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N. D. SEN GUPTA

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Section A :
Physique théorique.

On a perturbation expansion of the characteristic function of classical mechanics

by

N. D. SEN GUPTA
Tata Institute of Fundamental Research,
Bombay-5.

RÉSUMÉ. — L'objet de cet article est de développer une série de perturbation pour la fonction d'Hamilton-Jacobi d'un système mécanique dont le potentiel peut se développer en puissances d'un petit paramètre ε. La solution du problème non perturbé correspondant à $\varepsilon = 0$, est connue exactement. La série de perturbation est tout d'abord obtenue pour le mouvement d'une particule dans un champ de potentiel. Ensuite, on détermine la solution pour une particule chargée dans un champ électromagnétique. Finalement la méthode est généralisée pour un système mécanique général qui peut être décrit par un hamiltonien possédant un nombre arbitraire de variables canoniques.

ABSTRACT. — The object of the paper is to develop a perturbation expansion of the Hamilton-Jacobi characteristic function of a mechanical system, for which the potential function possesses an expansion in terms of a small parameter ε. The solution of the unperturbed problem corresponding to $\varepsilon = 0$ is assumed to be known exactly. The perturbation expansion is first obtained for the motion of a particle in a potential field. Next it is found for the motion of a charged particle in an external electromagnetic field. Finally the method is generalized for a general dynamical system which can be formulated in terms of a Hamiltonian with an arbitrary number of canonical variables.

1. — INTRODUCTION

The basic principle of the perturbation theory in classical mechanics, where the Hamiltonian can be expressed in a power series of a small parameter, consists of determining suitable canonical transformations, which reduce the problem to a stationary one, with successive degree of accuracy in the small parameter appearing in the Hamiltonian [*1*]. This, in practice, is to be attained by solving partial differential equations at each step. Alternatively, one has to express all the quantities in terms of time and each of the successive canonical variable may be obtained by quadrature [*2*]. The Lagrange-Poisson's method of variation of parameter [*3*] is also, in practice, equivalent to it. Though the aim of a dynamical problem is to express all the momenta and coordinates in terms of time, yet in many physical problems one is interested to know integral surfaces and orbits expressed in terms of the coordinates rather than expressed in parametric forms with time. In this respect it is very convenient if one can find a perturbation expansion of the Hamilton-Jacobi characteristic function. This is due to the unique role of the characteristic function (we use simply the characteristic function as abbreviation for the Hamilton-Jacobi characteristic function) in a dynamical problem, as the momenta, the integral surfaces and the evolution in time are obtained by partial differentiation of the characteristic function. Thus once the perturbation expansion for the characteristic function is developed, any dynamical variable may be determined to the desired degree of accuracy in the small parameter.

There is another important aspect of this investigation. In most of the problems of classical mechanics, e.g. astronomical problems, one is usually interested in the effect of small perturbations on an unperturbed simply or multiply periodic motion. As such, in many cases it is sufficient to know the time-average effects of the perturbation. Recently in many physical problems in laboratory, in the earth's neighbourhood as well as in extraterrestrial systems, one is not satisfied only with the knowledge of time-average effects of perturbation. But it is imperative to know its effect in the process of evolution (in time) of the motion. This necessitates the development of a perturbation theory which has general appeal.

In short the method consists of solving the Hamilton-Jacobi partial differential equation with successive degreee of approximation in powers of the small parameter in the Hamiltonian. It should be mentioned that such an expansion has already been developed previously by various authors [4]. But this is in terms of action and angle variables in case where the Hamiltonian (generally the unperturbed Hamiltonian) is separable in the variables. We have tried to develop the perturbation expansion of the characteristic function for a dynamical system without these limitations. We need not restrict ourselves to problems for which the unperturbed motion is simply or multiply periodic. An important point to be mentioned is that the unperturbed motion being known completely, i. e. the unperturbed characteristic function being known exactly, the problem of the determination of higher order terms is reduced to successive quadratures.

The method consists of developing the characteristic function corresponding to the motion of the system which satisfies given initial values of momenta and coordinates. It still represents a class of solution as the initial quantities may have arbitrary values. This programme is carried out for the motion of a particle in a potential field, in section 2. The convergence of the perturbation series for the characteristic function is discussed in the Appendix. In section 3, the method is extended to the case of a charged particle in an external electromagnetic field. Finally in section 4, it is generalized to the case of a dynamical system with an arbitray number of canonical variables.

2. — THE MOTION OF A PARTICLE IN A POTENTIAL FIELD

Let m be the mass of a particle moving in a potential field $U(\vec{r})$, which does not depend explicitly on time. It is assumed that $U(\vec{r})$ may be expanded in a power series of a small parameter ε, in a given region of space; thus

$$U(\vec{r}) = U_0(\vec{r}) + \varepsilon U_1(\vec{r}) + \varepsilon^2 U_2(\vec{r}) + \dots \qquad (1)$$

The characteristic function $S(\vec{r}, t)$ satisfies the Hamilton-Jacobi partial differential equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\vec{p}, \vec{p}) + U(\vec{r}) = 0\,; \qquad (2)$$

where $\vec{p}$, the momentum of the particle, is given by

$$m\frac{d\vec{r}}{dt} = \vec{p} = \vec{\nabla} S. \qquad (3)$$

Since $U(\vec{r})$ is independent of time, it follows from Eqs. (2) and (3) that $\mathcal{S}(\vec{r}, t)$ is of the form

$$\mathcal{S}(\vec{r}, t) = -Et + S(\vec{r})\,; \tag{4}$$

where E is the constant energy of the particle. The differential equation for time-independent part $S(\vec{r})$ of the characteristic function $\mathcal{S}(\vec{r}, t)$ is found to be

$$\frac{1}{2m}(\vec{\nabla}S.\vec{\nabla}S) = E - U(\vec{r}). \tag{5}$$

Initially, i. e. at $t = 0$ let the position of the particle be $\vec{r}_0$ and let its velocity be $\vec{V}_0$ there. E is an integral of the motion, it can be written as

$$E = E_0 + \varepsilon U_1^0 + \varepsilon^2 U_2^0 + \ldots, \tag{6}$$

where $E_0 = k_0 + U_0(\vec{r}_0)$, $k_0 = \frac{m}{2}(\vec{V}_0.\vec{V}_0)$ the initial kinetic energy and $U_n^0 = U_n(\vec{r}_0)$.

We assume that $S(\vec{r})$ can be expanded in a power series of ε, i. e.

$$S(\vec{r}) = S_0(\vec{r}) + \varepsilon S_1(\vec{r}) + \varepsilon^2 S_2(\vec{r}) + \ldots \tag{7}$$

On substituting the expressions for S, U and E from Eqs. (7), (1) and (6) respectively in Eq. (5) and equating the coefficients of ε^n ($n = 0, 1, 2, \ldots$), we get

$$\frac{1}{2m}(\vec{\nabla}S_0.\vec{\nabla}S_0) = E_0 - U_0(\vec{r}), \tag{8}$$

$$(\vec{\nabla}S_0.\vec{\nabla}S_n) = -\frac{1}{2}\sum_{l=1}^{n-1}(\vec{\nabla}S_l.\vec{\nabla}S_{n-l}) + m(U_n^0 - U_n). \tag{9}$$

(for $n \geqslant 1$)

It is clear from Eq. (7) that $S_0(\vec{r})$ is the time-independent part of the unperturbed characteristic function, which should satisfy Eq. (8) with the same initial condition. Since we assume that the solution of the unperturbed equation is known exactly, the complete integral of Eq. (8), i. e. $S_0(\vec{r};\, k_0, k_1, k_2)$ is already determined. k_1 and k_2 are two parameters to be determined from the initial condition,

$$[\vec{\nabla}S_0(\vec{r})]_{\vec{r}=\vec{r}_0} = m\vec{V}_0. \tag{10}$$

It follows from

$$\left.\begin{aligned}\left(\vec{\nabla} S_0 . \vec{\nabla} \frac{\partial S_0(\vec{r})}{\partial k_1}\right) = 0\\ \left(\vec{\nabla} S_0 . \vec{\nabla} \frac{\partial S_0(\vec{r})}{\partial k_2}\right) = 0\end{aligned}\right\} \tag{12}$$

that

$$\varphi \equiv \frac{\partial S_0(\vec{r})}{\partial k_1} = \text{constant}$$

and

$$\psi \equiv \frac{\partial S_0(\vec{r})}{\partial k_2} = \text{constant} \tag{13}$$

are the two integral surfaces of the unperturbed problem. We can take $\theta \equiv S_0(\vec{r})$, φ and ψ as curvilinear coordinates. This is admissible as the Jacobian of the transformation,

$$\frac{D(\theta, \varphi, \psi)}{D(x, y, z)} = \left(\vec{\nabla} S_0 . \vec{\nabla} \frac{\partial S_0}{\partial k_1} \times \vec{\nabla} \frac{\partial S_0}{\partial k_2}\right)$$

is not zero (excluding some trivial cases). They are not in general orthogonal since $(\vec{\nabla}\varphi . \vec{\nabla}\psi) \neq 0$, though both of the latter surfaces are orthogonal to θ = constant surface.

Since $\theta \equiv S_0(\vec{r})$ is known we can determine S_1, S_2, successively by integrating Eq. (9). For $n = 1$, Eq. (9) leads to

$$(\vec{\nabla} S_0 . \vec{\nabla} S_1) = m(U_1^0 - U_1). \tag{9'}$$

This determines S_1 except for an additive part which is a function φ and ψ. But this ambiguity is apparent, as what we need is a complete integral. Hence we can write

$$S_1 = m \int^{\theta} \frac{U_1^0 - U_1}{E_0 - U_1} d\theta + \alpha_1 \varphi + \beta_1 \psi, \tag{14}$$

where α_1 and β_1 are constants. It is clear that the above integral is a complete one, as α_1 and β_1 are essential parameters. The constants α_1 and β_1 are to be determined from the condition

$$[\vec{\nabla} S_1(\vec{r})]_{\vec{r} = \vec{r}_0} = 0 \tag{15}$$

as the initial condition is to be satisfied for all orders of the perturbation expansion. Thus α_1 and β_1 depend on k_1 and k_2. Finally, since $S_0(\vec{r}; k_0, k_1, k_2)$ is a complete integral of Eq. (8), $S_0 + \varepsilon S_1$ is also a complete integral which satisfies Eq. (5) upto first order terms in ε.

From Eq. (14) one observes that $\left(\frac{\partial S_1}{\partial \theta}\right)_{\vec{r}=\vec{r}_0} = 0$; hence Eq. (15) leads to two equations to determine α_1 and β_1; which are thus given by

$$m\left[\int^{\theta} \frac{\partial}{\partial \varphi} \frac{U_1^0 - U_1}{E_0 - U_0}\, d\theta\right]_{\vec{r}=\vec{r}_0} + \alpha_1 = 0 \tag{16}$$

$$m\left[\int^{\theta} \frac{\partial}{\partial \psi} \frac{U_1^0 - U_1}{E_0 - U_0}\, d\theta\right]_{\vec{r}=\vec{r}_0} + \beta_1 = 0. \tag{17}$$

This determines S_1 uniquely. For $n = 2$, Eq. (9) leads in the similar manner

$$S_2 = \int^{\theta} \left\{ m \frac{U_2^0 - U_2}{E_0 - U_0} - \frac{1}{2} \frac{(\vec{\nabla} S_1 . \vec{\nabla} S_1)}{E_0 - U_0} \right\} d\theta + \alpha_2 \varphi + \beta_2 \psi. \tag{18}$$

The integration can be carried out after substituting the expression for S_1 from Eq. (14) in the integrand. The constants α_2 and β_2 are given by

$$\left[\int^{\theta} \frac{\partial}{\partial \varphi} \left\{ m \frac{U_2^0 - U_2}{E_0 - U_0} - \frac{1}{2} \frac{(\vec{\nabla} S_1 . \vec{\nabla} S_1)}{E_0 - U_0} \right\} d\theta\right]_{\vec{r}=\vec{r}_0} + \beta_2 = 0 \tag{19}$$

$$\left[\int^{\theta} \frac{\partial}{\partial \psi} \left\{ m \frac{U_2^0 - U_2}{E_0 - U_0} - \frac{1}{2} \frac{(\vec{\nabla} S_1 . \vec{\nabla} S_1)}{E_0 - U_0} \right\} d\theta\right]_{\vec{r}=\vec{r}_0} + \beta_2 = 0. \tag{20}$$

Hence S_2 is determined uniquely. In general, from Eq. (9), one can write

$$S_n = \int^{\theta} \left\{ m \frac{U_n^0 - U_n}{E_0 - U_0} - \frac{1}{2} \sum_{l=1}^{n-1} \frac{(\vec{\nabla} S_l . \vec{\nabla} S_{n-l})}{E_0 - U_0} \right\} d\theta + \alpha_n \varphi + \beta_n \psi. \tag{21}$$

Thus if $S_1, S_2, \ldots, S_{n-1}$ are known the integration on the right hand side of Eq. (21) can be performed. One observes that the integrand on the right hand side is zero at $\vec{r} = \vec{r}_0$, hence $\left(\frac{\partial S_l}{\partial \theta}\right)_{\vec{r}=\vec{r}_0} = 0$ for $l = 1, 2, \ldots, n - 1$. So the constants α_n and β_n may be determined from the remaining two equations of

$$[\vec{\nabla} S_n]_{\vec{r}=\vec{r}_0} = 0.$$

Hence

$$\left[\int^{\theta} \frac{\partial}{\partial \varphi} \left\{ m \frac{U_n^0 - U_n}{E_0 - U_0} - \frac{1}{2} \sum_{l=1}^{n-1} \frac{(\vec{\nabla} S_l . \vec{\nabla} S_{n-l})}{E_0 - U_0} \right\} d\theta\right]_{\vec{r}=\vec{r}_0} + \alpha_n = 0 \tag{22}$$

$$\left[\int^{\theta} \frac{\partial}{\partial \psi} \left\{ m \frac{U_n^0 - U_n}{E_0 - U_0} - \frac{1}{2} \sum_{l=1}^{n-1} \frac{(\vec{\nabla} S_l . \vec{\nabla} S_{n-l})}{E_0 - U_0} \right\} d\theta\right]_{\vec{r}=\vec{r}_0} + \beta_n = 0 \tag{23}$$

Thus knowing the coefficients of the expansion for S, Eq. (7) upto $(n - 1)$ degree of ε, one can find the coefficients of ε^n by quadrature. Hence the perturbation expansion of $\mathcal{S}(\vec{r}, t)$ can be found successively to any desired degree in ε. The question of convergency of the series, (Eqs. (7) and (21)) for S is discussed in Appendix. Here we only make an observation that $E_0 - U_0$, which is equal to the unperturbed kinetic energy of the particle, appears in the denominator; so that the integrals in S_n $(n = 1, 2, \ldots)$ are undefined if velocity vanishes for the unperturbed motion. This is not unexpected as the perturbation method is justified only for small relative change of the kinematics of the particle. This situation is comparable to that due to small denominators in the theory of perturbation of multiply periodic motion. Finally it is clear that the more the initial kinetic energy, i. e. the more the magnitude of $E_0 - U_0$ in the interval, the more rapid is the convergence of the series as the degree of this factor in the denominators multiply in steps in each order of the perturbation expansion. The characteristic function $\mathcal{S}(\vec{r}, t)$ is thus given by

$$\mathcal{S}(\vec{r}, t) = -E_0 t + S_0 + \varepsilon(S_1 - U_1^0 t) + \ldots + \varepsilon^n(S_n - U_n^0 t) + \ldots \tag{24}$$

from Eqs. (4) and (7). The perturbation expansion of the momentum is obtained form Eq. (3) as

$$\vec{p} = \vec{\nabla} S_0 + \varepsilon \vec{\nabla} S_1 + \varepsilon^2 \vec{\nabla} S_2 + \ldots \tag{25}$$

The two integral surfaces in powers of ε are obtained from

$$\frac{\partial S_0}{\partial k_1} + \varepsilon \frac{\partial S_1}{\partial k_1} + \varepsilon^2 \frac{\partial S_2}{\partial k_1} + \ldots = \text{constant} \tag{26}$$

and

$$\frac{\partial S_0}{\partial k_2} + \varepsilon \frac{\partial S_1}{\partial k_2} + \varepsilon^2 \frac{\partial S_2}{\partial k_2} + \ldots = \text{constant.} \tag{27}$$

The constants on the right hand side of Eqs. (26) and (27) are the values of the left hand side at $\vec{r} = \vec{r}_0$. In order to find the evolution in time one observes that

$$\frac{\partial \mathcal{S}}{\partial k_0} = -t + \frac{\partial S}{\partial k_0} = \text{constant},$$

i. e.

$$\frac{\partial S_0}{\partial k_0} + \varepsilon \frac{\partial S_1}{\partial k_0} + \varepsilon^2 \frac{\partial S_2}{\partial k_0} + \ldots = t + \text{constant.} \tag{28}$$

The constant as before is the value of the left hand side at $\vec{r} = \vec{r}_0$. The coordinates may thus be expressed in terms of t, from Eqs. (26)-(28).

3. — THE MOTION OF A CHARGED PARTICLE IN AN ELECTROMAGNETIC FIELD

We consider the relativistic Hamilton-Jacobi equation for a particle of charge q and rest mass m in an external electromagnetic field. Because of its wider range of applicability and elegance of form we work with relativistic equation. One can pass conveniently to non-relativistic limit whenever needed. Let the electromagnetic field be derivable from the potential $(\vec{A}, \Phi)$. As before we restrict ourselves to time-independent fields so that we can take the potential to be also independent of time. Let the potential be expressed in power series of a small parameter ε, i. e.

$$\left.\begin{aligned}\vec{A}(\vec{r}) &= \vec{A}_0(\vec{r}) + \varepsilon\vec{A}_1(\vec{r}) + \varepsilon^2\vec{A}_2(\vec{r}) + \dots \\ \Phi(\vec{r}) &= \Phi_0(\vec{r}) + \varepsilon\Phi_1(\vec{r}) + \varepsilon^2\Phi_2(\vec{r}) + \dots\end{aligned}\right\} \tag{29}$$

The characteristic function $\mathcal{S}(\vec{r}, t)$ satisfies the Hamilton-Jacobi equation

$$\left(\frac{1}{c}\frac{\partial \mathcal{S}}{\partial t} + q\Phi\right)^2 - (\{\vec{\nabla}\mathcal{S} - q\vec{A}\}.\{\vec{\nabla}\mathcal{S} - q\vec{A}\}) = m^2c^2. \tag{30}$$

The momentum $\vec{p}$ of the particle is given by

$$\vec{p} = \vec{\nabla}\mathcal{S} - q\vec{A}. \tag{31}$$

As before since $\vec{A}$ and Φ are independent of time, $\mathcal{S}(\vec{r}, t)$ can be written in the form

$$\mathcal{S}(\vec{r}, t) = -Et + S(\vec{r}), \tag{32}$$

where E is the constant energy of the particle ([1]). The equation for the time-independent part $S(\vec{r})$ of $\mathcal{S}(\vec{r}, t)$ is given by

$$(\{\vec{\nabla}S - q\vec{A}\}.\{\vec{\nabla}S - q\vec{A}\}) = (E - q\Phi)^2 - m^2c^2. \tag{33}$$

Since E is an integral of the motion and at $t = 0$, the particle is at $\vec{r}_0$ with velocity $\vec{V}_0$, E is given from Eq. (33),

$$E = E_0 + \varepsilon q\Phi_1^0 + \varepsilon^2 q\Phi_2^0 + \dots, \tag{34}$$

([1]) $\mathcal{S}$ is not a gauge invariant quantity but the observable quantities are gauge invariant. This follows from the fact $\Phi \to \Phi - \frac{1}{c}\frac{\partial f}{\partial t}$ and $\vec{A} \to \vec{A} + \vec{\nabla}f$ lead from (Eq. (30)) to $\mathcal{S} \to \mathcal{S} + fq$ so that p and E are invariants from Eqs. (31) and (32). Next since the integral surfaces and the evolution in time are obtained by differentiating $\mathcal{S}$ with respect to the constants of integration the additional term namely f does not contribute anything.

where

$$E_0 = k_0 + q\Phi_0$$

with

$$k_0 = \frac{mc^2}{\sqrt{1 - \frac{(\vec{V}\vec{V})}{c^2}}} \tag{35}$$

and Φ_n^0 stands for $\Phi_n(\vec{r_0})$. We assume a power series expansion of $S(\vec{r})$ as in Eq. (7) of the previous problem. On substituting this expansion of $S(\vec{r})$ and the corresponding expressions for $(\vec{A}, \Phi)$ and E from Eqs. (29) and (34), in Eq. (33) and equating the coefficients of ε^n, we get

$$(\{\vec{\nabla}S_0 - q\vec{A}_0\}.\{\vec{\nabla}S_0 - q\vec{A}_0\}) = (E_0 - q\Phi_0)^2 - m^2c^2. \tag{36}$$

$$\begin{aligned}(\{\vec{\nabla}S_0 - q\vec{A}_0\}.\vec{\nabla}S_n) = (q\vec{A}_n.\{\vec{\nabla}S_0 - q\vec{A}_0\}) + qk_0(\Phi_n^0 - \Phi_n) \\ - \frac{1}{2}\sum_{l=1}^{n-1}(\{\vec{\nabla}S_l - q\vec{A}_l\}.\{\vec{\nabla}S_{n-l} - q\vec{A}_{n-l}\}) \\ + \frac{q^2}{2}\sum_{l=0}^{n}(\Phi_l^0 - \Phi_l)(\Phi_{n-l}^0 - \Phi_{n-l})\end{aligned} \tag{37}$$

(for $n = 1, 2, \ldots$).

$S_0(\vec{r})$ is again the time-independent part of the unperturbed characteristic function which satisfies the Eq. (36). The complete integral of Eq. (36) is known as we have assumed that the unperturbed problem is exactly solved. Let $S_0(\vec{r}; k_0, k_1, k_2)$ be the complete integral with two parameters k_1 and k_2 which may be assigned as before by the initial condition (Eqs. 10)). $\frac{\partial S_0}{\partial k_1}$ and $\frac{\partial S_0}{\partial k_2}$ are constants in this case also. We proceed exactly as before and introduce curvilinear coordinates $\theta \equiv S_0(\vec{r})$, $\varphi \equiv \frac{\partial S_0(\vec{r})}{\partial k_1}$ and $\psi \equiv \frac{\partial S_0(\vec{r})}{\partial k_2}$. It may be noted that $\left(\vec{\nabla}S_0.\vec{\nabla}\frac{\partial S_0}{\partial k_1} \times \vec{\nabla}\frac{\partial S_0}{\partial k_2}\right) \neq 0$ in general. Thus we can determine $S_1, S_2, \ldots,$ successively by integrating Eq. (37); for $n = 1$, the complete integral of Eq. (37) is

$$\begin{aligned}S_1 = \int^{\theta}[q(\vec{A}_1.\{\vec{\nabla}\theta - q\vec{A}_0\}) + qE_0(\Phi_1^0 - \Phi_1)]\frac{d\theta}{(\vec{\nabla}\theta.\{\vec{\nabla}\theta - q\vec{A}_0\})} \\ + \alpha_1\varphi + \beta_1\psi\,;\end{aligned} \tag{38}$$

where α_1 and β_1 are constants. They are to be determined from

$$[\vec{\nabla} S_1 - q\vec{A}_1]_{\vec{r}=\vec{r}_0} = 0. \tag{39}$$

Since $(\{\vec{\nabla}\theta - q\vec{A}_0\}.\{\vec{\nabla}S_1 - q\vec{A}_1\})_{\vec{r}=\vec{r}_0} = 0$, Eq. (39) determines α_1 and β_1 uniquely, in the same manner as Eqs. (16) and (17). In general for any n

$$S_n = \int^{\theta} F_n \frac{d\theta}{(\{\vec{\nabla}\theta - q\vec{A}_0\}.\vec{\nabla}\theta)} + \alpha_n\varphi + \beta_n\psi \tag{40}$$

where F_n stands for the right hand side of Eq. (37). The integration may be performed if S_l's, for all $l < n$, are known. α_n and β_n are given by

$$\left[\int^{\theta} \frac{\partial}{\partial\varphi} \frac{F_n}{(\{\vec{\nabla}\theta - q\vec{A}_0\}.\vec{\nabla}\theta)} d\theta - q\frac{(\vec{\nabla}\varphi.A_n)}{(\vec{\nabla}\varphi.\vec{\nabla}\varphi)}\right]_{\vec{r}=\vec{r}_0} + \alpha_n = 0 \tag{41}$$

and

$$\left[\int^{\theta} \frac{\partial}{\partial\psi} \frac{F_n}{(\{\vec{\nabla}\theta - q\vec{A}_0\}.\vec{\nabla}\theta)} d\theta - q\frac{(\vec{\nabla}\psi.\vec{A}_n)}{(\vec{\nabla}\psi.\vec{\nabla}\psi)}\right]_{\vec{r}=\vec{r}_0} + \beta_n = 0. \tag{42}$$

It is clear that

$$[\vec{\nabla} S_l - q\vec{A}_l]_{\vec{r}=\vec{r}_0} = 0 \tag{43}$$

is satisfied for $l = n$ if it is satisfied for all $l < n$; so that the initial condition is satisfied to any order of the perturbation expansion. Thus the coefficients of the perturbation series may be obtained by quadrature. The characteristic function $S(\vec{r}, t)$ is obtained as in Eq. (24). The perturbation expansion of the momentum is given by

$$\vec{p} = \vec{\nabla} S_0 - q\vec{A}_0 + \varepsilon(\vec{\nabla} S_1 - q\vec{A}_1) + \ldots \tag{44}$$

The expressions for the coordinates of the particle in terms of time may be obtained as before.

4. — A GENERAL DYNAMICAL SYSTEM

In this section we consider very briefly the perturbation expansion of the characteristic function of a general dynamical system with an arbitrary number of independent variables. Let q_i's ($i = 1, \ldots, N$) be the generalized coordinates and p_i's be the corresponding momenta. The canonical equations of motion are

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad ; \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \tag{45}$$

$$(i = 1, 2, \ldots, N)$$

where H is the Hamiltonian, which is a function of q_i's, p_i's and a set of functions A^μ's ($\mu = 1, \ldots, M$) which determine the external field. In the special cases considered above A^μ's are the potentials. As before we assume A^μ's as well as H do not explicitly contain time. Further let A^μ be expressed in a power series of a small parameter ε, i. e.

$$A^\mu(q_i) = A_0^\mu(q_i) + \varepsilon A_1^\mu(q_i) + \varepsilon^2 A_2^\mu(q_i) + \ldots \tag{46}$$

The characteristic function $\mathcal{S}(q_i, t)$ satisfies the Hamilton-Jacobi equation

$$\frac{\partial \mathcal{S}}{dt} + H(q_i, p_i, A^\mu) = 0. \tag{47}$$

We assume that H can be expanded in powers of p_i's and A^μ's. The generalized momenta are given by

$$p_i = \frac{\partial \mathcal{S}}{\partial q_i}. \tag{48}$$

As before, since H does not contain time explicitly, $\mathcal{S}(q_i, t)$ can be written in the form

$$\mathcal{S}(q_i, t) = -Et + S(q_i), \tag{49}$$

where E is a constant and corresponds to the energy. We assume the perturbation expansion of $S(q_i)$,

$$S(q_i) = S_0(q_i) + \varepsilon S_1(q_i) + \varepsilon^2 S_2(q_i) + \cdots, \tag{50}$$

so that from Eq. (49)

$$p_i = \frac{\partial S_0}{\partial q_i} + \varepsilon \frac{\partial S_1}{\partial q_i} + \varepsilon^2 \frac{\partial S_2}{\partial q_i} + \cdots \tag{51}$$

Let the initial condition be, at $t = 0$, $q_i = q_i^0$ and $p_i = p_i^0$ ($i = 1, 2, \ldots, N$); so that we can express E from Eqs. (47) and (49)

$$E = H^0_{\varepsilon=0} + \varepsilon \sum_{\mu=1}^{M} \left(\frac{\partial H^0}{\partial A^\mu}\right)_{\varepsilon=0} A_1^{\mu 0} + \frac{\varepsilon^2}{2}\left\{ \sum_{\mu,\nu=1}^{M} \left(\frac{\partial^2 H^0}{\partial A^\mu \partial A^\nu}\right)_{\varepsilon=0} A_1^{\mu 0} A_1^{\nu 0} + \sum_{\mu=1}^{M} 2\left(\frac{\partial H^0}{\partial A^\mu}\right)_{\varepsilon=0} A_2^{\mu 0} \right\} + \ldots \tag{52}$$

where $A^{\mu 0}$, H^0 stand for $A^{\mu}(q_i^0)$ and $H(q_i^0, p_i^0, A^{\mu 0})$. The summations for μ, ν are from 1 to M and those for i, j's are from 1 to N. With the help of Eqs. (46) and (51), one can expand H in the form

$$H(p_i, q_i, A^{\mu}) = H\left(\frac{\partial S_0}{\partial q_0}, q_i, A_0^{\mu}\right) + \varepsilon\left\{\sum_i \left(\frac{\partial H}{\partial p_i}\right)_{\varepsilon=0} \frac{\partial S_1}{\partial q_i} + \sum_{\mu}\left(\frac{\partial H}{\partial A^{\mu}}\right)_{\varepsilon=0} A_1^{\mu}\right\}$$
$$+\frac{\varepsilon^2}{2}\left\{2\sum_i\left(\frac{\partial H}{\partial p_i}\right)_{\varepsilon=0}\frac{\partial S_2}{\partial q_i} + 2\sum_{\mu}\left(\frac{\partial H}{\partial A^{\mu}}\right)_{\varepsilon=0} A_2^{\mu} + \sum_{i,j}\left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right)_{\varepsilon=0}\frac{\partial S_1}{\partial q_i}\frac{\partial S_1}{\partial q_j}\right.$$
$$\left. + \sum_{\mu,\nu}\left(\frac{\partial^2 H}{\partial A^{\mu}\partial A^{\nu}}\right)_{\varepsilon=0} A_1^{\mu}A_1^{\nu} + \sum_{i,\mu}\left(\frac{\partial^2 H}{\partial p_i \partial A^{\mu}}\right)_{\varepsilon=0}\frac{\partial S_1}{\partial q_i} A_1^{\mu}\right\} + \ldots \tag{53}$$

Substituting these expansions of E and H from Eqs. (52) and (53) in the Hamilton-Jacobi equation and equating the coefficients of the same powers of ε, we get

$$-H_0^0 + H\left(q_i, \frac{\partial S_0}{\partial q_i}, A_0^{\mu}\right) = 0 \tag{54}$$

$$\sum_i\left(\frac{\partial H}{\partial p_i}\right)_{\varepsilon=0}\frac{\partial S_1}{\partial q_i} + \sum_{\mu}\left\{\left(\frac{\partial H}{\partial A^{\mu}}\right)_{\varepsilon=0} A_1^{\mu} - \left(\frac{\partial H^0}{\partial A^{\mu 0}}\right)_{\varepsilon=0} A^{\mu 0}\right\} = 0 \tag{55}$$

$$2\sum_i\left(\frac{\partial H}{\partial p_i}\right)_{\varepsilon=0}\frac{\partial S_2}{\partial q_i} + \sum_{i,j}\left(\frac{\partial^2 H}{\partial p_i\partial p_j}\right)_{\varepsilon=0}\frac{\partial S_1}{\partial q_i}\frac{\partial S_1}{\partial q_j} + \sum_{i,\mu}\left(\frac{\partial^2 H}{\partial p_i \partial A^{\mu}}\right)_{\varepsilon=0} A_1^{\mu}\frac{\partial S_1}{\partial q_i}$$
$$+\sum\left[2\left\{\left(\frac{\partial H}{\partial A^{\mu}}\right)_{\varepsilon=0} A_2^{\mu} - \left(\frac{\partial H^0}{\partial A^{\mu}}\right)_{\varepsilon=0} A_2^{\mu 0}\right\}\right.$$
$$\left. + \left(\frac{\partial^2 H}{\partial A^{\mu}\partial A^{\nu}}\right)_{\varepsilon=0} A_1^{\mu}A_1^{\nu} - \left(\frac{\partial^2 H^0}{\partial A^{\mu 0}\partial A^{\nu 0}}\right)_{\varepsilon=0} A_1^{\mu 0}A_1^{\nu 0}\right] = 0. \tag{56}$$

Eq. (54) is the unperturbed Hamilton-Jacobi equation, hence S_0, the time-independent part of the unperturbed characteristic function, is known exactly. Let the complete integral of Eq. (54) be $S_0(q_i, H_0^0, k_i)$; k_i's are $N-1$ parameters, $i = 1, \ldots, N-1$. The $N-1$ integral surfaces are $\frac{\partial S_0}{\partial k_l}$. In order to integrate the remaining equations we proceed as before. Let us introduce curvilinear coordinate systems $\theta_0 \equiv S_0(q_i)$ and $\theta_i = \frac{\partial S_0}{\partial k_i}$ $(i = 1, 2, \ldots, N-1)$. It is easy to see that the Jacobian

$$\frac{D(\theta_0, \theta_1, \ldots \theta_{N-1})}{D(q_1, q_2, \ldots q_N)} \neq 0$$

in general. Now since

$$\sum_i \left(\frac{\partial H}{\partial p_i}\right)_{\varepsilon=0} \frac{\partial S_n}{\partial q_i} = \sum_i \left(\frac{dq_i}{dt}\right)_{\varepsilon=0} \frac{\partial S_0}{\partial q_i} \frac{\partial S_n}{\partial S_0} = Q \frac{\partial S_n}{\partial \theta_0} \tag{57}$$

where

$$Q = \sum_i \left(p_i \frac{dq_i}{dt}\right)_{\varepsilon=0},$$

we can integrate Eqs. (55) and (56) successively. The complete integral of Eq. (55) is

$$S_1 = \int^{\theta_0} \frac{1}{Q} \left\{ \left(\frac{\partial H^0}{\partial A^{\mu 0}}\right)_{\varepsilon=0} A_1^{\mu 0} - \left(\frac{\partial H}{\partial A^{\mu}}\right) A_1^{\mu} \right\} d\theta_0 + \sum_1^{N-1} \alpha_{l1} \theta_l \tag{58}$$

the constants α_{l1} are to be determined from

$$\left[\int^{\theta_0} \frac{\partial}{\partial \theta_i} \left\{ \left(\frac{\partial H^0}{\partial A^{\mu 0}}\right)_{\varepsilon=0} A_1^{\mu 0} - \left(\frac{\partial H}{\partial A^{\mu}}\right)_{\varepsilon=0} A_1^{\mu} \right\} \frac{d\theta_0}{Q}\right]_{q_i=q_i^0} + \alpha_{l1} = 0 \tag{59}$$

so taht $\left(\frac{\partial S_1}{\partial q_i}\right)_{q_i=q_i^0} = 0$. Similarly

$$S_2 = -\int^{\theta_0} \frac{F_2}{Q} d\theta_0 + \sum_{l=1}^{N-1} \alpha_{t2} \theta_i \tag{60}$$

where F_2 is the left hand side of Eq. (56), excluding its first term and

$$\alpha_{l2} = \left[\int^{\theta_0} \frac{\partial}{\partial \theta_i} \frac{F_2}{Q} d\theta_0\right]_{q_i=q_i^0}. \tag{61}$$

The integration can be performed if S_1 is known. In this way we can successively find S_3, S_4 and so on. Since the expressions are very much involved we omit the general term. The perturbation expansion of the momentun is obtained from Eq. (51) and the expressions for the coordinates may be obtained as in the previous cases.

It is clear from above that the procedure can also be applied to the case of more than one small parameter. The method can also be extended to the case of time-dependent field, which will be followed up.

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APPENDIX

The perturbation series for $\vec{\nabla}S$,

$$\vec{\nabla}S = \vec{\nabla}S_0 + \varepsilon\vec{\nabla}S_1 + \varepsilon^2\vec{\nabla}S_2 + \ldots \tag{A1}$$

clearly converges for $|\varepsilon| < 1$ if $|\vec{\nabla}S_n|$ is bounded for all n. It also converges even in case $|\vec{\nabla}S_n|$'s are not bounded but satisfy a kind of Lifschitz' condition; i. e. there exists a μ and K independent of n such taht

$$|\vec{\nabla}S_n| < n^{\mu}K \tag{A2}$$

for all $n > 0$. The convergence of the series follows form the fact that

$$|\vec{\nabla}S_0 + \varepsilon\vec{\nabla}S_1 + \varepsilon^2\vec{\nabla}S_2 + \ldots + \varepsilon^n\vec{\nabla}S_n| < |\nabla S_0| + \sum_{L=1}^{n} |\varepsilon^L| L^{\mu'}K, \tag{A3}$$

where μ' is an integer such that $\mu + 1 > \mu^1 \geqslant \mu$. Since the right hand side of (A3) converges as $n \to \infty$ for $|\varepsilon| < 1$, the series (A1) also converges. Hence so long as $|\vec{\nabla}S_n|$'s do not increase with n as $n^{\mu(n)}K$, where $\mu(n)$ increases without bound with n for large values of n, there is always a domain, in the neighbourhood of $\vec{r}_0$ (with $|\varepsilon| < 1$), in which the perturbation series converges. It is difficult to assert the condition for this. However, for the convergence of the series, it is necessary that $U_n^0 - U_n$ are analytic in θ, φ, ψ in the neighbourhood of θ_0, φ_0, ψ_0 (i. e. of $\vec{r}_0$) for all n and similar condition for A^{μ}' s and Φ. It is to be noted that since we have assumed the possibility of expansion of U_n in the given region of space $U_n^0 - U_n$'s satisfy an inequality of the form (A2).

Next it has already been mentioned that Q, which is proportional to the unperturbed kinetic energy of the particle in the first case, should be non-vanishing. In general it can be stated that in the domain of validity of the perturbation expansion the unperturbed characteristic function, during the evolution in time, should not be in the neighbourhood of any of its extrema. It is also expected from the very fact that we are only trying to study small quantitative changes due to the perturbation. The rate of convergence depends predominantly on the magnitude of Q.

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