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Section A :
Physique théorique.

Poisson brackets of the constraints in unified field theory

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ABSTRACT. — When Einstein's unified relativistic field theory is canonically formulated, proper dynamical variables are linked by some constraints. We calculate explicit Poisson Brackets of these constraints, between one another.

These brackets turn out to be very simple linear combinations of the constraints.

In a previous work [*1*] we have tried to adapt canonical methods to Lagrangians which are linear in the « velocities » (i. e. time derivatives of the field).

Then we have applied our results to Einstein-Schrödinger unified field theory.

Of course many other theories could be considered within the framework of linear Lagrangian formalism: Indeed, by a formal increase of the number of field variables, most well known Lagrangian densities can be rearranged as linear functions of the velocities. For instance Maxwell equations can be derived from a Lagrangian which is linear with respect to the derivatives of variationally independent A_μ and $F_{\rho\sigma}$.

Besides, Arnowitt, Deser and Misner [*2*] [*3*] have many times emphasized the occurence of a similar situation in General Relativity as a starting point for canonical investigations, metric and affinity being regarded as independent under variation. Nevertheless, unified field theory itself is not much

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more complicated, so far as canonical formulation is concerned. As shown by Einstein, variational principle can be applied to the scalar density

$$\mathfrak{R} = \mathfrak{G}^{\mu\nu} R_{\mu\nu}$$

which involves the assymmetric metric and the assymetric affinity Γ as independent quantities.

In $\mathfrak{R}$ the only « velocities » are the time derivatives of the affinity. They occur linearly, since the Ricci tensor is

$$R_{\mu\nu} = \partial_\rho \Gamma_\mu{}^\rho{}_\nu - \partial_\nu \Gamma_\mu{}^\rho{}_\rho + \Gamma_\mu{}^\lambda{}_\nu \Gamma_\lambda{}^\rho{}_\rho - \Gamma_\mu{}^\lambda{}_\rho \Gamma_\lambda{}^\rho{}_\nu$$

As well known [4], variational principle yields field equations

(1) $$R_{\mu\nu} = 0$$

(2) $$D_\rho \mathfrak{G}^{\overset{\mu\nu}{+-}} = 0$$

In the paper quoted above [1] we have examinated the canonical structure of equations (1) (2), assuming that a space-like 3-surface (Σ) is given.

With respect to evolution in time and existence of Poisson bracket, the notion of « *proper dynamical variable* » has been defined [5]. According to this point of view all the metric quantities $\mathfrak{G}^{\mu\nu}$ are proper dynamical variables, while two kinds of variables arise from the affinity:

The linear combinations

$$Y_\mu{}^j{}_k = \Gamma_\mu{}^j{}_k - \frac{1}{3}\delta_k{}^j \Gamma_\mu{}^l{}_l$$

and

$$Y_\mu{}^\rho = \Gamma_\mu{}^\rho{}_4$$

are not proper dynamical variables.

On the contrary, the general formalism stated in [1] leads to exhibit 16 proper dynamical variables

(3) $$\bar{y}_{\mu\nu} = \varepsilon_{\mu\nu}{}^\beta{}_\alpha{}^\gamma \Gamma_\beta{}^\alpha{}_\gamma$$

where

(4) $$\varepsilon_{\mu\nu}{}^\beta{}_\alpha{}^\gamma = \delta_\mu{}^\beta (n_\alpha \delta_\nu{}^\gamma - \delta_\alpha{}^\gamma n_\nu)$$

and vector n_α being the normal vector to surface (Σ).

In general the length of n_α is not unity, but the whole canonical formulation is homogeneous with respect to n_α.

Hence normalization of n_α does not matter and one may assume that (Σ) is a surface $x^4 = \text{const.}$, $n_\alpha = \delta_\alpha{}^4$ without loss of generality.

Explicitly equation (3) reads

(5) $$\bar{y}_{\mu\nu} = \Gamma_{\mu}{}^{4}{}_{\nu} - \delta_{\nu}{}^{4}\Gamma_{\mu}{}^{\alpha}{}_{\alpha}$$

In other words

$$\bar{y}_{\mu j} = \Gamma_{\mu}{}^{4}{}_{j} \qquad \bar{y}_{\mu 4} = -\Gamma_{\mu}{}^{j}{}_{j}$$

Finally, all the properly dynamical part of the affinity is included in these $\bar{y}_{\mu\nu}$.

As regards Cauchy problem, equation (1) and the four equations

(6) $$D_4 \mathcal{G}^{\underset{+}{\mu}\underset{-}{\nu}} = 0$$

are evolution equations insuring determination outside (Σ) of the $16 + 16 = 32$ proper dynamic variables $\mathcal{G}^{\mu\nu}$ and $\bar{y}_{\alpha\beta}$. The *improper variables* $Y_{\mu}{}^{j}{}_{k}$ and $Y_{\mu}{}^{\rho}$ have no time derivative involved at all in field equations (1) (2).

Then one could ask wether the remaining 48 equations

(7) $$D_j \mathcal{G}^{\underset{+}{\mu}\underset{-}{\nu}} = 0$$

simply express improper quantities Y in terms of proper dynamical variables.

Indeed there is 48 linearly independent quantities in the Y since $Y_{\mu}{}^{j}{}_{j}$ identically vanishes.

But if one tries to solve (7) with respect to improper variables, five of them (viz. $Y_k{}^k$ and $Y_4{}^j$) cannot be calculated [*6*].

As a result one may exhibit five combinations of (7) where only proper variables are involved. These five expressions are the only effective constraints of the theory.

They read as follows [*6*]

$$C_j \equiv D_j \mathcal{G}^{\underset{+}{4}\underset{-}{4}} = 0$$

$$B \equiv D_j \mathcal{G}^{\underset{+}{4}\underset{-}{j}} = 0$$

$$C \equiv D_j \mathcal{G}^{\underset{+}{j}\underset{-}{4}} = 0$$

with the identical expressions

(8) $$C_j \equiv \mathcal{G}^{44}{}_{j} + Q^{4\sigma}{}_{j\sigma} + Q^{\sigma 4}{}_{\sigma j}$$

(9) $$B \equiv \mathcal{G}^{4j}{}_{j} - Q^{4\sigma}{}_{4\sigma} + Q^{\sigma j}{}_{\sigma j}$$

(10) $$C \equiv \mathcal{G}^{j4}{}_{j} + Q^{j\sigma}{}_{j\sigma} - Q^{\sigma 4}{}_{\sigma 4}$$

according to the notations

$$Q^{\alpha\beta}{}_{\mu\nu} = \mathcal{G}^{\alpha\beta}\bar{y}_{\mu\nu} \tag{11}$$

$$\mathcal{G}^{\alpha\beta}{}_j = \partial_j \mathcal{G}^{\alpha\beta} \tag{12}$$

The five quantities C_j, B, C are purely spatial and properly dynamical variables. Hence they do have well defined Poisson brackets. But, although the field equations require C_j, B and C to vanish, in general Poisson brackets of these constraints with any proper dynamical variable are not zero.

Such a situation is common when canonical formulation concerns a theory which is invariant under an infinite dimensional group. This question has been widely investigated by P. G. Bergmann [7].

Constraints with non zero Poisson brackets seem scarcely to permit any further quantization. Actually, satisfactory results are possible when all the Poisson brackets of the constraints among themselves are equal to zero *modulo the constraints* (i. e. they are not identically zero but vanish provide the constraints do, viz. when field equations are satisfied).

This point has been emphasized by P. A. M. Dirac [8].

Now our aim is to check wether our five constraints C_j, B, C exhibit this nice property. Therefore the brackets

$$\{ C_j C_{k'} \} \qquad \{ B B' \} \qquad \{ C C' \} \qquad \{ B C' \} \qquad \{ C_j B' \} \qquad \{ C_j C' \}$$

must be calculated. (For compacity of writing $C_{k'}$ stands for $C_{k'}(x')$, B′ stands for $B_{(x')}$ and so on.) For this explicit computation we need the basic Poisson brackets given in I.

As explained above, the properly dynamical part of the field consists in the quantities $\mathcal{G}^{\mu\nu}$ and $\bar{y}_{\alpha\beta}$.

In I we have given the fundamental Poisson Brackets:

$$\{ \bar{y}_{\alpha\beta}\, \bar{y}_{\mu'\nu'} \} = 0 \qquad \{ \mathcal{G}^{\alpha\beta}\, \mathcal{G}^{\mu'\nu'} \} = 0 \tag{13}$$

$$\{ \mathcal{G}^{\alpha\beta}\, \bar{y}_{\mu'\nu'} \} = - \Delta^{\alpha\beta}{}_{\mu'\nu'} \tag{14}$$

where

$$\Delta^{\alpha\beta}{}_{\mu'\nu'} = n_\sigma \Delta^{\sigma\,\alpha\beta}{}_{\mu'\nu'}$$

$$\Delta^{\sigma\,\alpha\beta}{}_{\mu'\nu'} = \tau^{\alpha}{}_{\mu'}\tau^{\beta}{}_{\nu'}\delta^{\sigma}_{(\Sigma)}(x,x') \tag{15}$$

and $\delta^{\sigma}_{(\Sigma)}(x, x')$ is the Dirac vector-scalar density associated with surface (Σ) by the defining property

$$\int_{\Sigma} f_{(x)}\delta^{\mu}_{\Sigma}(x,x')d\sigma_{\mu} = f_{(x')} \tag{16}$$

Actually δ^{μ}_{Σ} is a vectorial density at point x, and is a scalar at point x'. (About bilocal objects, like tensors or densities and so on, refering to differents points, see R. Brehme and B. S. de Witt [9] or A. Lichnerowicz [10]).

The bitensor $\tau^{\alpha}{}_{\mu'}$ is arbitrary except for the condition

(17) $$(\tau^{\alpha}{}_{\mu'})_{(x'=x)} = \delta^{\alpha}{}_{\mu}$$

With convenient coordinates $(\Sigma) \equiv (x^4 = \text{const.})$.

$$d\sigma_i = 0 \quad , \quad d\sigma_4 = dx \wedge dy \wedge dz$$

$$\delta^4_{\Sigma} = \delta^{(3)}_{(x,x')} \equiv \delta(x^1 - x^{1\prime})\delta(x^2 - x'^2)\delta(x^3 - x'^3)$$

Due to well known properties of Dirac's distribution, $\Delta^{\alpha\beta}{}_{\mu'\nu'}$ does not actually depends on the choice of $\tau^{\alpha}{}_{\beta'}$. With the above special coordinates one gets

$$\Delta^{\alpha\beta}{}_{\mu'\nu'} = \delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu}\delta^{(3)}_{(x,x')}$$

The bi-scalar density

$$\delta_{\Sigma} = n_{\mu}\delta^{\mu}_{\Sigma}$$

has the following properties

(18) $$\partial_j\delta_{\Sigma} \equiv -\,\partial_{k'}(\tau^{k'}{}_j\delta_{\Sigma})$$

(19) $$\partial_j(f_{(x')}\delta_{\Sigma}) + \partial_{k'}(\tau^{k'}{}_j f_{(x)}\delta_{\Sigma}) \equiv (\partial_j f_{(x)})\delta_{\Sigma}$$

These formulae, being manifestly covariant in the 3-dimensional space (Σ), just have to be proved in a special coordinates system. The proof is obvious when $(\Sigma) \equiv (x^4 = \text{const.})$.

Then (18) and (19) reduce to

(18′) $$\partial_j\delta^{(3)} = -\,\partial_{j'}\delta^{(3)}$$

and

(19′) $$\partial_j(f_{(x')}\delta^{(3)}) + \partial_{j'}(f_{(x)}\delta^{(3)}) \equiv (\partial_j f)\delta^{(3)}$$

Property (18′) is trivial.

For (19) one just has to write down

$$\begin{aligned}\partial_j(f_{(x')}\delta^{(3)}) &= \partial_j(f_{(x)}\delta^{(3)}) \\ &= \partial_j f\,\delta^{(3)} + f\,\partial_j\delta^{(3)} \\ &= \partial_j f\,\delta^{(3)} - f\,\delta_{j'}\delta^{(3)}\end{aligned}$$

Then (19′) comes out.

Since the constraints are linear in $\mathcal{G}^{\alpha\beta}{}_j$ and $Q^{\mu\nu}{}_{\rho\sigma}$ we have better first computing the brackets of these expressions. By use of the expansion

$$\mathcal{G}^{\alpha\beta} = \int_\Sigma \Delta^{\mu\,\alpha\beta}{}_{\rho'\sigma'}\mathcal{G}^{\rho'\sigma'} d\sigma_\mu \tag{20}$$

it is easy to check the identity

$$\{\, \mathcal{G}^{\alpha\beta}{}_j, \mathrm{V} \,\} = \partial_j \{\, \mathcal{G}^{\alpha\beta}, \mathrm{V} \,\} \tag{21}$$

where V is any proper dynamical variable.

Moreover Jacobi identity holds, and also

$$\{\, \mathrm{U}, \mathrm{VW} \,\} \equiv \{\, \mathrm{U}, \mathrm{V} \,\}\, \mathrm{W} + \mathrm{V} \{\, \mathrm{U}, \mathrm{W} \,\} \tag{22}$$

By use of (21) and (13) one gets

$$\{\, \partial_j \mathcal{G}^{\alpha\beta} \quad , \quad \partial_{k'} \mathcal{G}^{\mu'\nu'} \,\} = 0 \tag{23}$$

As we have shown in I p. 30 (where eq. III-47 is meant for $\mu = 1, 2, 3$ only) identity (22) provides

$$\{\, \mathcal{G}^{\alpha\beta} Q^{\mu'\nu'}{}_{\rho'\sigma'} \,\} = -\, \mathcal{G}^{\mu'\nu'} \Delta^{\alpha\beta}{}_{\rho'\sigma'} \tag{24}$$

$$\{\, Q^{\alpha\beta}{}_{\gamma\delta} Q^{\mu'\nu'}{}_{\rho'\sigma'} \,\} = \mathcal{G}^{\alpha\beta} \bar{y}_{\rho'\sigma'} \Delta^{\mu'\nu'}{}_{\gamma\delta} - \mathcal{G}^{\mu'\nu'} \bar{y}_{\gamma\delta} \Delta^{\alpha\beta}{}_{\rho'\sigma'} \tag{25}$$

Since no confusion is possible we shall drop the indices and write

$$\{\, \mathcal{G} \;\; Q' \,\} = -\, \mathcal{G}'\Delta \tag{24'}$$

$$\{\, Q \;\; Q' \,\} = -\, \mathcal{G}\bar{y}'\Delta' - \mathcal{G}'\bar{y}\Delta \tag{25'}$$

By use of (14) and (21) one gets

$$\{\, \partial_j \mathcal{G}, \bar{y} \,\} = -\, \partial_j \Delta \tag{26}$$

Only (23), (24'), (25'), (26) are needed in order to achieve computations.

After very tedious calculations (see Appendix) one finds simply

$$\{\, C_j, C_{k'} \,\} = 0$$

$$\{\, B, B' \,\} = \{\, C, C' \,\} = \{\, C, B' \,\} = 0$$

$$\{\, C_j \;,\; B' \,\} = C_j \delta_{\Sigma(xx')}$$

$$\{\, C_j \quad C' \,\} = C_j \delta_{\Sigma(xx')}$$

This result is satisfactory since the right-handsides merely reproduce a trivial combination of the constraints. It may be stated that, according to our canonical formulation, the brackets of the constraints with the constraints are equivalent to zero, *modulo* the constraints themselves.

The simplicity of the brackets we got is encouraging as regards possible further calculations.

On the other hand, the fine behaviour of the brackets of constraints is a good test of the canonical procedure we have chosen.

This procedure may be expected to fit with General Relativity as well.

APPENDIX

1. — COMPUTATION OF $\{C_j C_{k'}\}$

By use of (23) only eight terms arise, viz.

$$\begin{aligned}\{C_j\, C_{k'}\} = &\{\mathcal{G}^{44}{}_j\ \ Q'^{4\sigma}{}_{k\sigma}\} + \{\mathcal{G}^{44}{}_j\ \ Q'^{\sigma 4}{}_{\sigma k}\}\\ &+\{Q^{4\sigma}{}_{j\sigma}\,\mathcal{G}'^{44}{}_k\} + \{Q^{4\sigma}{}_{j\sigma}\,Q'^{4\tau}{}_{k\tau}\}\\ &+\{Q^{4\sigma}{}_{j\sigma}\,Q'^{\tau 4}{}_{\tau k}\} + \{Q^{\sigma 4}{}_{\sigma j}\,\mathcal{G}'^{44}{}_k\}\\ &+\{Q^{\sigma 4}{}_{\sigma j}\,Q'^{4\tau}{}_{k\tau}\} + \{Q^{\sigma 4}{}_{\sigma j}\,Q'^{\tau 4}{}_{\tau k}\}\end{aligned}$$

But four terms vanish because

$$\Delta^{44}{}_{k'\sigma'} = \Delta^{44}{}_{\sigma' k'} = \Delta^{4'4'}{}_{j\sigma} = \Delta^{4'4'}{}_{\sigma j} = 0$$

And two other ones vanish because

$$\Delta^{4\sigma}{}_{k'\tau'} = \Delta^{4'\tau'}{}_{j\sigma} = \Delta^{\sigma 4}{}_{\tau' k'} = \Delta^{\tau' 4'}{}_{\sigma j} = 0$$

One is left with

$$\{C_j\, C_{k'}\} = \{Q^{4\sigma}{}_{j\sigma}\,Q'^{\tau 4}{}_{\tau k}\} + \{Q^{\sigma 4}{}_{\sigma j}\,Q'^{4\tau}{}_{k\tau}\}$$

Eq. (25′) yields

$$\{Q^{4\sigma}{}_{j\sigma}\,Q'^{\tau 4}{}_{\tau k}\} = (Q^{4\sigma}{}_{\tau k}\delta^{\tau}{}_j\delta^4{}_\sigma - Q^{\tau 4}{}_{j\sigma}\delta^4{}_\tau\delta^\sigma{}_k)\delta_\Sigma$$

This term is zero, as proved by summation over σ and τ.
Eq. (25′) also yields

$$\{Q^{\sigma 4}{}_{\sigma j}\,Q'^{4\tau}{}_{k\tau}\} = (Q^{\sigma 4}{}_{k\tau}\delta^4{}_\sigma\delta^\tau{}_j - Q^{4\tau}{}_{\sigma j}\delta^\sigma{}_k\delta^4{}_\tau)\delta_\Sigma$$

and summation over σ and τ shows that this term too cancels out. Hence finally

(27) $$\{C_j,\ C_{k'}\} = 0$$

2. — COMPUTATION OF $\{C_j C'\}$

By use of (23), only eight terms arise, but, since $\Delta^{44}{}_{k'\sigma'} = \Delta^{k'4'}{}_{\sigma j} = 0$ we have

$$\{\mathcal{G}^{44}{}_j\,Q'^{k\sigma}{}_{k\sigma}\} = 0 \qquad \{Q^{\sigma 4}{}_{\sigma j}\,\mathcal{G}'^{k4}{}_k\} = 0$$

Therefore one is actually left with six terms. In compact notations

(28) $$\{C_j\, C'\} = A_{j(xx')} + B_{j(xx')}$$

where

$$A_j = -\{\mathcal{G}'^{k4}{}_k\,Q^{4\sigma}{}_{j\sigma}\} - \{\mathcal{G}^{44}{}_j\,Q'^{\sigma 4}{}_{\sigma 4}\}$$

$$B_j = \{Q^{4\sigma}{}_{j\sigma}\,Q'^{k\tau}{}_{k\tau}\} - \{Q^{4\sigma}{}_{j\sigma}\,Q'^{\tau 4}{}_{\tau 4}\} + \{Q^{\sigma 4}{}_{\sigma j}\,Q'^{k\tau}{}_{k\tau}\} - \{Q^{\sigma 4}{}_{\sigma j}\,Q'^{\tau 4}{}_{\tau 4}\}$$

Of course, A_j can be written

$$A_j = -\partial_{k'}\left\{\mathfrak{g}'^{k4}Q^{4\sigma}{}_{j\sigma}\right\} - \partial_j\left\{\mathfrak{g}^{44}Q'^{\sigma4}{}_{\sigma4}\right\}$$

Then applying (24′) gives

$$A_j = \partial_{k'}(\mathfrak{g}^{4\sigma}\Delta^{k'4'}{}_{j\sigma}) + \partial_j(\mathfrak{g}^{\sigma'4'}\Delta^{44}{}_{\sigma'4'})$$

where

$$\mathfrak{g}^{4\sigma}\Delta^{k'4'}{}_{j\sigma} = \mathfrak{g}^{44}\tau^{k'}{}_j\delta_\Sigma \qquad \mathfrak{g}^{\sigma'4'}\Delta^{44}{}_{\sigma'4'} = \mathfrak{g}'^{44}\delta_\Sigma$$

Thus

$$A_j = \mathfrak{g}^{44}\partial_{k'}(\tau^{k'}{}_j\delta_\Sigma) + \mathfrak{g}^{4'4'}\partial_j\delta_\Sigma$$

Then identity (19) permits writing

(29) $$A_j = (\partial_j\,\mathfrak{g}^{44})\delta_\Sigma$$

Let us now calculate B_j.
By applying formula (25′) we get

$$\begin{aligned} B_j = {} & \mathfrak{g}^{4\sigma}\bar{y}'_{k\tau}\,\Delta^{k'\tau'}{}_{j\sigma} - \mathfrak{g}'^{k\tau}\bar{y}_{j\sigma}\,\Delta^{4\sigma}{}_{k'\tau'} \\ & - (\mathfrak{g}^{4\sigma}\bar{y}'_{\tau4}\,\Delta^{\tau'4'}{}_{j\sigma} - \mathfrak{g}'^{\tau4}\bar{y}_{j\sigma}\,\Delta^{4\sigma}{}_{\tau'4'}) \\ & + (\mathfrak{g}^{\sigma4}\bar{y}'_{k\tau}\,\Delta^{k'\tau'}{}_{\sigma j} - \mathfrak{g}'^{k\tau}\bar{y}_{\sigma j}\,\Delta^{\sigma4}{}_{k'\tau'}) \\ & - (\mathfrak{g}^{\sigma4}\bar{y}'_{\tau4}\,\Delta^{\tau'4'}{}_{\sigma j} - \mathfrak{g}'^{\tau4}\bar{y}_{\sigma j}\,\Delta^{\sigma4}{}_{\tau'4'}) \end{aligned}$$

First of all, two terms in the above expression of B_j vanish because

$$\Delta^{4\sigma}{}_{k'\tau'} = 0 \qquad \Delta^{\tau'4'}{}_{\sigma j} = 0$$

Then, remembering the properties of δ-function, we may use the identity

$$\mathfrak{g}^{\alpha\beta}\bar{y}_{\mu'\nu'}\delta_\Sigma = Q^{\alpha\beta}{}_{\rho\sigma}\Delta^{\rho\sigma}{}_{\mu'\nu'}\delta_\Sigma$$

which, in special co-ordinates, reduces to the trivial relation

$$\mathfrak{g}^{\alpha\beta}\bar{y}_{\mu'\nu'}\delta^{(3)}_{(x,x')} = Q^{\alpha\beta}{}_{\mu\nu}\delta^{(3)}_{(x,x')}$$

Finally, B_j turns out to have the following expression:

$$B_j = (Q^{4\sigma}{}_{j\sigma} - Q^{44}{}_{j4} + Q^{44}{}_{j4} + Q^{k4}{}_{kj} - Q^{k4}{}_{kj} + Q^{\tau4}{}_{\tau j})\delta_\Sigma$$

Thus

$$B_j = (Q^{4\sigma}{}_{j\sigma} + Q^{\sigma4}{}_{\sigma j})\delta_\Sigma$$

and coming back to (28) and (29) we find

$$\{C_j\ C'\} = (\partial_j\mathfrak{g}^{44} + Q^{4\sigma}{}_{j\sigma} + Q^{\sigma4}{}_{\sigma j})\delta_\Sigma$$

According to (8) we recognize

(30) $$\{C_j\ C'\} = C_j\delta_{\Sigma(xx')}$$

3. — COMPUTATION OF $\{ C_j, B' \}$

As previously, (23) makes one term immediately to vanish.

Separating « mixed » terms which involve $\mathcal{G}$ and Q from « pure » terms involving Q only, one gets

$$\{ C_j,\ B' \} = M_{j(xx')} + N_{j(xx')} \tag{31}$$

with

$$M_j = - \left\{ \mathcal{G}^{44}{}_j \quad Q'^{4\sigma}{}_{4\sigma} \right\} + \left\{ \mathcal{G}^{44}{}_j \quad Q'^{\sigma k}{}_{\sigma k} \right\} + \left\{ Q^{4\sigma}{}_{j\sigma} \ \mathcal{G}'^{4k}{}_k \right\} + \left\{ Q^{\sigma 4}{}_{\sigma j} \ \mathcal{G}'^{4k}{}_k \right\}$$

That is to say

$$M_j = \partial_j \left(\mathcal{G}'^{4\sigma} \Delta^{44}{}_{4'\sigma'} \right) - \partial_j \left(\mathcal{G}'^{\sigma k} \Delta^{44}{}_{\sigma' k'} \right) + \partial_{k'} \left(\mathcal{G}^{4\sigma} \ \Delta^{4'k'}{}_{j\sigma} \right) + \partial_{k'} \left(\mathcal{G}^{\sigma 4} \Delta^{4'k'}{}_{\sigma j} \right)$$

where

$$\Delta^{44}{}_{\sigma' k'} = \Delta^{4'k'}{}_{j\sigma} = 0$$

Thus, after summing over σ

$$M_j = \partial_j \left(\mathcal{G}'^{44} \delta_\Sigma \ \right) + \partial_{k'} \left(\mathcal{G}^{44} \tau^{k'}{}_j \delta_\Sigma \right)$$

Therefore, according to (19)

$$M_j = (\partial_j \mathcal{G}^{44}) \delta_\Sigma \tag{32}$$

On the other hand N_j must be expressed.

By definition:

$$N_j = - \left\{ Q^{4\sigma}{}_{j\sigma} \ Q'^{4\tau}{}_{4\tau} \right\} + \left\{ Q^{4\sigma}{}_{j\sigma} \ Q'^{\tau k}{}_{\tau k} \right\} - \left\{ Q^{\sigma 4}{}_{\sigma j} \ Q'^{4\tau}{}_{4\tau} \right\} + \left\{ Q^{\sigma 4}{}_{\sigma j} \ Q'^{\tau k}{}_{\tau k} \right\}$$

These brackets are to be calculated with the help of formula (25)—or equivaently formula (25').

But it is much more simple to use special co-ordinates, where, remembering trivial properties of distribution $\delta^{(3)}_{(xx')}$, one can give eq. (25) the following form:

$$\left\{ Q^{\alpha\beta}{}_{\gamma\delta} \ Q^{\mu'\nu'}{}_{\rho'\sigma'} \right\} = \left(Q^{\alpha\beta}{}_{\rho\sigma} \delta^{\mu}{}_{\gamma} \delta^{\nu}{}_{\delta} - Q^{\mu\nu}{}_{\gamma\delta} \delta^{\alpha}{}_{\rho} \delta^{\beta}{}_{\sigma} \right) \delta^{(3)}_{(xx')} \tag{25''}$$

Applying (25'') to the brackets involved in N_j, and performing a lot of summations, one finally gets

$$N_j = \left(Q^{4\sigma}{}_{j\sigma} + Q^{\sigma 4}{}_{\sigma j} \right) \delta_\Sigma \tag{33}$$

Then (31), (32) and (33) permit to exhibit expression (8) of C_j in the result:

$$\{ C_j,\ B' \} = C_j \delta_{\Sigma (xx')} \tag{34}$$

4. — COMPUTATION OF $\{C B'\}$

Direct calculation yields

$$\begin{aligned}\{C\ B'\} = &-\{\mathcal{G}^{j4}{}_j\ \ Q'^{4\sigma}{}_{4\sigma}\} + \{\mathcal{G}^{j4}{}_j\ \ Q'^{\tau k}{}_{\tau k}\}\\ &+\{Q^{j\sigma}{}_{j\sigma}\ \ \mathcal{G}'^{4k}{}_k\} - \{Q^{j\sigma}{}_{j\sigma}\ Q'^{4\tau}{}_{4\tau}\}\\ &+\{Q^{j\sigma}{}_{j\sigma}\ Q'^{\tau k}{}_{\tau k}\} - \{Q^{\sigma 4}{}_{\sigma 4}\ \mathcal{G}'^{4j}{}_j\}\\ &+\{Q^{\sigma 4}{}_{\sigma 4}\ Q'^{4\tau}{}_{4\tau}\} - \{Q^{\sigma 4}{}_{\sigma 4}\ Q'^{\tau k}{}_{\tau k}\}\end{aligned}$$

All the terms involving $\mathcal{G}$ turn out to vanish because $\delta^4_j = 0$. For the same reason $\{Q^{j\sigma}{}_{j\sigma}\,Q'^{4\tau}{}_{4\tau}\}$ and $\{Q^{\sigma 4}{}_{\sigma 4}\,Q'^{\tau k}{}_{\tau k}\}$ also vanish, and one is left with

$$\begin{aligned}\{C\ B'\} = &\ \mathcal{G}^{j\sigma}\bar{y}'_{\tau k}\ \Delta^{\tau' k'}{}_{j\sigma} - \mathcal{G}'^{\tau k}\bar{y}_{j\sigma}\,\Delta^{j\sigma}{}_{\tau' k'}\\ &+\mathcal{G}^{\sigma 4}\bar{y}_{4'\tau'}\,\Delta^{4'\tau'}{}_{\sigma 4} - \mathcal{G}'^{4\tau}\bar{y}_{\sigma 4}\,\Delta^{\sigma 4}{}_{4'\tau'}\end{aligned}$$

Using special co-ordinates, one simply finds

$$\begin{aligned}\{C\ B'\} = &\ (Q^{jk}{}_{jk} - Q^{jk}{}_{jk})\delta^{(3)}_{(xx')}\\ &+(Q^{44}{}_{44} - Q^{44}{}_{44})\delta^{(3)}_{(xx')}\end{aligned}$$

That is of course

(35) $$\{C\ B'\} = 0$$

5. — COMPUTATION OF $\{B, B'\}$

As above we can separate two kinds of terms and write

(36) $$\{B, B'\} = a + b$$

where a denotes « mixed » terms, while b is the sum of terms involving Q only.
Explicitly we have

$$\begin{aligned}a_{(x,x')} = &-\{\mathcal{G}^{4j}{}_j\ \ Q'^{4\sigma}{}_{4\sigma}\} + \{\mathcal{G}^{4j}{}_j\ \ Q'^{\sigma k}{}_{\sigma k}\}\\ &-\{Q^{4\sigma}{}_{4\sigma}\ \ \mathcal{G}'^{4k}{}_k\} + \{Q^{\sigma j}{}_{\sigma j}\ \mathcal{G}'^{4k}{}_k\}\end{aligned}$$

$$\begin{aligned}a = &\ \mathcal{G}'^{4\sigma}\partial_j\ \Delta^{4j}{}_{4'\sigma'} - \mathcal{G}'^{\sigma k}\partial_j\ \Delta^{4j}{}_{\sigma' k'}\\ &-\mathcal{G}^{4\sigma}\partial_{k'}\ \Delta^{4'k'}{}_{4\sigma} + \mathcal{G}^{\sigma j}\partial_{k'}\ \Delta^{4'k'}{}_{\sigma j}\end{aligned}$$

$$\begin{aligned}a = &\ \partial_j(\mathcal{G}'^{4\sigma}\ \Delta^{4j}{}_{4'\sigma'}) - \partial_j(\mathcal{G}'^{\sigma k}\ \Delta^{4j}{}_{\sigma' k'})\\ &-\partial_{k'}(\mathcal{G}^{4\sigma}\ \Delta^{4'k'}{}_{4\sigma} - \mathcal{G}^{\sigma j}\ \Delta^{4'k'}{}_{\sigma j})\end{aligned}$$

Summation over σ and k provides

$$a = 0$$

Also b must be computed. Direct calculation leads to

$$\begin{aligned}b_{(x,x')} = &\ \{Q^{4\sigma}{}_{4\sigma}\ Q'^{4\tau}{}_{4\tau}\} - \{Q^{4\sigma}{}_{4\sigma}\ Q'^{\tau k}{}_{\tau k}\}\\ &-\{Q^{\sigma j}{}_{\sigma j}\ Q'^{4\tau}{}_{4\tau}\} + \{Q^{\sigma j}{}_{\sigma j}\ Q'^{\tau k}{}_{\tau k}\}\end{aligned}$$

Expressing the brackets with the help of (25″) yields

$$b = \left(Q^{4\sigma}{}_{4\sigma} - Q^{4\sigma}{}_{4\sigma} - Q^{4k}{}_{4k} + Q^{4k}{}_{4k} - Q^{4j}{}_{4j} + Q^{4j}{}_{4j} + Q^{\sigma j}{}_{\sigma j} - Q^{\sigma j}{}_{\sigma j}\right)\delta_{\Sigma}$$

i.e. obviously

$$b = 0$$

Since a too has been found equal to zero, coming back to (36) we have

$$\{ B, B' \} = 0 \tag{37}$$

6. — COMPUTATION OF $\{ C, C' \}$

Without expanding the « mixed » brackets (involving both $\mathcal{G}$ and Q) one sees from general symmetry properties of Poisson bracket, that these terms cancel each other pairwise. Thus one is left with

$$\begin{aligned}\{ C, C' \} = {} & \{ Q^{j\sigma}{}_{j\sigma}\ Q'^{k\tau}{}_{k\tau} \} - \{ Q^{j\sigma}{}_{j\sigma}\ Q'^{\tau 4}{}_{\tau 4} \} \\ & - \{ Q^{\sigma 4}{}_{\sigma 4}\ Q'^{k\tau}{}_{k\tau} \} + \{ Q^{\sigma 4}{}_{\sigma 4}\ Q'^{\tau 4}{}_{\tau 4} \}\end{aligned}$$

Application of (25″) gives

$$\begin{aligned}\{ C, C' \} = {} & \left(- Q^{j\sigma}{}_{\tau 4}\delta^{\tau}_{j}\delta^{4}_{\sigma} + Q^{\tau 4}{}_{j\sigma}\delta^{j}_{\tau}\delta^{\sigma}{}_{4}\right)\delta^{(3)} \\ & + \left(- Q^{\sigma 4}{}_{k\tau}\delta^{k}_{\sigma}\delta^{\tau}{}_{4} + Q^{k\tau}{}_{\sigma 4}\delta^{\sigma}_{k}\delta^{4}{}_{\tau}\right)\delta^{(3)} \\ & + \left(Q^{\sigma 4}{}_{\tau 4}\delta^{\tau}_{\sigma}\delta^{4}{}_{4} - Q^{\tau 4}{}_{\sigma 4}\delta^{\sigma}_{\tau}\delta^{4}{}_{4}\right)\delta^{(3)}\end{aligned}$$

Performing all the summations one finds $\{ C, C' \} = 0$.

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[2] Stanley Deser and R. Arnowitt, Quantum theory of gravitation: General formulation and linearized theory. *Phys. Rev.*, t. 113, 1959, p. 745.

[3] R. Arnowitt, S. Deser and C. W. Misner, in *Recent advances in Relativity*, J. Wiley, editor, New York, 1963.

[4] Albert Einstein, A generalization of the relativistic theory of gravitation. *Ann. Math. Princeton*, t. 46, 1945, p. 578 and t. 47, 1945, p. 146 and 731. In equation (2), D_{ρ} means covariant derivation defined by the affinity

$$L_{\mu}{}^{\alpha}{}_{\nu} = \Gamma_{\mu}{}^{\alpha}{}_{\nu} + \frac{2}{3}\,\delta_{\mu}{}^{\alpha}\Gamma_{\nu}$$

where

$$\Gamma_{\nu} = \Gamma_{[\nu}{}^{\sigma}{}_{\sigma]}$$

About + and — derivations we just follow Einstein's notations.

[5] See Reference [1], p. 383.

[6] See Reference [1], p. 394.

[7] J. L. ANDERSON and P. G. BERGMANN, Constraints in covariant field theories. *Phys. Rev.*, t. 83, 1951, p. 1018.
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