as the *background quantities*. Furthermore, we limit ourselves to small disturbances of these background quantities:

$$\left.\begin{aligned} \tilde{\mathcal{N}}(x^\rho, u^\rho) &= \mathcal{N}(x^\rho, u^\rho) + \delta\mathcal{N}(x^\rho, u^\rho) \\ \tilde{g}(x^\rho) &= g_{\mu\nu}(x^\rho) + \delta g_{\mu\nu}(x^\rho) \end{aligned}\right\} \tag{1}$$

The variation of $\mathcal{N}$ should, of course, preserve the support of $\mathcal{N}$: $\delta\mathcal{N}$ satisfies the same support condition as $\mathcal{N}$.

In what follows we use the notations

$$Z(x^\rho, u^\rho) = \delta\mathcal{N}(x^\rho, u^\rho)\,; \qquad h_{\mu\nu}(x^\rho) = \delta g_{\mu\nu}(x^\rho). \tag{2}$$

Note that, corresponding to the variations (1), there exists a variation of the momentum-energy tensor $T_{\mu\nu}(x^\rho)$, which we denote by

$$\delta T_{\mu\nu}(x^\rho) = \chi^{-1} j_{\mu\nu}(x^\rho) \tag{3}$$

[See Eq. (2.20)]. Next assuming that the varied quantities $Z(x^\rho, u^\rho)$ and $h_{\mu\nu}(x^\rho)$ are small, *i. e.* that second order terms as Z^2, $(h_{\mu\nu})^{\otimes 2}$, or $h_{\mu\nu}Z$ are negligible compared to Z or $h_{\mu\nu}$, we are led to a new system of equations for these unknown quantities which are now *linear*. Of course, the solutions of this new system depend strongly on the background quantities chosen at the beginning.

It should be noted that we could linearize Einstein's equations only, express their solutions as a given function of $\mathcal{N}$ and introduce them in the one-particle Liouville equation (2.17). Doing so we should obtain a non-linear kinetic equation for $\mathcal{N}$ and we should face the same kind of problems as those occurring when dealing with Einstein's equations. Therefore it seems preferable to obtain directly a consistent linear kinetic equation.

Linearization of Einstein's equations.

Any arbitrary (although preserving the Riemannian property) metric disturbance $h_{\mu\nu}$ induces in the Riemannian affinity the first order variation [10], [11]

$$X_{\alpha}{}^{\gamma}{}_{\beta} \equiv \delta\Gamma_{\alpha}{}^{\gamma}{}_{\beta} = \frac{1}{2}\{ \nabla_{\alpha}h^{\gamma}_{\beta} + \nabla_{\beta}h^{\gamma}_{\alpha} - \nabla^{\gamma}h_{\alpha\beta}\}. \tag{4}$$

In Eq. (4) and throughout this paper, ∇ denotes the covariant differentiation operator defined with the help of the *background* Riemannian affinity (Christoffel symbols corresponding to $g_{\mu\nu}$) while indices are raised or lowered with the background metric only. For instance,

$$h^{\alpha}_{\mu}(x^{\rho}) = g^{\alpha\beta}(x^{\rho}).h_{\mu\beta}(x^{\rho}), \text{ etc.} \tag{5}$$

The change in the Ricci tensor follows from Eq. (4) in a straightforward way

$$\delta R_{\mu\nu} = -\frac{1}{2}\Delta h_{\mu\nu} + \frac{1}{2}\{\nabla_{\alpha}I_{\beta} + \nabla_{\beta}I_{\alpha}\} \tag{6}$$

where I_{α} is defined by

$$I_{\alpha} = \nabla_{\rho}h^{\rho}_{\alpha} - \frac{1}{2}\nabla_{\alpha}h, \qquad h \equiv h^{\alpha}{}_{\alpha} \tag{7}$$

and Δ is (up to the sign) the de Rham's Laplacian operator extended to symmetric tensors by A. Lichnerowicz [13]. With our notations, it reads

$$\Delta h_{\mu\nu} = \nabla_{\rho}\nabla^{\rho}h_{\mu\nu} - R_{\mu\rho}h^{\rho}{}_{\nu} - R_{\nu\rho}h_{\mu}{}^{\rho} + 2R_{\mu\nu\rho\sigma}h^{\rho\sigma}. \tag{8}$$

From this definition, it follows that Δ reduces to the usual Dalembertian operator when the background manifold is flat (*i. e.* when $R_{\mu\nu\rho\sigma} = 0$).

Let us now return to the linearization of Einstein's equations. According the definition (2.19) of the Einstein tensor; we have

$$R_{\mu\nu} = S_{\mu\nu} - \frac{1}{2}g_{\mu\nu}.\{S - 2\lambda\} \tag{9}$$

so that Einstein's equations read

$$R_{\mu\nu} = \chi\left\{T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}.T\right\} + \lambda g_{\mu\nu} \tag{10}$$

Since we have

$$\delta T = -h^{\mu\nu}.T_{\mu\nu} + g^{\mu\nu}.\delta T_{\mu\nu}, \tag{11}$$

where we used [14]

$$\delta g^{\alpha\beta} = - h^{\alpha\beta}, \qquad \delta g_{\alpha\beta} = h_{\alpha\beta} \tag{12}$$

variation of Eq. (10) yields

$$\delta R_{\mu\nu} = - \frac{1}{2} h_{\mu\nu} S + \frac{1}{2} g_{\mu\nu} h_{\alpha\beta} S^{\alpha\beta} + \lambda h_{\mu\nu} + \left\{ j_{\mu\nu} - \frac{1}{2} g_{\mu\nu} . j \right\} \tag{13}$$

where we have set

$$j \equiv g^{\alpha\beta} . j_{\alpha\beta}. \tag{14}$$

In Eq. (13), $\delta R_{\mu\nu}$ is given by Eq. (6). Therefore, Eq. (6) and (13) provide a linear partial differential equation for $h_{\mu\nu}$ whose source term is

$$\left\{ j_{\mu\nu} - \frac{1}{2} g_{\mu\nu} j \right\}. \tag{15}$$

Alternatively we might as well consider that the actual source is just the contribution of the disturbance in the *contravariant* components of the momentum-energy tensor. Therefore, the « new » source term would involve

$$k^{\mu\nu} = \chi \delta T^{\mu\nu} \tag{16}$$

rather than *j*-terms. Note that $j_{\mu\nu}$ and $k^{\mu\nu}$ are interrelated through

$$j_{\mu\nu} = k_{\mu\nu} + h_{\mu\alpha} S^{\alpha}{}_{\nu} + h_{\nu\alpha} S^{\alpha}{}_{\mu}. \tag{17}$$

Obviously, when the spacetime manifold is empty, $S^{\mu}{}_{\nu} = 0$ and $j_{\mu\nu} = k_{\mu\nu}$.

Anyway, once source terms have been separated, we are left with second-order partial differential equations which have to be solved with various techniques and more particularly with the help of Green functions m-thods.

Formal solution of the linearized Einstein equation.

Starting from Eq. (6) and (13) we get

$$\triangle h_{\mu\nu} = - 2 \left\{ - \frac{1}{2} h_{\mu\nu} S + \frac{1}{2} g_{\mu\nu} h_{\alpha\beta} S^{\beta\alpha} + \lambda h_{\mu\nu} + \left[j_{\mu\nu} - \frac{1}{2} g_{\mu\nu} . j \right] \right\} \tag{18}$$

or equivalently

$$\begin{aligned} &\triangle h_{\mu\nu} + 2\lambda h_{\mu\nu} - h_{\mu\nu} S + g_{\mu\nu} h_{\alpha\beta} S^{\alpha\beta} \\ &+ 2[h_{\mu\alpha} S^{\alpha}{}_{\nu} + h_{\nu\alpha} S^{\alpha}{}_{\mu}] - g_{\mu\nu}[2 g^{\sigma\rho} S^{\alpha}{}_{\rho} h_{\sigma\alpha}] \equiv L h_{\mu\nu} = - 2\left[k_{\mu\nu} - \frac{1}{2} g_{\mu\nu} . k \right] \end{aligned} \tag{19}$$

where use has been made of Eq. (17).

[Note that Eq. (6) may be simplified further by imposing the usual gauge condition $I_\alpha = 0$, or

$$\nabla_\rho h^\rho_\alpha - \frac{1}{2}\nabla_\alpha h = 0.$$

This gauge condition can be cast into a form similar to the common Lorentz condition of electromagnetism (see Appendix)].

The formal solution of Eq. (19) may be written as

$$h_{\mu\nu} = \int \eta' \mathrm{H}_{\mu\nu\alpha'\beta'} k^{\alpha'\beta'} \tag{20}$$

where the « Green function » $\mathrm{H}_{\mu\nu\alpha'\beta'}$ is a bi-tensor distribution which has to be specified further by giving conditions on its support (retarded, advanced conditions, etc.). The primed indices of $\mathrm{H}_{\mu\nu\alpha'\beta'}$ are related to the variable x' occurring implicitly: $\mathrm{H}_{\mu\nu\alpha'\beta'} \equiv \mathrm{H}_{\mu\nu\alpha'\beta'}(x, x')$. In fact we have

$$\mathrm{H}_{\mu\nu\alpha'\beta'} = \mathrm{K}_{\mu\nu\alpha'\beta'} - \frac{1}{2} g^{\rho'\sigma'} \mathrm{K}_{\mu\nu\rho'\sigma'} . g_{\alpha'\beta'}$$

where $\mathrm{K}_{\mu\nu\alpha'\beta'}$ is a Green function of the operator L acting on $h_{\mu\nu}$ in Eq. (19); *i. e.* we have

$$\mathrm{LK}_{\mu\nu\alpha'\beta'} = (\tau_{\mu\alpha'} . \tau_{\nu\beta'} + \tau_{\mu\beta'} . \tau_{\nu\alpha'})\delta(x, x').$$

In the case where the background space is empty, $\mathrm{S}_{\mu\nu} = 0$ and hence Eq. (19) reduces to a simple Klein-Gordon-like equation. Accordingly, the Green function $\mathrm{K}_{\mu\nu\alpha'\beta'}$ reduces to the Lichnerowicz propagators (upto the sign).

Let us now evaluate the source term $k^{\alpha\beta}(x)$ occurring in Eq. (20). This term is a functional of the distribution function given by

$$k^{\alpha\beta}(x) = \chi m \delta \int d_4 u u^\alpha u^\beta \mathcal{N}(x^\rho, u^\rho) \tag{21}$$

where

$$d_4 u = \sqrt{g} du^0 \wedge du^1 \wedge du^2 \wedge du^3. \tag{22}$$

Accordingly, Eq. (21) reduces to

$$k^{\alpha\beta}(x) = \chi m \int d_4 u u^\alpha u^\beta \left\{ \delta \mathcal{N}(x^\rho, u^\rho) + \mathcal{N}(x^\rho, u^\rho) \frac{\delta\sqrt{g}}{\sqrt{g}} \right\} \tag{23}$$

Let us now consider the second term of the r. h. s. of Eq. (23). The well-known formula of Riemannian geometry

$$d\{\log . g\} \equiv g^{\alpha\beta} dg_{\alpha\beta}$$

provides immediately

$$\delta\sqrt{g} = \frac{1}{2}h\sqrt{g}. \tag{24}$$

Consequently Eq. (23) becomes

$$k^{\alpha\beta}(x) = \chi m \int d_4 u u^\alpha u^\beta \left\{ Z(x^\rho, u^\rho) + \frac{1}{2} h(x^\rho) \mathcal{N}(x^\rho, u^\rho) \right\}. \tag{25}$$

It follows that Eq. (20) no longer appears to be an explicit solution of Eq. (19) but rather of the integral equation

$$h_{\mu\nu} = \chi m \iint \eta' d_4 u' \mathrm{H}_{\mu\nu\alpha'\beta'} u^{\alpha'} u^{\beta'} Z + \frac{\chi m}{2} \iint \eta' d_4 u' \mathrm{H}_{\mu\nu\alpha'\beta'} u^{\alpha'} u^{\beta'} \mathcal{N} h. \tag{26}$$

Setting now

$$\mathrm{F} = \mathrm{Z} + \frac{1}{2} h \mathcal{N} \tag{27}$$

one may rewrite Eq. (26)

$$h_{\mu\nu} = \chi m \iint \eta' d_4 u' u^{\alpha'} u^{\beta'} \mathrm{H}_{\mu\nu\alpha'\beta'} \mathrm{F}'. \tag{28}$$

In the next paragraph we shall see that we do not need the explicit solution of Eq. (26) for $h_{\mu\nu}$ and that the change of function (27) is extremely useful.

The self-consistent kinetic equation for the gravitating gas

Let us now rewrite Eq. (2.17) taking into account the linearization procedure. Assuming that $\mathcal{N}$ is a solution of

$$u^\mu \partial_\mu \mathcal{N} - \Gamma_\alpha{}^\mu{}_\beta u^\alpha u^\beta \frac{\partial}{\partial u^\mu} \mathcal{N} = 0, \tag{29}$$

i. e. that $\mathcal{N}$ is a « background quantity », we can rewrite Eq. (2.17) as

$$u^\mu \partial_\mu \mathrm{Z} - \mathrm{X}_\alpha{}^\mu{}_\beta u^\alpha u^\beta \frac{\partial}{\partial u^\mu} \mathcal{N} - \Gamma_\alpha{}^\mu{}_\beta u^\alpha u^\beta \frac{\partial}{\partial u^\mu} \mathrm{Z} = 0. \tag{30}$$

Using now Eq. (27), we find

$$u^\mu \partial_\mu \mathrm{F} - \Gamma_\alpha{}^\mu{}_\beta u^\alpha u^\beta \frac{\partial}{\partial u^\mu} \mathrm{F} - \mathrm{X}_\alpha{}^\mu{}_\beta u^\alpha u^\beta \frac{\partial}{\partial u^\mu} \mathcal{N} - \frac{1}{2} \mathcal{N} u^\mu \partial_\mu h = 0. \tag{31}$$

With the help of Eq. (4) and (28), the explicit expression for $X_\alpha{}^\mu{}_\beta$ may be obtained and we get

$$X_\alpha{}^\mu{}_\beta = \frac{\chi m}{2} \iint \eta' d_4u' \{ \nabla_\alpha H^\mu_{\beta\rho'\sigma'} + \nabla_\beta H^\mu_{\alpha\rho'\sigma'} - \nabla^\mu H_{\alpha\beta\rho'\sigma'} \} u^{\rho'} u^{\sigma'} F'. \quad (32)$$

Finally, the kinetic equation looked for is

$$u^\mu \partial_\mu F - \Gamma_\alpha{}^\mu{}_\beta u^\alpha u^\beta \frac{\partial}{\partial u^\mu} F = \frac{\chi m}{2} \iint \eta' d_4u' u^{\rho'} u^{\sigma'} F'$$
$$\times [\{ \nabla_\alpha H^\mu_{\beta\rho'\sigma'} + \nabla_\beta H^\mu_{\alpha\rho'\sigma'} - \nabla^\mu H_{\alpha\beta\rho'\sigma'} \} u^\alpha u^\beta \frac{\partial}{\partial u^\mu} \mathcal{N} + \mathcal{N} u^\mu \partial_\mu \{ g^{\alpha\beta} H_{\alpha\beta\rho'\sigma'} \}]. \quad (33)$$

This equation is an integrodifferential linear equation, as we expected from the beginning. It may be called a « linearized Vlasov equation for the gravitational plasma ».

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APPENDIX 1

Let us briefly show that Eq. (2.20) is actually consistent with Eq. (2.21). We have

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\sigma} T^{\sigma\nu} + \Gamma^\nu_{\mu\sigma} T^{\mu\sigma}. \tag{1}$$

Deriving Eq. (2.20) we get

$$\partial_\mu T^{\mu\nu} = m \int u^\mu u^\nu \partial_\mu (\mathcal{N}\eta) \tag{2}$$

$$(\text{with } \eta = \sqrt{g(x)}\, du^0 \wedge du^1 \wedge du^2 \wedge du^3).$$

A simple calculation shows that

$$\partial_\mu(\mathcal{N}\eta) = \partial_\mu \mathcal{N} . \eta + \mathcal{N} \Gamma_\mu{}^\sigma{}_\sigma \cdot \eta \tag{3}$$

so that Eq. (2) is rewritten as

$$\partial_\mu T^{\mu\nu} = m \int \eta \left\{ u^\nu u^\mu \partial_\mu \mathcal{N} + u^\mu u^\nu \mathcal{N} \Gamma_\mu{}^\sigma{}_\sigma \right\}. \tag{4}$$

Using the one-particle Liouville equation (2.17), one can rewrite Eq. (4):

$$\partial_\mu T^{\mu\nu} = m \int \eta \left\{ u^\nu u^\alpha u^\beta \Gamma_\alpha{}^\mu{}_\beta \frac{\partial}{\partial u^\mu} \mathcal{N} + u^\mu u^\nu \Gamma_\mu{}^\sigma{}_\sigma \mathcal{N} \right\}. \tag{5}$$

Integrating by parts the first term of the r. h. s. of this last equation, we obtain

$$\partial_\mu T^{\mu\nu} = I^\nu - m \int \eta \mathcal{N} \left\{ \Gamma_\alpha{}^\nu{}_\beta u^\alpha u^\beta + u^\nu u^\mu \Gamma_\mu{}^\sigma{}_\sigma \right\} \tag{6}$$

where the *surface integral* I^ν is given by

$$I^\nu = \oint \Gamma_\alpha{}^\mu{}_\beta u^\nu u^\beta u^\beta \mathcal{N} d\sigma_\mu \tag{7}$$

(Actually, I^ν is a 2-dimensional integral since the support of $\mathcal{N}$ is in fact 3-dimensional.)

This integral vanishes when $\mathcal{N}$ decreases rapidly enough in the hyperblood $u^\mu u_\mu = 1$, $u^0 > 0$. It follows that Eq. (6) finally reduces to

$$\nabla_\mu T^{\mu\nu} = 0. \qquad \text{Q. E. D.} \tag{8}$$

APPENDIX 2

In this appendix we consider only the case of an empty background, *i. e.*

$$R_{\mu\nu} = \lambda g_{\mu\nu} \tag{1}$$

or equivalently $S_{\mu\nu}$ vanishes and $k_{\mu\nu}$ reduces to $j_{\mu\nu}$ (see Eq. (3.17)). Moreover the variational formula

$$\delta(\nabla_\mu S^{\mu\nu}) = \nabla_\mu(\delta S^{\mu\nu}) + X_\alpha{}^\alpha{}_\sigma T^{\sigma\nu} + X_\alpha{}^\nu{}_\sigma T^{\sigma\alpha} \tag{2}$$

is simplified into

$$\delta\left(\nabla_\mu S^{\mu\nu}\right) = \nabla_\mu k^{\mu\nu} \tag{3}$$

since the well-known conservation relation

$$\nabla_\mu S^{\mu\nu} = 0$$

is preserved by any arbitrary variations. Hence we also have

$$\nabla_\mu j^{\mu\nu} = 0 \tag{4}$$

since $j^{\mu\nu} = k^{\mu\nu}$.

Using Eq. (1) we can write the basic equation (3.13) in the simpler form

$$\delta R_{\mu\nu} = j_{\mu\nu} - \frac{1}{2} g_{\mu\nu} j + \lambda h_{\mu\nu}. \tag{5}$$

Taking into account Eq. (3.6) for $\delta R_{\mu\nu}$, we find that the linear operator acting on $h_{\mu\nu}$ is still complicated and very little is known about its Green functions. However we may impose the conditions

$$I_\alpha = 0, \tag{6}$$

which in terms of the more suitable quantities

$$q_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h \tag{7}$$

may be written as

$$\nabla_\mu q^{\mu\nu} = 0 \tag{8}$$

which is much reminiscent of the Lorentz condition in so far as $q^{\mu\nu}$ is analogous to an electromagnetic potential. Lichnerowicz has shown how, at least in the vacuum, Eq. (6) is a kind of gauge condition [Besides, when the background metric is a *flat* one, Eq. (8) is equivalent to the socalled harmonicity condition.]

Using our auxiliary conditions, we can writhe the equation for $h_{\mu\nu}$ as

$$(\Delta + 2\lambda) h_{\mu\nu} = -2\left(j_{\mu\nu} - \frac{1}{2} g_{\mu\nu} . j\right). \tag{9}$$

Since the Laplacian operator commutes with the contraction by $g_{\mu\nu}$, a simple linear combination provides (see Eq. (7))

$$(\Delta + 2\lambda) q_{\mu\nu} = -2 j_{\mu\nu} \tag{10}$$

which is, of course, equivalent to Eq. (9). We now have to deal with a familiar Klein-Gordon-like operator in curved space. Green functions of the operator $(\Delta + 2\lambda)$ have been studied by Lichnerowicz [*16*] and by DeWitt [*17*] who have exhibited some of their interesting properties.

Green functions.

Let $G^{(0)\pm}(x, x')$ be the scalar kernel and $G^{(1)\pm}_{\alpha\lambda'}(x, x')$ be the vectorial kernel associated to the operator $(\Delta + 2\lambda)$. In other words, $\delta(x, x')$ being the Dirac biscalar distribution, $G^{(0)\pm}$ and $G^{(1)\pm}$ are defined by support conditions [*18*] and equations:

$$(\Delta + 2\lambda) G^{(0)\pm}(x, x') = \delta(x, x')$$

$$(\Delta + 2\lambda) G^{(1)\pm}_{\alpha\lambda'}(x, x') = \tau_{\alpha\lambda'} \delta(x, x')$$

where $\tau_{\alpha\lambda'}$ is supposed to be any bitensor satisfying

$$\tau_{\alpha\lambda'}(x, x' = x) = g_{\alpha\lambda}(x).$$

From well-known properties of the Dirac distribution, products like $\tau\delta(x, x')$, $\tau \otimes \tau\delta$, etc., do not actually depend on τ. For example, in the Minkowskian case (which requires λ to vanish) the scalar Green functions $G^{(0)+}$ and $G^{(0)-}$ would reduce to the usual $D^{\text{Adv.}}$ and $D^{\text{Ret.}}$ respectively.

Let $K^{\pm}_{\alpha\beta\lambda'\mu'}$ be the so-called second-order symmetric kernel associated with the operator $(\Delta + 2\lambda)$. In other words the bitensor-distribution $K^{\pm}_{\alpha\beta\lambda'\mu'}$ satisfies

$$(\Delta + 2\lambda)K^{\pm}_{\alpha\beta\lambda'\mu'} = \{ \tau_{\alpha\lambda'}\tau_{\beta\mu'} + \tau_{\alpha\mu'}\tau_{\beta\lambda'} \} \, \delta(x, x')$$

and the following support condition:

The support of K^+ (resp. K^-) is required to be inside (and possibly on) $\Gamma^+(x')$ [resp. $\Gamma^-(x')$] where $\Gamma^{\pm}(x')$ are the half past or future characteristic conoids whose vertex is at point x' [*i. e.*, $\Gamma^+(x')$ is determined by the lightlike paths directed towards the future and having x' as origin while $\Gamma^-(x')$ is defined in a similar way] [When we set $\mu = -2\lambda$, the present Green functions are Lichnerowicz's $G^{(0)\pm}$ and $K^{\pm}$ up to the sign.] In the particular case of a flat background $K^{\pm}$ is given by

$$K^{\pm}_{\alpha\beta\lambda'\mu'}(x, x') = \{ g_{\alpha\lambda}g_{\beta\mu} + g_{\beta\lambda}g_{\alpha\mu} \} \, D^{\substack{\text{Adv.}\\ \text{Ret.}}}(x - x')$$

in any inertial frame of reference.

Provided the source term is regular enough (*i. e.* so that the following integrals have a sense) the advanced or retarded solution of Eq. (9) is given by (*)

$$q^{\pm}_{\mu\nu} = - \int \eta' K^{\pm}_{\mu\nu\alpha'\beta'} j^{\alpha'\beta'} \tag{11}$$

with $\eta' = \eta(x', dx')$. For physical purposes we shall be concerned with the retarded solution $q^{-}_{\mu\nu}$ only.

Compatibility of Lorentz-like conditions.

Now we have to check that the above solution of Eq. (9) actually satisfies the coordinate condition (8). Contracted differentiation of Eq. (11) yields

$$\nabla^{\mu}\underset{=}{q}^{\pm}_{\mu\nu} = - \int \eta' \nabla^{\mu} K^{\pm}_{\mu\nu\alpha'\beta'} j^{\alpha'\beta'} \; ;$$

the background space satisfies

$$\nabla_{\alpha} R_{\beta\gamma} = 0,$$

therefore Lichnerowicz's formulas [17] [19]

$$- \nabla^{\lambda'} K^{\pm}_{\alpha\beta\mu'\lambda'} = \nabla_{\alpha} G^{(1)\pm}_{\beta\mu'} + \nabla_{\beta} G^{(1)\pm}_{\alpha\mu'}$$

$$- \nabla^{\mu} K^{\mp}_{\mu\nu\alpha'\beta'} = \nabla_{\alpha'} G^{(1)\mp}_{\nu\beta'} + \nabla_{\beta'} G^{(1)\mp}_{\nu\alpha'}$$

(*) If $\mathcal{V}^4$ is not flat, it seems to be necessary that $j^{\alpha'\beta'}$ has a compact support in the past of x, in order that the integral (11) makes sense.

are valid and imply

$$\nabla^{\mu} q^{\pm}_{\mu\nu} = \int \eta' j^{\alpha'\beta'} \{ \nabla_{\alpha'} G^{(1)\pm}_{\nu\beta'} + \nabla_{\beta'} G^{(1)\pm}_{\nu\alpha'} \}.$$

Since $j^{\alpha'\beta'}$ is divergence-free, and integration by parts shows that Eq. (8) holds provided the flux

$$\oint_{(s)} G^{(1)\pm}_{\nu\beta'} j^{\alpha'\beta'} \eta_{\alpha'} \to 0$$

when the closed 3-surface (s) surrounding the point x goes to infinity. It is noteworthy that nowhere, the explicit structure of the source term has been used. Therefore the whole linearization procedure presented above is relevant for the general case of an unspecified $j^{\mu\upsilon}$.

The linearized kinetic equation.

The kinetic equation may be obtained either from the above considerations or from Eq. (3.33). We get

$$u^{\mu} \partial_{\mu} Z - \Gamma_{\alpha}{}^{\mu}{}_{\beta} u^{\alpha} u^{\beta} \frac{\partial}{\partial u^{\mu}} Z = 0 \tag{12}$$

since $\mathcal{N} \equiv 0$ in the case of an empty background. Therefore, we see that, *in this particular case*, the kinetic equation obtained has the form of the one-particle Liouville equation for a gas embedded in an *external* gravitational field. However the main difference is that this external gravitational field should be a *free field.* Physically this means that the background gravitational field, if not flat, may be interpreted as being constituted by gravitational radiation or as a given cosmological background.

When the background spacetime manifold is flat, then there is no source at all and $\Gamma_{\alpha}{}^{\mu}{}_{\beta}$ may be chosen to be zero. If follows that Eq. (12) reduces to a trivial equation

$$u^{\mu} \partial_{\mu} Z = 0.$$

It is interesting to note that Eq. (12) is identical to the one-particle Liouville equation although their physical interpretations are completely different.

REFERENCES

[*1*] R. Hakim, *Einstein's Random Equations*, to be published.
[*2*] E. G. Tauber and J. W. Weinberg, *Phys. Rev.*, t. **122**, 1961, p. 1342.
N. A. Chernikov, *Soviet Phys. Dokl.*, t. **1**, 1956, p. 103; t. **2**, 1957, p. 248; t. **5**, 1960, p. 764; t. **5**, 1960, p. 786; t. **7**, 1962, p. 397; t. **7**, 1962, p. 428; *Phys. Letters*, t. **5**, 1963, p. 115; *Acta Phys. Polonica*, t. **23**, 1963, p. 629; t. **26**, 1964, p. 1069; t. **27**, 1964, p. 465.
R. W. Lindquist, *Ann. Phys.*, t. **37**, 1966, p. 487.
[*3*] R. Hakim, *J. Math. Phys.*, t. **8**, 1967, p. 1153; *Ibid.*, t. **8**, 1967, p. 1379.
See also, *Ann. Inst. H. Poincaré*, t. **6**, 1967, p. 225.
[*4*] Phase space is always the tangent fibre bundle of the manifold configuration space.
[*5*] Actually Σ is 6-dimensional if we bear in mind the constraint (2).
[*6*] Since μ is the tangent bundle of a metric manifold (*i. e.* $\mathcal{V}^4$), then on this space one can construct a canonical metric tensor G^{AB}. See the article by Lindquist (Ref. [*2*]) and references quoted therein.

[*7*] By « effective volume » we mean a 6-dimensional volume. This conservation law, *i. e.* the Liouville theorem, means that if $\Delta_1 \subset \Sigma_1$ is such a 6-dimensional volume, then mes $(\Delta_1) =$ mes (Δ_2) where Δ_2 is the « volume » in Σ_2 obtained from the transformation of Δ_1 under the group motion (*i. e.* Eq. (3)).

[*8*] Such as those given by P. HAVAS and J. N. GOLDBERG, *Phys. Rev.*, t. **128**, 1962, p. 398.

[*9*] This would be only a simple generalization of previous results where the electromagnetic radiation was dealt with (See Ref. [*3*] and also R. HAKIM and A. MANGENEY, *Relativistic kinetic equations including radiation effects I. Vlasov approximation* (to appear in *J. Math. Phys.*, **9**, 116 (1968)).

[*10*] A. LICHNEROWICZ, *Propagateurs et commutateurs en relativité générale* (Publications Mathématiques n° 10 de l'I. H. E. S.), p. 40.

[*11*] We mainly use the notations of Ref [*10*].

[12] Ref. [*10*], p. 43.

[*13*] Ref. [*10*], p. 27.

[*14*] Ref. [*10*], p. 39.

[*15*] In the same way as neglecting correlations of electromagnetic field or of particles amounts to dealing with a kinetic equation valid at order $\sim e^2$, neglecting correlations of the gravitational field is expected to provide a kinetic equation valid at order χ. We verify this statement on the resulting equation.

[*16*] Ref. [*10*], p. 33.

[*17*] B. DE WITT, *Ann. Phys.*, t. **9**, 1960, p. 220.

[*18*] Ref. [*10*], p. 30.

[*19*] Ref. [*10*], p. 35.

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