ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 7, nº 1 (1967), p. 77-94
<http://www.numdam.org/item?id=AIHPA_1967__7_1_77_0>

Ann. Inst. Henri Poincaré,
Vol. VII, n° 1, 1967, p. 77-94.

Section A :
Physique théorique.

Brownian motions and quantum mechanics

by

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SUMMARY. — It is shown how Schrödinger's equation can be transformed into a set of Kolmogorov's equations. The time dependent Green function of Schrödinger's eqn goes over into the transition probability function of Kolmogorov's equation for a Markov process whose stationary asymptotic distribution is the square of the ground state wave function of the Schrödinger equation one started with. Planck's constant is found to be connected to the spectral density of the Brownian noise.

The case of Q. M. harmonic oscillator is examined in detail and an interpretation of Planck's constant postulated by Stern is derived on classical grounds.

RIASSUNTO. — Si dimostra come l'equazione di Schrödinger possa essere trasformata in un sistema di equazioni di Kolmogorov. La funzione di Green dipendente dal tempo diventa la probabilità di transizione delle equazioni di Kolmogorov per un processo di Markov la cui distribuzione asintotica stazionaria di probabilità è il quadrato della funzione d'onda dello stato fondamentale della equazione di Schrödinger di partenza.

Si trova che la costante di Planck è legata alla densità spettrale del disturbo Browniano.

Si esamina poi in dettaglio il caso dell'oscillatore armonico quantistico e si deduce, su basi puramente classiche, una interpretazione della costante di Planck che è stata assunta come postulato da Stern.

§ 1. — INTRODUCTION

There have been in the past [*1* ... *8*] several attempts to derive Quantum Mechanics from Classical Mechanics. In all these attempts diffusion processes or Brownian motions play a very important role in giving to Q. M. its statistical character.

The purpose of the present paper is to point out a direct connection between Schrödinger's equation with imaginary time and the Kolmogorov Fokker Planck (1) equation for Markov processes.

The two equations go into each other by means of a trivial change of the unknown function, and the quantum mechanical Green's function is transformed into the transition probability of a Markov process.

The potentials which are involved in the two equations are not the same but are related to each other in a simple way. Moreover the stationary probability distribution of the Markov process is the squared modulus of the ground state wave function of the Schrödinger equation.

These results are somewhat different from those of Fenyes and Weizel. Namely Fenyes [*1*] has proved analogies between Quantum processes and Markov processes, while Weizel [*2*] has proved that Schrödinger's eqn may be manipulated in such a way as to obtain an equation of diffusion type for the squared modulus of the wave function.

The connection we have just mentioned between Schrödinger's and K. F. P. eqns brings to conclusions which are similar to those of the De Broglie-Bohm-Vigier theory [*5*] [*6*] in what concerns the interpretation of the Quantum Mechanical probability distribution as the stationary asymptotic probability distribution for a Markov process.

Moreover our results have some aspects in common with those of Bopp. This author has found [*7*] [*3*] that there exist several statistical theories whose description of physical facts does not differe appreciably from the Quantum Mechanical one for a duration of about ten billion years. In our case, at least for what concerns the ground state, the Quantum Mechanical description and that in terms of Markov processes differ from each other during a very small initial interval of time of about 10^{-13} seconds.

Some of our results, namely formula (21) and a slightly different form of formula (25), have been also derived in a very recent paper by Della Riccia and Wiener [*21*] starting with a quite different approach.

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(1) From now on : K. F. P.

The imaginary time plays a role in our derivation only in the asymptotic behaviour of the transformed Green's function when $t \to \infty$, but the set of eigenfunctions of the Schrödinger eqn is just the same as that of the K. F. P. eqns, neglecting a trivial one-to-one transformation. When a particle is in the ground state the same results which are obtained by describing the movement of the particle in terms of the Schrödinger eqn can be also obtained with a description in terms of classical random processes provided one transforms accordingly the external field of force.

Another consequence of the previously mentioned connection between the Schrödinger and the K. F. P. eqns is that the constant ($\hbar/2$ m) plays the role of a diffusion coefficient in the K. F. P. eqns. It is known that the diffusion coefficient is proportional to the spectral density of the force of the Wiener Levy type which causes the random movement of the particles in the Brownian motions. If this spectral density vanishes, h also vanishes and then we are concerned with a perfectly deterministic process of classical mechanics. The fact that h is finite entails that the spectral density of the « noise force » is finite. The last paragraph deals with the particular case of the Schrödinger eqn with an harmonic oscillator potential. If one defines the temperature of a particle as the temperature of the radiation which is in equilibrium with the particle inside a large cavity, then the entropy of the particle (= degree of unpredictability in the sense of Shannon's information theory) is given essentially by the logarithm of h. This last result of course is well known but the following facts are new :

1) the entropy is the logarithm of the spectral density of the random force as a consequence of the connection existing between this spectral density and h;

2) it is a consequence of a classical statistical description in terms of random processes that at very small temperatures the entropy tends to a finite limit, or, what is the same, the position of the particle in phase space cannot be predicted in an arbitrarily small region but only inside a region whose volume has a prescribed lower bound.

In a recent paper [9] Stern expressed his belief that Q. M. should be directly derivable from Nernst's theorem supplemented by some additional assumptions. According to Stern the zero point limit of the entropy in Nernst's theorem should be essentially the logarithm of a finite phase volume and this phase volume should be h. The above connection between the spectral density and the zero point entropy proves, at least in one case, that there is no need for postulating at zero temperature a finite phase volume containing the representative point of a particle: the finite phase volume is a classical consequence of the existence of a random force which acts on the particle also at small temperatures.

§ 2. — MARKOV PROCESSES AND K. F. P. EQUATIONS

We want now to recall some essential properties of the Markov processes which will be used later.

It is assumed here that the reader is familiar with Markov processes, in any case he is advised to ref. (16) (17) (18).

A complete characterization of a Markov process is obtained by giving a function $P(x, t; y, \tau)$. If we are describing the movement of a particle along a line, the quantity:

$$\int_{-\infty}^{z} P(x, t; y, \tau)dy = F(x, t; z, \tau) \qquad \tau > t \tag{1}$$

represents the probability that starting at time t from the point x the particle is found at time τ in any position $y < z$.

Of course:

$$\left.\begin{array}{ll} (2a) & \lim\limits_{z\to-\infty} F(x, t; z, \tau) = 0 \\ (2b) & \lim\limits_{z\to+\infty} F(x, t; z, \tau) = 1 \end{array}\right\} \tau > t$$

The following is the well-known Markov equation

$$P(x, t; y, \tau) = \int_{-\infty}^{\infty} P(x, t; \xi, \vartheta)P(\xi, \vartheta; y, \tau)d\xi \qquad t < \vartheta < \tau \tag{3}$$

Under certain conditions which are stated for instance in ref (17) eq. 3 implies that the function P satisfies the equations (K. F. P. eqns)

$$(4) \quad \begin{array}{ll} \text{I)} & A(x, t)\dfrac{\partial P}{\partial x} + \dfrac{B(x, t)}{2}\dfrac{\partial^2 P}{\partial x^2} = \dfrac{\partial P}{\partial t} \\ \text{II)} & \dfrac{1}{2}\dfrac{\partial^2}{\partial y^2}(B(y, \tau)P) - \dfrac{\partial}{\partial y}(A(y, \tau)P) - \dfrac{\partial P}{\partial \tau} = 0 \end{array}$$

$$P = P(x, t; y, \tau)$$

$$A(x, t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{\infty}(y - x)\, d_y F(t - \Delta t, x; t, y)$$

$$B(x, t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{\infty}(y - x)^2 d_y F(t - \Delta t, x; t, y)$$

A and B are known as diffusion and drift coefficients respectively. The function P is obtained from eqns 4 with the following boundary conditions

(5*a*) $\quad P(x, t\,; y, \tau) \geqslant 0 \quad$ for all t, x, τ, y

(5*b*) $\quad \int_{-\infty}^{a} P(x, t\,; y, \tau)dy = F(x, t\,; a, \tau)$ exists for every $a(\tau > t)$

(5*c*) $\quad \lim_{a\to\infty} F(x, t\,; a\tau) = 1 \qquad (\tau > t)$

(5*d*) $\quad \lim_{\tau\to t+0} P(x, t\,; y, \tau) = \delta(x-y)$

When P depends on t and τ only through $(\tau - t)$ then the Markov process is said to be « stationary ». It can be proved ([*18*], VI, § 2 ; [*10*] [*11*]) that for stationary Markov processes there exists a limit stationary distribution $\Phi(y)$ such that

(6*a*) $$\Phi(y) = \lim_{\tau\to\infty} P(x, t\,; y, \tau)$$

(6*b*) $$\int \Phi(y)dy = 1$$

(6*c*) $$\Phi(y) = \int P(x, t\,; y, \tau)\Phi(x)dx$$

§ 3. — THE SMOLUCHOWSKY EQUATION AND THE BROWNIAN MOTIONS

In the following sections we shall be concerned with the eqn

(7) $$\frac{\partial P}{\partial t} = -\frac{1}{f}\frac{\partial}{\partial x}(K(x)P) + D\frac{\partial^2 P}{\partial x^2}$$

which is known as Smoluchowsky [*10*] eqn and is a special case of the second K. F. P. eqn.

D is the diffusion coefficient and is given by $D = S^2/f^2$, where S^2 is the spectral density of a random force of which we shall speak soon and is equal to fkT; f is a friction coefficient, k is Boltzmann's constant and T is the absolute temperature. According to Smoluchowsky this eqn is satisfied by the transition probability density which describes the Brownian motion of a particle which is subjected to a field of force $K(x)$ a frictional force $-fx$

force and a random F(t) of Wiener Levy type [16] [17] [18]. The Langevin eqn for this particle is [10] [11]

$$mx'' = -fx' + K(x) + F(t). \tag{8}$$

Brownian Motions are Markov processes [11] [12] [16] [19].

As is known Markov processes in one random variable are connected with ordinary differential eqns of first order, hence it may be strange that the Markov process described by (7) is associated to eqn (8) which is of second order. If we observe that eqn 8 may be transformed in a system of two eqns of first order by setting $x' = u$ we should expect to be concerned with a vector Markov process in the two random variables x and u so that eqn (7) should be replaced by a more complicated one describing the evolution of the transition functions S $(x_0, p_0, t_0; x, p, t)$.

This is really so, as Kramers proved in ref. [13], where it is also proved that eqn (9) is still a good eqn and governs the space part of the transition probability S when we are in presence of a large viscosity.

In these conditions the function S, for large, t becomes approximately factorized as follows, since a Maxwell distribution in the velocities is soon established:

$$S(x_0, p_0, t_0\,; x, p, t) \sim e^{-p^2/2mkT}\, P(x_0, t_0, x, t). \tag{9}$$

It is easy to see that we should obtain the same eqn (7) by neglecting the left hand side in eqn (8); i. e. the Smoluchowsky eqn is correct until we are dealing with very small masses so that the inertial forces are negligible with respect to the frictional and random forces.

§ 4. — THE THEOREMS OF KAC AND THE PROPERTIES OF THE PRINCIPAL SOLUTION

We need now some theorems in order to prove that the Green's function of the Scrhödinger eqn with imaginary time is connected to the space part of the transition probability of a stationary Markov process in phase space.

In 1949 Kac [15] proved that the principal solution $G_{x_0,t_0}(x, t)$ of the partial differential equation (2)

$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial x^2} - V(x)G \tag{10}$$

(2) There is no restriction in considering the one dimensional case only, since Rosenblatt [15] has proved that the same properties hold even in the case of several dimensions.

has the following properties:

1) It exists and is unique under broad conditions for the function $V(x)$ (for example to be bounded from below).

2) It is expressible as on integral over « conditional Wiener Measure » (formally similar to Feynman's « integrals over histories »; see appendix B) of the functional $\exp\left[\int_0^t V(x(\tau))d\tau\right]$.

3) It is always positive.

The last property is not explicitly stated in Kac's work but it is a direct consequence of property n° 2 since G is the limit of a sum of positive terms (exponentials) and hence is itself a positive quantity (see append. B).

The function G has furthermore the following properties (12).

4) $$\int dx G_{x_0t_0}(x, t)G_{x,t}(\xi, \tau) = G_{x_0t_0}(\xi, \tau) \qquad (t_0 < t < \tau)$$

5) $$\lim_{t\to t_0+0} G_{x_0t_0}(x, t) = \delta(x - x_0)$$

6) $$\lim_{t\to t_0+0} \frac{1}{t-t_0}\left\{ \int dx G_{x_0t_0}(x, t) - 1 \right\} = V(x_0)$$

The property n° 4 is the well-known semigroup property of Markov processes and one could think at first sight that G is the probability density connected with some Markov process.

This is wrong since in order to describe a Markov process G should satisfy the condition

$$\int dx G_{x_0t_0}(x, t) = 1 \tag{11}$$

which is not satisfied, but is replaced by property n° 6. However one can ask wether it is possible to obtain from the function G another function $P(x_0, t_0; x, t)$ through multiplication of the first one by some function $M(x_0t_0, x, t)$ such that P has the prerequisites of the conditional distribution function characteristic of Markov processes. We shall prove in the following that this is possible and that the function P satisfies, in the variables x and x_0, equations which are exactly of the form of K. F. P.

We remember that G may be expressed as a series of eigenfunctions of eqn (10) (in general as a Stjeltjes integral over the spectrum of the operator at the r. h. s. of (10) (3)). Let us consider only the case of discrete spectrum: we have then

$$G_{x_0 t_0}(x, t) = \sum_{n=0}^{\infty} e^{-E_n(t-t_0)} \varphi_n(x_0)\varphi_n(x) \tag{12}$$

where $\varphi_n(x)$ satisfy the equations

$$\frac{\partial^2\varphi_n}{\partial x^2} - V(x)\varphi_n(x) = -E_n\varphi_n(x) \tag{13}$$

$$\int |\varphi_n(x)|^2 dx = 1 \tag{14}$$

Whatever the potential $V(x)$ (provided it belongs to the class specified in the footnote (3) the « energy spectrum » has always a lowei bound hence there existe always an eigenvolue E_0 such that

$$E_0 < E_n \qquad (n = 1, 2, \ldots).$$

It follows then from (12)

$$G_{x_0 t_0}(x, t) \underset{t\to\infty}{\sim} O(e^{-E_0 t}) \qquad (E_0 \gtrless 0) \tag{15}$$

This is the dominant behaviour also in presence of a continuous spectrum, since $G_{x_0 t_0}(x, t)$ is the inverse Laplace transform of the operator

$$\Omega = \left(\frac{\partial^2}{\partial x^2} - V - E\right)^{-1}$$

of which E_0 is the singularity with the greatest real part (Ω has singularities on a « left » half plane).

§ 5. — THE RELATION BETWEEN G AND P

Let us consider the function

$$e^{E_0(t-t_0)} \frac{\varphi_0(x)}{\varphi_0(x_0)} G_{x_0 t_0}(x, t) = P(x_0, t_0 ; x, t) \tag{16}$$

$$\left(\int |\varphi_0(x)|^2 dx = 1\right)$$

(3) We suppose throughout that the stationary Schrödinger eqn with the potential $V(x)$ admits at least one discrete eigenstate.

It is very easy to see that P may be interpreted as a probability density in a Markov Process which is stationary since, according to (12), P depends only on $t - t_0$. Namely we have, by (12) and the orthogonality relations

$$\int \varphi_0(x)\varphi_n(x)dx = 0 \qquad n \neq 0$$

$$\int e^{E_0(t-t_0)} \frac{\varphi_0(x)}{\varphi_0(x_0)} G_{x_0 t_0}(x, t)dx = \int |\varphi_0(x)|^2 dx = 1 \tag{17}$$

The function P is still positive and the δ function property still holds when $t \to t_0 + 0$ and besides P satisfies also the semigroup Markov property

$$P(x_0, t_0 ; x, t) = \int P(x_0, t_0 ; y, \tau)P(y, \tau ; x, t)dy \qquad t_0 < \tau < t \tag{18}$$

Moreover the function P is the only one which can be constructed as a product of G and some unknown function of x, x_0, t and t_0. We sketch the proof in the appendix A. The reason for the above uniqueness is that $\varphi_0(x)$ is the ground state of a Schrödinger equation, and has no zero at finite distances.

The Kolmogorov equations which are satisfied by P are

$$\frac{\partial^2 P}{\partial x_0^2} + B(x_0) \frac{\partial P}{\partial x_0} + \frac{\partial P}{\partial t_0} = 0 \tag{19}$$

$$\frac{\partial^2 P}{\partial x^2} - \frac{\partial}{\partial x} (B(x)P) - \frac{\partial P}{\partial t} = 0 \tag{20}$$

where $B(x)$ is defined by

$$B(x) = \frac{1}{2} \frac{\partial}{\partial x} \lg \varphi_0(x) = \frac{1}{4} \frac{\partial}{\partial x} \lg |\varphi_0(x)|^2 \tag{21}$$

and satisfies the equation

$$\frac{1}{2} B' + B^2 + (E_0 - V) = 0 \tag{22}$$

E_0 is the energy of the lowest bound state of the Schrödinger equation.

§ 6. — CONCLUSIONS FROM THE PRECEDING SECTIONS

The transition from the Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = i\hbar \frac{\partial \psi}{\partial t} \tag{23}$$

to (10) is only a matter of a trivial change of variables, the most important of which is $t \to -i\,t'$ and the resetting $t' = t$, so that our $G_{x_0 t_0}(x, t)$ is merely the analytical continuation of the usual Q. M. « G » on the imaginary t axis.

It is rather difficult to think in terms of imaginary times so it would perhaps be better to speak of a mapping between the solutions of Schrödinger's equation and those of (10) in the sense that they have the same set of space eigenfunctions.

This mapping entails, at least for what concerns the ground state, a mapping between Markov processes and Quantum processes, and the whole situation is summarised in what follows.

1) Every Schrödinger equation with « reasonable » potential can be transformed by simple algebraic manipulations into a K. F. P. set of equations describing a stationary Markov process or better Brownian motion of a particle in a conservative external field of force, whose potential is essentially the logarithm of the probability distribution in the ground state of the originary Schrödinger equation (Eq. (21) (22)).

The diffusion coefficient D has the value $D = \frac{\hbar}{2m}$ (see next paragraph 31 *a*) and the drift coefficient is a function of Planck's constant, and depends on $V(x)$ through (22).

2) The transition function $P(x_0, t_0; x, t)$ is connected to the Green's function G by the equation

$$P(x_0, t_0 ; x, t) = e^{E_0(t_0 - t_0)} \frac{\varphi_0(x)}{\varphi_0(x_0)} G_{x_0 t_0}(x, t) \tag{24}$$

Since the Markov process is stationary according to (6 *a*) there exists a well defined limit distribution $\Phi(x)$ for $t \to \infty$ (see § 2).

3) Writing explicitly in the above equation $G_{x_0 t_0}(x, t)$ in terms of its series expansion we have

$$\lim_{t \to \infty} P(x_0, t_0 ; x, t) = |\,\varphi_0(x)\,|^2 = \Phi(x) \tag{25}$$

Hence we have proved that the stationary distribution of the Brownian motion is exactly the probability distribution that the corresponding Quantum Mechanical system would have in its ground state.

Let us see the order of magnitude of the time which is necessary so that the particle reaches the stationary probability distribution corresponding to the Q. M. ground state starting at $t = t_0$ with a well defined position.

Suppose that the potential is such that the energy difference between the ground state and the first excited state of the particle is given by

$$\frac{\Delta E}{\hbar} = \frac{c}{5.10^{-5}\ \text{cm}}.$$

This corresponds to $\Delta E \sim 1$ e. v. and if this energy is emitted as electromagnetic radiation the wavelength is about 5 000 A.

Now let us evaluate the time which is needed in order to have the second term of the series in (12) e^{-20} times smaller than the first one. Taking into account the fact that E_n in (12) must be replaced by $E_n/\hbar$, for evaluation, we must have $\frac{E - E_0}{\hbar}(t - t_0) = \frac{\Delta E}{\hbar}\Delta t = 20$ i. e. $\Delta t \sim 3.10^{-14}$ sec.

4) The stationary probability distribution which is solution of a K. F. P. equation with diffusion coefficient 1 and assigned $B(x)$ is given by

$$\Phi(x) = N \exp\left[4\int_a^x B(x)dx\right] \tag{26}$$

(where N is a normalization constant such that $\int \Phi(x)dx = 1$) provided $\int_a^x B(x)dx \to -\infty$ when $|x| \to \infty$ otherwise (26) is meaningless.

§ 7. — THE SPECIAL CASE OF THE LINEAR HARMONIC OSCILLATOR

If we start with the Schrödinger eqn for a linear harmonic oscillator

$$-\frac{\hbar^2}{2m}\frac{\partial^2 G}{\partial x^2} + \frac{\lambda}{2}x^2 G = i\hbar\frac{\partial G}{\partial t} \tag{27}$$

the 2^{nd} K. F. P. eqn which is obtained with the suggested manipulations is the following:

$$\frac{\hbar}{2m}\frac{\partial^2 P}{\partial y^2} + \frac{\partial}{\partial y}\left(\sqrt{\frac{\lambda}{m}}\,yP\right) = \frac{\partial P}{\partial t} \tag{28}$$

This equation has just the same structure as that of Smoluchowsky for particles which undergo a Brownian motion in an elastic field of force and in presence of friction (see [*10*] p. 833). Suppose that the equilibrium has been reached by this last physical system, hence the probability distribution of the particles is just the same as that in the ground state of the corresponding Schrödinger equation. This means that if we want to identify the two systems (classical oscillator and the Q. M. one) we must admit that in the Q. M. system the only occupied state is the ground state $\varphi_0(x)$ and this implies that our classical system has a very law « temperature » $T \approx 0$: otherwise there would be other excited quantum states with a non zero probability of occupation.

Since there is one oscillator only the concept of temperature here has to be understood in a sense which is not the usual one.

Let us assume here that the word « temperature » means temperature of the e. m. radiation which is in equilibrium with the oscillator inside a large volume.

« Low temperature » then means that a transition from the ground state to an excited state is highly improbable.

With these premises we can compare equation (28) and (29). The last one is just that corresponding to a damped oscillator which undergoes a Brownian motion ([*2*], p. 833)

$$\frac{\partial P}{\partial t} = \frac{\omega_0^2}{\beta} \frac{\partial}{\partial x}(xP) + D \frac{\partial^2 P}{\partial x^2} \tag{29}$$

D is the diffusion coefficient, ω_0 the frequency of oscillation and $\beta = \frac{f}{m}$; f is the friction constant and m the mass of the particle. In usual Brownian motions the random force $F(t)$ is caused by the thermal agitation of the other molecules and this implies that D has the well-known expression

$$D = \frac{kT}{f} \tag{30}$$

Here we ignore this last eqn because it would imply that the Brownian motion is caused by other similar oscillators whose average kinetic energy is related to temperature by Boltzmann's relation. We stress again that we are dealing with one oscillator only and consequently the Brownian motion does not arise from thermal agitation of other particles: all we can say is that the oscillator is subject to a random force.

However the relation

$$D = \frac{S^2}{f^2} \tag{30,1}$$

is quite general (see [*10*] [*11*] [*19*]). S^2 is the spectral density of the random force. In the ordinary Brownian motions $S^2 = fKT$ and so we get (30).

By identifying the coefficients of (32) and (33) we have

(31*a*) $$D = \frac{\hbar}{2m}$$

(31*b*) $$\frac{\omega_0^2}{\beta} = \sqrt{\frac{\lambda}{m}}$$

i. e. taking into account that $\beta = \frac{f}{m}$ and $D = \frac{S^2}{f^2}$ it follows

(32*a*) $$f^2 = \lambda m$$

(32*b*) $$S^2 = \frac{\lambda \hbar}{2}$$

S^2 is defined by

(33) $$M[F(t)F(t+\tau)] = S^2\delta(\tau).$$

The average is taken over the « realizations » [*19*] of the random function $F(t)$. Eq (32*b*) proves that the spectral density of the random noise is proportional to h; it is also dependent on the strength of the external field of force. Notice however that the relation $D = \frac{S^2}{f^2} = \frac{\hbar}{2m}$ does not depend on the type of potential which is emploied.

A relation of the type $S^2 = \alpha\hbar$ holds in every case provided we assume that the friction constant is proportional to the square root of the mass: in the case of the harmonic oscillator the constant 2α is the elastic constant. We now calculate the entropy of the particle: according to Shannon's information theory it measures the degree of unpredictability for the localization of the particle in the phase space. According to Kramers'results and the remark n° 3 of § 6 when t is very large the probability density in phase space is given by

(34) $$P(x, p) = \frac{1}{\sqrt{2\pi m f D}} e^{-\frac{p^2}{2mfD}} \sqrt{\frac{m\lambda}{\hbar^2\pi^2}} e^{-(\sqrt{m\lambda}/\hbar)x^2}$$

where D is given by (31*a*) and the two last factors give the probability distribution in the ground state of the quantum harmonic oscillator. Remembering that if:

(35) $$p(z) = \sqrt{\frac{\mu}{\pi}} e^{-\mu z^2}$$

then

(36) $$\int_{-\infty}^{\infty} p(z) \ln p(z) dz = \frac{1}{2} \ln \frac{\mu}{\pi e}$$

we have for the entropy of the distribution P(x, p) (Boltzmann's function)

$$H = -K\int P \ln P = K \ln \frac{e\hbar}{2} \tag{37}$$

« e » is the base of natural logarithms.

With the previous definition of temperature eq. 37 tells that for small temperatures the entropy approaches a constant which is essentially the logarithm of h, neglecting an additive constant whose purpose is that of making the argument of the logarithm a dimensionless quantity. h is thus proportional to the smallest phase volume which for $T \to 0$ can enclose the representative point of the state of the system, and this is just the hypotesis of Stern in ref. [*9*].

The only relevant assumption in our reasoning is that the system is always subject to a random force of Wiener Levy type whose spectral density is the universal constant h times the elastic constant.

We have furthermore

$$\sqrt{\overline{F^2}} \sim 0(\lambda^{1/2}) \text{ (neglecting the } \delta \text{ function)}$$

If λ is small the elastic forces [which are $0(\lambda)$] are negligible with respect to the « Brownian » forces which are $0(\sqrt{\lambda})$ and with these conditions the Smoluchowsky diffusion equation is obtained from that of Kramers.

Summarizing we can say that at low temperatures what Q. M. calls harmonic oscillator may be described classically by means of an oscillator which is subjected to a frictional force with friction constant $f = \lambda m$ and to a random force of special type (not due to random collisions) which is responsible for the lack of determinacy in localizing the representative point of the particle in its phase space.

At this point a question would be quite natural: which should be the cause of this random force ?

The reply to such a question does not seem very simple.

However if the existence of such a random force could be proved, this force could play the role of a hidden variable in Q. M.

We quote as references on this subject the papers of Bell and Bohm and Bub (*Rev. Mod. Phys.*, 38, nº 3, 1966, p. 447-475) and also ref. [*3*] [*4*] [*5*] [*8*] in which all important preceding references may be found.

ACKNOWLEDGEMENTS

The author is indebted to Prof. M. Verde for ref. [*9*] and for discussions and also to Prof. T. Regge for criticism.

APPENDIX A

In this appendix we want to show how K. F. P. equations become Schrödinger equations with imaginary time after elementary manipulations. We write the two equations of K. F. P. with a diffusion coefficient 1. This is not restrictive, since it is immediate to prove that if the variables x and t are required to maintain the same meaning in both Quantum and Classical statistical descriptions, the diffusion coefficient must be a constant in order to be possible a direct transition from Schrödinger eqn with imaginary time to an eq K. F. P. tipe.

$$\text{I} \qquad \frac{\partial^2(\mathrm{P})}{\partial x^2} + \mathrm{A}(x, t)\frac{\partial \mathrm{P}}{\partial x} + \frac{\partial \mathrm{P}}{\partial t} = 0$$

$$\text{II} \qquad \frac{\partial^2(\mathrm{P})}{\partial x^2} - \frac{\partial}{\partial y}(\mathrm{A}(y, \tau)\mathrm{P}) - \frac{\partial \mathrm{P}}{\partial \tau} = 0$$

with $\mathrm{P} = \mathrm{P}(x, t; y, \tau)$.

Let us consider the second equation and set $\mathrm{P} = \Lambda \mathrm{G}$. We want to determine Λ so that the new equation for G does not contain $\frac{\partial \mathrm{G}}{\partial y}$ and besides it becomes the Schrödinger equation with a preassigned potential $\mathrm{V}(x)$.

We have for II

$$(\text{A},1) \quad \Lambda''\mathrm{G} + 2\Lambda'\mathrm{G}' + \Lambda\mathrm{G}'' - \mathrm{A}(\Lambda\mathrm{G}' + \Lambda'\mathrm{G}) - \mathrm{G}\mathrm{A}'\Lambda - \Lambda\frac{\partial \mathrm{G}}{\partial \tau} - \mathrm{G}\frac{\partial \Lambda}{\partial \tau} = 0$$

The condition that the new equation does not contain G is

$$2\Lambda' + \mathrm{A}\Lambda = 0$$

$$(\text{A},2) \qquad \mathrm{A} = 2\frac{\Lambda'}{\Lambda}$$

and the new equation becomes

$$(\text{A},3) \qquad \mathrm{G}'' + \mathrm{G}\left(-\frac{\Lambda''}{\Lambda} - \frac{1}{\Lambda}\frac{\partial \Lambda}{\partial \tau}\right) - \frac{\partial \mathrm{G}}{\partial \tau} = 0$$

If this must be the Schrödinger eqn with imaginary time and potential $\mathrm{V}(x)$ we must have :

$$(\text{A},4) \qquad \mathrm{V}(x) = \frac{\Lambda''}{\Lambda} + \frac{1}{\Lambda}\frac{\partial \Lambda}{\partial \tau}$$

i. e. Λ must itself satisfy the equation

$$(\text{A},5) \qquad \Lambda'' - \mathrm{V}(x)\Lambda + \frac{\partial \Lambda}{\partial \tau} = 0$$

which must be solved with the following condition

1) Λ is always positive and never vanishes so that $\mathrm{A} = \frac{1}{2}\frac{\Lambda'}{\Lambda}$ has no singularities for finite x.

2) $\mathrm{A}(t, x)$ is independent on t i. e. $\frac{\Lambda'}{\Lambda}$ is only a function of x; this because we want that the process is stationary (see [*18*]).

Condition 2 implies that $\Lambda(x, t) = \Lambda_0(t)\,\Lambda_1(x)$, which is a consequence of eq (A, 5), and consequently $\Lambda_0(t)$ is of the form $e^{\alpha t}$ where α is to be determined by condition 1.

(A, 6) $$\Lambda_1'' - V(x)\Lambda_1 + \alpha\Lambda_1 = 0.$$

The only case in which condition 1 is satisfied is when Λ_1 is the ground state of the stationary Schrödinger equation (A, 6) because in this case Λ has no zeros.

Hence we have $\alpha = E_0$ = lowest bound state energy, and

(A, 7) $$\Lambda = N e^{E_0 \tau} \varphi_0(y)$$

where $\varphi_0(y)$ is the ground state wave function of eq (4, 6) normalized so that $\int | \varphi_0(x) |^2 dx = 1$ and N is a function which may still depend on t and x. The precise form of this dependence may be determined by substituting $P = \Lambda G$ with Λ expressed by (A, 7) in the 1ST Kolmogorov equation. It will be found that $N = \dfrac{e^{-E_0 t}}{\varphi_0(x)}$ so that

(A,8) $$P(x, t\,;\, y, \tau) = G_{x,t}(y, \tau) e^{E_0(t-\tau)} \frac{\varphi_0(y)}{\varphi_0(x)}$$

APPENDIX B

INTEGRALS OVER WIENER MEASURE

We think that an example will be illuminating better than a rigorous definition. Suppose the functional

(B,1) $$F(x(t)) = e^{-\alpha \int_{t_0}^{t} g(\tau) x^2(\tau) d\tau}$$

is given. We want to evaluate its « integral over ordinary Wiener measure ». We replace the integral in the exponent in (B1) with the following sum

(B,2) $$-\alpha \sum_{n=0}^{N} g(\tau_n) x^2(\tau_n) \Delta\tau = F_N(x_0 x_1 \ldots x_n)$$

where $\tau_n (n = 0 \ldots N)$ are $N + 1$ equally spaced points between $t = 0$ and $t = N\Delta\tau$ (t is fixed)
$\tau_n = n\Delta\tau$, and $x_i = x(\tau_i)$.
Let us consider now the following integral

(B,3) $$W_N^{(t)}(F) = \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_N \times e^{-\left[\frac{x_0^2}{\Delta\tau} + \sum_{i=0}^{N-1} \frac{(x_{i+1} - x_i)^2}{\Delta\tau}\right]} \times F_n^*(x_0 x_1 \ldots x_N)$$

The limit

$$\lim_{N \to \infty} W_N^{(t)}(F) \equiv W^{(t)}(F)$$

is defined to be the integral over ordinary Wiener measure of the functional $F(x(\tau))$.

If in (B, 3) we fix the value $x(\tau_N)$ and set $x_N = x(\tau_N) = x$ and drop down the integration over x_N in (B, 3) the limit for $N \to \infty$ of the resulting integral is defined to be the « integral over conditional Wiener measure » of the functional $F(x(\tau))$.

For more details the reader may see the paper of Gelfand and Yaglom [*12*] and also the original work of Wiener [*20*].

NOTE ADDED IN PROOF

A few days after this paper has been submitted for publication, the author has been aware of results similar to those of Fenyes and Weizel which have been found also by Dr. E. Nelson (*Phys. Rev.*, **150**, nº 4, 1079, 1966).

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Manuscrit reçu le 13 décembre 1966.