Annales de l'I. H. P., section A

Vu Thanh Khiêt

On the sign of energy of the plane gravitational waves

Annales de l'I. H. P., section A, tome 2, nº 3 (1965), p. 261-281
<http://www.numdam.org/item?id=AIHPA_1965__2_3_261_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. II, n° 3, 1965, p. 261-281. *Physique théorique.*

On the sign of energy of the plane gravitational waves

by

VU THANH KHIÊT
(University of Moscow).

ABSTRACT. — On the basis of the relative gravitational field conception advanced by Yu. Rylov, the Synge's plane gravitational waves energy relative to the separate base points has been calculated. It is found that the sign and the value of the energy depend on the base point position.

I. — INTRODUCTION

The plane gravitational waves have been investigated by a series of authors. Taub [*4*] and McVittie [*5*] showed that the non-polarized plane waves could not exist. Robinson and later Bondi [*6*, *7*] discovered that the field equations permitted the existence of « sandwich waves » bounded on both sides by hyperplanes in flat space-time. Synge investigated the « thick » plane gravitational waves [*3*].

The Robinson-Bondi plane waves energy has been calculated by Kuchar, Langer [*8*] and Cahen [*9*] by means of Møller's pseudotensor [*10*, *11*, *12*] and by Langer [*13*] by means of Plebanski's pseudotensor; these authors proved that energy of these waves is equal to zero. Araki [*14*] proved the existence and the uniqueness of the gravitational equation solution in the linear approximation in the case, when the spacial part of the metric is nonsingular and sufficient close to the flat metric; and he proved that this solution determines the gravitational waves, energy of which is positive definite.

In this paper, on the basis of the relative gravitational field conception advanced by Yu. Rylov [1] we shall calculated the relative energy of Synge's plane gravitational waves [3].

In [1], it is shown that: by a description of the gravitational field within the limits of Einstein's theory only a relative gravitational field, i. e. the gravitational field at the point x with respect to one at the base point x', is physically essential.

According to [1] the relative field of gravitation is described by the two-point tensor

$$Q^{\alpha}_{\beta\gamma}(x, x') = \gamma^{\alpha}_{\beta\gamma}(x) - \Gamma^{\alpha}_{\beta\gamma}(x, x') \qquad (1)$$

where $\gamma^{\alpha}_{\beta\gamma}(x)$ are Christoffel symbols in the space-time V_4 at the point x in the coordinate system K, and $\Gamma^{\alpha}_{\beta\gamma}(x, x')$ are Christoffel symbols in the flat four-dimension space $E_{X'}$ tangent to V_4 at the point x' in the coordinate system K' in the representation V_4 into $E_{X'}$. The method of representation V_4 into $E_{X'}$ depends on the choice of the base point x'. In the representation V_4 into $E_{X'}$ geodesics passing through x' in V_4 are represented by straight lines passing through x' in $E_{X'}$, and, moreover, angles between geodesics at the point x' and lengths of geodesics passing through x' are unchanged. The metric tensor $G_{\mu\nu}$ of the space $E_{X'}$ is connected to the Synge's world function closely. On the basis of the relative gravitational field conception, the integral conservation laws for energy-momentum are obtained, the energy-momentum being a true relative tensor, i. e. tensor depending on two points: x and x'. All values connected with a gravitational field are relative what is interpreted as the presence of some general relativity in the gravitational field.

II. — THE RELATIVE ENERGY OF SYNGE'S PLANE GRAVITATIONAL WAVES

According to [1], the 4-momentum of gravitational field relative to the base point x' is defined by the relation

$$\mathcal{P}_{\beta'} = \mathcal{P}_{\beta'}(X') = \int_{\Sigma} \theta^{\alpha}_{g\beta'} \sqrt{-\mathfrak{D}_X}\, dS_{\alpha}$$

where $\mathcal{P}_{\beta'}$ is the relative 4-momentum of gravitational field, being a vector at the point x', $\mathfrak{D}_X = \det \| G_{\mu\nu} \|$, Σ is an infinite spacelike hypersurface:

$$\theta^{\alpha}_{g\beta'} = P_{\beta'}^{\cdot\,\cdot\,\gamma} \theta^{\alpha}_{g\gamma},$$

$P_{\beta'}^{\cdot\gamma}$ is the tensor of the parallel transport in $E_{x'}$ and $\theta^{\alpha}_{g\gamma}$ is the energy-momentum tensor of the gravitational field relative to the base point x' and is defined by the relation

$$\Lambda\theta^{\alpha}_{g\beta} = -\frac{1}{2x}\{g^{\rho\sigma}(g^{\mu\nu}g^{\alpha\delta} - g^{\mu\delta}g^{\alpha\nu})(Q_{\sigma\nu\beta}Q_{\delta\mu\rho} + Q_{\nu\sigma\beta}Q_{\delta\mu\rho} + Q_{\sigma\rho\beta}Q_{\delta\mu\nu}) - \delta^{\alpha}_{\beta}L_g\}, \tag{2}$$

$$\Lambda = \sqrt{\frac{\mathfrak{D}_x}{g}}, \qquad g = \det \| g_{\mu\nu} \|, \qquad Q_{\sigma\nu\beta} = g_{\sigma\mu}Q^{\mu}_{\nu\beta},$$

where L_g is a Lagrangian of the gravitational field relative to the point x' and is taken in the form

$$L_g = L_g(x, x') = g^{\mu\beta}(Q^{\alpha}_{\gamma\beta}Q^{\gamma}_{\alpha\mu} - Q^{\alpha}_{\mu\beta}Q^{\gamma}_{\alpha\gamma}), \tag{3}$$

$x = \frac{8\gamma\pi}{c^4}$, γ is the gravitational constant of Newton.

In the case, if Σ we choose as the hypersurface $x^0 = \text{const.}$ we get

$$\mathfrak{I}_{\beta'} = \int_{\Sigma} \Lambda\theta^{0}_{g\beta'}\sqrt{-g}\,d^3x, \qquad d^3x = dx^1\,dx^2\,dx^3. \tag{4}$$

The « thick » gravitational waves in the space-time [3] are shown in fig. 1. Two three-dimension hypersurface divide the space-time into three regions: I, II and III. The matter is absent in all space and, everywhere, we have

$$R_{ij} = 0.$$

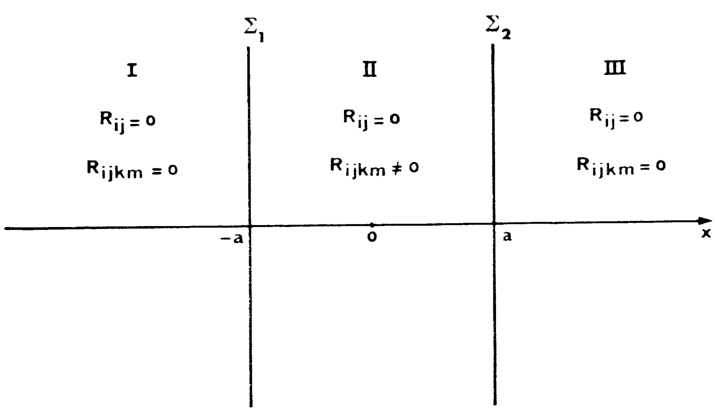

FIG. 1.

The gravitational field is absent in the region I and III, and hence, here

$$R_{ijkm} = 0.$$

The region II is the « thick » gravitational waves. Here, at least, one of the Riemann curvature tensor's components is not equal to zero and we take notice of it, in writing

$$R_{ijkm} \neq 0.$$

We remind that for the permissible coordinates $g_{\mu\nu}$ and their first derivatives are continuous, and their second derivatives can suffer rupture (Lichnerowicz's conditions [*15*]). The essential peculiarity of the « thick » gravitational waves consists in the existence of a non-flat region pressed in between two flat ones. In order to construct the gravitational waves with the metric

$$ds^2 = dt^2 - e^{2P}(dx^1)^2 - e^{2Q}(dx^2)^2 - (dx^3)^2 \tag{5}$$

where $x_0 = t$, the light speed is unit, P and Q are arbitrary functions of

$$x = \frac{1}{\sqrt{2}}(x^3 - x^0),$$

in the regions I and III it is necessary to satisfy the following equations:

$$P'' + (P')^2 = 0, \qquad Q'' + (Q')^2 = 0, \tag{6}$$

where the primed indices signify derivative, so

$$P = \ln m(x + \alpha), \qquad Q = \ln n(x + \beta), \tag{7}$$

where α, β, m and n are constants (different in the regions I and III), and in the region II it is necessary to satisfy the equation

$$P'' + (P')^2 + Q'' + (Q')^2 = 0, \tag{8}$$

without satisfying equations (6).

We shall calculate the energy-momentum tensor of these plane gravitational waves. The quantity $Q^{\alpha}_{\beta\gamma}$ can be found on the basis of the relation (*1*). $\Gamma^{\alpha}_{\beta\gamma}$ are connected with Synge's world function by the following manner [*2*]:

$$\Gamma^{\alpha}_{\beta\gamma} = G^{\alpha\mu'} . \frac{\partial G_{\beta\mu'}}{\partial x^{\gamma}}, \tag{9}$$

where $G = G(x, x')$ is Synge's world function, $G_{\alpha\mu'} = \frac{\partial^2 G}{\partial x^{\mu'} \partial x^{\alpha}}$, and $G^{\alpha\mu'}$ is the tensor with the matrix inverse to $G_{\alpha\mu'}$. Here the primed indices refer to the point x'. The world function is determined by the relation:

$$G(x, x') = \frac{1}{2} S^2(x, x'), \qquad S(x, x') = \int_{x'}^{x} \sqrt{g_{\mu\nu} dx^{\mu} dx^{\nu}}.$$

Here x and x' are coordinates of two arbitrary points of a Riemann space V_4, $S(x, x')$ is a finite interval between them, with the integral taken along the geodesic connecting the points x and x'. The tensors of the parallel transport in $E_{x'}$ have the form [2]

$$P_{\alpha'\cdot}^{\cdot\beta} = -g_{\alpha'\sigma'}(x')G^{\alpha'\beta}, \qquad P_{\beta\cdot}^{\cdot\alpha'} = -g^{\alpha'\sigma'}(x')G_{\sigma'\beta}, \tag{10}$$

where $P_{\beta+\cdot}^{\cdot\alpha'}$ is the tensor inverse to $P_{\alpha'\cdot}^{\cdot\beta}$. From (9) and (10) it follows

$$\Gamma^{\alpha}_{\beta\gamma} = P_{\mu'\cdot}^{\cdot\alpha} \frac{\partial}{\partial x^{\gamma}} (P_{\beta\cdot}^{\cdot\mu'}). \tag{11}$$

The world function for the metric (5) has the form

$$G(x, x') = \frac{1}{2}(\xi^0 - \xi^3) \left\{ (\xi^0 + \xi^3) + \frac{(\xi^1)^2}{\sqrt{2}I_1} + \frac{(\xi^2)^2}{\sqrt{2}I_2} \right\}. \tag{12}$$

Where

$$\xi^i = x^i - x^{i'}, \qquad I_1 = \int_{x'}^{x} e^{-2P(x)}dx, \qquad I_2 = \int_{x'}^{x} e^{-2Q(x)}dx. \tag{13}$$

From this we shall get for $P_{\beta\cdot}^{\cdot\alpha'}$

$$P_{0\cdot}^{\cdot 0'} = 1 + P_{0\cdot}^{\cdot 3'},$$

$$P_{0\cdot}^{\cdot 3'} = -P_{3\cdot}^{\cdot 0'} = \frac{(\xi^1)^2(m_1 + m_1' - 2m_1m_1')}{4I_1(x - x')} + \frac{(\xi^2)^2(m_2 + m_2' - 2m_2m_2')}{4I_2(x - x')},$$

$$P_{3\cdot}^{\cdot 3'} = 1 - P_{0\cdot}^{\cdot 3'}, \qquad P_{1\cdot}^{\cdot 1'} = m_1', \qquad P_{2\cdot}^{\cdot 2'} = m_2',$$

$$P_{0\cdot}^{\cdot 1'} = -P_{3\cdot}^{\cdot 1'} = \frac{(\xi^1)m_1'(m_1 - 1)}{\sqrt{2}(x - x')}, \qquad P_{0\cdot}^{\cdot 2'} = -P_{3\cdot}^{\cdot 2'} = \frac{(\xi^2)m_2'(m_2 - 1)}{\sqrt{2}(x - x')},$$

$$P_{1\cdot}^{\cdot 0'} = P_{1\cdot}^{\cdot 3'} = \frac{(\xi^1)(1 - m_1')}{\sqrt{2}I_1}, \qquad P_{2\cdot}^{\cdot 0'} = P_{2\cdot}^{\cdot 3'} = \frac{(\xi^2)(1 - m_2')}{\sqrt{2}I_2},$$

$$P_{1\cdot}^{\cdot 2'} = P_{2\cdot}^{\cdot 1'} = 0, \tag{14}$$

where

$$x - x' = \frac{1}{\sqrt{2}}(\xi^3 - \xi^0),$$

$$m_1 = \frac{(x - x')e^{-2P(x)}}{I_1}, \qquad m_1' = \frac{(x - x')e^{-2P(x')}}{I_1},$$

$$m_2 = \frac{(x - x')e^{-2Q(x)}}{I_2}, \qquad m_2' = \frac{(x - x')e^{-2Q(x')}}{I_2}.$$

After having calculated $P_{\alpha'}^{\cdot\beta}$, $\Gamma^{\alpha}_{\beta\gamma}$ and $\gamma^{\alpha}_{\beta\gamma}$ for the relative gravitational field we get:

$$Q^i_{j0} = -Q^i_{j3}, \qquad Q^3_{ij} = Q^0_{ij}, \qquad Q^i_{12} = Q^1_{2i} = Q^2_{1i} = 0 \qquad (i, j = 0, 1, 2, 3)$$

$$Q^0_{00} = \frac{(\xi^1)^2 m_1}{2\sqrt{2}(x-x')^2 I_1}\{ m_1(1+m_1') + P'(x-x')(1+2m_1') - 2 \} + \frac{(\xi^2)^2 m_2}{2\sqrt{2}(x-x')^2 I_2}\{ m_2(1+m_2') + Q'(x-x')(1+2m_2') - 2 \},$$

$$Q^0_{10} = \frac{(\xi^1)(m_1 m_1' - 1)}{2(x-x')I_1}, \qquad Q^0_{20} = \frac{(\xi^2)(m_2 m_2' - 1)}{2(x-x')I_2}, \tag{15}$$

$$Q^1_{00} = \frac{(\xi^1)m_1}{(x-x')^2}\{ 1 - m_1 - P'(x-x') \},$$

$$Q^2_{00} = \frac{(\xi^2)m_2}{(x-x')^2}\{ 1 - m_2 - Q'(x-x') \},$$

$$Q^1_{10} = \frac{1-m_1}{\sqrt{2}(x-x')} - \frac{P'}{\sqrt{2}}, \qquad Q^2_{20} = \frac{1-m_2}{\sqrt{2}(x-x')} - \frac{Q'}{\sqrt{2}},$$

$$Q^0_{11} = \frac{m_1'-1}{\sqrt{2}I_1} - \frac{P'e^{2P}}{\sqrt{2}}, \qquad Q^0_{22} = \frac{m_2'-1}{\sqrt{2}I_2} - \frac{Q'e^{2Q}}{\sqrt{2}},$$

$$Q^1_{11} = Q^2_{22} = 0,$$

where

$$P' = \frac{dP}{dx}, \qquad Q' = \frac{dQ}{dx}.$$

From (3) and (15) we have:

$$L_g = 0. \tag{16}$$

Therefore form (2) and (15) for the energy-momentum tensor we have:

$$\Lambda\theta^0_{g\beta} = -\frac{g^{00}}{2x}\{ (g^{11})^2 Q_{11\beta}(3Q_{011} - Q_{110}) + (g^{22})^2 Q_{22\beta}(3Q_{022} - Q_{220}) + g^{11}g^{22}[Q_{11\beta}(Q_{022} - Q_{220}) + Q_{22\beta}(Q_{011} - Q_{110})] \}. \tag{17}$$

Thus, from the relation $\Lambda\theta^0_{g\beta'} = P_{\beta'}^{\cdot\nu}\Lambda\theta^0_{g\nu}$ and (10) we shall get

$$\Lambda\theta^0_{g0'} = \Lambda\theta^0_{g0} = -\Lambda\theta^0_{g3'}, \qquad \Lambda^0_{g1'} = \Lambda\theta^0_{g2'} = 0, \tag{18}$$

$$\Lambda\theta^0_{g0'} = \frac{1}{4x}\left\{ \left(\frac{m_1-1}{x-x'} + P'\right)\left(\frac{4m_1 + 2m_2 - 3m_1m_1' - m_2m_2' - 2}{x-x'} + (4P' + 2Q')\right) + \left(\frac{m_2-1}{x-x'} + Q'\right)\left(\frac{4m_2 + 2m_1 - 3m_2m_2' - m_1m_1' - 2}{x-x'} + (4Q' + 2P')\right) \right\}, \tag{19}$$

where

$$x - x' = \frac{1}{\sqrt{2}}(\xi^3 - \xi^0), \qquad \xi^i = x^i - x^{i'},$$

$$\mathrm{P} = \mathrm{P}(x) = \mathrm{P}\left[\frac{1}{\sqrt{2}}(x^3 - x^0)\right], \qquad \mathrm{P}' = \frac{d\mathrm{P}}{dx},$$

$$\mathrm{Q} = \mathrm{Q}(x) = \mathrm{Q}\left[\frac{1}{\sqrt{2}}(x^3 - x^0)\right], \qquad \mathrm{Q}' = \frac{d\mathrm{Q}}{dx},$$

$$m_1 = \frac{(x - x')e^{-2\mathrm{P}}}{\mathrm{I}_1}, \qquad \mathrm{I}_1 = \int_{x'}^{x} e^{-2\mathrm{P}}dx, \qquad m_1' = \frac{(x - x')e^{-2\mathrm{P}(x')}}{\mathrm{I}_1},$$

$$m_2 = \frac{(x - x')e^{-2\mathrm{Q}}}{\mathrm{I}_2}, \qquad \mathrm{I}_2 = \int_{x'}^{x} e^{-2\mathrm{Q}}dx, \qquad m_2' = \frac{(x - x')e^{-2\mathrm{Q}(x')}}{\mathrm{I}_2},$$

Now we shall calculate $\mathcal{T}_{\beta'}$. According to (4) and (18) for $\mathcal{T}_{\beta'}$ we have:

$$\mathcal{T}_{1'} = \mathcal{T}_{2'} = 0,$$

$$\mathcal{T}_{0'} = \mathcal{T}_{3'} = \int_{\Sigma} \Lambda\theta^0_{g0'}\sqrt{-g}\,d^3x = \sqrt{2}\int dx^1dx^2\int_{-\infty}^{+\infty} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx \tag{20}$$

$$\mathcal{T}_{0'} = \mathcal{T}_{3'} = \sqrt{2}\,\mathrm{S}\int_{-\infty}^{+\infty} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx,$$

where

$$\mathrm{S} = \int dx^1dx^2,$$

for the expression $\Lambda\theta^0_{g0'}\sqrt{-g}$ does not depend on x^1 and x^2 and:

$$x = \frac{1}{\sqrt{2}}(x^3 - x^0).$$

We shall examine the integral

$$\mathrm{A} = \int_{-\infty}^{+\infty} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx. \tag{21}$$

At first, we remark that if this integral is finite, then:

$$\int_{-\infty}^{-a} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx = 0$$

for the base point being in the region I (see fig. 1), and

$$\int_{a}^{\infty} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx = 0$$

for the base point being in the region III, because in these cases, from (7) and (19) we have:

$$\frac{m_1 - 1}{x - x'} + P' = \frac{m_2 - 1}{x - x'} + Q' = 0.$$

These results are physically obvious, for the regions I and III are flat. But

$$\int_a^\infty \Lambda \theta^0_{g0'} \sqrt{-g}\, dx \neq 0$$

for the base point being in the region I, and

$$\int_{-\infty}^{-a} \Lambda \theta^0_{g0'} \sqrt{-g}\, dx \neq 0$$

for the base point being in the region III.

Now we shall find the finiteness condition of the integral (21). This integral depends on the base point position $x' = d$. Generally, for any base point (for any d) this integral is infinite. This integral is finite only for the separate base points. We take notice of the fact that all reasonings quoted above are correct for all functions P and Q, satisfying the equations (6) and (8).

Let us have the equations of Σ_1 and Σ_2, $x = -a$ and $x = a$ (a is constant). Then the region II is determined by the inequality $-a < x < a$ and in this region we satisfy (8) requiring the satisfaction of equations

$$P'' + (P')^2 = -k^{-2}, \qquad Q'' + (Q')^2 = k^{-2}, \qquad k = \frac{4a}{\pi}. \tag{22}$$

As particular solution, we choose

$$P = \ln \cos \frac{x}{k}, \qquad Q = \frac{x}{k}. \tag{23}$$

Now it is necessary to determine the functions P and Q in the regions I and III in the form (7), requiring the continuity of these functions and their first derivatives on Σ_1 and Σ_2. We get

in the region I

$$e^P = \frac{1}{k\sqrt{2}}(x + a + k), \qquad e^Q = e^{-\frac{\pi}{4}} . k^{-1}(x + a + k), \tag{24}$$

in the region III

$$e^P = \frac{1}{k\sqrt{2}}(a + k - x), \qquad e^Q = e^{\frac{\pi}{4}} . k^{-1}(x - a + k). \tag{25}$$

So we see that there are formal singularities at the points

$$x = -a - k \quad \text{in the region I}$$

and

$$x = a + k \quad \text{in the region III.} \tag{26}$$

Let us consider the case in which $x' = d < -a$ (the base points is in the region I). Then for A we have:

$$A = \int_{-\infty}^{+\infty} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx = \int_{-a}^{\infty} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx$$
$$= C + \lim_{x\to\infty} \{ B_1 \ln (x + \alpha) + B_2 \ln (x + \beta) \},$$

where C, B_1, B_2 are constants depending on d and α, β are constants not depending on d. In our case (the functions P and Q have the form (23), (24) and (25)) form the finiteness condition of A $B_1 + B_2 = 0$ we shall get the equation for d

$$\left(\frac{d}{a}\right)^7 + 1{,}530\left(\frac{d}{a}\right)^6 - 0{,}500\left(\frac{d}{a}\right)^5 + 0{,}526\left(\frac{d}{a}\right)^4 - 0{,}793\left(\frac{d}{a}\right)^3$$
$$+ 0{,}294\left(\frac{d}{a}\right)^2 - 0{,}078\left(\frac{d}{a}\right) + 0{,}020 = 0.$$

Resolving this equation we get for its solution satisfying the condition $d < -a$:

$$d = -2{,}022\,a. \tag{27}$$

Analogously, in the case in which $x' = d > a$ (the base point is in the region III) form the finiteness condition of A we have the equation for d

$$-\left(\frac{d}{a}\right)^7 + 1{,}530\left(\frac{d}{a}\right)^6 + 0{,}500\left(\frac{d}{a}\right)^5 + 0{,}526\left(\frac{d}{a}\right)^4 + 0{,}793\left(\frac{d}{a}\right)^3$$
$$+ 0{,}294\left(\frac{d}{a}\right)^2 + 0{,}078\left(\frac{d}{a}\right) + 0{,}020 = 0,$$

and, from this,

$$d = 2{,}022\,a. \tag{28}$$

In the case in which $x' = d$, $|d| < a$ (the base point is in the region II) we have:

$$A = \int_{-\infty}^{+\infty} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx = \int_{-\infty}^{-a} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx + \int_{-a}^{+a} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx$$
$$+ \int_{a}^{\infty} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx = C + \lim_{x\to-\infty} \{ B_1 \ln (x + \alpha_1) + B_1' \ln (x + \beta_1) \}$$
$$+ \lim_{x\to+\infty} \{ B_3 \ln (x + \alpha_3) + B_3' \ln (x + \beta_3) \},$$

where C, B_1, B_1', B_3, B_3' are constants depending on d, and α_1, β_1, α_3, β_3 are constants not depending on d. From the finiteness condition of this integral

$$B_1 + B_1' + B_3 + B_3' = 0$$

we obtain the equation for d:

$$\left\{k\frac{1+\operatorname{tg}\frac{d}{k}}{1-\operatorname{tg}\frac{d}{k}} - a - d\right\}\left\{2 - \frac{3}{\cos^2\frac{d}{k}\left(1-\operatorname{tg}\frac{d}{k}\right)^2} - \frac{2\exp\left(-\frac{\pi}{2}-\frac{2d}{k}\right)}{\left[1+\exp\left(-\frac{\pi}{2}-\frac{2d}{k}\right)\right]^2}\right\}$$
$$+\left\{\frac{2k}{1+\exp\left(-\frac{\pi}{2}-\frac{2d}{k}\right)} - (a+k+d)\right\}$$
$$\left\{2 - \frac{3\exp\left(-\frac{\pi}{2}-\frac{2d}{k}\right)}{\left[1+\exp\left(-\frac{\pi}{2}-\frac{2d}{k}\right)\right]^2} - \frac{1}{\cos^2\frac{d}{k}\left(1-\operatorname{tg}\frac{d}{k}\right)^2}\right\}$$
$$+ e^{\frac{\pi}{2}}\left\{k\frac{\operatorname{tg}\frac{d}{k}-1}{1+\operatorname{tg}\frac{d}{k}} + a - d\right\}$$
$$\left\{2 - \frac{3}{\cos^2\frac{d}{k}\left(1+\operatorname{tg}\frac{d}{k}\right)^2} - \frac{2\exp\left(\frac{\pi}{2}-\frac{2d}{k}\right)}{\left[1+\exp\left(+\frac{\pi}{2}-\frac{2d}{k}\right)\right]^2}\right\}$$
$$+ e^{\frac{\pi}{2}}\left\{\frac{2k}{1+\exp\left(\frac{\pi}{2}-\frac{2d}{k}\right)} + a - k - d\right\}$$
$$\left\{2 - \frac{3\exp\left(\frac{\pi}{2}-\frac{2d}{k}\right)}{\left[1+\exp\left(\frac{\pi}{2}-\frac{2d}{k}\right)\right]^2} - \frac{1}{\cos^2\frac{d}{k}\left(1+\operatorname{tg}\frac{d}{k}\right)^2}\right\} = 0.$$

From this we get:

$$d = 0{,}122\,a. \tag{29}$$

(We note that the base points $d = 2{,}022\,a$ and $d = -2{,}022\,a$ are near the formal singularities of the metric (see (26))).

Now we shall calculate the energy density $E = \frac{\mathcal{T}_0'}{S}$ (the energy on unit area in the plane $x^1 x^2$) for the separate base points (27), (28), (29). From (19), (20), (23), (24) and (25) we get the following results:

1. In the case in which $d = -2{,}022\, a$ (the base point is in the region I)

$$\int_a^{\infty} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx = -\frac{19{,}928}{a.x},$$

$$\int_{-a}^{a} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx = +\frac{1{,}06}{ax},$$

$$\mathrm{E} = \sqrt{2}\int_{-\infty}^{+\infty} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx = -\frac{26{,}85}{a.x}. \qquad (30)$$

2. In the case in which $d = 0{,}122\, a$ (the base point is in the region II)

$$\int_{-\infty}^{-a} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx + \int_a^{\infty} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx = +\frac{15{,}661}{ax},$$

$$\int_{-a}^{a} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx = +\frac{0{,}06}{ax},$$

$$\mathrm{E} = +\frac{22{,}09}{a.x}. \qquad (31)$$

3. In the case in which $d = 2{,}022\, a$ (the base point is in the region III)

$$\int_{-\infty}^{-a} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx = +\frac{1{,}366}{a.x}, \qquad (32)$$

$$\int_{-a}^{a} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx = +\frac{0{,}32}{ax},$$

$$\mathrm{E} = +\frac{2{,}30}{ax}.$$

In all three cases of approximate calculation for the integral:

$$\int_{-a}^{a} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx,$$

we use Simson's formula (The expression under the integral sign contains the power, exponential and trigonometrical functions and does not become infinity in the integration limits) (see [*17*]).

We see that $\int_a^{+\infty} \Lambda\theta^0_{g0'}.\sqrt{-g}\,dx \neq 0$ for the base points being in the region I and $\int_{-\infty}^{-a} \Lambda\theta^0_{g0'}.\sqrt{-g}\,dx \neq 0$ for the base points being in the region III. It shows that the region I is « curved » in regard to the region III and vice versa—the region III is « curved » in regard to the region I (in spite of the flatness of both the regions I and III). If we consider two normal coordinate systems (see for example [3]) K_I and K_{III} in the regions I and III accordingly, then we see that the straight lines in K_I (their equations have the form $x_i = u_i s + x_{i0}$ where u_i, x_{i0} are constants and s is their length), generally, turn into the curves (their equations are non-linear) in the passage from K_I to K_{III}. It is interesting that for the base point being in the region I the relative energy is negative, but for the base points being in the regions I and III the relative energy is positive. It is the relativity of the gravitational field energy: the sign and the value of the gravitational field energy are relative, i. e. they depend on the base point X′.

III. — THE LOCAL ENERGY OF THE PLANE GRAVITATIONAL WAVES

Let us consider the relative local energy of the Synge's plane gravitational waves. We know that unlike the other fields (for example the electromagnetic field) the gravitational field is not localized. It signifies that the gravitational at the given point of the space-time can be zero or non-zero depending on in what coordinate system this field is examined. However the field description basing on the relative gravitational field conception allows to localize the gravitational field at all the points in regard to the base point in a sense. The valuable property of this localization is that it does not contradict the equivalence principle [1].

The local energy is determined in the following way.

$$\mathcal{E} = \int_\Sigma \Lambda\theta^0_{g0'}\sqrt{-g}\,d^3x, \qquad d^3x = dx^1\,dx^2\,dx^3,$$

where Σ is the hypersurface $x^0 =$ const. bounding the 3-volume of the examined region, and, $x^3 = z$.

In our case we determine

$$\mathcal{E} = \int_{z_1}^{z_2} \Lambda\theta^0_{g0'}\sqrt{-g}\,dz = \sqrt{2}\int_{\frac{(z_1-t)}{\sqrt{2}}}^{\frac{z_2-t}{\sqrt{2}}} \Lambda\theta^0_{g0'}\sqrt{-g}\,dx.$$

For convenience we shall calculate the local energies being inside the examined regions having the dimension equal to that of the region II (at some time instant t); then we put:

$$z_2 = z + t + a\sqrt{2}, \qquad z_1 = z + t - a\sqrt{2}.$$

From here

$$\mathcal{E} = \sqrt{2}\int_{\frac{z}{\sqrt{2}}-a}^{\frac{z}{\sqrt{2}}+a} \Lambda\theta^{0}_{g0'}\sqrt{-g}\,dx.$$

For the local energy of the region II E_{II} ($z = 0$) we shall get:

— when $x' = -\,2{,}022\,a$ (the base point is in the region I)

$$ax.\mathcal{E}_{II} = +\,1{,}50;$$

— when $x' = 0$ (the base point is in the region II)

$$ax.\mathcal{E}_{II} = +\,0{,}40;$$

— when $x' = 0{,}122\,a$ (the base point is in the region II)

$$ax.\mathcal{E}_{II} = +\,0{,}08;$$

— when $x' = 2{,}022\,a$ (the base point is in the region III)

$$ax.\mathcal{E}_{II} = +\,0{,}45.$$

(We see that $\mathcal{E}_{II}$ is positive and different from zero).

For the local energy of the examined regions being in the region III $\mathcal{E}_{III}$ (z is arbitrary and greater than $2a\sqrt{2}$) we have:

— when $x' = -\,10\,a$ (the base point is in the region I)

$$ax.\mathcal{E}_{III} =$$
$$-\frac{304{,}839a^2}{\left(\frac{z}{\sqrt{2}}+10a\right)^2-a^2}+\frac{1{,}172a^2}{\left(\frac{z}{\sqrt{2}}-1{,}179a\right)^2-a^2}+\frac{0{,}509a^2}{\left(\frac{z}{\sqrt{2}}+0{,}193a\right)^2-a^2}$$
$$-\frac{0{,}677\left(\frac{z}{\sqrt{2}}-1{,}179a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}}-1{,}179a\right)^2-a^2\right]^2}-22{,}192\ln\left|\frac{\frac{z}{\sqrt{2}}+11a}{\frac{z}{\sqrt{2}}+9a}\right|$$
$$-2{,}979\ln\left|\frac{\frac{z}{\sqrt{2}}-0{,}179a}{\frac{z}{\sqrt{2}}-2{,}179a}\right|-2{,}265\ln\left|\frac{\frac{z}{\sqrt{2}}+1{,}193a}{\frac{z}{\sqrt{2}}-0{,}807a}\right|;$$

— when $x' = -5a$ (the base point is in the region I)

$$ax.\mathcal{E}_{\text{III}} = -\frac{33{,}016a^2}{\left(\frac{z}{\sqrt{2}}+5a\right)^2-a^2} - \frac{3{,}109a^2}{\left(\frac{z}{\sqrt{2}}-1{,}40a\right)^2-a^2} - \frac{0{,}135a^2}{\left(\frac{z}{\sqrt{2}}+0{,}28a\right)^2-a^2}$$

$$-\frac{1{,}921\left(\frac{z}{\sqrt{2}}-1{,}40a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}}-1{,}40a\right)^2-a^2\right]^2} + \frac{0{,}001\left(\frac{z}{\sqrt{2}}+0{,}28a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}}+0{,}28a\right)^2-a^2\right]^2}$$

$$-8{,}624\ln\left|\frac{\frac{z}{\sqrt{2}}+6a}{\frac{z}{\sqrt{2}}+4a}\right| - 3{,}355\ln\left|\frac{\frac{z}{\sqrt{2}}-0{,}40a}{\frac{z}{\sqrt{2}}-2{,}40a}\right| - 1{,}860\ln\left|\frac{\frac{z}{\sqrt{2}}+1{,}28a}{\frac{z}{\sqrt{2}}-0{,}72a}\right|;$$

— when $x' = -3a$ (the base point is in the region I)

$$ax.\mathcal{E}_{\text{III}} = -\frac{57{,}518a^2}{\left(\frac{z}{\sqrt{2}}+3a\right)^2-a^2} + \frac{5{,}612a^2}{\left(\frac{z}{\sqrt{2}}-1{,}1811a\right)^2-a^2}$$

$$-\frac{0{,}568a^2}{\left(\frac{z}{\sqrt{2}}+0{,}298a\right)^2-a^2} - \frac{3{,}810\left(\frac{z}{\sqrt{2}}-1{,}811a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}}-1{,}811a\right)^2-a^2\right]^2}$$

$$+\frac{0{,}011\left(\frac{z}{\sqrt{2}}+0{,}298a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}}+0{,}298a\right)^2-a^2\right]^2} - 10{,}172\ln\left|\frac{\frac{z}{\sqrt{2}}+4a}{\frac{z}{\sqrt{2}}+2a}\right|$$

$$+2{,}287\ln\left|\frac{\frac{z}{\sqrt{2}}-0{,}811a}{\frac{z}{\sqrt{2}}-2{,}811a}\right| - 4{,}389\ln\left|\frac{\frac{z}{\sqrt{2}}+1{,}298a}{\frac{z}{\sqrt{2}}-0{,}702a}\right|;$$

— when $x' = -a-k = -2{,}273a$ (the base point is in the region I)

$$ax.\mathcal{E}_{\text{III}} = 10{,}896 - \frac{28{,}573a^2}{\left(\frac{z}{\sqrt{2}}+2{,}273a\right)^2-a^2} - \frac{0{,}906a^2}{\left(\frac{z}{\sqrt{2}}+0{,}273a\right)^2-a^2}$$

$$+\frac{106{,}768a^2}{\left(\frac{z}{\sqrt{2}}-2{,}273a\right)^2-a^2} + 23{,}350\ln\left|\frac{\frac{z}{\sqrt{2}}+3{,}273a}{\frac{z}{\sqrt{2}}+1{,}273a}\right|$$

$$+ 3{,}173 \ln \left| \frac{\frac{z}{\sqrt{2}} - 1{,}273a}{\frac{z}{\sqrt{2}} - 3{,}273a} \right| + 2{,}490 \ln \left| \frac{\frac{z}{\sqrt{2}} + 1{,}273a}{\frac{z}{\sqrt{2}} - 0{,}727a} \right| ;$$

— when $x' = -2a$ (the base point is in the region I)

$$ax.\mathcal{E}_{\mathrm{III}} = -\frac{17{,}916a^2}{\left(\frac{z}{\sqrt{2}} + 2a\right)^2 - a^2} + \frac{47{,}231a^2}{\left(\frac{z}{\sqrt{2}} - 2{,}621a\right)^2 - a^2}$$

$$- \frac{4{,}849a^2}{\left(\frac{z}{\sqrt{2}} + 0{,}260a\right)^2 - a^2} + \frac{15{,}294\left(\frac{z}{\sqrt{2}} - 2{,}621a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} - 2{,}621a\right)^2 - a^2\right]^2}$$

$$- \frac{0{,}065\left(\frac{z}{\sqrt{2}} + 0{,}260a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} + 0{,}260a\right)^2 - a^2\right]^2} - 8{,}015 \ln \left| \frac{\frac{z}{\sqrt{2}} + 3a}{\frac{z}{\sqrt{2}} + a} \right|$$

$$+ 12{,}580 \ln \left| \frac{\frac{z}{\sqrt{2}} - 1{,}621a}{\frac{z}{\sqrt{2}} - 3{,}621a} \right| - 3{,}127 \ln \left| \frac{\frac{z}{\sqrt{2}} + 1{,}260a}{\frac{z}{\sqrt{2}} - 0{,}740a} \right| ;$$

— when $x' = 0$ (the base point is in the region II)

$$ax.\mathcal{E}_{\mathrm{III}} = -\frac{1{,}339a^2}{\left(\frac{z}{\sqrt{2}}\right)^2 - a^2} + \frac{2{,}822a^2}{\left(\frac{z}{\sqrt{2}} + 0{,}273a\right)^2 - a^2} + \frac{2{,}406a^2}{\left(\frac{z}{\sqrt{2}} - 0{,}165a\right)^2 - a^2}$$

$$+ \frac{0{,}001\left(\frac{z}{\sqrt{2}} + 0{,}273a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} + 0{,}273a\right)^2 - a^2\right]^2} - \frac{0{,}173\left(\frac{z}{\sqrt{2}} - 0{,}165a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} - 0{,}165a\right)^2 - a^2\right]^2}$$

$$- 7{,}759 \ln \left| \frac{\frac{z}{\sqrt{2}} + a}{\frac{z}{\sqrt{2}} - a} \right| + 6{,}865 \ln \left| \frac{\frac{z}{\sqrt{2}} + 1{,}273a}{\frac{z}{\sqrt{2}} - 0{,}727a} \right| + 0{,}633 \ln \left| \frac{\frac{z}{\sqrt{2}} + 0{,}835a}{\frac{z}{\sqrt{2}} - 1{,}165a} \right| ;$$

— when $x' > a$ (the base point is in the region III)

$$\mathcal{E}_{\mathrm{III}} = 0.$$

Thus we saw that for the base points being in the iegion II and for the most of the base points being in the region I (for which $x' < -a - k$, where $x = -a - k$ is the formal singularity of the metric (see (26))), the local eneigies $\mathcal{E}_{\text{III}}$ are negative, and for those base points in the region I, for which $-a - k \leqslant x' < -a$, $\mathcal{E}_{\text{III}}$ at first are negative, then become positive when z increases. For the fixed base point the absolute values of $\mathcal{E}_{\text{III}}$ decrease when z increases, i. e. when the examined region goes away farther from the base point and the region II (except the examined regions being in that small part of the region III which applies to the region II). The value and even the sign of the local energy $\mathcal{E}_{\text{III}}$ of the fixed examined region depend also on the base point position.

For the local energy of the examined regions being in the region I $\mathcal{E}_{\text{I}}$ (z is arbitrary and smaller than $-2a\sqrt{2}$) we have:

— when $x' < -a$ (the base point is in the region I)

$$\mathcal{E}_{\text{I}} = 0;$$

— when $x' = 0$ (the base point is in the region II)

$$ax.\mathcal{E}_{\text{I}} = +\frac{2{,}934a^2}{\left(\frac{z}{\sqrt{2}}\right)^2 - a^2} + \frac{3{,}184a^2}{\left(\frac{z}{\sqrt{2}} - 0{,}273a\right)^2 - a^2} + \frac{2{,}867a^2}{\left(\frac{z}{\sqrt{2}} + 0{,}165a\right)^2 - a^2}$$

$$-\frac{0{,}149\left(\frac{z}{\sqrt{2}} - 0{,}273a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} - 0{,}273a\right)^2 - a^2\right]^2} + \frac{0{,}177\left(\frac{z}{\sqrt{2}} + 0{,}165a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} + 0{,}165a\right)^2 - a^2\right]^2}$$

$$+ 27{,}144 \ln \left|\frac{\frac{z}{\sqrt{2}} + a}{\frac{z}{\sqrt{2}} - a}\right| - 6{,}131 \ln \left|\frac{\frac{z}{\sqrt{2}} + 0{,}727a}{\frac{z}{\sqrt{2}} - 1{,}273a}\right|$$

$$- 21{,}064 \ln \left|\frac{\frac{z}{\sqrt{2}} + 1{,}165a}{\frac{z}{\sqrt{2}} - 0{,}835a}\right|;$$

— when $x' = 1{,}50a$ (the base point is in the region III)

$$ax.\mathcal{E}_{\text{I}} = +\frac{8{,}528a^2}{\left(\frac{z}{\sqrt{2}} - 1{,}50a\right)^2 - a^2} - \frac{61{,}249a^2}{\left(\frac{z}{\sqrt{2}} + 4{,}242a\right)^2 - a^2}$$

$$- \frac{3{,}982a^2}{\left(\frac{z}{\sqrt{2}} - 0{,}225a\right)^2 - a^2} + \frac{61{,}065\left(\frac{z}{\sqrt{2}} + 4{,}242a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} + 4{,}242a\right)^2 - a^2\right]^2}$$

$$+ \frac{0{,}288\left(\frac{z}{\sqrt{2}} - 0{,}225a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} - 0{,}225a\right)^2 - a^2\right]^2} - 2{,}396 \ln \left| \frac{\frac{z}{\sqrt{2}} - 0{,}5a}{\frac{z}{\sqrt{2}} - 2{,}5a} \right|$$

$$+ 9{,}980 \ln \left| \frac{\frac{z}{\sqrt{2}} + 5{,}242a}{\frac{z}{\sqrt{2}} + 3{,}242a} \right| + 2{,}169 \ln \left| \frac{\frac{z}{\sqrt{2}} + 0{,}775a}{\frac{z}{\sqrt{2}} - 1{,}225a} \right| ;$$

— when $x' = 2a$ (the base point is in the region III)

$$ax.\mathcal{E}_{\rm I} = + \frac{9{,}914a^2}{\left(\frac{z}{\sqrt{2}} - 2a\right)^2 - a^2} - \frac{1{,}578a^2}{\left(\frac{z}{\sqrt{2}} + 2{,}621a\right)^2 - a^2}$$

$$+ \frac{6{,}279a^2}{\left(\frac{z}{\sqrt{2}} - 0{,}260a\right)^2 - a^2} + \frac{3{,}424\left(\frac{z}{\sqrt{2}} + 2{,}621a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} + 2{,}621a\right)^2 - a^2\right]^2}$$

$$+ \frac{2{,}528\left(\frac{z}{\sqrt{2}} - 0{,}260a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} - 0{,}260a\right)^2 - a^2\right]^2} - 2{,}881 \ln \left| \frac{\frac{z}{\sqrt{2}} - a}{\frac{z}{\sqrt{2}} - 3a} \right|$$

$$+ 3{,}575 \ln \left| \frac{\frac{z}{\sqrt{2}} + 3{,}621a}{\frac{z}{\sqrt{2}} + 1{,}621a} \right| - 0{,}823 \ln \left| \frac{\frac{z}{\sqrt{2}} + 0{,}740a}{\frac{z}{\sqrt{2}} - 1{,}260a} \right| ;$$

— when $x' = a + k = 2{,}273a$ (the base point is in the region III)

$$ax.\mathcal{E}_{\rm I} = + \frac{3{,}912a^2}{\left(\frac{z}{\sqrt{2}} - 0{,}273a\right)^2 - a^2} + \frac{11{,}654a^2}{\left(\frac{z}{\sqrt{2}} - 2{,}273a\right)^2 - a^2}$$

$$+ \frac{0{,}237\left(\frac{z}{\sqrt{2}} - 0{,}273a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} - 0{,}273a\right)^2 - a^2\right]^2} + 0{,}961 \ln \left| \frac{\frac{z}{\sqrt{2}} + 3{,}273a}{\frac{z}{\sqrt{2}} + 1{,}273a} \right|$$

$$- 2{,}564 \ln \left| \frac{\frac{z}{\sqrt{2}} - 1{,}273a}{\frac{z}{\sqrt{2}} - 3{,}273a} \right| + 0{,}891 \ln \left| \frac{\frac{z}{\sqrt{2}} + 0{,}727a}{\frac{z}{\sqrt{2}} - 1{,}273a} \right| ;$$

— when $x' = 5a$ (the base point is in the region III)

$$ax.\mathcal{E}_1 = + \frac{76{,}650a^2}{\left(\frac{z}{\sqrt{2}} - 5a\right)^2 - a^2} + \frac{0{,}762a^2}{\left(\frac{z}{\sqrt{2}} + 1{,}40a\right)^2 - a^2}$$

$$+ \frac{1{,}647a^2}{\left(\frac{z}{\sqrt{2}} + 0{,}774a\right)^2 - a^2} + \frac{0{,}206\left(\frac{z}{\sqrt{2}} + 1{,}40a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} + 1{,}40a\right)^2 - a^2\right]^2}$$

$$+ \frac{0{,}017\left(\frac{z}{\sqrt{2}} + 0{,}774a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} + 0{,}774a\right)^2 - a^2\right]^2} - 1{,}775 \ln \left| \frac{\frac{z}{\sqrt{2}} - 4a}{\frac{z}{\sqrt{2}} - 6a} \right|$$

$$- 1{,}361 \ln \left| \frac{\frac{z}{\sqrt{2}} + 2{,}40a}{\frac{z}{\sqrt{2}} + 0{,}40a} \right| + 1{,}333 \ln \left| \frac{\frac{z}{\sqrt{2}} + 1{,}774a}{\frac{z}{\sqrt{2}} - 0{,}226a} \right| ;$$

— when $x' = 10a$ (the base point is in the region III)

$$ax.\mathcal{E}_1 = + \frac{78{,}782a^2}{\left(\frac{z}{\sqrt{2}} - 10a\right)^2 - a^2} + \frac{0{,}772a^2}{\left(\frac{z}{\sqrt{2}} + 1{,}179a\right)^2 - a^2}$$

$$+ \frac{1{,}019a^2}{\left(\frac{z}{\sqrt{2}} + 0{,}902a\right)^2 - a^2} + \frac{0{,}106\left(\frac{z}{\sqrt{2}} + 1{,}179a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} + 1{,}179a\right)^2 - a^2\right]^2}$$

$$+ \frac{0{,}009\left(\frac{z}{\sqrt{2}} + 0{,}902a\right)a^3}{\left[\left(\frac{z}{\sqrt{2}} + 0{,}902a\right)^2 - a^2\right]^2} - 28{,}302 \ln \left| \frac{\frac{z}{\sqrt{2}} - 9a}{\frac{z}{\sqrt{2}} - 11a} \right|$$

$$- 2{,}172 \ln \left| \frac{\frac{z}{\sqrt{2}} + 2{,}179a}{\frac{z}{\sqrt{2}} + 0{,}179a} \right| - 0{,}366 \ln \left| \frac{\frac{z}{\sqrt{2}} + 1{,}902a}{\frac{z}{\sqrt{2}} - 0{,}098a} \right| .$$

Thus we see that for the base points being in the region II and for the most of the base points being in the region I (for which $x' > a + k$, where $x = a + k$ is the formal singularity of the metric (see (26))), the local energies $\mathcal{E}_{\mathrm{I}}$ are positive, and for those base points in the region I, for which $a < x' \leqslant a + k$, $\mathcal{E}_{\mathrm{I}}$ at first are positive, and then become negative when z decreases. For the fixed base point the absolute values of $\mathcal{E}_{\mathrm{I}}$ decrease when z decreases, i. e. when the examined region goes away from the base point and the region II (except the examined regions in that small part of the region I which applies to the region II). The value and even the sign of the local energy $\mathcal{E}_{\mathrm{I}}$ of the fixed examined region depend also on the base point position.

On the figure 2 the dependence of $\mathcal{E}_{\mathrm{III}}$ on z (on the examined region position) is represented for the different base points: when $x' = -5a$ (the curve I), $x' = -2a$ (the curve 2) and $x' = 0$ (the curve 3).

On the figure 3 the dependence of $\mathcal{E}_{\mathrm{I}}$ on z is represented for the different base points: when $x' = 5a$ (the curve 1), $x' = 2a$ (the curve 2) and $x' = 0$ (the curve 3).

Thus we see, as in the case of energy, the value and the sign of Synge's plane gravitational waves depend on the position of the base point.

Pirani [*16*] calculated the average value of the canonical energy-momentum tensor density

$$t_{\nu}^{\mu} = \mathrm{L}\delta_{\nu}^{\mu} - g_{\rho\sigma,\mu}\frac{\partial \mathrm{L}}{\partial g_{\rho\sigma,\nu}},$$

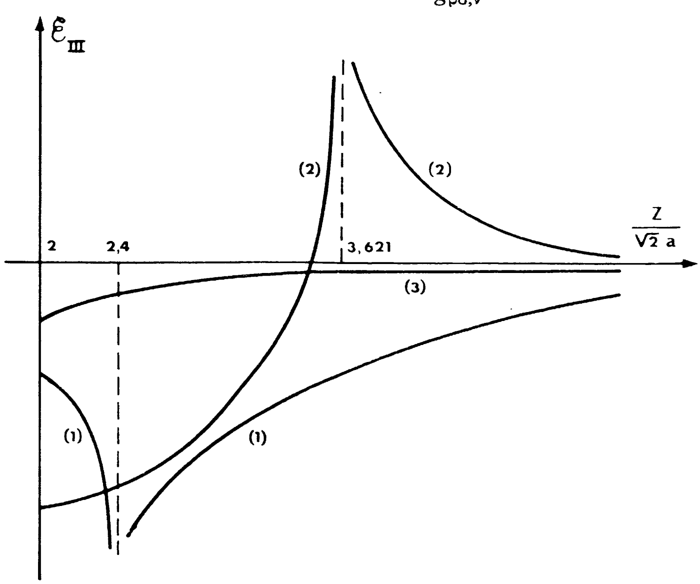

FIG. 2.

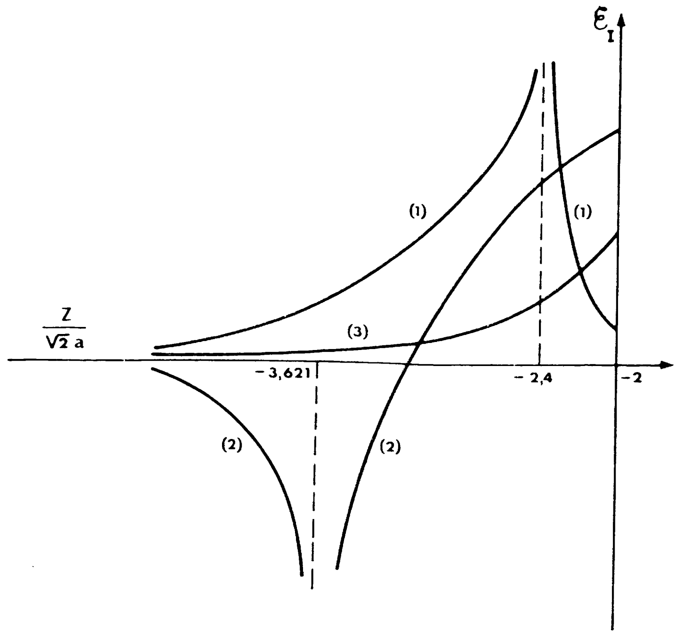

FIG. 3.

where

$$L = (-g)^{1/2} g^{\mu\nu} \left(\gamma^{\alpha}_{\mu\beta} \gamma^{\beta}_{\nu\alpha} - \gamma^{\alpha}_{\mu\nu} \gamma^{\beta}_{\alpha\beta} \right),$$

in the neighbourhood of the point of the normal coordinate system, which can always be choosen as that at the arbitrary space-time point O:

$$x^{\mu} = 0, \qquad g_{\mu\nu} = \eta_{\mu\nu},$$

$$\gamma^{\rho}_{\mu\nu} = g_{\mu\nu}, \qquad \rho = 0, \qquad g_{\mu\nu,\rho\sigma} = \frac{1}{3} (R_{\rho\mu\nu\sigma} + R_{\rho\nu\mu\sigma}).$$

Averaging it on the small 2-dimension sphere he obtained the invaıiant expression for the average

$$\bar{t}^{\mu}_{\nu} = \lim_{r \to 0} \left[(4\pi r^4)^{-1} \int t^{\mu}_{\nu} d^2 S \right].$$

This average, unfortunately, has not the dimension of energy density. It is a structure similar to the energy and characterizes the measure of energy which can be operated by the observer moving with the velocity U^{ν} satisfying the conditions for the normal coordinate system. For our plane waves this energy average is positive in the region II and is equal to zero in the regions I and III. In some sense these results coincide with some of our results. But our results are more clear, because we have investigated the local energy (but we have not investigated the quantity similar to energy). We proved that the value and even the sign of the local energy depend on the base point position. We also proved that the local energy of the region II is different from zero and positive independent of the base point position (but it is obvious that its value depends on the base point position), and, apparently, it is connected with the fact that the Riemann curvature tensor is different from zero in the region II.

IV. — CONCLUSION

All the obtained results show the relativity character of the gravitational field energy (total energy as well as local energy): the sign and the value of the gravitational field energy are relative, i. e. they depend on the base point position. The obtained results also show that: with the same basis the gravitational field energy can be considered a positive as well as a negative quantity.

I wish to express my genuine thanks to Professor Ya. P. Terletsky for his care and interest to the work and to Yu. A. Rylov for suggesting the problem and for many helpful discussions.

LITERATURE

[*1*] Yu. Rylov, *Ann. Physik* t. 7, 1964, p. 12.
[*2*] Ю. А. Рылов, « Изв. Вузов С.С.С.Р. ». Матем. сер. Вып. з, 131 (1962).
[*3*] J. L. Synge, *Relativity : The general theory*, Amsterdam, 1960.
[*4*] A. H. Taub, *Ann. d. Math.*, t. 53, 1951, p. 472.
[*5*] G. C. McVittie, *Journ. Rat. Mech. a. Anal.*, t. 4, 1955, p. 201.
[*6*] H. Bondi, *Nature*, t. 179, 1957, p. 1072.
[*7*] H. Bondi, F. A. E. Pirani et I. Robinson, *Proc. Roy. Soc.* (London), t. A 251, 1959, p. 519.
[*8*] K. Kuchar et J. Langer, *Czech. J. Phys.*, t. (B) 13, 1963, p. 233.
[*9*] M. Cahen et J. Sengier-Diels, *Bull. Cl. Sci. Acad. roy. Belg.*, t. 49, 1, 1963, p. 43.
[*10*] C. Møller, *Ann. Physik* (U. S. A.), t. 4, 1958, p. 347.
[*11*] C. Møller, *Kgl. Danske Vidensk. Selsk. Mat.-fys. Medd.*, t. 31, nº 14, 1959.
[*12*] C. Møller, *Ann. Physik* (U. S. A.), t. 12, 1961, p. 118.
[*13*] J. Langer, *Czech. J. Phys.* t. (B) 13, 1963, p. 789.
[*14*] H. Araki, *Ann. Physik* (U. S. A.), t. 7, 1959, p. 456.
[*15*] A. Lichnerowicz, *Théories relativistes de la Gravitation et de l'Électromagnétisme*, Paris, 1955.
[*16*] F. A. E. Pirani, *Phys. Rev.*, t. 105, 1957, p. 1089.
[*17*] И. С. Березин, Н. П. Жидков, Методы вычислений, москва (1959).

Manuscrit reçu le 10 décembre 1964.