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On the unitary representations of $SU(1,1)$ and $SU(2,1)$

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by

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I. — INTRODUCTION

The present paper is concerned with the explicit construction of all unitary representations of the two non-compact groups SU(1, 1) and SU(2, 1) as ancillary to the study of the family of groups SU(p, q).

There is much current interest for investigating the SU(p, q) groups, since some of them have been proposed as having physical content.

The group SU(1, 1) occurs as the little group, for a spacelike momentum, of the quantum mechanical Poincaré group (covering of the Poincaré group). It was in this context that Bargmann [1] discussed this group exhaustively. The group SU(2, 2) is locally isomorphic to SO(4, 2); as such it is the group of conformal relativity and the largest invariance group of the Maxwell equations. The unitary representations of this group have been discussed recently by Esteve and Sona [2] (by methods quite different than those used in the present paper); physical interpretations associated with this group have been discussed in many places (a good summary as

well as a guide to the extensive literature may be found in the review of Fulton, Rohrlich and Witten [3]).

More speculative, but certainly no less interesting, is the proposal of the group SU(6, 6) as the symmetry group of strong interaction physics [4]. It is at this stage not very clear—at least group theoretically—just what significance attaches to the imposition of « equations of motion » as, for example, in Salam's approach. Nevertheless knowledge of the infinite unitary representations of SU(6, 6) could quite possibly be of importance in the further development of the theory.

Our emphasis on unitary representations stems from physics. Only such representations have a direct physical content [5]. Non-unitary finite dimensional representations of the Lorentz group appear only as an intermediate step to obtain unitary representations of the Poincaré group. In the present stage of the SU(6, 6) theory, such an interpretation is probably more suitable but, to repeat, we do not find any clear group theoretical interpretation of the equations of motion as currently applied.

Finally, let us give a class of groups which may also be of physical interest: the groups $SU(p, q)$ with $p + q = 6$ which appear as little groups of the inhomogeneous $SL(6, C)$ group. Such groups have been suggested as physical models by many authors [6].

Let us summarize now previous work on the $SU(p, q)$ groups $(p \geqslant q)$. As already mentioned, the work of Bargmann [1] contains a comprehensive treatment of the SU(1, 1) group and was in fact the precursor to the study of unitary representations of non-compact *real* Lie groups. The study of unitary representations of *complex* Lie groups is relatively well developed— as, for example, in the book of Gelfand and Neumark [7]—in contrast to the investigation of $SU(p, q)$. Graev [8], in two short papers, has shown, using the techniques of Gelfand and Neumark, that for the groups $SU(p, q)$ there exist $q + 1$ fundamental series of irreducible unitary representations and has stated briefly the function spaces in which these representations are to be realized. The $q + 1$ fundamental series are characterized by d_0: $p + q - 1$ integers, d_1: $p + q - 2$ integers and 1 real number, ..., d_q: $p - 1$ integers and q real numbers. Harish-Chandra [9] has given some general results for real Lie groups; the theorem quoted in the Appendix is a basic result taken from this work.

Very recently physically motivated investigations of SU(1, 1), SU(2, 1) and SU(2, 2) have been carried out by members of the Trieste group [10] (For SU(2, 2) see also Ref. [2]).

The methods used in the present work are quite distinct from these earlier investigations with the possible exception of Ref. [*1*].   They are purely algebraic, that is, we work almost entirely with the Lie algebra, invoking global properties only when essential.   The basis for our method is the fact that the Lie algebras of SU($p + q$) and SU(p, q) have the same complex extension. We may thereby take over directly many results and techniques from the compact case [*11*].   One principal advantage of this procedure is that it leads to explicit results for the matrix elements of the generators, and does, in fact, lead to all irreducible representations for both SU(p, q) and its covering group as well.

A concluding Section discusses the possible extensions of these methods to the general case.

II. — THE SU(1, 1) CASE
AS A MODEL TO BE GENERALIZED

The SU(2) group, the group of isospin or angular momentum, is the model of the algebraic approach to the study of the unitary representations of semi-simple compact Lie groups. This was first made mathematically complete in the work of Casimir and Van der Waerden [*12*].

The algebraic method generalizes directly to the non-compact groups. We shall first illustrate this method by solving completely for the unitary representations of the SU(1, 1) group, the group of transformations in a two-dimensional complex plane (C_2) which leaves the following Hermitian form invariant ([1]):

$$\overline{\psi}\psi = \psi^{+}\sigma_3\psi = \psi_1^{*}\psi_1 - \psi_2^{*}\psi_2.$$ (II.1)

This group has been discussed extensively in a paper of Bargmann [*1*] as incidental to his discussion of the Lorentz group.

([1]) For SU(p, q) the invariant Hermitian form is $\psi^{+}M\psi$ where:

$$M_{ij} = \varepsilon_i\delta_{ij} \begin{cases} \varepsilon_i = 1 & : 1 \leqslant i \leqslant p \\ \varepsilon_i = -1 & : p + 1 \leqslant i \leqslant p + q. \end{cases}$$

From the definition of the group it follows that it consists of 2×2 complex matrices S, satisfying the relation:

$$\sigma_3 S^+ \sigma_3 = S^{-1} \tag{II.2}$$

and hence of the general form:

$$S = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \tag{II.3}$$

$$|\alpha|^2 - |\beta|^2 = 1.$$

The group U(1) generated by:

$$S(\alpha) = \begin{pmatrix} \exp (i\alpha/2) & 0 \\ 0 & \exp (-i\alpha/2) \end{pmatrix} \tag{II.4}$$

is obviously a maximal compact subgroup of SU(1, 1). The corresponding Lie algebra su(1, 1) is:

$$su(1, 1) = \{\, A \mid \sigma_3 A^+ \sigma_3 = A \,\} \tag{II.5}$$

and has the explicit basis:

$$k_1 = \frac{i}{2} \sigma_1, \qquad k_2 = \frac{i}{2} \sigma_2, \qquad k_3 = \frac{1}{2} \sigma_3 \tag{II.6}$$

which, it should be noted, are the generators of the 2-dimensional non-unitary representation defining SU(1, 1).

In a unitary representation of SU(1, 1) the corresponding basis elements are self-adjoint operators:

$$K_1, K_2, K_3 \tag{II.7}$$

satisfying the commutation relations :

$$\begin{aligned} [K_1, K_2] &= -iK_3, \\ [K_3, K_1] &= iK_2, \\ [K_2, K_3] &= iK_1. \end{aligned} \tag{II.8}$$

Hence, the Casimir operator:

$$I = -K_1^2 - K_2^2 + K_3^2 \tag{II.9}$$

is self-adjoint but *indefinite*.

Let us now consider a unitary representation of SU(1, 1) in a Hilbert space $\mathfrak{H}$, which is necessary because all unitary irreducible representations (except the identity) of non-compact Lie groups are infinite dimensional. We assume that the representation is continuous in the sense that:

$$(\Psi, U(S)\Phi) \tag{II.10}$$

is continuous in S.

This representation decomposes with respect to the U(1) subgroup into a direct sum of 1-dimensional unitary representations $\mathfrak{H}_{[m]}$:

$$\mathfrak{H} = \oplus_m \mathfrak{H}_{[m]}, \tag{II.11}$$

where:

$$U(\alpha)\psi_m = e^{im\alpha}\psi_m; \qquad \psi_m \in \mathfrak{H}_{[m]}. \tag{II.12}$$

Since:

$$\alpha = 0 \,(\mathrm{mod}\, 4\pi) \tag{II.13}$$

is the identity (E) in the group, it follows that m is integer or half integer.

It is essential to recognize that the quantization of m does not follow from the Lie algebra but only from a global property. This reflects the fact that, unlike SU(2) which is its own universal covering group, SU(1, 1) is not simply connected. As Bargmann [1] shows, the group manifold (parameter space) is homeomorphic to the direct product of the circle with a 2-dimensional Euclidean space. The group manifold of the universal covering group of SU(1, 1) is then the 3-dimensional Euclidean space and covers SU(1, 1) infinitely many times.

In the decomposition (II.11) each m appears at most once. This is a special case of the following

Theorem. — Every irreducible representation of the maximal compact subgroup U(p) of SU(p, 1) occurs at most once in every irreducible unitary representation of SU(p, 1).

This theorem is an immediate consequence of the branching rule of Weyl [16] and a theorem due to Godement [17].

Consider now the raising, lowering operator:

$$K_{\pm} = K_1 \pm iK_2, \tag{II.14}$$

which obey the identities:

$$\begin{aligned} K_3 K_- \psi_m &= (m-1)K_- \psi_m \\ K_3 K_+ \psi_m &= (m+1)K_+ \psi_m \end{aligned} \tag{II.15}$$

$$K_+K_- = -I + K_3(K_3 - 1) \qquad (II.16)$$

$$K_-K_+ = -I + K_3(K_3 + 1). \qquad (II.17)$$

In the Hilbert space $\mathfrak{H}$ of the representation corresponding to a given I the norm of the states $| K_\pm \psi_m \rangle$ must be positive or zero:

$$\| K_\pm \psi_m \|^2 = (\psi_m, K_\mp K_\pm \psi_m) \geqslant 0. \qquad (II.18)$$

It follows that the operators K_+K_- and K_-K_+ must be positive Hermitian and hence that the eigenvalues of:

$$-I + K_3(K_3 + 1), \qquad (II.19)$$

$$-I + K_3(K_3 - 1), \qquad (II.20)$$

must be positive or zero   (In the compact case these eigenvalues must be negative or zero).

From Eq. (II.12):

$$K_3\psi_m = m\psi_m \qquad (II.21)$$

and clearly for sufficiently large $| m |$ the positivity condition on (II.19, II.20) are both satisfied.   Assume that among the allowed m there exists one positive m and its corresponding state ψ_m (Negative m's are treated analogously).   The infinite chain of states:

$$\psi_m, \psi_{m+1}, \cdots \qquad (II.22)$$

obtained by applying K_+ repeatedly on ψ_m all exist and are different from zero since:

$$\begin{aligned} \| K_+\psi_m \|^2 &= (\psi_m, K_-K_+\psi_m) \\ &= (\psi_m, (2K_3 + K_+K_-)\psi_m) \qquad (II.23) \\ &= 2m \| \psi_m \|^2 + \| K_-\psi_m \|^2 > 0. \end{aligned}$$

Applying the lowering operator K_- repeatedly to ψ_m, one of the alternatives occurs:

a) the chain never terminates,

b) the chain terminates.

Consider case *a)*; the positivity conditions (II.19, II.20) in terms of eigenvalues for all m imply either:

$$I < 0 \qquad \text{if } m \text{ is integral.} \qquad (II.24)$$

$$I < -\frac{1}{4} \text{ if } m \text{ is half integral.} \qquad (\text{II.}25)$$

In case *b)*, let ψ_{m_0} be the last non-vanishing state in the descending chain. This means:

$$K_-\psi_{m_0} = 0 \qquad (\text{II.}26)$$

and:

$$K_+K_-\psi_{m_0} = [-I + K_3(K_3 - 1)]\psi_{m_0} = 0, \qquad (\text{II.}27)$$

or:

$$I = m_0(m_0 - 1). \qquad (\text{II.}28)$$

Equation (II.23) applied to ψ_{m_0} shows that m_0 must be positive and non zero.

The case $m_0 = 0$ occurs only in the identity representation for which:

$$K_+\psi_0 = K_-\psi_0 = 0 \qquad (\text{II.}29)$$

and $I = 0$ [Note that there are three representations corresponding to $I = 0$ ($a = 0$) (See fig. 1)].

The representations obtained under cases *a)* and *b)* are unitary (by construction) and irreducible (again by construction since every vector of these representations is cyclic ([2]). It can be shown that the representations so constructed constitute all the irreducible unitary representations of SU(1, 1). The representations with m integral are representations of SO(2, 1) [the Lorentz group with signature $(+, +, -)$] the half-integral m correspond to « double valued » (projective) representations of SO(2, 1) [1].

It is useful to recapitulate these results in a notation which will prove convenient for more general cases.

The first step is to factorize the Casimir invariant in terms of two symmetric variables, that is,

$$I = -\frac{1}{4} ab. \qquad (\text{II.}30)$$

In terms of these variables the operator K_+K_- takes on the symmetric form on ψ_m :

$$K_-K_+ = \left(m - \frac{a}{2}\right)\left(m - \frac{b}{2}\right), \qquad (\text{II.}31)$$
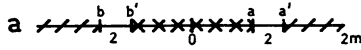

([2]) A vector is cyclic if upon applying the enveloping algebra one generates the whole space of the representation.
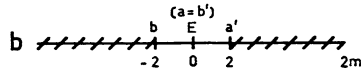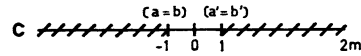
Types of Representations for SU(2) and SU(1,1)

$\times\times\times\times\times\times\times$ representations of SU(2)

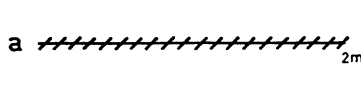$/\!/\!/\!/\!/\!/\!/$ representations of SU(1,1)

1. Discrete Set $(-)^{2m} = (-)^{2a}$

$a > 0$ integer

$a = 0 \begin{bmatrix} E = \begin{cases} \text{identity} \\ \text{both for} \\ \text{SU(2)} + \text{SU(1,1)} \end{cases} \end{bmatrix}$

$a = -1$

2. Continuous Set

a $\begin{cases} a = -1 + i\alpha \\ \alpha > 0 \\ m \text{ integer or half-integer} \end{cases}$
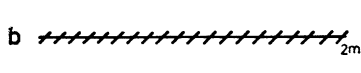
b $\begin{cases} -1 < a < 0 \\ (-)^{2m} = +1 \end{cases}$

FIG. 1.

which requires:

$$a + b = -2. \tag{II.32}$$

Similarly:

$$K_+ K_- = \left(m - \frac{a'}{2}\right)\left(m - \frac{b'}{2}\right). \tag{II.33}$$

We may identify the variables in four ways, the most useful being:

$$\begin{aligned} a' &= a + 2, \\ b' &= b + 2. \end{aligned} \tag{II.34}$$

The purpose of this notation is to make the boundaries for the raising, lowering operators clear; thus (a, b) represent boundaries for the raising operator K_+ and (a', b') for the lowering operator K_-.

The different representations of both SU(2) and SU(1, 1) may now be plotted as in fig. 1. The critical condition which separates the cases is the coincidence of two of the four variables ([3]) (a, b, a', b').

III. — UNITARY REPRESENTATIONS OF SU(2, 1)

The group SU(2, 1) is defined to be the set of transformations in a 3-dimensional complex space leaving invariant the following quadratic form:

$$\overline{\psi}\psi = \psi^+ M \psi = \psi_1^* \psi_1 + \psi_2^* \psi_2 - \psi_3^* \psi_3. \qquad \text{(III.1)}$$

From the definition of the group it follows that it consists of 3×3 complex matrices S, satisfying the relation:

$$MS^+ M = S^{-1}. \qquad \text{(III.2)}$$

The corresponding Lie algebra su(2, 1) is:

$$su(2, 1) = \{ A \mid MA^+ M = A \} \qquad \text{(III.2)}$$

and has, in terms of Gell-Mann's $\lambda's$ [13], the explicit basis:

$$\begin{aligned} k_i &= \lambda & i &= (1, 2, 3, 8), \\ k &= i\lambda & i &= (4, 5, 6, 7). \end{aligned} \qquad \text{(III.4)}$$

In a unitary representation of SU(2, 1) the corresponding basis elements are self-adjoint operators $K_i(i = 1 \ldots 8)$ satisfying commutation relation which can be deduced immediately from the definition (III.4). Let us consider the complex extension of the Lie algebra of SU(2, 1) which is the same as for the compact group SU(3), that is to say the algebra A_2. The Casimir invariants for A_2 are well known and using the notation of Gell-Mann have the from:

$$12 I_2 = \lambda.\lambda \qquad \text{(III.5)}$$

$$I_3 = d^{ijk} \lambda_i \lambda_j \lambda_k. \qquad \text{(III.6)}$$

([3]) It is useful in generalizing this procedure to make explicit the three following points:

i) the boundary points a, b (a', b') separate positive and negative regions for $K_- K_+ (K_+ K_-)$;

ii) motion toward the left terminates, if at all, at a' or b', toward the right at a or b;

iii) the motion is discrete; one may jump over unallowed regions.

To obtain from this the Casimir invariants of SU(2, 1) one uses the substitution (III.4).

For a unitary representation $(k_i \to K_i)$, I_2 and I_3 are indefinite self-adjoint operators, as will be verified later.

The group U(2) generated by $k_1 k_2 k_3 k_8$ and consisting of the following matrices:

$$
\begin{aligned}
S(\alpha, \beta, \varphi) &=\\
|\alpha|^2 + |\beta|^2 = 1 &
\end{aligned}
\begin{pmatrix}
\alpha e^{i\varphi/6} & \beta & 0 \\
-\bar{\beta} & \bar{\alpha} e^{i\varphi/6} & 0 \\
0 & 0 & e^{-i\varphi/3}
\end{pmatrix}
\tag{III.7}
$$

generates a maximal compact subgroup of SU(2, 1).

A unitary representation decomposes with respect to the U(2) group into a direct sum of finite dimensional representations:

$$
\mathfrak{H} = \bigoplus_{I,m} \mathfrak{H}_{[I,m]},
\tag{III.8}
$$

where:

$$
\mathfrak{J}\psi_{[I,I_z,m]} = \frac{1}{4}(K_1^2 + K_2^2 + K_3^2)\psi_{[I,I_z,m]} = I(I+1)\psi_{[I,I_z,m]},
$$

$$
M_2\psi_{[I,I_z,m]} = \frac{1}{2\sqrt{3}} K_8\psi_{[I,I_z,m]} = m\psi_{[I,I_z,m]},
\tag{III.9}
$$

$$
M_1\psi_{[I,I_z,m]} = \frac{1}{2} K_3\psi_{[I,I_z,m]} = I_z\psi_{[I,I_z,m]}, \quad \psi_{[I,I_z,m]} \in \mathfrak{H}_{[I,m]}.
$$

Note that the relation between the hypercharge Y and m is given by:

$$
m = Y/2.
\tag{III.10}
$$

It is essential to remark that because the group is U(2) and only locally SU(2) $\otimes$ U(1) the isospin and hypercharge are related by:

$$
6m = 3Y \equiv 2I \pmod{2}.
\tag{III.11}
$$

U(2) is a maximal compact subgroup of SU(2, 1) and each representation [I, m] occurs at most once in the decomposition (III.8).

Condition (III.11) does not follow from the Lie algebra alone but requires the global property that SU(2, 1) contains U_2. Just as in the SU(1, 1) case this reflects the fact that the group manifold of SU(2, 1) is not simply connected (see Appendix).

The method we shall use to calculate the irreducible representations of SU(2, 1) is the analogue of that used for SU(1, 1). There we were able to obtain the non-compact case directly from the formal results of the compact case by replacing the positivity condition on $J_{12}J_{21}$ and $J_{21}J_{12}$ with a negativity condition which means a positivity condition of $K_{12}K_{21}$ (II.16, II.17, II.6). This procedure generalizes to all SU(p, q) groups because the complex extension of the Lie algebras is the same in both SU($p + q$) and SU(p, q).

Let us define the following operators ([4]) :

$$K_{12} = \frac{1}{2\sqrt{6}} (K_1 + iK_2),$$

$$K_{13} = \frac{1}{2\sqrt{6}} (K_4 + iK_5), \qquad \text{(III.12)}$$

$$K_{23} = \frac{1}{2\sqrt{6}} (K_6 + iK_7).$$

and their Hermitian conjugate K_{21}, K_{31}, K_{32} (remember that K_1, ..., K_8 are Hermitian operators).

To construct an irreducible representation let us assume that there exists one vector denoted by $\psi_{[I,I_z,m]}$. All other vectors are to be obtained by applying the set K_{ij} repeatedly to $\psi_{[I,I_z,m]}$. Since the norm must be positive (unitary representation) this shows that:

$$\| K_{ji}\psi_{[I,I_z,m]} \|^2 = (\psi_{[I,I_z,m]}, K_{ij}K_{ji}\psi_{[I,I_z,m]}) \geqslant 0, \qquad \text{(III.13)}$$

for all ij, and for all I, I_z, m belonging to the representation. The next step is to express this result in terms of conditions on the Casimir operators I_2 and I_3 of SU(2, 1), I of SU(2) and the diagonal operators M_1 and M_2. This required form, however, can be obtained directly from the equivalent condition on the compact group SU(3), using once again the fact that the extended (complex) Lie algebras are the same.

It is an interesting fact that one may construct explicitly eight linear raising, lowering operators which accomplish all desired shifts in the subgroup quantum number space I, I_z, m. Using matrix elements of these operators

---

([4]) These operators belong to the Cartan basis, and require the use of the complex extension of the Lie algebra.

one may then carry out for SU(2, 1) the same analysis employed for SU(1, 1) using $K_\pm$.

To be precise we define the following eight operators { V }:

$$V(\pm, \pm, \pm) : \left(I \to I \pm \frac{1}{2}, I_z \to I_z \pm \frac{1}{2}, m \to m \pm \frac{1}{2}\right). \quad \text{(III.14)}$$

We assume the generators in the Cartan-Weyl basis of A_2:

$$[E_{ij}, E_{kl}] = \frac{1}{\sqrt{6}} [\delta_{jk} E_{il} - \delta_{il} E_{kj}], \quad \text{(III.15)}$$

$$H_1 = \frac{1}{\sqrt{2}} (E_{11} - E_{22}), \quad \text{(III.16)}$$

$$H_2 = \frac{1}{\sqrt{6}} (E_{11} + E_{22} - 2E_{33}), \quad \text{(III.17)}$$

and introduce the following operators:

$$\Lambda^2 = 3(E_{12}E_{21} + E_{21}E_{12} + H_1^2), \quad \text{(III.18)}$$

$$T_\tau = \begin{pmatrix} E_{13} \\ E_{23} \end{pmatrix} \qquad \tilde{T}_\tau = \begin{pmatrix} E_{32} \\ E_{31} \end{pmatrix}, \quad \text{(III.19)}$$

$$Q_\tau = \begin{cases} 3\left(\sqrt{\dfrac{2}{3}} E_{12}E_{23} + \sqrt{\dfrac{1}{3}} H_1 E_{13}\right) \\ 3\left(\sqrt{\dfrac{2}{3}} E_{21}E_{13} - \sqrt{\dfrac{1}{3}} H_1 E_{23}\right) \end{cases}$$

$$\tilde{Q}_\tau = \begin{cases} 3\left(\sqrt{\dfrac{2}{3}} E_{31}E_{12} - \sqrt{\dfrac{1}{3}} E_{32}H_1\right) \\ 3\left(\sqrt{\dfrac{2}{3}} E_{32}E_{21} + \sqrt{\dfrac{1}{3}} E_{31}H_1\right) \end{cases} \quad \text{(III.20)}$$

Here Λ^2 is the Casimir invariant for the sub-algebra $A(1)$ [consisting of E_{12}, E_{21} and H_1]; $T^\pm$ and $Q^\pm$ form two spin $\frac{1}{2}$ tensor operators for this sub-algebra.

These operators satisfy the following relations:

$$[\Lambda^2, T_\tau] = -\frac{3}{4} T_\tau + Q_\tau, \quad \text{(III.21)}$$

$$[\Lambda^2, Q_\tau] = T_\tau\left(\Lambda^2 - \frac{3}{4}\right) + \frac{5}{4} Q_\tau, \qquad \text{(III.22)}$$

as can be verified directly and easily from the commutation relations. Henceforth we drop the τ index.

Diagonalization brings these equations to the following form:

$$[\Lambda^2, V_\pm] = V_\pm x_\pm \qquad \text{(III.23)}$$

with:

$$V_\pm = T\left(x_\pm - \frac{5}{4}\right) + Q, \qquad \text{(III.24)}$$

and:

$$x_\pm = \frac{1}{4} \pm \sqrt{\Lambda^2 + \frac{1}{4}}. \qquad \text{(III.25)}$$

These equations to be meaningful require that $\Lambda^2 + \frac{1}{4}$ be positive or zero [Verified if $A(1)$ corresponds to SU(2)].

The $V_\pm^\tau$ operators correspond to shift operators:

$$V(\pm, \tau, +).$$

To prove this remark consider a state $\psi_{\{I, I_z, m\}}$, apply the operator $V(\pm, \tau, +)$ and use Eq. (III.23). Similarly the operator $V(\pm, \tau, -)$ is related to $\tilde{T}$ and $\tilde{Q}$ and assumes the form:

$$V(\pm, \tau, -) = \tilde{T}_\tau\left(x_\pm + \frac{1}{4}\right) - \tilde{Q}_\tau. \qquad \text{(III.26)}$$

For the compact subgroup U(2) the operators V^+V are diagonal in all quantum numbers and hence can be expressed as functions of I_2, I_3, Λ^2, H_1, H_2. For example the operator $V(+, +, -)$ has the form:

$$
\begin{aligned}
&V^+(+, +, -)V(+, +, -) \\
&= \frac{\sqrt{\Lambda^2 + 1/4} + \sqrt{3}H_1 + 1/2}{2(\sqrt{\Lambda^2 + 1/4} + 1/2)} \Big[3I_2(\sqrt{\Lambda^2 + 1/4} - H_2 + 1/2) \\
&\quad - 6I_3 - (\sqrt{\Lambda^2 + 1/4} - H_2 - 1/2)(\sqrt{\Lambda^2 + 1/4} - H_2 + 1/2) \\
&\qquad\qquad \left(\sqrt{\Lambda^2 + 1/4} - H_2 + \frac{3}{2}\right)\Big].
\end{aligned}
\qquad \text{(III.27)}
$$

To carry over these results from the compact case SU(3) to the non-compact group SU(2, 1) one need only make the appropriate substitutions:

$$E_{jk} \to \frac{1}{\sqrt{6}}\left(\frac{1}{i}\right)K_{jk}$$

which is equivalent to imposing the proper reality conditions.

From the expressions for V^+V one gets immediately the matrix elements given below. Imposing positivity conditions on the norms $\| V\psi \|^2$ is equivalent to positivity conditions on these matrix elements.

$$|\langle I - 1/2, I_z + 1/2, m - 1/2 \,|\, K_{32} \,|\, I, I_z, m \rangle |^2$$
$$= \frac{I - I_z}{2I(2I + 1)} [-6I_3 - (I + m)(3I_2 + 1) + (I + m)^3], \quad \text{(III.28)}$$

$$|\langle I + 1/2, I_z + 1/2, m - 1/2 \,|\, K_{32} \,|\, I, I_z, m \rangle |^2 = \frac{I + I_z + 1}{(2I + 1)(2I + 2)} \quad \text{(III.29)}$$
$$[6I_3 - 3I_2(I - m + 1) + (I - m)(I - m + 1)(I - m + 2)].$$

It is convenient to introduce the symmetrical variables a, b, c instead of I_2, I_3 in the form:

$$I_2 + 1/3 = -\frac{1}{3^3}(ab + bc + ac), \quad \text{(III.30)}$$

$$I_3 = \frac{1}{2 \cdot 3^4} abc, \quad \text{(III.31)}$$

$$a + b + c = 0.$$

Equations (III.28) and (III.29) assume then a complete symmetrical form, namely:

$$|\langle I - 1/2, I_z + 1/2, m - 1/2 \,|\, K_{32} \,|\, I, I_z, m \rangle |^2$$
$$= \left[\frac{I - I_z}{2 \cdot 3^3 I(2I + 1)}\right] [3(I + m) - a][3(I + m) - b][3(I + m) - c], \quad \text{(III.33)}$$

$$|\langle I + 1/2, I_z + 1/2, m - 1/2 \,|\, K_{32} \,|\, I, I_z, m \rangle |^2$$
$$= \left[\frac{I + I_z + 1}{3^3(2I + 1)(2I + 2)}\right] [3(I - m) - \overline{a}'][3(I - m) - \overline{b}'][3(I - m) - \overline{c}'], \quad \text{(III.34)}$$

with the notation:

$$\overline{\alpha}' = -\alpha - 3 \qquad \alpha = (a, b, c). \quad \text{(III.35)}$$

From (III.33) and (III.34) one can deduce easily analogous relations for K_{23}, the Hermitian conjugate of K_{32},:

$$|\langle I + 1/2, I_z - 1/2, m + 1/2 \,|\, K_{23} \,|\, I, I_z, m \rangle |^2$$
$$= \left[\frac{I - I_z + 1}{3^3(2I + 1)(2I + 2)}\right] \prod_{\alpha'} [3(I + m) - \alpha'], \quad \text{(III.36)}$$

$$|\langle I - 1/2, I_z - 1/2, m + 1/2 \,|\, K_{23} \,|\, I, I_z, m \rangle |^2$$
$$= \left[\frac{I + I_z}{2 \cdot 3^3 I(2I + 1)}\right] \prod_{\overline{\alpha}} [3(I - m) - \overline{\alpha}], \quad \text{(III.37)}$$

with:

$$\alpha' = \alpha - 3 \tag{III.38}$$

$$\overline{\alpha} = -\alpha. \tag{III.39}$$

It is perhaps useful to make the connection of these results to the compact case explicit. Firstly, one makes the substitution from K_{ij} to λ_{ij} in the left-hand side of Eqs. (III.33, III.34, III.36, III.37) and, secondly, one changes the over-all sign of the right-hand side. In the resulting expression the variables a, b, c are related to the Young pattern labels of $SU_3[p, q, 0]$ by the equations:

$$\begin{aligned}
a &= 2p - q + 3 \\
b &= 2q - p \\
c &= -p - q - 3,
\end{aligned} \tag{III.40}$$

or:

$$\begin{aligned}
p &= \frac{1}{3}(a - c - 6) \\
q &= \frac{1}{3}(a - b - 3).
\end{aligned} \tag{III.41}$$

The condition for the compact case [p, q integers and $p \geqslant q \geqslant 0$] implies that:

$$a \equiv b \equiv c \,(\mathrm{mod}\,3), \tag{III.42}$$

$$a \geqslant b + 3 \geqslant c + 6. \tag{III.43}$$

We will show below that condition (III.42) is also valid for the non-compact case (cf., however, the Appendix).

To obtain conditions on the variables a, b, c—which will completely characterize all unitary representations of SU(2, 1)—we simply have to impose the positivity requirements contained in Eqs. (III.33, III.34, III.36, III.37). The factor in curly brackets in each of these equations is always positive and may be discarded. (This follows from the properties of the SU_2 subgroup).

A convenient way to discuss the problem further is to plot the states of the representation not as usually in weight space (I_z, m) but as lattice points in two-dimensional space with ordinate $3I$ and abscissa $3m$ (and ignore the I_z co-ordinate). The operators in Eqs. (III.33, III.34, III.36, III.37) correspond to shift operators in ($3I$, $3m$) space. A cubic form is associated to each of these operators, that is:

$$\boxed{\!\!\!\diagup\!\!\!} = \prod_{\alpha} [3(I + m) - \alpha], \tag{III.44}$$

$$\boxed{\diagdown} = \prod_{\bar\alpha} [3(I - m) - \bar\alpha], \tag{III.45}$$

$$\boxed{\diagup} = \prod_{\alpha'} [3(I + m) - \alpha'], \tag{III.46}$$

$$\boxed{\diagdown} = \prod_{\bar\alpha'} [3(I - m) - \bar\alpha']. \tag{III.47}$$

These positivity relations divide the (I, m) plane into allowed and unallowed regions with straight line boundaries.

To enumerate all allowed regions one observes first that for given α and m, sufficiently large I is always allowed. As one proceeds inward, using the two operators $\boxed{\diagdown}$ and $\boxed{\diagup}$, one necessarily hits a boundary in either direction (This follows because a cubic equation with real coefficients has always at least one real root). This allowed region will correspond to a representation iff the boundary lines are lattice lines whose lattice points are points of the representation.

The points of the lattice obey the conditions stated earlier [see (III.11)], namely:

$$\begin{aligned}
3I \pm 3m &\equiv 0 \,(\text{mod } 1), \\
2I &\equiv 0 \,(\text{mod } 1).
\end{aligned} \tag{III.48}$$

A. — Three real roots (discrete set of representations).

The notation we shall employ is illustrated in figure 2, which has been drawn for the special case a-3 $> b > 0$. In this figure solid lines represent the Eqs. (III.44, III.45) and are labelled α and $\bar\alpha$, respectively; similarly dotted lines represent (III.46, III.47) (α', $\bar\alpha'$). Note that [by (III.39)] the solid line α intersects its partner $\bar\alpha$ on the I $= 0$ axis. The dotted lines [by (III.35, III.36)] are (vertically) three units below the corresponding solid lines ($\alpha \rightarrow \alpha'$, $\bar\alpha \rightarrow \bar\alpha'$). Allowed regions are easily seen to be separated from each other by two dotted and two solid lines and are hatched (m fixed and I large being always allowed). The allowed region for the compact case is shown cross-hatched and corresponds to a change of sense in the inequalities.
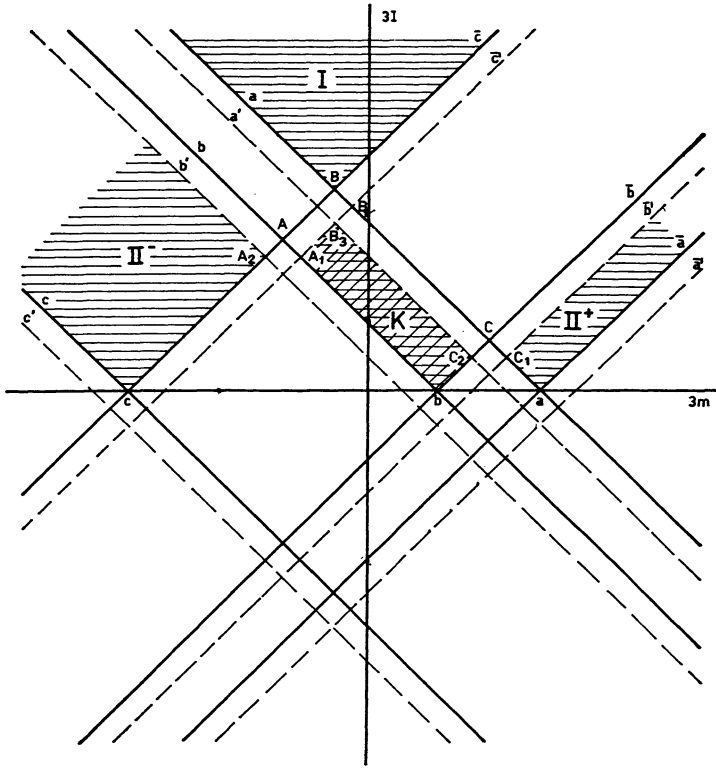
$$a-3>b>0$$

FIG. 2.

The figure shows that for every $a\text{-}3 > b > 0$ there are three distinct non-compact regions I,II$\pm$ and one compact region K. These regions may be characterized by the co-ordinates (I, m) of a representative point:

$$\begin{cases} \text{type I} & \text{point B} & \left(\dfrac{a-c}{6}, \dfrac{a+c}{6}\right) \\[2mm] \text{type II}^+ & \text{point C}_1 & \left(\dfrac{a-b}{6} - \dfrac{1}{2}, \dfrac{a+b}{6} + \dfrac{1}{2}\right) \\[2mm] \text{type II}^- & \text{point A}_2 & \left(\dfrac{b-c}{6} - \dfrac{1}{2}, \dfrac{b+c}{6} - \dfrac{1}{2}\right) \\[2mm] \text{type K} & \text{point B}_3 & \left(\dfrac{a-c}{6} - 1, \dfrac{a+c}{6}\right). \end{cases} \qquad \text{(III.49)}$$

In order that any of these regions correspond to a representation it is necessary and sufficient that the representative point be a lattice point (III.48). It can be easily seen that, whenever one of the four points above is a lattice point, all are. In terms of a, b, c one finds that (independent of the special conditions of fig. 2):

$$a \equiv b \equiv c \equiv 0 \,(\text{mod } 1) \qquad\qquad (\text{III}.50)$$

$$a \equiv b \equiv c \,(\text{mod } 3). \qquad\qquad (\text{III}.51)$$
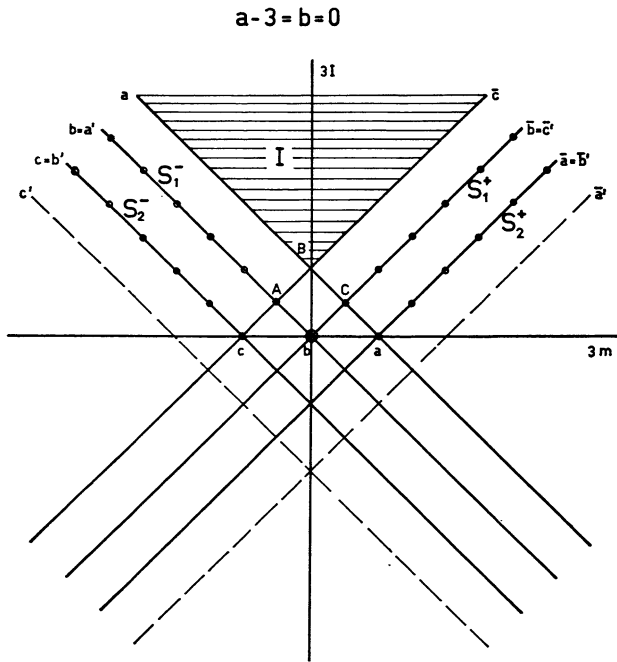

FIG. 3.

This result justifies our designating this set of representations as discrete.

Using the above result we can now enumerate and discuss completely all the representations of the discrete series. Since the three variables (a, b, c) occur symmetrically we may always choose:
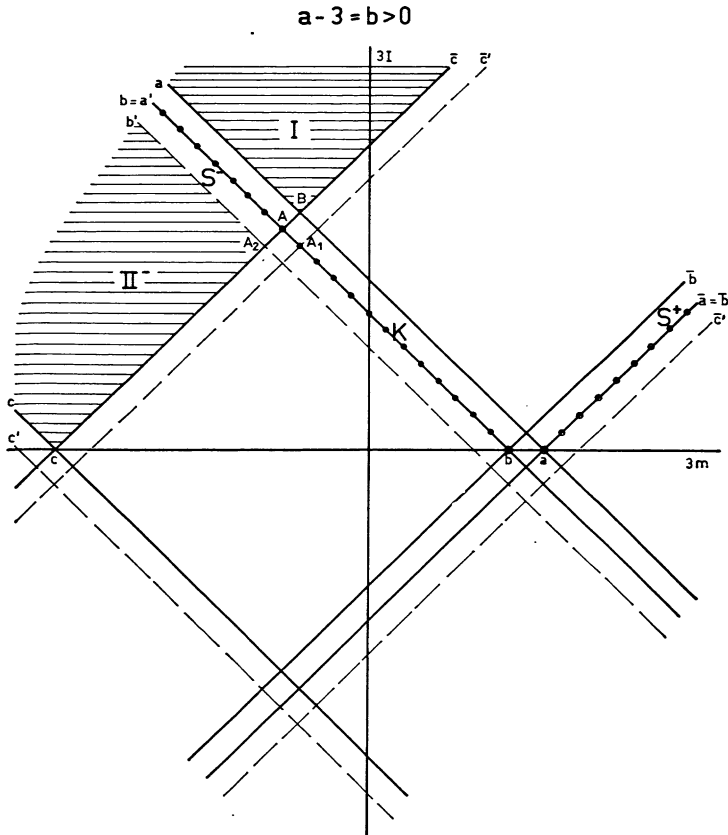
$$a \geqslant b \geqslant c. \qquad\qquad (\text{III}.52)$$

FIG. 4.

Since $a + b + c = 0$, there are now three possibilities $b > 0$, $b = 0$ and $b < 0$. By the following transformation:

$$a \rightarrow -c$$
$$b \rightarrow -b \qquad \text{(III.53)}$$
$$c \rightarrow -a$$

which carries a representation into its complex conjugate (and reflects the I, m plane in the line $m = 0$) one may always associate the representations for the case $b < 0$ with those of $b > 0$. We are then left with $b \geqslant 0$.

Singular cases occur at the coincidence of boundary lines in I, m plane. Owing to the relation (III.50), one needs only to consider:

$$a > b + 3(b \geqslant 0 : \text{fig. 2}), \qquad \text{(III.54)}$$

$$a = b + 3(b = 0 \colon \text{fig. 3}; \, b > 0 \colon \text{fig. 4}), \qquad \text{(III.55)}$$

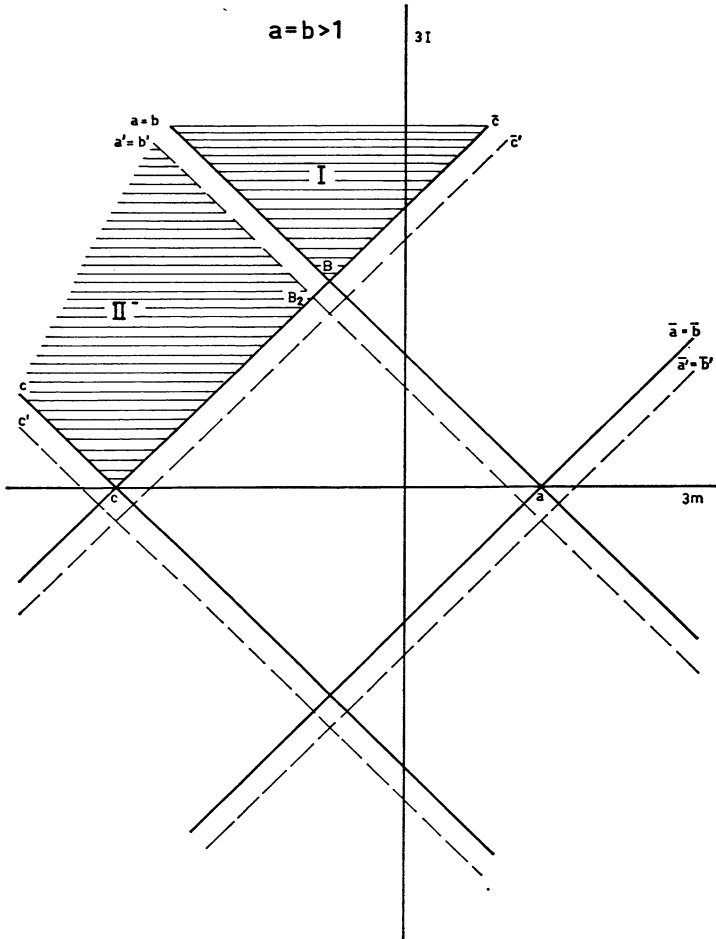$$a = b(b > 1 \colon \text{fig. 5}; \, b = 1 \colon \text{fig. 6}). \qquad \text{(III.56)}$$

FIG. 5.

In these latter cases a new type of representation appears, the degenerate cases designated S and characterized by their lowest lattice point. These degenerate representations correspond to the « triangle » representations of SU(3)   (Note that $a = b = c = 0$ which seems to belong to case (III.56) is a limiting case of the continuum series of representations).
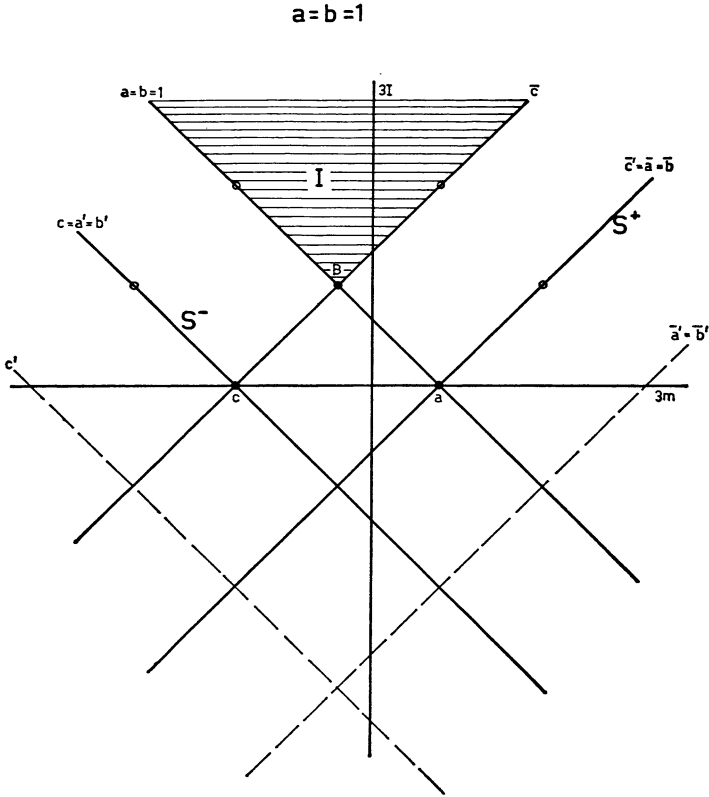
$$a = b = 1$$

FIG. 6.

B. — One real root (continuum set of representations).

With no loss of generality one may choose a to be the real root and $b^* = c$.
Thus

$$b = -a/2 + i\rho, \qquad (III.57)$$

$$c = -a/2 - i\rho, \qquad (III.58)$$

$$\rho > 0. \qquad (III.59)$$

The condition (III.59) holds in every case except $a = b = c = 0$ where the limit $\rho = 0$ must be considered as part of the continuous series.

The figure for this continuous series consists only of two solid lines which intersect on the $I = 0$ axis and two dotted lines. The subgroup

condition now shows that a must be an integer.   This series is illustrated in figure 7.

$$b = -\frac{a}{2} + i\varrho \qquad c = -\frac{a}{2} - i\varrho \qquad \varrho > 0$$
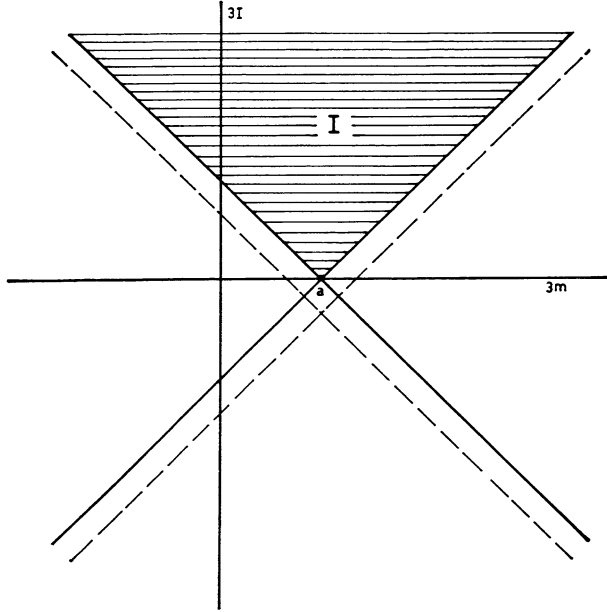

FIG. 7.

C. — Three real roots (continuous set of representations).

For $c = -(2n + 1)(n$ integer) it is easy to see that the region I of figure 8 is allowed for any real a satisfying (^{6}):

$$\begin{aligned} n + 1/2 &\leqslant a < n + 2, \\ n - 1 &< b = -a - c \leqslant n + 1/2. \end{aligned} \qquad (\text{III}.60)$$

This results from the fact that no lattice points lie in the unallowed region between the lines a and b.

(^{6}) Recall that there are three disconnected sets of lattice points separated by the triality $3y = 6m \equiv 0,1$ or 2 (mod. 3).   Each representation belongs to one and only one of these three sets.

$$c = -(2n + 1)$$
$$n + \frac{1}{2} \leqslant a < n + 2$$
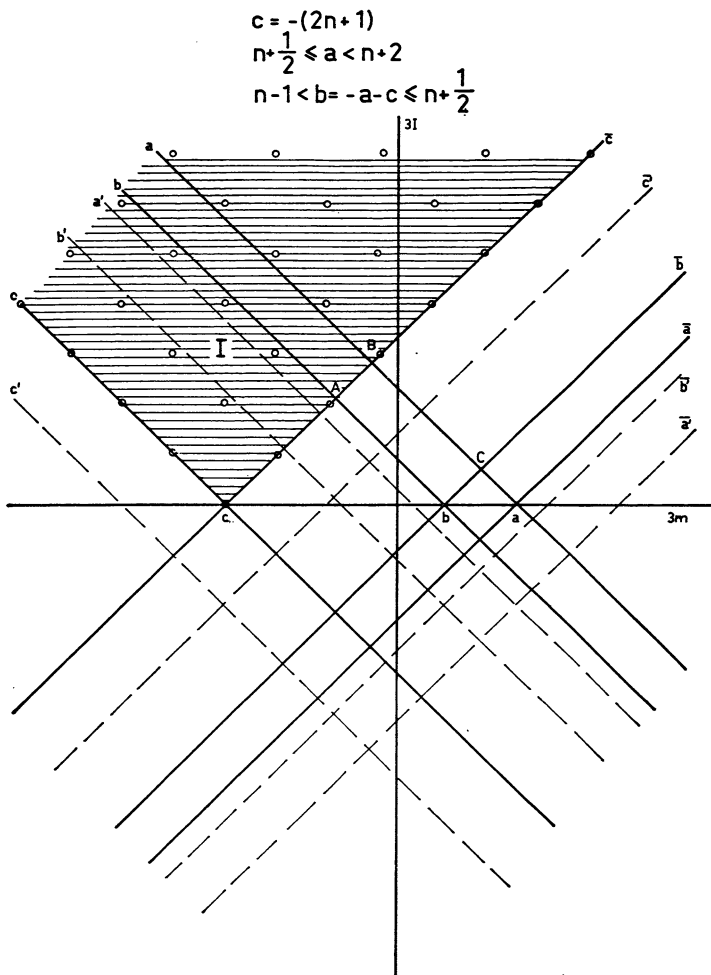$$n - 1 < b = -a - c \leqslant n + \frac{1}{2}$$

FIG. 8.

Clearly there is also the conjugate case given by the reflection of the (I, m) plane already used in Eq. (III.53).

Finally, for $b = 0$ a continuous set of representations appears for:

$$0 < a = -c < 3. \tag{III.61}$$

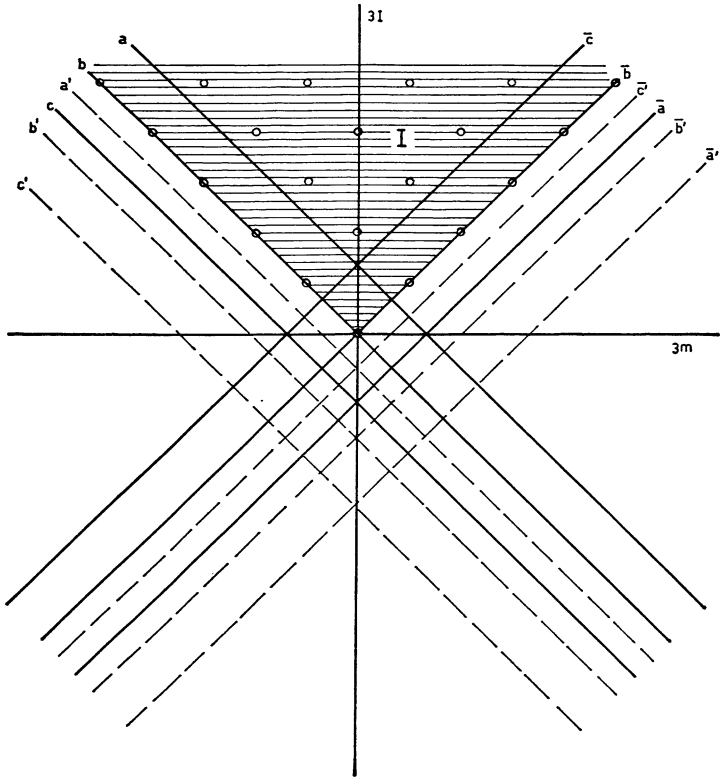The corresponding typical picture is given in figure 9.

b = 0

0 < a = - c < 3

Fig. 9.

IV. — CONCLUDING REMARKS

The generalization of the preceding methods to SU(p, 1) is immediate. The subgroup U(p) is maximal, compact, and provides a unique labelling for a canonical basis. Moreover, the analogues to the positivity conditions Eqs. (III.44, III.47)—that is restrictions on the matrix elements of the generalized raising, lowering operators—are already explicitly available (from the compact SU(p + 1) case) in factored form for this U(p) basis [11].

The generalization to SU(p, q), $q > 1$ is more difficult. The essential

complication—the lack of a « unique compact labelling »—can already be studied when one attempts to label SU(2,1) with respect to the non-compact subgroup U(1,1).

We intend to discuss these problems in detail subsequently.

ACKNOWLEDGEMENTS

APPENDIX

It is necessary to discuss the topology of the group manifold of SU(p, q) in order to show that the representations of the Lie algebra which can be constructed by the methods of this paper are true representations of the group and not projective representations (this means representations up to a factor).

The required topological properties of SU(p, q) can be found in the book of Helgason [*14*]. From lemma (4.1) and (4.3) in Chapter 9 it is easy to conclude that the group manifold of SU(p, q) as a topological space is homeomorphic [1] to the topological product of the group manifolds of SU(p), SU(q), a circle S^1, and a $2pq$ dimensional Euclidean space R^{2pq}:

$$SU(p, q) \sim SU(p) \times SU(q) \times S' \times R^{2pq}. \qquad (A.1)$$

The group manifold of the universal covering group of SU(p, q) can be read directly from this formula: one simply has to replace S$'$ by its universal covering space R.

Each representation of the Lie algebra can be extended to a representation of the universal covering group and thus in general to a projective representation of SU(p, q). In order to obtain true representations of SU(p, q) one has only to impose the discreteness condition coming from S^1.

To obtain explicitly all representations of the universal covering groups of SU(1,1) one has simply to relax the integer condition on the U(1) quantum number m. For SU(2,1) one relaxes the two conditions:

$$3I \pm 3m \equiv 0 \ (\mathrm{mod}.\ 1), \qquad (A.2)$$

and secondly:

$$6m \equiv 0 \ (\mathrm{mod}.\ 1). \qquad (A.3)$$

In effect one allows m to be an arbitrary real number.

At this point it is perhaps useful to quote the following result [*15*] one has for infinite dimensional representations of semi-simple Lie algebras:

THEOREM. — If $\mathcal{M}$ is the vector space (possibly infinite dimensional) of a representation with a dominant weight Λ [2] for a semi-simple Lie *algebra* $\mathfrak{L}$, then $\mathcal{M}$ is a direct sum of weight spaces $\mathcal{M}_M$ which are all finite dimensional and $\mathcal{M}_\Lambda$ has dimension 1. All other weights have the form $M = \Lambda - \Sigma m_i \alpha_i$ with integers $m_i \geqslant 0$. Conversely, for *each linear form* Λ on the Cartan algebra $\mathfrak{H}$ of $\mathfrak{L}$, there exists one and up to an isomorphism only one irreducible representation (not necessarily Hermitian) with dominant weight Λ.

The requirement in this theorem that there exists a dominant weight is very restrictive, as can be seen already from the explicit examples of irreducible infinite dimensional representations without a dominant weight constructed in Sections II and III (representations of type I).

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[1] Two topological spaces are homeomorphic if there exists a one-to-one mapping between the two spaces continuous in both directions.

[2] $\mathcal{M}$ has a dominant vector x if $x \in \mathcal{M}_\Lambda$ for a certain weight Λ, x is annihilated by all E_{α_i} (α_i simple roots) and x generates $\mathcal{M}$. In this case the corresponding weight Λ is called dominant.

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(Manuscrit reçu le 11 mai 1965).