ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 8, nº 3 (1968), p. 301-309
<http://www.numdam.org/item?id=AIHPA_1968__8_3_301_0>

Ann. Inst. Henri Poincaré,
Vol. VIII, n° 3, 1968, p. 301-309.

Section A :
Physique théorique.

Possible derivation of some SO(p, q) group representations by means of a canonical realization of the SO(p, q) lie algebra

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ABSTRACT. — Starting from the usual realization of the inhomogeneous SO($p - 1$, $q - 1$) algebra in terms of quantum mechanical operators, we get a realization of the SO(p, q) algebra by introducing supplementary variables and arbitrary parameters. In this way we show how we can derive some representations of the SO(p, q) group ($pq \neq 0$).

SOMMAIRE. — En partant de la réalisation habituelle de l'algèbre de Lie du groupe inhomogène SO($p - 1$, $q - 1$) écrite en termes d'opérateurs de la mécanique quantique, nous obtenons une réalisation de l'algèbre de Lie des groupes de rotation non compacts SO(p, q) par addition de constantes arbitraires et de variables canoniques supplémentaires. Nous montrons alors comment on peut déduire de cette façon certaines représentations des groupes SO(p, q) ($pq \neq 0$).

I. — INTRODUCTION

In this note, we investigate a realization of the Lie algebra of the non-compact rotation groups SO(p, $n - p$) ($n > p$, $p \neq 0$) in terms of quantum mechanical operators. Writing the elements of the inhomogeneous

SO($p-1, n-p-1$) subalgebra as functions of the canonical variables x_i and p_i ($i = 1, 2, \ldots, n-2$), we could express all the SO($p, n-p$) generators in terms of the x_i's and p_i's. But in this way we derive only some representations of SO($p, n-p$). Indeed, if we want to get all the unitary irreducible ones, we know [1] that the SO($p, n-p$) Lie algebra must be expressed with

$$\text{(1)} \qquad N = 1/2 \{ n(n-1)/2 - [n/2] \}$$

pairs of conjugate variables and $[n/2]$ arbitrary parameters [2] so that in the general case we have to use, in addition to the x_i's and p_i's new canonical variables [3].

In fact, we show in Section II that we have to consider two cases, one when $n \leq 4$, the other when $n > 4$. In this later case, we derive the general expression of the SO($p, n-p$) generators and deduce in Section III the most degenerate series of representations.

In Section IV, we investigate the case $n \leq 4$ and show as an example how we can derive the representations of the Lorentz group.

II. — THE REALIZATION OF THE SO($p, n-p$) ALGEBRA ($n > 4$)

Let us consider the SO($p, n-p$) group for which the generators $M_{\mu\nu}$ satisfy the following commutation rules:

$$\text{(2)} \qquad [M_{\mu\nu}, M_{\rho\sigma}] = i \{ g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho} \}$$

where $\mu, \nu, \rho, \sigma = 1, 2, \ldots, p, p+1, \ldots, n$, and $g_{\mu\mu} = 1$ if $\mu \leq p$, $g_{\mu\mu} = -1$ if $\mu \geqslant p+1$, $g_{\mu\nu} = 0$ if $\mu \neq \nu$.

We know that the generalized Wigner rotations [5] [6] leaving invariant a given light-like vector, generate a subgroup of SO($p, n-p$), isomorphic to the inhomogeneous SO($p-1, n-p-1$) group. The generators of this subgroup expressed in terms of the $M_{\mu\nu}$ are chosen to be [7]

$$\text{(3)} \qquad \Gamma_i^- = M_{ip} - M_{in} \qquad \text{and} \qquad M_{ij}$$

with [8] $i, j = 1, 2, \ldots, p-1, p+1, \ldots, n-1$.

We can enlarge this group by adding a supplementary generator namely the dilatation operator D and get in this way the similitude group which we define as the group leaving invariant *the direction* of a given light-like vector. It is easily seen that the dilatation operator D is

$$D = M_{pn} \tag{4}$$

Moreover, we put

$$\Gamma_i^+ = M_{ip} + M_{in} \tag{5}$$

which can be interpreted as the « special conformal transformations » acting in an euclidian space of signature $(p - 1, n - p - 1)$. The commutation rules between the generators (3) (4) (5) are

$$(a)\quad [D, M_{ij}] = 0 \qquad (b)\quad [D, \Gamma_i^\pm] = \mp i\Gamma_i^\pm \tag{6}$$

$$[M_{ij}, M_{kl}] = i\{g_{il}M_{jk} + g_{jk}M_{il} - g_{ik}M_{jl} - g_{jl}M_{ik}\} \tag{7}$$

$$[M_{ij}, \Gamma_i^\pm] = i(g_{jk}\Gamma_i^\pm - g_{ik}\Gamma_i^\pm) \tag{8}$$

$$[\Gamma_i^\pm, \Gamma_j^\pm] = 0 \tag{9}$$

$$[\Gamma_i^-, \Gamma_j^+] = -2i(M_{ij} + g_{ij}D) \tag{10}$$

Let us choose now the usual canonical realization of the inhomogeneous $SO(p - 1, n - p - 1)$ subalgebra, namely

$$M_{ij} = x_i p_j - x_j p_i \tag{11}$$

$$\Gamma_i^- = p_i \tag{12}$$

with

$$[x_i, p_j] = -ig_{ij}$$

Following the comment of Section I, we are led to introduce in addition to the $(n - 2)$ pairs of variables x_i, p_i

$$N_\xi = N - (n - 2) = 1/2\{n(n - 1)/2 - [n/2] - 2(n - 2)\} \tag{13}$$

new pairs of canonical variables which we shall denote by ξ_μ and π_μ with $[\xi_\mu, \pi_\nu] = i\delta_{\mu\nu}$. The values of N and N_ξ [Formulas (1) and (13)] with respect to the type of non-compact rotation group are given in table I. Let us note that for $n \leq 4$ only, the number of variables x_i, p_i introduced in the generators (11) (12) of the inhomogeneous subgroup is sufficient to get all the representations of $SO(p, n - p)$. So, we have to consider as special cases the groups for which $n \leq 4$, and we treat them in a complete way in Section IV.

Now, let us give the expression of the dilatation operator (4). From (6(*a*)) and (6(*b*)), we deduce that if $n \leq 4$, the only possible expression for D is $-\frac{1}{2}g_{ij}(x_ip_j + p_jx_i)$. In the general case $n > 4$, we shall add the term $1/2(\vec{\xi}\vec{\pi} + \vec{\pi}.\vec{\xi})$ where $\xi_\mu\pi_\mu$ are conjugate variables with $\mu = 1, 2, \ldots, N_\xi$. Finally, the expression of D is chosen to be:

$$(14) \qquad D = 1/2\, g_{ij}(x_ip_j + p_jx_i) + 1/2\,(\vec{\xi}\cdot\vec{\pi} + \vec{\pi}\cdot\vec{\xi})$$

From (8), we see that the Γ_i^+ defined in (5) transform like a vector under the M_{ij} and then are of the type $Sp_i + S'x_i$ where S and S′ are scalar functions. It is easy to determine the expression of these two quantities with the aid of (6(*b*)) and (10). Finally, we are led to

$$(15) \qquad \Gamma_i^+ = -1/2\,[p_i, g_{jk}x_jx_k]_+ - [x_i, D]_+ - f(\vec{\xi}, \vec{\pi})p_i$$

which satisfy (9), the only condition on $f(\vec{\xi}, \vec{\pi})$ being

$$(16) \qquad [D, f(\vec{\xi}, \vec{\pi})] = -2if(\vec{\xi}, \vec{\pi})$$

So, the $[n/2]$ arbitrary parameters we have to introduce must be put in $f(\vec{\xi}, \vec{\pi})$. For instance, it can be shown that this function is of type $A\xi^2 + A'[\xi^2, \pi^2]_+$ for the SO(4, 1) and SO(3,2) groups [*4*], the vector ξ being one-dimensional and A, A′ two arbitrary real parameters (see table I).

TABLE I

Groups	SO(2,1)	SO(3,1) SO(2,2)	SO(4,1) SO(3,2)	SO(4,2) SO(5,1)	SO(p, n−p)	
					n even > 4	n odd > 4
rank	1	2	2	3	$\frac{n}{2}$	$(n-1)/2$
N	1	2	4	6	$n(n-2)/4$	$[(n-1)/2]^2$
N_ξ	0	0	1	2	$(n-2)(n-4)/4$	$[(n-1)(n-5)/2]-1$

III. — THE MOST DEGENERATE REPRESENTATIONS OF SO(p, $n-p$) FOR $n > 4$

If we want to find the most degenerate series of representations of SO(p, $n - p$), it is not useful in the general case to introduce the required number of variables of type ξ but only one and one arbitrary parameter. So, we suppose that in (14) and (15) we introduce only one pair of variables ξ, π and take $f(\xi, \pi)$ in the following form

$$f(\xi, \pi) = -A^2\xi^2 \tag{17}$$

which satisfy (16). A being an arbitrary real number.

Then, all the invariant operators are fixed except the second order Casimir operator:

$$c_2 = \frac{1}{2} g_{ij}g_{kl}M_{ik}M_{lj} - D^2 + \frac{1}{2} g_{ij}(\Gamma_i^+\Gamma_j^- + \Gamma_j^-\Gamma_i^+) \tag{18}$$

which becomes

$$c_2 = -\frac{1}{4}(\xi\pi + \pi\xi)^2 + A^2\xi^2 g_{ij}p_ip_j - \left(\frac{n-2}{2}\right)^2 \tag{19}$$

By solving the eigenvalue problem, we are led to the following results: if $A \neq 0$, we get the discrete energy spectrum for which the eigen-functions are of type

$$\Phi(x_i, \xi) = e^{ig_{ij}p_ix_j}\xi^{-1/2}J_\nu(mA\xi) \tag{20}$$

corresponding to the eigen-values

$$c_2 = \nu^2 - \left(\frac{n-2}{2}\right)^2 \tag{21}$$

$g_{ij}p_ip_j = m^2$ being the translational invariant operator of the inhomogeneous SO($p - 1$, $n - p - 1$) subgroup, and J_ν a Bessel function of which the order ν is an integer when n even and half integer when n is odd.

If $A = 0$, the eigenfunctions Φ are of the form:

$$\Phi = e^{ig_{ij}p_ix_j}\xi^{\alpha-1/2} \tag{22}$$

where α is a number. The eigen-values of the Casimir operator (19) becomes:

$$c_2 = \alpha^2 - \left(\frac{n-2}{2}\right)^2 \tag{23}$$

and then we get the continuous spectrum. We see that the case for which no variable of type ξ is introduced correspond to a particular value of (23), namely $c_2 = -[(n-2)/2]^2$.

Of course, these results are in accordance with those of Raczka Niederle Limic [9] and L. Castell [10] who obtains them in the case of the conformal group SO(4,2) by an analogous way.

IV. — THE REALIZATION OF THE SO(p, $n-p$) LIE ALGEBRA WHEN $n \leq 4$

The case of the SO(2,1) group has been investigated in paper of ref [11] and we do not talk about.

So, let us consider the SO(3,1) and SO(2,2) groups for which we have to introduce two arbitrary parameters in their Lie algebra and no supplementary variable of type ξ (see table I). In the SO(3,1) case, a simple calculus leads to the following expressions of D and $\Gamma_i^{\pm}$:

$$D = \frac{1}{2}[x, p_x]_+ + \frac{1}{2}[y, p_y]_+$$

$$(24) \quad \Gamma_1^- = p_x + \frac{Ax + By}{x^2 + y^2} \qquad \Gamma_1^+ = \frac{1}{2}[p_x, x^2 + y^2]_+ - [x, D]_+ + Ax - By$$

$$\Gamma_2^- = p_y + \frac{Ay - Bx}{x^2 + y^2} \qquad \Gamma_2^+ = \frac{1}{2}[p_y, x^2 + y^2]_+ - [y, D]_+ + Ay + Bx$$

where A and B are real arbitrary parameters.

By analytic continuation, we get the SO(2,2) case, namely

$$D = \frac{1}{2}[x, p]_+ - \frac{1}{2}[x_0, p_0]_+$$

$$(25) \quad \Gamma_0^- = p_0 + \frac{\alpha x_0 - \beta x}{x^2 - x_0^2} \qquad \Gamma_0^+ = \frac{1}{2}[p_0, x^2 - x_0^2]_+ - [x_0, D]_+ + \alpha x_0 + \beta x$$

$$\Gamma_1^- = p + \frac{\alpha x - \beta x_0}{x^2 - x_0^2} \qquad \Gamma_1^+ = \frac{1}{2}[p, x^2 - x_0^2]_+ - [x, D]_+ + \alpha x + \beta x_0$$

α being real and β pure imaginary. The angular momentum J_3 is taken in its usual form.

The fondamental invariants take the values:

for SO(3,1)

(26)
$$F = \vec{J}^2 - \vec{K}^2 = B^2 - A^2 - 1$$
$$W = \vec{J}.\vec{K} = - AB$$

for SO(2,2)

(27)
$$F = \beta^2 - \alpha^2 - 1$$
$$W = - \alpha\beta$$

The Hilbert space of representations will be given by solving the eigenvalue problem

(28)
$$\vec{\Gamma}^{-2}\Phi = c\Phi$$

$\vec{\Gamma}^{-2}$ being the translational invariant operator of the inhomogeneous subgroup. Thus, we shall get conditions on the parameters A, B and α, β.

For example, let us point out the possible derivation of the Lorentz group representations. In this case, the expression of $\vec{\Gamma}^{-2}$ becomes

(29)
$$\vec{\Gamma}^{-2} = \vec{p}^2 + \frac{2A}{\vec{x}^2}(\vec{x}\vec{p}) + \frac{A^2 + B^2}{\vec{x}^2} - \frac{2B}{\vec{x}^2} J_3$$

Solving the equation (28), we find the general solution

(30)
$$\Phi = e^{im\varphi}(r\sqrt{c})^{-iA} J_\nu(r\sqrt{c})$$

where (r, φ) are the polar coordinates in the plane. J_ν is a Bessel function of arbitrary real order ν and m the eigenvalues of the angular momentum ($m = 0, \pm 1, \pm 2, \ldots$). The parameter B is related to the order ν by

(31)
$$\nu^2 = (m + B)^2$$

The constant A is left arbitrary. A more detailled discussion in which the construction of an orthonormal basis is considered, would give the usual values [*12*] of the invariants (26).

V. — CONCLUSION

We have shown that, starting with the usual canonical realization of the inhomogeneous SO(p − 1, n − p − 1) group, we can get the SO(p, n − p) Lie algebra with which it should be possible to reduce the irreducible unitary representations of the SO(p, n − p) group with respect to the inhomogeneous subgroup ones. In particular, this is of physical interest for the conformal group SO(4,2) since it contains as inhomogeneous subgroup the Poincaré group.

ACKNOWLEDGMENTS

The author is indebted to Prof. H. Bacry who has suggested this work and thanks him for valuable discussions.

REFERENCES AND FOOTNOTES

[1] A general formula is given in H. Bacry, *Space time and degrees of freedom of the elementary particle, Comm. Math. Phys.*, t. 5, 1967, p. 97, for the semi-simple Lie groups. Let us recall it: $1/2(d - r) = g$ where d, r, g denote respectively the dimensionality of the Lie group, the number of its fundamental invariants, and the number of the generators of a maximal abelian subalgebra of the enveloping algebra. This formula comes from a more general framework in Gelfand and Kirillov, *Sur les corps liés avec algèbres enveloppantes des algèbres de Lie*, *Publications Mathématiques*, n° 31 (I. H. E. S.).

[2] $[n/2]$ denotes the rank of the group, namely $n/2$ if n is even, $(n - 1)/2$ if n is odd.

[3] Let us note the approach of A. Kihlberg in which he considers as subgroup the maximal compact group and enters in similar considerations (A. Kihlberg, *Arkiv für Fysik*, t. 30, 1965, p. 121). Many references on the study of particular non compact rotation groups are given in this paper. We recall some particular works in ref. [4].

[4] The De Sitter group SO(4,1) has been treated thoroughly by Dixmier, in *J. Bull. Soc. Math. France*, t. 89, 1961, p. 9. The SO(3,2) group has been investigated by J. B. Ehrman, in *Proc. Camb. Phil. Soc.*, t. 53, 1957, p. 290.

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[*12*] I. M. GEL'FAND, R. A. MINLOS et Z. Y. SHAPIRO, *Representations of the rotation and Lorentz groups and their applications*, Pergamon Press, 1963. For other references, see [*6*].

Manuscrit reçu le 19 décembre 1967.