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On the coupling of 3 angular momenta

by

A. CHAKRABARTI

SUMMARY. — In the problem of coupling of three angular momenta (j_1, j_2, j_3) , it is proposed to classify the states according to the eigenvalues of the operator $Z = (\vec{J}_1 \times \vec{J}_2) \cdot \vec{J}_3$, along, of course, with those of $\vec{J}_1^2, \vec{J}_2^2, \vec{J}_3^2, (\vec{J}_1 + \vec{J}_2 + \vec{J}_3)^2, (J_1^z + J_2^z + J_3^z)$. Z replaces the usual choice of one of the operators $(\vec{J}_a + \vec{J}_b)^2 (a, b = 1, 2, 3)$, connected through the Racah coefficients.

The great advantage of our method lies in the fact that the eigenstates thus obtained possess remarkable symmetry properties under particle permutations as is shown in the detailed discussion of Sec. 4. This fact practically eliminates the problem of constructing final statevectors of required symmetries in the 3-particle problem.

SOMMAIRE. — Les états obtenus par le produit tensoriel de trois états de moments cinétiques j_1, j_2, j_3 sont classés suivant les états propres des opérateurs $\vec{J}^2 = (\vec{J}_1 + \vec{J}_2 + \vec{J}_3)^2, \vec{J}_z$ et l'opérateur $Z = (\vec{J}_1 \times \vec{J}_2) \cdot \vec{J}_3$; ce dernier au lieu de l'utilisation habituelle de l'un des opérateurs $(\vec{J}_a + \vec{J}_b)^2 (a, b = 1, 2, 3)$ les différents choix possibles des valeurs pour a et b étant reliés entre eux par l'utilisation de coefficients de Racah.

Le grand avantage de notre méthode est de faire apparaître toutes les propriétés de symétrie pour les permutations (voir spécialement Section 4). Cela élimine pratiquement le problème de construire les états finaux possédant une symétrie d'un type donné pour les états à trois particules.

1. **Introduction.** — In the usual coupling scheme for three angular momenta [1] [2], in order to completely classify the states of definite total angular momentum the quantum numbers:

$$j_1 m_1; j_2 m_2; j_3 m_3$$

are replaced by the set

$$j_1, j_2, j_3; j, m$$

and any one of the following three:

$$j', j'', j'''$$

where

$$(1.1) \quad \begin{aligned} \vec{J} &= \vec{J}_1 + \vec{J}_2 + \vec{J}_3 \\ \vec{J}' &= \vec{J}_1 + \vec{J}_2, \quad \vec{J}'' = \vec{J}_2 + \vec{J}_3, \quad \vec{J}''' = \vec{J}_3 + \vec{J}_1. \end{aligned}$$

As is well known, this involves the use of 3- j and 6- j symbols, the three equivalent couplings (accordingly as j' , j'' or j''' is used) being expressible in terms of one another with the help of Racah coefficients.

It is however, also well-known that the sets of ortho-normal statevectors thus obtained do not behave in a simple way under permutations of the particles and the problem of construction of states of desired symmetry becomes more and more complicated for the higher j values.

To remedy this defect, we propose in this article a different classification of the states. Instead of taking one of the three operators (1.1), we take the operator (hermitian)

$$(1.2) \quad Z = (\vec{J}_1 \times \vec{J}_2) \cdot \vec{J}_3.$$

That is instead of using (apart from the squares of the \vec{J} 's) one of the three scalars

$$\vec{J}_1 \cdot \vec{J}_2, \quad \vec{J}_2 \cdot \vec{J}_3, \quad \vec{J}_3 \cdot \vec{J}_1$$

we propose to use the scalar whose classical analogue is the volume generated by the three angular momentum vectors. Thus our eigenstates will be labelled by the quantum numbers ($j_1, j_2, j_3, j, m, \zeta$), where the ζ 's are the eigenvalues of Z .

As will be shown in the following sections, the interesting symmetry properties of Z offer a remarkable advantage in the construction of states of definite symmetries (under permutations) in the 3-particle problem.

2. **Connection with the usual coupling.** — Before considering in detail the expansion of the proposed states in terms of products of individual particle states (see Sec. 3), we consider in this section certain general properties of Z and its eigenstates $|\zeta\rangle$ (We suppress for the moment the other quantum numbers for the sake of brevity).

a) We have,

$$(2.1) \quad Z = (\vec{J}_1 \times \vec{J}_2) \cdot \vec{J}_3$$

$$(2.2) \quad = \det \begin{vmatrix} J_1^x & J_1^y & J_1^z \\ J_2^x & J_2^y & J_2^z \\ J_3^x & J_3^y & J_3^z \end{vmatrix}$$

$$(2.3) \quad = \frac{i}{2} [(J_2^+ J_3^- - J_2^- J_3^+) J_1^z + (J_3^+ J_1^- - J_3^- J_1^+) J_2^z + (J_1^+ J_2^- - J_1^- J_2^+) J_3^z]$$

$$(2.4) \quad = \frac{i}{4} [\vec{J}'_2, \vec{J}''_2] = \frac{i}{4} [\vec{J}''_2, \vec{J}'''_2] = \frac{i}{4} [\vec{J}'''_2, \vec{J}'_2].$$

The different forms of Z given above permit us to directly deduce many important properties.

The discussion of the important symmetry properties will be taken up in Sec. 4. Here, we take up the relation with the standard coupling scheme.

b) Let us briefly denote (suppressing again the other quantum numbers) the eigenstates of J'^2, J''^2 by $|j'\rangle, |j''\rangle$ respectively. Then for a particular value of j' (say \underline{j}), we have, in a subspace (j, m) ,

$$(2.5) \quad |\underline{j}'\rangle = \sum_{j''} R_{j''j'} |j''\rangle$$

where the R 's are well-known and proportional to the Racah coefficients [1].

From (2.4) we have

$$(2.6) \quad \begin{aligned} Z |j'\rangle &= \frac{i}{4} [\vec{J}'_2 \vec{J}''_2 - \vec{J}''_2 \vec{J}'_2] |j'\rangle \\ &= \frac{i}{4} \sum_{j''} \{j'(j'+1) - j''(j''+1)\} \langle j'' | \vec{J}''_2 | j'\rangle |j''\rangle \\ &= \frac{i}{4} \sum_{j''} \left[\{j'(j'+1) - j''(j''+1)\} \left\{ \sum_{j'''} j''(j''+1) R_{j''j'''} R_{j''j'''} \right\} \right] |j''\rangle. \end{aligned}$$

The above equation shows that Z induces a linear transformation in the subspace spanned by the vectors j'' for a fixed value of (j, m) , such that the matrix of transformation, say M , is imaginary, antisymmetric and of course hermitian.

Such a matrix is always diagonalizable through a similarity transformation with a *unitary* matrix, say U .

Let

$$(2.7) \quad U \cdot M U^{-1} = \text{diag} (\zeta_1, \zeta_2, \dots, \zeta_p)$$

where M is a $p \times p$ matrix, supposing the number of ortho-normalized eigenstates of J^2 in the subspace (i, m) to be p . Then

$$(2.8) \quad \begin{pmatrix} |\zeta_1\rangle \\ \vdots \\ |\zeta_p\rangle \end{pmatrix} = U \cdot \begin{pmatrix} |j'_1\rangle \\ \vdots \\ |j'_p\rangle \end{pmatrix}$$

are the p orthonormalized eigenstates of Z in the same subspace.

As for the eigenvalues ζ_1, \dots, ζ_p of Z we note that they are the roots of the equation

$$(2.9) \quad \det | M - \zeta I | = 0.$$

Since from (2.6) M is antisymmetric, we have (tr. denoting the transposed matrix),

$$(2.10) \quad \begin{aligned} \det | M - \zeta I | &= \det \{ | M - \zeta I |^{\text{tr}} \} \\ &= \det | -M - \zeta I | = (-1)^p \det | M + \zeta I |. \end{aligned}$$

Thus the l. h. s. of (2.9) remains invariant under the substitution $\zeta \rightarrow -\zeta$ when p is even and changes sign when p is odd. Hence it contains only even powers of ζ when p is even and only odd powers of ζ when p is odd. Thus in both cases the eigenvalues ζ of Z occur in pairs $(\zeta_1 - \zeta)$, one pair coinciding to zero for odd p .

In order that the classification of the states with the help of the quantum numbers $(j_1, j_2, j_3, j, m, \zeta)$ be complete the eigenvalues ζ_1, \dots, ζ_p in (2.7) must all be distinct. This is established in App. C. Assuming that result we see that the zero eigenvalue of Z can occur only when p is odd and then only once in a subspace (j, m) .

3. The coefficients $\langle m_1 m_2 m_3 | j m \zeta \rangle$. — In this section we consider the properties of the coefficients connecting the sets of total angular momentum eigenstates and those formed by the direct product of the individual particle states.

Let,

$$(3.1) \quad \begin{pmatrix} j & m & \zeta \\ j_1 & j_2 & j_3 \end{pmatrix} = \sum_{m_1 + m_2 + m_3 = m} A \begin{pmatrix} j_1 & j_2 & j_3 & j & \zeta \\ m_1 & m_2 & m_3 & m \end{pmatrix} | j_1 m_1, j_2 m_2, j_3 m_3 \rangle.$$

In the above the subscript of the summation sign indicates that the summation is to be carried out only over values of m_1, m_2, m_3 satisfying $m_1 + m_2 + m_3 = m$.

We should have added, along with j_1, j_2, j_3 other quantum numbers, say $\alpha_1, \alpha_2, \alpha_3$, such as mass, charge, etc., of the individual particles which complete

the description. We will however suppress not only them, but as many of others as possible without creating confusion (e. g. j_1, j_2, j_3).

Applying in turn Z and \vec{J}^2 to both sides of (3.1) and taking scalar products with $|m_1, m_2, m_3\rangle$ we can derive, in the usual fashion the following recursion relations :

$$(3.2) \quad \begin{aligned} -2i\zeta A_{m_1, m_2, m_3} = & m_1 \lambda_{m_2-1}^+ \lambda_{m_3+1}^- A_{m_1, m_2-1, m_3+1} - m_1 \lambda_{m_2+1}^- \lambda_{m_3+1}^+ A_{m_1, m_2+1, m_3-1} \\ & + m_2 \lambda_{m_2-1}^+ \lambda_{m_1+1}^- A_{m_1+1, m_2, m_3-1} - m_2 \lambda_{m_2+1}^- \lambda_{m_1-1}^+ A_{m_1-1, m_2, m_3+1} \\ & + m_3 \lambda_{m_1-1}^+ \lambda_{m_2+1}^- A_{m_1-1, m_2+1, m_3} - m_3 \lambda_{m_1+1}^- \lambda_{m_2-1}^+ A_{m_1+1, m_2-1, m_3} \end{aligned}$$

$$(3.3) \quad \begin{aligned} [j(j+1) - j_1(j_1+1) - j_2(j_2+1) - j_3(j_3+1) - 2(m_1 m_2 + m_2 m_3 + m_3 m_1)] A_{m_1, m_2, m_3} \\ = \lambda_{m_2-1}^+ \lambda_{m_3+1}^- A_{m_1, m_2-1, m_3+1} + \lambda_{m_2+1}^- \lambda_{m_3-1}^+ A_{m_1, m_2+1, m_3-1} \\ + \lambda_{m_2-1}^+ \lambda_{m_1+1}^- A_{m_1+1, m_2, m_3-1} + \lambda_{m_2+1}^- \lambda_{m_1-1}^+ A_{m_1-1, m_2, m_3+1} \\ + \lambda_{m_1-1}^+ \lambda_{m_2+1}^- A_{m_1-1, m_2+1, m_3} + \lambda_{m_1+1}^- \lambda_{m_2-1}^+ A_{m_1+1, m_2-1, m_3} \end{aligned}$$

In the above

$$\lambda_{m_k}^\pm = [(j_k \mp m_k)(j_k \pm m_k + 1)]^{1/2} \quad (k = 1, 2, 3)$$

and we have condensed the notation further by suppressing j, m, ζ .

As will be shown in more detail in the Appendix (3.2) (3.3) are sufficient in principle for a step by step evaluation of the A 's, though except for the simplest cases, the calculations involved tend to become tedious.

Also it is to be noted that due to the relation

$$\lambda_m^\pm = \lambda_{-m}^\pm$$

the coefficient in (3.2) (3.3) remain unchanged if we make the substitutions,

$$A_{m_1, m_2, m_3} \rightarrow A_{-m_1, -m_2, -m_3}$$

$$A_{m_1, m_2-1, m_3+1} \rightarrow A_{-m_1, -(m_2-1), -(m_3+1)}, \text{ etc.}$$

Hence we need calculate the coefficient for non-negative values of m only. The remaining ones can then be written down directly.

From (3.2) it is evident that the A 's will in general be complex. Also it is to be noted that since ζ appears through $v = -i\zeta$ and all the other coefficients are real, we have for proper normalisation,

$$(3.4) \quad A^*(\zeta) = A(-\zeta)$$

(the other quantum numbers remaining unchanged). The A 's can be made real only for $\zeta = 0$. Thus in our scheme the operations of permutation and

complex conjugation are intimately related. The condition for the orthonormality of the states $|jm\zeta\rangle$ not only gives us

$$(3.5) \quad \sum_{m_1+m_2+m_3=m} A^* \begin{pmatrix} j' & m & \zeta' \\ m_1 & m_2 & m_3 \end{pmatrix} A \begin{pmatrix} j & m & \zeta \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{j'j} \delta_{\zeta'\zeta} \quad (m = m' \text{ always})$$

but also,

$$(3.6) \quad \sum_{m_1+m_2+m_3=m} A^2 \begin{pmatrix} j & m & \zeta \\ m_1 & m_2 & m_3 \end{pmatrix} = 0 \quad (\text{for } \zeta \neq 0)$$

the real and imaginary parts vanishing separately. The expansion (3.1) can be inverted in the usual fashion as

$$(3.7) \quad |m_1 m_2 m_3\rangle = \sum_{j,\zeta} A^* \begin{pmatrix} j & m & \zeta \\ m_1 & m_2 & m_3 \end{pmatrix} |j m \zeta\rangle \quad (m = m_1 + m_2 + m_3)$$

$$(3.8) \quad = \sum_{j,\zeta} A \begin{pmatrix} j & m & -\zeta \\ m_1 & m_2 & m_3 \end{pmatrix} |j m \zeta\rangle.$$

Equation (3.7) supplies the condition for orthonormality of the states $|m_1, m_2, m_3\rangle$ as

$$(3.8) \quad \sum_{j,\zeta} A^* \begin{pmatrix} j & m & \zeta \\ m_1 & m_2 & m_3 \end{pmatrix} A \begin{pmatrix} j & m & \zeta \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} = \delta_{m_1\mu_1} \delta_{m_2\mu_2} \delta_{m_3\mu_3}.$$

We now discuss the relations between the A's and the D-matrices providing the irreducible representations of the rotation group.

We have from (3.7) (applying a rotation transformation to both sides)

$$(3.9) \quad \sum_{\mu'_1, \mu'_2, \mu'_3} D_{\mu'_1 m_1}^{j_1} D_{\mu'_2 m_2}^{j_2} D_{\mu'_3 m_3}^{j_3} |\mu'_1, \mu'_2, \mu'_3\rangle = \sum_{j,\zeta} A^* \begin{pmatrix} j & m & \zeta \\ m_1 & m_2 & m_3 \end{pmatrix} D_{\mu' m}^j |j \mu' \zeta\rangle.$$

Taking the scalar product with $|\mu_1, \mu_2, \mu_3\rangle$, we have

$$(3.10) \quad D_{\mu_1 m_1}^{j_1} D_{\mu_2 m_2}^{j_2} D_{\mu_3 m_3}^{j_3} = \sum_{j,\zeta} A^* \begin{pmatrix} j & m & \zeta \\ m_1 & m_2 & m_3 \end{pmatrix} A \begin{pmatrix} j & \mu & \zeta \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} D_{\mu m}^j \quad (\mu = \mu_1 + \mu_2 + \mu_3, m = m_1 + m_2 + m_3).$$

Similarly from (3.1) again,

$$(3.11) \quad D_{\mu m}^j = \sum_{\substack{m_1+m_2+m_3=m \\ \mu_1+\mu_2+\mu_3=\mu}} A \begin{pmatrix} j & m & \zeta \\ m_1 & m_2 & m_3 \end{pmatrix} A^* \begin{pmatrix} j & \mu & \zeta \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} \times D_{\mu_1 m_1}^{j_1} D_{\mu_2 m_2}^{j_2} D_{\mu_3 m_3}^{j_3}.$$

From (3.10), we have

$$(3.12) \quad \int d\Omega D_{\mu m}^{j*} D_{\mu_1 m_1}^{j_1} D_{\mu_2 m_2}^{j_2} D_{\mu_3 m_3}^{j_3} = \frac{8\pi^2}{8j+1} \left[\sum_{\zeta} A^* \left(\begin{matrix} j & m & \zeta \\ m_1 & m_2 & m_3 \end{matrix} \right) A \left(\begin{matrix} j & m & \zeta \\ \mu_1 & \mu_2 & \mu_3 \end{matrix} \right) \right]$$

since

$$\int d\Omega D_{\mu' m'}^{j'*} D_{\mu m}^j = \frac{8\pi^2}{2j+1} \delta_{\mu' \mu} \delta_{m' m} \delta_{j' j}.$$

As a special case of (3.12) we obtain the integral of four spherical harmonics.

4. Symmetry properties. — In this section we discuss the behaviour of the eigenstates of Z under particle permutations. Let (ab) denote the operator transposing the particles a and b .

We have (most directly from (2.2))

$$(4.1) \quad [(ab)Z + Z(ab)] = 0.$$

Since \vec{J}^2, J^z commute with (ab) , we have, for example

$$(4.2) \quad (12) \left\langle \begin{matrix} j_1 j_2 j_3 \\ j m \zeta \end{matrix} \right\rangle = \eta_{12} \left\langle \begin{matrix} j_2 j_1 j_3 \\ j m - \zeta \end{matrix} \right\rangle$$

where η_{12} is some constant factor.

Supposing all the eigenstates of ζ to be normalized to unity, the unitarity of the permutation operators (ab) implies that the η_{ab} 's have all modulus unity (Additional quantum numbers necessary to complete the description of the states, say $\alpha_1, \alpha_2, \alpha_3$, are always supposed to be permuted along with j_1, j_2, j_3 respectively. They are not written explicitly).

CASE I. — $j_1 = j_2 = j_3$.

Since the quantum numbers j_1, j_2, j_3 are permuted among the particles under (ab) if we want to construct states of definite symmetry which are eigenstates, not only of \vec{J}^2, J^z , but also of $\vec{J}_1^2, \vec{J}_2^2, \vec{J}_3^2$, we have to confine ourselves to the cases where

$$j_1 = j_2 = j_3 = k, \quad \text{say.}$$

The particles in this case may said to be *equivalent*. For such cases suppressing all the quantum numbers which remain unchanged under permutations, we can write

$$(4.3) \quad (ab) | \zeta \rangle = \eta_{ab} | - \zeta \rangle$$

where the factor η_{ab} must be a function of the $A\left(\begin{smallmatrix} j & m & \zeta \\ m_1 m_2 m_3 \end{smallmatrix}\right)$'s appearing on the r. h. s. of (3.1). Since exactly the same function of the $A\left(\begin{smallmatrix} j & m & -\zeta \\ m_1 m_2 m_3 \end{smallmatrix}\right)$'s is, from (3.4), just η_{ab}^* , we have

$$(4.4) \quad (ab) | -\zeta \rangle = \eta_{ab}^* | \zeta \rangle$$

This is in accordance with the fact that,

$$(4.5) \quad | \zeta \rangle = (ab)^2 | \zeta \rangle = \eta_{ab} \eta_{ab}^* | \zeta \rangle$$

$$(4.6) \quad \text{Thus, } \eta_{ab} = e^{i\theta_{ab}}$$

such that θ_{ab} (real) changes sign with ζ (the other quantum numbers being kept unchanged).

In particular for

$$(4.7) \quad \zeta = 0, \quad e^{i\theta_{ab}} = \pm 1.$$

Thus the states corresponding to a zero eigenvalue of Z are always completely symmetric or antisymmetric (Two simple but non-trivial examples are given by (A.16) and (A.17) which show that for $j_1 = j_2 = j_3 = 1$, $|100\rangle$ is symmetric and $|000\rangle$ is antisymmetric). In such cases no further problems arise.

Confining our attention now only to non-zero eigenvalues, we can write

$$(4.8) \quad (ab) | \zeta \rangle = e^{i\theta_{ab}} | -\zeta \rangle$$

where θ_{ab} is real, non-zero and a function of the quantum numbers labelling the states.

In App. B, we discuss in detail the simplest case, namely

$$j_1 = j_2 = j_3 = \frac{1}{2}.$$

It is shown that the θ_{ab} 's corresponding to the three transpositions are

$$(4.9) \quad 0, \pm \frac{2\pi}{3}$$

and as a consequence the states corresponding to $\zeta = \pm \frac{\sqrt{3}}{4}$ are the basic vectors of a two-dimensional representation of S_3 . The eventual construction of a completely antisymmetric state is also discussed.

For an arbitrary integral or half integral value k ($= j_1 = j_2 = j_3$) we may

verify easily from (A. 7) and (A. 10), that exactly the same θ 's arise for non zero values in both cases, namely, for

$$(4.10) \quad \begin{aligned} j &= \alpha - 1, \\ j &= \alpha - 2, \end{aligned}$$

where now $\alpha = 3k$ (this behaviour is, of course, independent of m).

Thus in each case the states ($|\zeta\rangle, |-\zeta\rangle$) form a basis for a two-dimensional representation of S_3 .

Starting from $j = \alpha - 3$, we have, for each value of $j(\alpha - 3, \alpha - 4, \text{etc.})$ more than one pair ($\pm \zeta$) of non-zero values of ζ . From this state a new phenomenon arises. Let us first illustrate it by the simplest example.

Let us consider the case

$$(4.11) \quad j = 3k - 3 = m.$$

The corresponding ζ values are given in (A. 12). Writing down the states explicitly we find that for

$$(4.12) \quad \begin{aligned} (i) \quad \zeta &= + \sqrt{3k(k-1)} \\ |\zeta\rangle &= N^{-1} \left[\left(\frac{k}{k-1} \right)^{1/2} (\omega |300\rangle + \omega^* |030\rangle + |003\rangle) \right. \\ &\quad \left. + i \{ \omega(|012\rangle - |021\rangle) + \omega^*(|201\rangle - |102\rangle) + (|120\rangle - |210\rangle) \} \right] \end{aligned}$$

where:

$$\omega = e^{i2\pi/3} \quad \text{and} \quad N^2 = \frac{3(3k-2)}{(k-1)}.$$

The vector $|-\zeta\rangle$ is obtained as usual by replacing the coefficients by their complex conjugates.

For this case we have exactly as in the examples previously mentioned

$$(4.13) \quad (12) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad (23) = \begin{vmatrix} 0 & \omega^* \\ \omega & 0 \end{vmatrix}, \quad (31) = \begin{vmatrix} 0 & \omega \\ \omega^* & 0 \end{vmatrix}$$

as the permutation matrices acting on the two dimensional space provided by ($|\zeta\rangle, |-\zeta\rangle$).

But let us now consider the other case, namely,

$$(ii) \quad \zeta = + \{ 3k(k-1)(9k^2 - 9k + 2) \}^{1/2}$$

we have

$$(4.14) \quad \begin{aligned} |\zeta\rangle &= N^{-1} \left[\frac{\sqrt{2k}}{\sqrt{6k(k-1)+1}} \{ |300\rangle + |030\rangle + |003\rangle \} \right. \\ &\quad - \{ e^{-i\theta/2} (|210\rangle + |021\rangle + |102\rangle) + e^{i\theta/2} (|120\rangle + |201\rangle + |012\rangle) \} \\ &\quad \left. + \sqrt{\frac{(2k-1)}{k}} (e^{i\theta/2} + e^{-i\theta/2}) |111\rangle \right] \end{aligned}$$

where N is the normalizing constant and

$$(4.15) \quad e^{i\theta} = \frac{3k(k-1) - i\zeta}{3k(k-1) + i\bar{\zeta}}.$$

In this case we find, considering the space $(|\zeta\rangle, |-\zeta\rangle)$, that

$$(4.16) \quad (12) = (23) = (31) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

Such a degenerate 2-dimensional representation of S_3 can be reduced to two 1-dimensional ones one sym. and one antisym. (see (4.30) (4.31)). The unitary matrix of transformation is

$$(4.16') \quad \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}.$$

Now, having illustrated by a simple example how the two possibilities (4.13) (4.16) are realized in practice, we proceed to show that no other essentially different behaviour is possible under particle permutations for arbitrary k .

This fact is indeed assured by the results in the theory of projective representations of finite groups (see Hamermesh [4]). Since inequivalent projective representations can arise from S_4 onward only, for S_3 we can have either the standard 2-dimensional case (4.13) or something which is reducible, i. e. essentially (4.16). Hence after the initial normalization (3.4), no further ambiguity due to phase factors is possible. Nevertheless, we prefer to establish these facts explicitly in what follows, not only to show exactly how the alternative (4.16) arises for $\zeta \neq 0$, but also, incidentally, to derive explicitly all the restrictions implied on the A -coefficients.

We first note that in order to study the behaviour of under particle permutations, it is sufficient to consider one set of direct product states which are permuted among themselves. If a, b, c are all distinct $|a, b, c\rangle$ will be included in a set of six direct product states, closed under particle permutation. Let

$$(4.17) \quad \begin{aligned} |\zeta\rangle = & \dots + [A_{abc}|a, b, c\rangle + A_{cab}|c, a, b\rangle + A_{bca}|b, c, a\rangle \\ & + A_{bac}|b, a, c\rangle + A_{acb}|a, c, b\rangle + A_{cba}|c, b, a\rangle \\ & + \dots \end{aligned}$$

where we may ignore the other terms for our present objective. Since

the corresponding coefficients of $|- \zeta \rangle$ are just A_{abc}^* , etc., we must have from (4.3) (4.6)

$$(4.18) \quad \left\{ \begin{aligned} e^{i\theta_{12}} &= \frac{A_{bac}^*}{A_{abc}} = \frac{A_{acb}^*}{A_{cab}} = \frac{A_{cba}^*}{A_{bca}} \\ e^{i\theta_{23}} &= \frac{A_{acb}^*}{A_{abc}} = \frac{A_{cba}^*}{A_{cab}} = \frac{A_{bac}^*}{A_{bca}} \\ e^{i\theta_{31}} &= \frac{A_{cba}^*}{A_{abc}} = \frac{A_{bac}^*}{A_{cab}} = \frac{A_{acb}^*}{A_{bca}} \end{aligned} \right.$$

These equations give immediately

$$(4.19) \quad \left\{ \begin{aligned} e^{i(\theta_{12}-\theta_{23})} &= e^{i(\theta_{23}-\theta_{31})} = e^{i(\theta_{31}-\theta_{12})} \\ e^{i3\theta_{12}} &= e^{i3\theta_{23}} = e^{i3\theta_{31}} \end{aligned} \right.$$

Evidently all the A's belonging to such a set must have the same modulus, say R.

Also we note that by a proper choice of phase we can always reduce one of the $e^{i\theta_{ab}}$'s (say $e^{i\theta_{12}}$) to unity. For if

$$(4.20) \quad \frac{A_{bac}^*}{A_{abc}} = e^{i\varphi} \neq 1.$$

Then multiplying $|\zeta \rangle$ by the phase factor $e^{i\varphi/2}$ we at once obtain the desired result. Thus we can, without real loss of generality, put

$$(4.21) \quad e^{i\theta_{12}} = 1.$$

Then from (4.19) we have,

$$(4.22) \quad e^{i\theta_{23}} = e^{-i\theta_{31}} = 1, \quad \omega, \quad \omega^*$$

where 1, ω , ω^* , are the three cube roots of unity. Thus either,

$$(4.23) \quad e^{i\theta_{12}} = e^{i\theta_{23}} = e^{i\theta_{31}} = 1$$

or

$$(4.24) \quad e^{i\theta_{12}} = 1, \quad e^{i\theta_{23}} = \omega^* = e^{-i\theta_{31}}.$$

Replacing ω^* by ω in (4.24) simply amounts to reversing the two components of $(|\zeta \rangle, |- \zeta \rangle)$.

(It may seem that since from (4.18)

$$e^{i\theta_{12}} = \frac{(A_{bac} + A_{acb} + A_{cba})^*}{(A_{abc} + A_{cab} + A_{bca})} = e^{i\theta_{23}} = e^{i\theta_{31}}.$$

(4.23) is the only possible case. However, the alternative (4.24) is possible since in that case the above ratio is not defined, being of the form $\frac{0}{0}$).

Thus we have established that (4.13) and (4.16) are the only two possibilities.

$$a) \text{ For } e^{i\theta_{12}} = e^{i\theta_{23}} = e^{i\theta_{31}} = 1$$

we can rewrite (4.17) as

$$(4.25) \quad |\zeta\rangle = \dots + [A_{abc}(|a, b, c\rangle + |c, a, b\rangle + |b, c, a\rangle) + A_{abc}^*(|b, a, c\rangle + |a, c, b\rangle + |c, b, a\rangle)] + \dots$$

$$b) \text{ For } e^{i\theta_{12}} = 1, \quad e^{i\theta_{23}} = \omega^*, \quad e^{i\theta_{31}} = \omega$$

we can write,

$$(4.26) \quad |\zeta\rangle = \dots + [A_{abc}(|a, b, c\rangle + \omega|c, a, b\rangle + \omega^*|b, c, a\rangle) + A_{abc}^*(|b, a, c\rangle + \omega^*|a, c, b\rangle + \omega|c, b, a\rangle)] + \dots$$

When two of the subscripts a, b, c are equal, we have a set of only three direct product states closed under permutations and we can write (putting $a = b$, say)

$$(4.27) \quad a) \quad |\zeta\rangle = \dots + [A_{aac}(|aac\rangle + |caa\rangle + |aca\rangle)] + \dots$$

for (4.23) and

$$(4.28) \quad b) \quad |\zeta\rangle = \dots + [A_{aac}(|a, a, c\rangle + \omega|c, a, a\rangle + \omega^*|a, c, a\rangle)] + \dots$$

for (4.24).

It is easy to see that in these two cases A_{aac} must be real for the choice of phase factor implied by (4.21).

For $a = b = c$, we have only one term to consider, namely (for both (4.23) and (4.24))

$$(4.29) \quad A_{aaa} |a, a, a\rangle$$

where again A_{aaa} must be real. In fact it will be seen that such a symmetric term cannot occur in the case (4.24), but only in the case (4.23).

All these general features discussed are illustrated by the two typical examples (4.12), (4.14). Thus for each set closed under permutations we have effectively to calculate only one coefficient.

Naturally, the symmetry considerations by themselves cannot give any information about the ratio of two coefficients A_{abc} , $A_{a'b'c'}$ which belong to different sets, not related through permutations. To determine these ratios we have to use the recursion relations (3.2), (3.3), which also finally provide, separately for each particular case, the relevant values of ζ and information for classification according to (4.23) or (4.24) as the case may be.

Let us now consider the problem of the construction of total eigenstates belonging to different symmetry classes.

We will call our eigenstates $|\zeta\rangle$ to be of type I or type II ($|\zeta\rangle_{I,II}$) accordingly as the corresponding permutation matrices are given by (4.13) or (4.16) respectively.

It has been seen that for $j = \alpha, \alpha - 1, \alpha - 2$ we have only type I states. Incidentally, these cases include completely the usual spin and isopin functions ($k = 0, 1/2, 1$).

The classification of the product of two such states is given explicitly in App. B ((B.5), (B.6)).

The type II states arise from $k = 2$ onwards (for $k = 1, j = 3 - 3 = 0$ is completely antisymmetric) and hence can occur, unless we consider spin or isospin greater than 1, for the orbital functions only.

We note, that a given pair of vectors $|\zeta\rangle_{II}, |-\zeta\rangle_{II}$ can be directly combined into the completely symmetric and antisymmetric states (mutually orthogonal)

$$(4.30) \quad |s\rangle_{II} = \frac{1}{\sqrt{2}}(|\zeta\rangle_{II} + |-\zeta\rangle_{II})$$

$$(4.31) \quad |a\rangle_{II} = \frac{1}{\sqrt{2}}(|\zeta\rangle_{II} - |-\zeta\rangle_{II}).$$

The operator (Z/ζ) interchanges $|s\rangle_{II}$ and $|a\rangle_{II}$ just as the permutations interchange $|\zeta\rangle_{II}$ and $|-\zeta\rangle_{II}$. The problem of classifying the product of two states (or more) of type II, is thus trivial. If we consider the product of a state of type I with type II, we note that the pairs

$$(4.32) \quad \left| \begin{array}{l} |\zeta\rangle_I |s\rangle_{II} \\ |-\zeta\rangle_I |s\rangle_{II} \end{array} \right|, \quad \left| \begin{array}{l} |\zeta\rangle_I |a\rangle_{II} \\ |-\zeta\rangle_I |a\rangle_{II} \end{array} \right|$$

both transform as type I, namely (4.13) is applicable to both. We may, of course, replace (4.32) by the pairs $(|\zeta\rangle_I |\zeta'\rangle_{II}, |-\zeta\rangle_I |-\zeta'\rangle_{II})$ and $(|\zeta\rangle_I |-\zeta'\rangle_{II}, |-\zeta\rangle_I |\zeta'\rangle_{II})$ if so desired.

In considering the product of orbital, spin and isospin functions we get either a product of 3 states of type I or a product of two states of type I will one of type II (for spin and isospin $\frac{1}{2}$ and 1).

Of the three possibilities, for a totally antisymmetric state, listed in (B.7) the first two can arise (for non-zero values of ζ) when $|\pm\zeta\rangle_{or}$ is of type II and the third one when $|\pm\zeta\rangle_{or}$ is of type I. Thus the possible totally antisymmetric states are (using an obvious notation)

$$(4.33) \quad |\psi^s\rangle_{II} |\xi^a\rangle_I, \quad |\psi^a\rangle_{II} |\xi^s\rangle_I, \quad \frac{1}{\sqrt{2}} \{ |\psi'\rangle_I |\zeta''\rangle_I - |\psi''\rangle_I |\zeta'\rangle_I \}.$$

Similarly, the possible totally symmetric states are

$$(4.34) \quad |\psi^s\rangle_{\text{II}} |\xi^s\rangle_{\text{I}}, \quad |\psi^a\rangle_{\text{II}} |\xi^a\rangle_{\text{I}}, \quad \frac{1}{\sqrt{2}} \{ |\psi'\rangle |\zeta''\rangle_{\text{I}} + |\psi''\rangle |\zeta'\rangle_{\text{I}} \}.$$

The prescription for dealing with states belonging to $\zeta = 0$ is simple and obvious.

CASE 2. — j_1, j_2, j_3 not all equal (non equivalent particles).

In this case we have to go back to equation (4.2) and the corresponding symmetry restrictions are naturally much weaker than in the previous case.

The equations analogous to (4.18) are given by (using the somewhat fuller notation)

$$(4.35) \quad \begin{aligned} \left. \begin{matrix} j_1 & j_2 & j_3 \\ i & j & -r\zeta \end{matrix} \right\rangle &= \sum_{a+b+c=r} \mathbf{A} \left(\begin{matrix} j_1 & j_2 & j_3 \\ j_1-a & j_2-b & j_3-c \end{matrix} \middle| \begin{matrix} j_1 & j_2 & j_3 \\ j_1-a & j_2-b & j_3-c \end{matrix} \right\rangle \\ e^{i\theta_{11}} &= \frac{\mathbf{A}^* \left(\begin{matrix} j_2 & j_1 & j_3 \\ j_2-b & j_1-a & j_3-c \end{matrix} \right)}{\mathbf{A} \left(\begin{matrix} j_1 & j_2 & j_3 \\ j_1-a & j_2-b & j_3-c \end{matrix} \right)} = \frac{\mathbf{A}^* \left(\begin{matrix} j_2 & j_1 & j_3 \\ j_2-a & j_1-c & j_3-b \end{matrix} \right)}{\mathbf{A} \left(\begin{matrix} j_1 & j_2 & j_3 \\ j_1-c & j_2-a & j_3-b \end{matrix} \right)} = \frac{\mathbf{A}^* \left(\begin{matrix} j_2 & j_1 & j_3 \\ j_2-c & j_1-b & j_3-a \end{matrix} \right)}{\mathbf{A} \left(\begin{matrix} j_1 & j_2 & j_3 \\ j_1-b & j_2-c & j_3-a \end{matrix} \right)} \\ &= \frac{\mathbf{A}^* \left(\begin{matrix} j_2 & j_1 & j_3 \\ j_2-a & j_1-b & j_3-c \end{matrix} \right)}{\mathbf{A} \left(\begin{matrix} j_1 & j_2 & j_3 \\ j_1-b & j_2-a & j_3-c \end{matrix} \right)} = \frac{\mathbf{A}^* \left(\begin{matrix} j_2 & j_1 & j_3 \\ j_2-c & j_1-a & j_3-b \end{matrix} \right)}{\mathbf{A} \left(\begin{matrix} j_1 & j_2 & j_3 \\ j_1-a & j_2-c & j_3-b \end{matrix} \right)} = \frac{\mathbf{A}^* \left(\begin{matrix} j_2 & j_1 & j_3 \\ j_2-b & j_1-c & j_3-a \end{matrix} \right)}{\mathbf{A} \left(\begin{matrix} j_1 & j_2 & j_3 \\ j_1-c & j_2-b & j_3-a \end{matrix} \right)} \end{aligned}$$

with two similar sets for $e^{i\theta_{22}}$, $e^{i\theta_{33}}$ obtained by cyclic displacements of the j_i 's and m_i 's in the numerators.

(It should be remarked that for any particular set of values of the parameters, all the above ratios may not be defined, unless

$$0 \leq a, b, c \leq 2j_1, 2j_2, 2j_3.$$

Otherwise, for some of the ratios both the denominator and the numerator would vanish simultaneously. Such ratios are to be rejected).

It is not possible in this case, as in the previous one, to determine all the θ_{ab} 's explicitly.

But we may note that adjusting the phases properly, we can always put (as in 4.21)

$$(4.36) \quad e^{i\theta_{11}} = 1.$$

It is not difficult to verify that the above relations among the A's do indeed assure that the $e^{i\theta_{ab}}$'s must have values compatible (without ambiguity of phase factors) with the operator relations of S_3 , namely

$$\begin{aligned} a) & \quad (12)^2 = (23)^2 = (31)^2 = I \\ b) & \quad (31) = (12)(23)(12), \text{ etc.} \end{aligned}$$

These conditions impose the following restrictions on the phase factors:

$$(4.37) \quad a) \quad \begin{cases} \text{For } j_a \rightleftharpoons j_b, & e^{i\theta_{ab}} \rightarrow e^{i\theta_{ab}} \\ \text{and For } \zeta \rightleftharpoons -\zeta, & e^{i\theta_{ab}} \rightarrow e^{-i\theta_{ab}} \end{cases}$$

i. e. θ_{ab} must be symmetric in j_a, j_b and change sign with ζ (It is to be remembered that ζ is always symmetric in j_1, j_2, j_3).

b) With the convention (4.36)

$$(4.38) \quad \text{for } j_1 \rightleftharpoons j_2, \quad e^{i\theta_{23}} \rightleftharpoons e^{-i\theta_{21}}.$$

c) Suppose that for a particular set of values of the parameters

$$(4.39) \quad e^{i\theta_{23}} = e^{i\varphi}$$

where φ may be written as a function $\varphi(j_1, j_2, j_3; \zeta)$.

Let

$$(4.39') \quad \begin{aligned} \varphi' & \equiv \varphi(j_2, j_1, j_2; \zeta) \\ \varphi'' & \equiv \varphi(j_3, j_2, j_1; \zeta) \end{aligned}$$

Then the relation

$$(31)(23)(12)(23) = I$$

implies

$$(4.40) \quad e^{i(\varphi + \varphi' + \varphi'')} = 1.$$

As a simple illustration of the above properties we may consider the example (A.7), where

$$(4.41) \quad e^{i\theta_{12}} = 1, \quad e^{i\theta_{23}} = \left(\frac{j_2 j_3 - i\zeta}{j_2 j_3 + i\zeta} \right), \quad e^{i\theta_{31}} = \left(\frac{j_3 j_1 + i\zeta}{j_3 j_1 - i\zeta} \right).$$

To verify (4.40) we have to use (A.8).

It will be seen in fact, that in absence of such an equation as (A.8), we may use (4.40) to determine the relevant values of ζ . For more complicated cases (from $j = \alpha - 3$ onwards) the equations corresponding to (4.41) will involve higher powers of ζ , providing an adequate number of eigenvalues.

Let us now proceed to the explicit construction of the matrices providing a representation of S_3 in the general case.

In the previous case $|\pm \zeta\rangle$ formed a two-dimensional subspace closed

under S_3 . In the general case we have to start with a 12-dimensional space (for $\zeta \neq 0$).

Denoting $\left| \begin{matrix} j_1 j_3 & j_3 \\ j & m \pm \zeta \end{matrix} \right\rangle$ briefly by $|j_1 j_2 j_3\rangle_{\pm}$ respectively, the twelve basic vectors are:

$$(4.42) \quad |j_1 j_2 j_3\rangle_{\pm}, |j_3 j_1 j_2\rangle_{\pm}, |j_2 j_3 j_1\rangle_{\pm}, |j_2 j_1 j_3\rangle_{\pm}, |j_1 j_3 j_2\rangle_{\pm}, |j_3 j_2 j_1\rangle_{\pm}.$$

Fortunately, the strong restrictions on the transformation properties of these vectors ((4.37), (4.40)) makes the problem of reduction of this 12-dimensional space into the subspaces corresponding to the irreducible representations quite an easy one.

In fact the required linear combinations of the basic vectors (4.42), which provide the carrier spaces for the 2-dimensional representations can be written down directly, by inspection, as the following orthonormalized set:

$$(4.43) \quad \left\{ \begin{array}{l} \frac{1}{\sqrt{3}} \left(|j_1 j_2 j_3\rangle_{+} + e^{i(\varphi+\theta)} |j_3 j_1 j_2\rangle_{+} + e^{-i(\theta+\varphi')} |j_2 j_3 j_1\rangle_{+} \right) \\ \frac{1}{\sqrt{3}} \left(|j_2 j_1 j_3\rangle_{-} + e^{i(\varphi+\theta)} |j_1 j_3 j_2\rangle_{-} + e^{-i(\theta+\varphi')} |j_3 j_2 j_1\rangle_{-} \right) \\ \frac{1}{\sqrt{3}} \left(|j_2 j_1 j_3\rangle_{-} + e^{i(\varphi-\theta)} |j_1 j_3 j_2\rangle_{-} + e^{i(\theta-\varphi')} |j_3 j_2 j_1\rangle_{-} \right) \\ \frac{1}{\sqrt{3}} \left(|j_1 j_2 j_3\rangle_{+} + e^{i(\varphi-\theta)} |j_3 j_1 j_2\rangle_{+} + e^{i(\theta-\varphi')} |j_2 j_3 j_1\rangle_{+} \right) \\ \frac{1}{\sqrt{3}} \left(|j_1 j_2 j_3\rangle_{+} + e^{i\varphi} |j_3 j_1 j_2\rangle_{+} + e^{-i\varphi'} |j_2 j_3 j_1\rangle_{+} \right) \\ \frac{1}{\sqrt{3}} \left(|j_2 j_1 j_3\rangle_{-} + e^{i\varphi} |j_1 j_3 j_2\rangle_{-} + e^{-i\varphi'} |j_3 j_2 j_1\rangle_{-} \right) \\ \frac{1}{\sqrt{3}} \left(|j_2 j_1 j_3\rangle_{+} + e^{-i(\varphi+\theta)} |j_1 j_3 j_2\rangle_{+} + e^{i(\varphi'+\theta)} |j_3 j_2 j_1\rangle_{+} \right) \\ \frac{1}{\sqrt{3}} \left(|j_1 j_2 j_3\rangle_{-} + e^{-i(\varphi+\theta)} |j_3 j_1 j_2\rangle_{-} + e^{i(\varphi'+\theta)} |j_2 j_3 j_1\rangle_{-} \right) \\ \frac{1}{\sqrt{3}} \left(|j_1 j_2 j_3\rangle_{-} + e^{-i(\varphi-\theta)} |j_3 j_1 j_2\rangle_{-} + e^{i(\varphi'-\theta)} |j_2 j_3 j_1\rangle_{-} \right) \\ \frac{1}{\sqrt{3}} \left(|j_2 j_1 j_3\rangle_{+} + e^{-i(\varphi-\theta)} |j_1 j_3 j_2\rangle_{+} + e^{i(\varphi'-\theta)} |j_3 j_2 j_1\rangle_{+} \right) \\ \frac{1}{\sqrt{3}} \left(|j_2 j_1 j_3\rangle_{+} + e^{-i\varphi} |j_1 j_3 j_2\rangle_{+} + e^{i\varphi'} |j_3 j_2 j_1\rangle_{+} \right) \\ \frac{1}{\sqrt{3}} \left(|j_1 j_2 j_3\rangle_{-} + e^{-i\varphi} |j_3 j_1 j_2\rangle_{-} + e^{i\varphi'} |j_2 j_3 j_1\rangle_{-} \right) \end{array} \right.$$

where φ, φ' are defined by (4.39), (4.39') and $\theta = \frac{2\pi}{3}$ ($e^{i\theta} = \omega$).

The explicit expression for φ (which gives φ') is the *only unknown element* among the coefficients and must be determined for any particular case, from other sources, such as a solution of the corresponding set of recurrence equations. It will be seen, however, that for arbitrary values of φ, φ' , the permutation matrices acting on the space provided by the above vectors are completely reduced to the form:

$$(4.44) \quad (ab) = \begin{bmatrix} T_{ab} & 0 \\ 0 & T_{ab} \end{bmatrix}, \quad \text{with} \quad T_{ab} = \begin{bmatrix} I_{ab} & 0 & 0 \\ 0 & I_{ab} & 0 \\ 0 & 0 & \Pi_{ab} \end{bmatrix}$$

where I_{ab} , II_{ab} represent symbolically the 2×2 matrices corresponding to (ab) in type I (i. e. (4.13) and type II (i. e. (4.16)) respectively.

A 2-dimensional subspace of type II can, of course, immediately be broken into two 1-dimensional subspaces providing one symmetric and one anti-symmetric state respectively (see (4.30), (4.31)). We have provisionally kept it in a 2-dimensional form to show the relation between the $|\rangle_+$ and $|\rangle_-$ states explicitly. In fact we have four 2-dimensional and four 1-dimensional representations of the standard forms.

Having once got these representation we may proceed exactly as in case 1, to construct the final state vectors of required symmetry. Apart from the incidental complications, the chief difference from case 1 is that in case 2, the states thus constructed will no longer be eigenstates of J_i^2 ($i = 1, 2, 3$).

When any two of the parameters j_1, j_2, j_3 are equal but not all three it may easily be seen that we have, to start with, a 6-dimensional space, which can be reduced to three 2-dimensional spaces (two type I and one type II).

When all the j_i 's are equal we have only one 2-dimensional representation (either type I or type II) as discussed previously.

So far we have been discussing the case $\zeta \neq 0$. When in the $\zeta = 0$ general case, we must have

$$\eta_{ab} = e^{i\theta_{ab}} = \pm 1.$$

(It is easy to see that $\eta_{12}, \eta_{23}, \eta_{31}$ all must simultaneously be $+1$ or -1).

We note directly, that the ortho-normalized pair,

$$(4.45) \quad \frac{1}{\sqrt{6}} [(|j_1 j_2 j_3 \rangle_0 + |j_3 j_1 j_2 \rangle_0 + |j_2 j_3 j_1 \rangle_0) \\ \pm (|j_2 j_1 j_3 \rangle_0 + |j_1 j_3 j_2 \rangle_0 + |j_3 j_2 j_1 \rangle_0)]$$

give respectively the symmetric and antisymmetric (antisym. and sym.) states for $\eta_{ab} = +1$ ($\eta_{ab} = -1$). When two of the j 's are equal we have (say for $j_2 = j_1$)

$$(4.46) \quad \frac{1}{\sqrt{3}} [|j_1 j_1 j_3 \rangle_0 + |j_1 j_3 j_1 \rangle_0 + |j_3 j_1 j_1 \rangle]$$

as either the symmetric ($\eta_{ab} = +1$) or the antisymmetric ($\eta_{ab} = -1$) state. Similarly for $j_1 = j_2 = j_3 = k$ we have again only the symmetric or antisymmetric state $|k k k \rangle_0$.

5. Conclusion. — In many important respects our discussion remains incomplete. We have not obtained a general expression for the A-coefficients. Nor have we constructed explicitly the spectrum of the eigenvalues

of Z . In 2 *b*) we have demonstrated that our scheme is connected to the usual one through a unitary transformation but we have not given an explicit form for this unitary matrix.

However, the detailed discussion of the symmetry properties in 4, are sufficient to demonstrate the remarkable advantages of our method. In fact these symmetry considerations prove powerful enough to deduce the general features even in absence of a complete solution for the A 's. Not only for the simple (but important) case $j_1 = j_2 = j_3$, but also for the most general one we have completely solved the problem of construction of states belonging to the different symmetry classes. The simplicity and the generality of the solution given in (4.43) and (4.44) may be compared with an attempt such as Koba's [3]. In the « effective angular momentum » method of Koba, not only is the technique tied to a complicated solution of non-relativistic Schrodinger equation (excluding half integral angular momenta), but the task of obtaining the component irreducible representations become more and more complicated (as Λ the « e. a. m. » increases) and is to be tackled afresh in each case. Our method is based finally on the usual direct product states and the complete reduction is obtained once for all and quite simply, for the most general case.

For the particular case $j_1 = j_2 = j_3 = k$, we may compare our method with the usual group theoretical methods familiar in nuclear shell theory [4].

We note that from as early as f -shell ($k = 3$) onwards the usual introduction of the symmetry classes supplemented by Racah's « seniority quantum number » does not prove adequate and we require a further reduction with respect to another subgroup of R_7 containing in turn R_3 as a subgroup. Even this, of course, does not solve the problem of obtaining a complete classification for higher values of k .

Our method (which of course applies only to the case of 3 particles) gives at once (*for arbitrary values of k*) a complete set of states, which are already broken up into irreducible one- and two-dimensional subspaces with respect to S_3 . Moreover, we obtain such a complete reduction for arbitrary unequal values of j_1, j_2, j_3 . We obtain all these results, without any specialized group theoretical techniques, only considering the familiar algebra of the operators of the rotation group R_3 .

Further we note that in considering scattering processes, we do not, in general, deal (as so often in shell theory) with equal component angular momenta. Thus our relatively simple result (4.43) for nonequivalent particles is of particular interest in such cases.

Finally we add a few remarks on the fact already noted (see (3.4) and

what follows that the effects of the *unitary* permutation operators has a close connection with the complex conjugation of the (*c*-number) *A*-coefficients which is typical of *antiunitary* operators. This is most direct for $j_1 = j_2 = j_3 = k$. The antiunitary time inversion operator anticommutes both with *Z* and J^z and the transition $\zeta \rightarrow -\zeta$ is brought about through the direct complex conjugation of the *A*'s. The permutation operator (*ab*) anticommutes with *Z* but commutes with J^z and of course there is no complex conjugation of the *c*-numbers involved. But it just so happens that the equations (3.3) and (3.4) namely, the fact that we can put

$$(5.1) \quad \begin{cases} A(m_1 m_2 m_3 | j m \zeta) = A(-m_1 - m_2 - m_3 | j - m \zeta) & \text{and} \\ A(m_1 m_2 m_3 | j m \zeta) = A^*(m_1 m_2 m_3 | j m - \zeta) \end{cases}$$

together imply that after both the operations we have a state in which the coefficients are just the complex conjugates of the previous ones.

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APPENDIX A

In this section we stretch a straight forward method for the evaluation of the A's. Let,

$$(A.1) \quad \left. \begin{aligned} j_1 + j_2 + j_3 &= \alpha \\ j_1 j_2 + j_2 j_3 + j_3 j_1 &= \beta \\ j_1 j_2 j_3 &= \gamma \end{aligned} \right\}$$

The values of j are of the form $(\alpha - r)$ ($r = 0, 1, 2, \dots$). Corresponding to each partition of r ,

$$r = a + b + c,$$

we will have a set of simultaneous linear equations in the A's, the different sets being interconnected. The number of equations will be restricted in each particular case by the actual values j_1, j_2, j_3 considered—even among the permissible values of r , certain partitions may be suppressed.

Let us consider all the A's for a fixed value of $m = (\alpha - r)$ [when j can have values $\alpha, (\alpha - 1), \dots, (\alpha - r)$] and let us denote (for $m_1 = j_1 - a, m_2 = j_2 - b, m_3 = j_3 - c$)

$$A \left(\begin{matrix} j, & \alpha - r, & \zeta \\ j_1 - a, & j_2 - b, & j_3 - c \end{matrix} \right) \text{ and } |j_1, (j_1 - a); j_2, (j_2 - b); j_3, (j_3 - c)\rangle$$

briefly as

$$A_{a,b,c}, |a, b, c\rangle \text{ respectively.}$$

Also let

$$(A.2) \quad \left\{ \begin{aligned} \mu &= j(j + 1) - \alpha(\alpha + 1) + 2r\alpha \\ \nu &= -i\zeta. \end{aligned} \right.$$

From (3.2) (3.3), we have for:

Partition ($r, 0, 0$)

$$(A.3) \quad \left\{ \begin{aligned} (\mu - 2rj_1)A_{r,0,0} &= \lambda_{j_1-r+1}^- (\lambda_{j_2-1}^+ A_{r-1,1,0} + \lambda_{j_3-1}^+ A_{r-1,0,1}) \\ 2\nu A_{r,0,0} &= \lambda_{j_1-r+1}^- (-j_3 \lambda_{j_2-1}^+ A_{r-1,1,0} + j_2 \lambda_{j_3-1}^+ A_{r-1,0,1}) \end{aligned} \right.$$

and two similar pairs of equations for $A_{0,r,0}, A_{0,0,r}$ and for :

Partition ($r - 1, 1, 0$)

$$(A.4) \quad \left\{ \begin{aligned} [\mu - 2\{(r-1)j_1 + j_2 + (r-1)\}]A_{r-1,1,0} &= \lambda_{j_2}^- (\lambda_{j_1-r}^+ A_{r,0,0} + \lambda_{j_3-1}^+ A_{r-1,0,1}) + \lambda_{j_1-r+2}^- (\lambda_{j_2-2}^+ A_{r-2,2,0} + \lambda_{j_3-1}^+ A_{r-2,1,1}) \\ 2\nu A_{r-1,1,0} &= \lambda_{j_2}^- (j_3 \lambda_{j_1-r}^+ A_{r,0,0} - (j_1 - r + 1) \lambda_{j_3-1}^+ A_{r-1,0,1}) \\ &\quad + \lambda_{j_1-r+2}^- (-j_3 \lambda_{j_2-2}^+ A_{r-2,2,0} + (j_2 - 1) \lambda_{j_3-1}^+ A_{r-2,1,1}) \end{aligned} \right.$$

and five similar pairs of equations for $A_{r-1,0,1}, A_{1,r-1,0}$,

$$A_{0,r-1,1}, A_{1,0,r-1}, A_{0,1,r-1} \text{ respectively.}$$

In each case the remaining equations (two equations for (A.3) and five for (A.4)) can be written down directly by applying the proper permutation on both sides (i. e. the suffixes 1, 2, 3 and a, b, c are to be given the same permutation). It must be noted however that ν undergoes a change of sign corresponding to each

odd permutation. This is, of course, a consequence of the property of Z discussed in (4.1).

Similarly we can go on writing the equations corresponding to the partitions $(r - 2, 2, 0)$ $(r - 2, 1, 1)$ and so on. There being at most six equations for each case, which may reduce to three (or even one for special values of j_1, j_2, j_3) in some cases.

It is easy to verify that if we start from (A.3) and substitute the results in the successive sets, at each stage we have to solve effectively two simultaneous equations in two unknowns (ratios of the A's being only needed due to the normalization conditions). This is certainly possible, though the coefficients involved tend to become more and more complicated as one advances.

For any particular value of r , the partitions stop at a certain stage completing the system of equations we have to consider for that case. The consistency condition for these (homogeneous) equations in the A's give us an equation for the values of ν corresponding to any given value of j (or μ). The values of ζ (or ν) being independent of m may be evaluated for any one of the m values for a particular j . Usually the choice $m = j$ is the most convenient one.

For particular values of j_1, j_2, j_3 all of the equations corresponding to a particular set may not be present, truncating the corresponding symmetry. Only for $j_1 = j_2 = j_3$ ($= k$, say), the sets are always complete.

Next we present the actual solutions for some simple special cases.

Particular cases (In what follows the quantum numbers of the state vectors on the L. H. S. will refer to j, m, ζ):

a) for $r = 0$ (i. e. $m = \alpha$), we have evidently

$$(A.5) \quad | \alpha, \alpha, 0 \rangle = | 0 0 0 \rangle$$

(where 0, 0, 0 on the R. H. S. of course means $m_1 = j_1 - 0$, etc.) ;

b) $r = 1$ (i. e., $m = \alpha - 1$)

1. $j = \alpha, \zeta = 0$

$$(A.6) \quad | \alpha, \alpha - 1, 0 \rangle = N^{-1} [j_1^{1/2} | 1, 0, 0 \rangle + j_2^{1/2} | 0, 1, 0 \rangle + j_3^{1/2} | 0, 0, 1 \rangle]$$

where $N^2 = \alpha$

2. $j = \alpha - 1, \zeta = \pm (\gamma\alpha)^{1/2}$

$$(A.7) \quad | \alpha - 1, \alpha - 1, \zeta \rangle = N^{-1} \left[\frac{1}{j_1^{1/2}(j_2 j_3 + i\zeta)} | 1, 0, 0 \rangle + \frac{1}{j_2^{1/2}(j_3 j_1 - i\zeta)} | 0, 1, 0 \rangle \right. \\ \left. + \frac{(j_3 j_1 + i\zeta)}{j_3^{1/2}(j_1 j_2 - i\zeta)(j_3 j_1 - i\zeta)} | 0, 0, 1 \rangle \right]$$

where
$$N^2 = \sum \frac{1}{j_1} \cdot \frac{1}{(j_2^2 j_3^2 + \zeta^2)}$$

We may also note that ν satisfies

$$(A.8) \quad (j_1 j_2 + \nu)(j_2 j_3 + \nu)(j_3 j_1 + \nu) = (j_1 j_2 - \nu)(j_2 j_3 - \nu)(j_3 j_1 - \nu) \\ = -(j_1 + j_2)(j_2 + j_3)(j_3 + j_1)\nu$$

This is useful in considering the symmetry properties.

$$c) \underline{r = 2(m = \alpha - 2)}$$

$$(A.9) \quad \begin{cases} j = \alpha, & \zeta = 0 \\ j = \alpha - 1, & \zeta = \pm (\gamma\alpha)^{1/2} \\ j = \alpha - 2, & \zeta = 0, \pm (4\gamma\alpha - 3\gamma - \beta\alpha + \beta)^{1/2}. \end{cases}$$

The general expression is (writing as one the equations for the 3 values of j)

$$(A.10) \quad \begin{aligned} |j, \alpha - 2, \zeta\rangle = N^{-1} & \left[2((2j_1 - 1)j_2j_3)^{1/2}(j_2 + j_3) \frac{\{j_1(\mu - 4j_3) - 2\nu\}}{\{j_3(\mu - 4j_1) + 2\nu\}} |2, 0, 0\rangle \right. \\ & + 2((2j_2 - 1)j_3j_1)^{1/2}(j_3 + j_1) \frac{\{j_2(\mu - 4j_3) + 2\nu\}}{\{j_3(\mu - 4j_2) - 2\nu\}} |0, 2, 0\rangle \\ & + 2((2j_3 - 1)j_1j_2)^{1/2}(j_1 + j_2) |0, 0, 2\rangle \\ & + j_1^{1/2} \{j_2(\mu - 4j_3) + 2\nu\} |0, 1, 1\rangle + j_2^{1/2} \{j_1(\mu - 4j_3) - 2\nu\} |1, 0, 1\rangle \\ & \left. + j_3^{1/2} \frac{\{j_1(\mu - 4j_2) + 2\nu\} \{j_2(\mu - 4j_3) + 2\nu\}}{\{j_3(\mu - 4j_2) - 2\nu\}} |1, 1, 0\rangle \right]. \end{aligned}$$

In the above μ, ν are determined by (A.2) and we have,

$$(A.11) \quad \begin{aligned} & \{j_1(\mu - 4j_2) + 2\nu\} \{j_2(\mu - 4j_3) + 2\nu\} \{j_3(\mu - 4j_1) + 2\nu\} \\ & = \{j_2(\mu - 4j_1) - 2\nu\} \{j_3(\mu - 4j_2) - 2\nu\} \{j_1(\mu - 4j_3) - 2\nu\} \end{aligned}$$

where ν being imaginary can have its sign changed throughout.

It is to be noted that the *normalizing constant* N is *not* (unlike that in (A.7)) *symmetrical* in j_1, j_2, j_3 , but is only so in j_1, j_2 , in the forms we have chosen. This is important while considering the effect of permutations.

d) $\underline{r = 3}$ There are ten coefficients. At this stage the results already tend to become relatively complicated for the general case and we give only the ζ values for the special case $j_1 = j_2 = j_3 = k$ (i. e. $\alpha = 3k, \beta = 3k^2, \gamma = k^3$). We have (repeating the values for $j = \alpha, \alpha - 1, \alpha - 2$)

$$(A.12) \quad \begin{cases} j = \alpha, & \zeta = 0 \\ j = \alpha - 1, & \zeta = \pm \sqrt{3}k^2 \\ j = \alpha - 2, & \zeta = 0, \pm \sqrt{3}k(2k - 1) \\ j = \alpha - 3, & \zeta = \pm \sqrt{3}k(k - 1), \\ & \pm \sqrt{3} \{k(k - 1)(9k^2 - 9k + 2)\}^{1/2}. \end{cases}$$

e) Finally we write down explicitly the complete solutions (for non-negative m -values only) for two of the simplest but particularly important cases. Also now we put on the r. h. s. the values of m_1, m_2, m_3 directly instead of $(j_1 - m_1)$, etc., as we were doing before.

$$1. \underline{j_1 = j_2 = j_3 = 1/2}$$

$$(A.13) \quad \begin{aligned} \left| \begin{matrix} 3 & 3 & 0 \\ 2 & 2 & 0 \end{matrix} \right\rangle &= \left| \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{matrix} \right\rangle \\ \left| \begin{matrix} 3 & 1 & 0 \\ 2 & 2 & 0 \end{matrix} \right\rangle &= \frac{1}{\sqrt{3}} \left[\left| -\frac{1}{2} \frac{1}{2} \frac{1}{2} \right\rangle + \left| \frac{1}{2} - \frac{1}{2} \frac{1}{2} \right\rangle + \left| \frac{1}{2} \frac{1}{2} - \frac{1}{2} \right\rangle \right] \\ \left| \begin{matrix} 1 & 1 & \zeta \\ 2 & 2 & \zeta \end{matrix} \right\rangle &= \frac{1}{\sqrt{3}} \left[\left(\frac{1 + i4\zeta}{1 - i4\zeta} \right) \left| -\frac{1}{2} \frac{1}{2} \frac{1}{2} \right\rangle + \left(\frac{1 - i4\zeta}{1 + i4\zeta} \right) \left| \frac{1}{2} - \frac{1}{2} \frac{1}{2} \right\rangle + \left| \frac{1}{2} \frac{1}{2} - \frac{1}{2} \right\rangle \right] \end{aligned}$$

where

$$\zeta = \pm \frac{\sqrt{3}}{4}.$$

$$2. \underline{j_1 = j_2 = j_3 = 1}$$

$$(i) \underline{(j = 3, \zeta = 0)}$$

$$\begin{aligned} |330\rangle &= |111\rangle \\ |320\rangle &= \frac{1}{\sqrt{3}} [|011\rangle + |101\rangle + |110\rangle] \\ (A.14) \quad |310\rangle &= \frac{1}{\sqrt{15}} \{ [| -111\rangle + |1-11\rangle + |11-1\rangle] \\ &\quad + 2 [|100\rangle + |010\rangle + |001\rangle] \} \\ |300\rangle &= \frac{2}{5} \left[\frac{1}{2} [|000\rangle + | -101\rangle + | -110\rangle + |0-11\rangle \right. \\ &\quad \left. + |1-10\rangle + |0-11\rangle + |10-1\rangle \right] \end{aligned}$$

$$(ii) \underline{(j = 2, \zeta = \pm \sqrt{3})}$$

$$\begin{aligned} |22\zeta\rangle &= \frac{1}{\sqrt{3}} \left[\frac{1+i\zeta}{1-i\zeta} |011\rangle + \frac{1-i\zeta}{1+i\zeta} |101\rangle + |110\rangle \right] \\ |21\zeta\rangle &= \frac{1}{\sqrt{6}} \left[\left\{ \frac{1+i\zeta}{1-i\zeta} | -111\rangle + \frac{1-i\zeta}{1+i\zeta} |1-11\rangle + |11-1\rangle \right\} \right. \\ (A.15) \quad &\quad \left. - \left\{ \frac{1+i\zeta}{1-i\zeta} |100\rangle + \frac{1-i\zeta}{1+i\zeta} |010\rangle + |001\rangle \right\} \right] \\ |20\zeta\rangle &= \frac{1}{\sqrt{6}} \left[\frac{1+i\zeta}{1-i\zeta} (|01-1\rangle - |0-11\rangle) \right. \\ &\quad \left. + \frac{1-i\zeta}{1+i\zeta} (| -101\rangle - |10-1\rangle) + (|1-10\rangle - | -110\rangle) \right]. \end{aligned}$$

$$(iii) \underline{(j = 1, \zeta = 0, \pm \sqrt{3})}$$

$$\begin{aligned} |110\rangle &= \frac{1}{\sqrt{15}} [2(| -111\rangle + |1-11\rangle + |11-1\rangle) \\ &\quad - (|100\rangle + |010\rangle + |001\rangle)] \\ |11\zeta\rangle &= \frac{1}{\sqrt{6}} \left[\left\{ \frac{1-i\zeta}{1+i\zeta} | -111\rangle + \frac{1+i\zeta}{1-i\zeta} |1-11\rangle + |11-1\rangle \right\} \right. \\ (A.16) \quad &\quad \left. + \left\{ \frac{1-i\zeta}{1+i\zeta} |100\rangle + \frac{1+i\zeta}{1-i\zeta} |010\rangle + |001\rangle \right\} \right] (\zeta = \pm \sqrt{3}) \\ |100\rangle &= \frac{1}{\sqrt{55}} \left[-\frac{1}{3} [|000\rangle + | -101\rangle + | -110\rangle + |0-11\rangle \right. \\ &\quad \left. + |1-10\rangle + |10-1\rangle + |10-1\rangle \right] \\ |10\zeta\rangle &= \frac{1}{\sqrt{6}} \left[\frac{1-i\zeta}{1+i\zeta} (|01-1\rangle + |0-11\rangle) \right. \\ &\quad \left. + \frac{1+i\zeta}{1-i\zeta} (| -101\rangle + |10-1\rangle) + (|1-10\rangle + | -110\rangle) \right] \end{aligned}$$

$$(iv) \underline{(j = 0, \zeta = 0)}$$

$$(A.17) \quad |000\rangle = \frac{1}{\sqrt{6}} \{ (|01-1\rangle - |0-11\rangle) + (| -101\rangle - |10-1\rangle) \\ + (|1-10\rangle - | -110\rangle) \}.$$

APPENDIX B

In this section we consider in detail the effect of permutation for the simplest case $j_1 = j_2 = j_3 = 1/2$. The special importance of this case arises in the 3-fermion problem in considering the couplings of spins and possibly of iso-spins. We will compare our treatment with the usual procedure [5].

The eigenstates for this case are listed in App. A (e) 1.).

Considering the only distinct non-trivial case

$$j = \frac{1}{2}, \quad m = \frac{1}{2}, \quad \zeta = \pm \frac{\sqrt{3}}{4}$$

we have

$$(B.1) \quad \begin{aligned} \chi' &= \left[\begin{array}{ccc} 1 & 1 & \sqrt{3} \\ 2 & 2 & 4 \end{array} \right] = \frac{1}{\sqrt{3}} [\omega | - + + \rangle + \omega^* | + - + \rangle + | + + - \rangle] \\ \chi'' &= \left[\begin{array}{ccc} 1 & 1 & -\sqrt{3} \\ 2 & 2 & 4 \end{array} \right] = \frac{1}{\sqrt{3}} [\omega^* | - + + \rangle + \omega | + - + \rangle + | + + - \rangle] \end{aligned}$$

$$\text{where } \omega = \frac{1 + i\sqrt{3}}{1 - i\sqrt{3}} = e^{i2\pi/3}, \text{ i. e. } 1, \omega, \omega^* (= \omega^2)$$

are the three cube roots of unity.

Thus (χ', χ'') transform under S_3 according to the representation

$$(B.2) \quad \left\{ \begin{array}{l} (12) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad (23) = \begin{vmatrix} 0 & \omega^* \\ \omega & 0 \end{vmatrix}, \quad (31) = \begin{vmatrix} 0 & \omega \\ \omega^* & 0 \end{vmatrix} \\ \text{impeying} \\ (312) = \begin{vmatrix} \omega^* & 0 \\ 0 & \omega \end{vmatrix}, \quad (231) = \begin{vmatrix} \omega & 0 \\ 0 & \omega^* \end{vmatrix} \end{array} \right.$$

The above representation by hermitian, unitary matrices is connected to the usual two-dimensional real representation of S_3 corresponding to the Young tableau:

$$(B.3) \quad \begin{array}{|c|c|} \hline & \\ \hline \hline & \\ \hline \end{array} : (12) = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \quad (23) = \begin{vmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}, \quad (31) = \begin{vmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$$

Through a unitary transformation by the matrix

$$(B.4) \quad \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 & i \\ & 1 & -i \end{vmatrix}.$$

Let us consider the 3-nucleon problem in particular.

Let (χ', χ'') be functions in spin-space and let (ζ', ζ'') denote the corresponding

functions in the iso-space. In the product space (considering the only non-trivial case $S = 1/2, T = 1/2$), we have

$$(B.5) \quad \begin{array}{l} \left[\begin{array}{|c|c|} \hline | & | \\ \hline \end{array} \right] : \xi^s = \frac{1}{\sqrt{2}} (\chi' \zeta'' + \chi'' \zeta') \\ \left[\begin{array}{|c|} \hline | \\ \hline \end{array} \right] : \xi^a = \frac{1}{\sqrt{2}} (\chi' \zeta'' - \chi'' \zeta') \end{array}$$

as the symmetric and antisymmetric functions respectively. Moreover

$$(B.6) \quad \left[\begin{array}{|c|c|} \hline | & | \\ \hline \end{array} \right] : \begin{array}{l} |\xi'| \\ |\xi''| \end{array} = \begin{array}{l} |\chi'' \zeta''| \\ |\chi' \zeta'| \end{array}$$

again transform under S_3 according to (B.2).

If we can construct orbital eigenfunctions (ψ) transforming again according to the same representation, the final totally antisymmetric state vectors can be constructed directly. Thus for example for $S = 1/2, T = 1/2$ with a similar notation for the ψ 's, we have

$$(B.7) \quad \psi^s \xi^a, \quad \psi^a \xi^s, \quad \frac{1}{\sqrt{2}} (\psi' \xi'' - \psi'' \xi')$$

as the possible totally antisymmetric combinations. The different possible behaviours (under permutations) of the orbital functions is discussed in detail in Sec. 4.

Comparing our treatment with that of Verde [5], we note that his symmetry operators (changing his notation slightly)

$$(B.7) \quad \left\{ \begin{array}{l} T^s = (12) + (23) + (31) \\ T' = \frac{\sqrt{3}}{2} [(23) - (31)] \\ T'' = - (12) + \frac{1}{2} [(23) + (31)] \end{array} \right.$$

which correspond exactly to the conventional coupling $[(j_1 j_2) \rightarrow j', (j' j_3) \rightarrow j$ are replaced in our case by the set

$$(B.8) \quad \left\{ \begin{array}{l} \tau_s = (12) + (23) + (31) \\ \tau' = (12) + \omega(23) + \omega^*(31) \\ \tau'' = (12) + \omega^*(23) + \omega(31) \end{array} \right.$$

1, ω, ω^* ($= \omega^2$) being the cube-roots of unity. The coefficients of τ', τ'' are related through complex conjugation.

These operators, like the corresponding eigenstates in the respective schemes are related as

$$(B.9) \quad \left\{ \begin{array}{l} \tau^s = T^s \\ \frac{1}{2} (\tau' + \tau'') = - T'' \\ \frac{1}{2i} (\tau' - \tau'') = T' \end{array} \right.$$

If in order to separate out the motion of the centre of mass (non-relativistic) we apply, instead of the set (B.7) [5], the set (B.8) to the particle coordinates $\vec{r}_1, \vec{r}_2, \vec{r}_3$, the two (complex) internal coordinates $\frac{1}{3}(\vec{r}_1 + \omega\vec{r}_2 + \omega^*\vec{r}_3), \frac{1}{3}(\vec{r}_1 + \omega^*\vec{r}_2 + \omega\vec{r}_3)$ become complex conjugates of one another (and we have added a factor 1/3 to the τ 's). The internal contributions to the K. E. and angular momentum separate out and are proportional to $\vec{\nabla}', \vec{\nabla}'^*, (\vec{r}' \times \vec{\nabla}' + \text{cong.})$ respectively, where we have denoted the internal coordinates as \vec{r}', \vec{r}'^* respectively. For a potential proportional to $(\vec{r}'^2 + \vec{r}_2'^2 + \vec{r}_3'^2)$ the internal contribution becomes proportional to \vec{r}', \vec{r}'^* .

Separating out the real and imaginary parts in each case we, of course, get back the results of Verde [5].

APPENDIX C

In this section we show that the eigenvalues ζ_1, \dots, ζ_p in (2.7) are necessarily all distinct.

If any characteristic root ζ of M is repeated twice (or more) then in (2.8) we will have two (or more) mutually orthogonal eigenstates of Z , belonging to the same eigenvalue ζ . This would imply that in (3.1) we can find two (or more) distinct sets of solutions for the A -coefficients corresponding to the same eigenvalues ζ, j, m . But from a study of the chain of equations of which (A.3) and (A.4) are the beginning it is evident that given (j, m, ζ) we have explicit and *unique* solutions for the A 's (up to a common factor which is fixed by the normalization).

Thus the eigenvalues ζ_1, \dots, ζ_p must all be distinct since the operators \vec{J}^2, M, Z furnish just enough equations to determine the A -coefficients uniquely. Thus in a subspace (j, m) there are as many distinct eigenstates of Z as of \vec{J}^2 . Hence, just like the set $(j_1, j_2, j_3, j, m, j')$ the set $(j_1, j_2, j_3, j, m, \zeta)$ also gives a complete classification of the states.

NOTE. — Prof. Racah has kindly pointed out that the matrix elements of Z between eigenstates of $(\vec{J}_1 + \vec{J}_2)^2$ (as given in (2.6)) can be expressed in a more compact form by using the usual technique to obtain the matrix elements of tensor products of operators.

In the usual notation (see sections 14 and 15 of « nuclear shell theory » by de-Shalit and Talmi; Academic Press, 1963), we can write:

$$\langle J_{12}JM | Z | J'_{12}JM \rangle = -i\sqrt{6}(-1)^{J'_{12} + j_3 + J} \begin{Bmatrix} J_{12} & j_3 & J \\ j_3 & J'_{12} & 1 \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_1 & j_2 & J'_{12} \\ 1 & 1 & 1 \end{Bmatrix} \\ \{j_1(j_1+1)(2j_1+1)j_2(j_2+1)(2j_2+1)j_3(j_3+1)(2j_3+1)(2J_{12}+1)(2J'_{12}+1)\}^{1/2}$$

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