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dimorphy, and ARI/GARI**

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Multizetas, perinomal numbers, arithmetical dimorphy, and ARI/GARI^(*)

JEAN ECALLE^(†)

ABSTRACT. — In a sprawling field like *multizeta arithmetic*, connected with intricate new structures and teeming with ‘special objects’ (functions, moulds etc), there is room for expositions of all formats: short, medium-sized, huge. Here is a survey on the tiniest scale possible, based on a talk given at the 2002 Luminy conference on Resurgent Analysis.

RÉSUMÉ. — Le texte qui suit, aussi ramassé que possible, reprend un exposé fait à Luminy en novembre 2002. Il présente un panorama des récents progrès en *arithmétique ‘dimorphique’ des multizêtas* et esquisse les théories (ARI/GARI, objets périnomaux, moules spéciaux) qui ont permis ces progrès.

Contents

0	Overview. Some notations	684
1	Multizeta numbers and numerical dimorphy	685
2	Generating series/functions	687
3	Perinomal objects: equations, functions, numbers	689
4	The adequate structure: ARI/GARI and AXI/GAXI	692
5	Multizeta arithmetic: the main steps	694
6	The general scheme	695
7	The bisymmetral bimoulds pal^\bullet/pil^\bullet and tal^\bullet/til^\bullet	697
8	From the atomic to subatomic level. Free generation and subgeneration	698
9	Construction of $loma^\bullet/lomi^\bullet$ and $roma^\bullet/romi^\bullet$: the easy steps	699
10	Singularities and the removal of singularities: the tricky steps	701

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11	Explicit formulae for $\text{loma}^\bullet/\text{lomi}^\bullet$	702
12	Explicit-canonical decomposition of multizetas into irreducibles	703
13	‘Impartial’ expression of the irreducibles as perinomial numbers	704
14	Conclusion. Looking back/ahead/sideways	706
15	Bibliography	708

0. Overview. Some notations

We begin (§1, §2) with a few reminders about *arithmetical dimorphy* and then focus on the prototypical instance of dimorphy: the \mathbb{Q} -ring of *multizetas*. The next two sections (§3, §4) outline, for future use, two special theories – one called forth by the study of dimorphy, the other predating it. They are the theory of *perinomial objects*; and the *flexion structure* – mainly the Lie algebra ARI and its group GARI. Next, we try (§5) to order the field of multizeta arithmetic as a hierarchy of increasingly arduous tasks, with a red thread running through everything: the search for *canonical irreducibles*. We then (§7 through §11) develop the tools (special moulds etc) which make it possible not only to explicitly decompose all multizetas into irreducibles (§12), but also to express these irreducibles *directly* and in a way that truly reflects their *neutral* position, half-way between the two natural bases of multizetas (§13). But before getting started, we must get a few definitions (about moulds, mould operations, and mould symmetries) out of the way.

Moulds $A^\bullet = \{A^\omega\} = \{A^{\omega_1, \dots, \omega_r}\}$ are simply functions of a *variable number of variables*. These variables are noted as upper indices, with bold face reserved for sequences, which often get subsumed as a simple dot \bullet . Mould addition is trivially defined, but mould multiplication is non-commutative and involves the breaking-up of sequences:

$$\{C^\bullet = A^\bullet \times B^\bullet\} \iff \{C^\omega = \sum_{\omega=\omega^1\omega^2} A^{\omega^1} B^{\omega^2}\} \quad (0.1)$$

Depending on the context, many other secondary operations may come into play. Moreover, useful moulds tend to fall into one or the other of a few *symmetry types*, which are either preserved by the basic operations, or transformed in transparent manner. Only six symmetry types will be relevant here, to wit: *symmetræl/alternæl*, *symmetrel/alternel*, *symmetril/alternil*.

A mould A^\bullet is said to be symmetral (resp alternal) or again symmetrel (resp. alternel) if the following identities hold for all ω^1, ω^2 :

$$\sum_{\omega \in \text{sha}(\omega^1, \omega^2)} A^\omega = A^{\omega^1} A^{\omega^2} \text{ (resp = 0)} \quad (\text{symmetral/alternel}) \quad (0.2)$$

$$\sum_{\omega \in \text{she}(\omega^1, \omega^2)} A^\omega = A^{\omega^1} A^{\omega^2} \text{ (resp = 0)} \quad (\text{symmetrel/alternel}) \quad (0.3)$$

with $\text{sha}(\omega^1, \omega^2)$ (resp. $\text{she}(\omega^1, \omega^2)$) denoting the set of all ordinary (resp. contracting) shufflings² of ω^1, ω^2 . The last pair *symmetrel/alternel* applies only to moulds with a double-storeyed indexation, i.e. with indices of the form $w_i = \binom{u_i}{v_i}$. It resembles *symmetrel/alternel*, except that the straight-forward addition $(\omega_i, \omega_j) \mapsto \omega_i + \omega_j$ makes way for the subtler contractions :

$$\left(A^{\binom{\dots u_i \dots}{\dots v_i \dots}}, A^{\binom{\dots u_j \dots}{\dots v_j \dots}} \right) \mapsto \frac{1}{v_i - v_j} \left(A^{\binom{\dots u_i + u_j \dots}{\dots v_i, \dots}} - A^{\binom{\dots u_i + u_j \dots}{\dots v_j, \dots}} \right) \quad (0.4)$$

Throughout, we shall use the following abbreviations for sums/differences :

$$u_{i,j} := u_i + u_j \quad , \quad u_{i,j,k} := u_i + u_j + u_k \text{ etc} \quad ; \quad v_{i,j} := v_i - v_j \quad (0.5)$$

1. Multizeta numbers and numerical dimorphy

Some extremely important \mathbb{Q} -rings of transcendental numbers are *dimorphic*, i.e. possess two *natural* \mathbb{Q} -bases³ $\{\alpha_m\}, \{\beta_n\}$ with a simple *conversion rule* and two independent *multiplication tables*, all of which involve only rational coefficients and finite sums :

$$\begin{aligned} \alpha_m &= \sum^* H_m^n \beta_n \quad , \\ \beta_n &= \sum^* K_n^m \alpha_m \quad (H_m^n, K_n^m \in \mathbb{Q}) \\ \alpha_{n_1} \alpha_{n_2} &= \sum^* A_{n_1, n_2}^{n_3} \alpha_{n_3} \quad , \\ \beta_{n_1} \beta_{n_2} &= \sum^* B_{n_1, n_2}^{n_3} \beta_{n_3} \quad (A_{n_1, n_2}^{n_3}, B_{n_1, n_2}^{n_3} \in \mathbb{Q}) \end{aligned}$$

The simplest, most basic of all such rings is *Zeta*, which is not only *multiplicatively* generated but also *linearly* spanned by the so-called *multizetas*.⁴

(2) under *ordinary/contracting* shufflings, adjacent indices ω_i, ω_j stemming from different sequences are *forbidden/allowed* to merge into $\omega_{i,j} := \omega_i + \omega_j$. Thus, for a pair $\omega^1 = (\omega_1)$ and $\omega^2 = (\omega_2, \omega_3)$, we have $\text{sha}(\omega^1, \omega^2) = \{(\omega_1, \omega_2, \omega_3), (\omega_2, \omega_1, \omega_3), (\omega_2, \omega_3, \omega_1)\}$ but $\text{she}(\omega^1, \omega^2) = \{(\omega_1, \omega_2, \omega_3), (\omega_2, \omega_1, \omega_3), (\omega_2, \omega_3, \omega_1), (\omega_1 + \omega_2, \omega_3), (\omega_1, \omega_2 + \omega_3)\}$.

(3) with some natural countable indexation $\{m\}, \{n\}$, not necessarily on \mathbb{N} or \mathbb{Z} .

(4) or MZV, short for *multiple zeta values*.

In the *first basis*, the multizetas are given by polylogarithmic integrals :

$$Wa_*^{\alpha_1, \dots, \alpha_l} := (-1)^{l_0} \int_0^1 \frac{dt_1}{(\alpha_l - t_l)} \cdots \int_0^{t_3} \frac{dt_2}{(\alpha_2 - t_2)} \int_0^{t_2} \frac{dt_1}{(\alpha_1 - t_1)} \quad (1.1)$$

with indices α_j that are either 0 or unit roots⁵.

In the *second basis*, multizetas are expressed as familiar-looking sums :

$$Ze_*^{\binom{\epsilon_1 \dots \epsilon_r}{s_1 \dots s_r}} := \sum_{n_1 > \dots > n_r > 0} n_1^{-s_1} \dots n_r^{-s_r} e_1^{-n_1} \dots e_r^{-n_r} \quad (1.2)$$

with $s_j \in \mathbb{N}^*$ and unit roots $e_j := \exp(2\pi i \epsilon_j)$ with ‘logarithms’ $\epsilon_j \in \mathbb{Q}/\mathbb{Z}$.

The stars * means that the integrals or sums are provisionally assumed to be convergent or semi-convergent : for Wa_*^α this means that $\alpha_1 \neq 0$ and $\alpha_l \neq 1$, and for $Ze_*^{\binom{\epsilon}{s}}$ this means that $\binom{\epsilon_1}{s_1} \neq \binom{0}{1}$ i.e. $\binom{\epsilon_1}{s_1} \neq \binom{1}{1}$.

The corresponding moulds Wa_*^\bullet and Ze_*^\bullet turn out to be respectively *symmetral* and *symmetrel*:

$$Wa_*^{\alpha^1} Wa_*^{\alpha^2} = \sum_{\alpha \in \text{sha}(\alpha^1, \alpha^2)} Wa_*^\alpha \quad \forall \alpha^1, \forall \alpha^2 \quad (1.3)$$

$$Ze_*^{\binom{\epsilon^1}{s^1}} Ze_*^{\binom{\epsilon^2}{s^2}} = \sum_{\binom{\epsilon}{s} \in \text{she}(\binom{\epsilon^1}{s^1}, \binom{\epsilon^2}{s^2})} Ze_*^{\binom{\epsilon}{s}} \quad \forall \binom{\epsilon^1}{s^1}, \forall \binom{\epsilon^2}{s^2} \quad (1.4)$$

These are the so-called “*quadratic relations*”, which express dimorphy. As for the conversion rule, it reads:⁶

$$Wa_*^{e_1, 0^{[s_1-1]}, \dots, e_r, 0^{[s_r-1]}} := Ze_*^{\binom{\epsilon_r \dots \epsilon_{r-1}, r \dots \epsilon_1, 2}{s_r \dots s_{r-1} \dots s_1}} \quad (1.5)$$

$$Ze_*^{\binom{\epsilon_1 \dots \epsilon_2 \dots \epsilon_r}{s_1 \dots s_2 \dots s_r}} =: Wa_*^{e_1 \dots e_r, 0^{[s_r-1]}, \dots, e_1 e_2, 0^{[s_2-1]}, e_1, 0^{[s_1-1]}} \quad (1.6)$$

There happen to be unique extensions $Wa_*^\bullet \rightarrow Wa^\bullet$ and $Ze_*^\bullet \rightarrow Ze^\bullet$ to the divergent case that keep our moulds symmetral/symmetrel while conforming to the ‘initial conditions’ $Wa^0 = Wa^1 = 0$ and $Ze^{\binom{0}{1}} = 0$. The only price to pay is a slight modification of the conversion rule : see §2 *infra*.

⁽⁵⁾ l_0 is the number of zeros in the sequence $\{\alpha_1, \dots, \alpha_l\}$.

⁽⁶⁾ with the usual shorthand for differences : $\epsilon_{i,j} := \epsilon_i - \epsilon_j$.

Basic gradations/filtrations: Four parameters dominate the discussion :

- the *weight* $s := \sum s_i$ (in the Ze^\bullet -encoding) or $:= l$ (in the Wa^\bullet -encoding)
- the *length* $r :=$ number of ϵ_i 's or s_i 's or non-zero α_i 's.
- the *degree* $d := s - r =$ number of zeros in the Wa^\bullet -encoding.¹⁵
- the *root order* $p :=$ smallest p such that all ϵ_i are in $\frac{1}{p}\mathbb{Z}/\mathbb{Z}$.

Only s defines an (additive and multiplicative) gradation; the other parameters merely induce filtrations.

2. Generating series/functions

The natural encodings Wa^\bullet and Ze^\bullet being unwieldy and too heterogeneous in their indexations, we must replace them by suitable *generating series*, so chosen as to preserve the simplicity of the two quadratic relations and that of the conversion rule. This essentially *imposes* the following definitions :

$$\text{Zag}^{\binom{u_1 \dots u_r}{\epsilon_1 \dots \epsilon_r}} := \sum_{1 \leq s_j} \text{Wa}^{e_{1,0}^{[s_1-1]}, \dots, e_{r,0}^{[s_r-1]}} u_1^{s_1-1} u_{12}^{s_2-1} \dots u_{12\dots r}^{s_r-1} \quad (2.1)$$

$$\text{Zig}^{\binom{\epsilon_1 \dots \epsilon_r}{v_1 \dots v_r}} := \sum_{1 \leq s_j} \text{Ze}^{\binom{\epsilon_1 \dots \epsilon_r}{s_1 \dots s_r}} v_1^{s_1-1} \dots v_r^{s_r-1} \quad (2.2)$$

These power series are actually convergent: they define *generating functions*⁷ that are meromorphic, with multiple poles at simple locations. These functions, in turn, verify simple difference equations, and admit an elementary mould factorisation :

$$\text{Zag}^\bullet := \lim_{k \rightarrow \infty} (\text{doZag}_k^\bullet \times \text{coZag}_k^\bullet) \quad (2.3)$$

$$\text{Zig}^\bullet := \lim_{k \rightarrow \infty} (\text{coZig}_k^\bullet \times \text{doZig}_k^\bullet) \quad (2.4)$$

with dominant parts $\text{doZag}^\bullet / \text{doZig}^\bullet$ that carry the \mathbf{u}/\mathbf{v} -dependence⁸:

$$\text{doZag}_k^{\binom{u_1 \dots u_r}{\epsilon_1 \dots \epsilon_r}} := \sum_{1 \leq m_i \leq k} e_1^{-m_1} \dots e_r^{-m_r} P(m_1 - u_1) P(m_{1,2} - u_{1,2}) \dots P(m_{1\dots r} - u_{1\dots r}) \quad (2.5)$$

$$\text{doZig}_k^{\binom{\epsilon_1 \dots \epsilon_r}{v_1 \dots v_r}} := \sum_{k \geq n_1 > n_2 > \dots > n_r \geq 1} e_1^{-n_1} \dots e_r^{-n_r} P(n_1 - v_1) P(n_2 - v_2) \dots P(n_r - v_r) \quad (2.6)$$

(15) d is called *degree*, because under the correspondence *scalars* \rightarrow *generating series*, the multizetas become coefficients of monomials of total degree d . See (2.1),(2.2).

(7) still denoted by the same symbols.

(8) with the usual abbreviations $m_{i,j} := m_i + m_j, m_{i,j,k} := m_i + m_j + m_k$ etc.

and corrective parts $coZag^\bullet/coZig^\bullet$ that reduce to constants :

$$coZag_k^{\binom{u_1 \dots u_r}{0 \dots 0}} := (-1)^r \sum_{1 \leq m_i \leq k} P(m_1)P(m_{1,2})\dots P(m_{1\dots r}) \quad (2.7)$$

$$coZig_k^{\binom{0 \dots 0}{v_1 \dots v_r}} := (-1)^r \sum_{k \geq n_1 \geq n_2 \geq \dots n_r \geq 1} \mu^{n_1, \dots, n_r} P(n_1)P(n_2)\dots P(n_r) \quad (2.8)$$

$$coZag_k^{\binom{u_1 \dots u_r}{\epsilon_1 \dots \epsilon_r}} := 0 \quad \text{if } (\epsilon_1, \dots, \epsilon_r) \neq (0, \dots, 0) \quad (2.9)$$

$$coZig_k^{\binom{\epsilon_1 \dots \epsilon_r}{v_1 \dots v_r}} := 0 \quad \text{if } (\epsilon_1, \dots, \epsilon_r) \neq (0, \dots, 0) \quad (2.10)$$

with $P(t) := 1/t$ (here and throughout) and with $\mu^{n_1, n_2, \dots, n_r} := \frac{1}{r_1! r_2! \dots r_l!}$ if the non-increasing sequence (n_1, \dots, n_r) attains r_1 times its highest value, r_2 times its second highest value, etc.

Setting $Mini_k^\bullet := Zig_k^\bullet|_{v=0}$ we find :⁹

$$Mini_k^{\binom{0 \dots 0}{v_1 \dots v_r}} := \sum_{\substack{[1 \leq l \leq r/2 \\ 1 \leq n_i \leq k] \\ [2 \leq r_1 \leq r_2 \leq r_l \\ r_1 + r_2 + \dots r_l = r]}} (-1)^{(r-l)} \mu^{r_1, \dots, r_l} \frac{(P(n_1))^{r_1}}{r_1} \dots \frac{(P(n_l))^{r_l}}{r_l} \quad (2.11)$$

$$Mini_k^{\binom{\epsilon_1 \dots \epsilon_r}{v_1 \dots v_r}} := 0 \quad \text{if } (\epsilon_1, \dots, \epsilon_r) \neq (0, \dots, 0) \quad (2.12)$$

We have an exact equivalence between old and new symmetries :

$$\{\text{Wa}^\bullet \text{ symmetrical}\} \iff \{\text{Zag}^\bullet \text{ symmetrical}\} \quad (2.13)$$

$$\{\text{Ze}^\bullet \text{ symmetrical}\} \iff \{\text{Zig}^\bullet \text{ symmetrical}\} \quad (2.14)$$

and the old conversion rule for scalar multizetas¹⁰ becomes :

$$Zig^\bullet = Mini^\bullet \times \text{swap}(Zag)^\bullet \quad (2.15)$$

$$(\iff \text{swap}(Zig)^\bullet = Zag^\bullet \times \text{Mana}^\bullet) \quad (2.16)$$

with the involution $swap$ defined as in (4.1) *infra* and with elementary moulds $Mana^\bullet/Mini^\bullet := \lim_{k \rightarrow \infty} Mana_k^\bullet/Mini_k^\bullet$ whose only non-zero components :

$$\text{Mana}^{\binom{u_1 \dots u_r}{0 \dots 0}} \equiv \text{Mini}^{\binom{0 \dots 0}{v_1 \dots v_r}} \equiv \text{Mono}_r \quad (2.17)$$

⁽⁹⁾ if we had no factor μ^{n_1, \dots, n_r} in (2.8), we would have $Zig_k^\bullet|_{v=0} = 0$ and therefore no $Mini_k^\bullet$ terms. But the mould Zig_k^\bullet would fail to be *symmetrical*, as required. Here lies the origin of the corrective terms in the conversion rule.

⁽¹⁰⁾ some modified form of (1.5),(1.6).

due to (2.11) are expressible in terms of monozetas :

$$1 + \sum_{r \geq 2} \text{Mono}_r t^r := \exp \left(\sum_{s \geq 2} (-1)^{s-1} \zeta(s) \frac{t^s}{s} \right) \quad (2.18)$$

To these relations one must add the so-called *self-consistency* relations :

$$\text{Zag}^{(u_1, \dots, u_r)}_{(q \epsilon_1, \dots, q \epsilon_r)} \equiv \sum_{q \epsilon_i^* = q \epsilon_i} \text{Zag}^{(q u_1, \dots, q u_r)}_{(\epsilon_1^*, \dots, \epsilon_r^*)} \quad (2.19)$$

$$\forall q|p, \forall u_i \in \mathbb{C}, \forall \epsilon_i, \epsilon_i^* \in \frac{1}{p} \mathbb{Z}/\mathbb{Z}$$

which merely reflect trivial identities between unit roots of order p .

3. Perinomial objects: equations, functions, numbers

Let $Sl_r(\mathbb{Z})$ denote the ‘special group’ (integer entries, unit determinant) with its natural action $M : f \mapsto \underline{M} f$ on functions of r variables :

$$\underline{M} f(\dots, x_i, \dots) := f(\dots, \sum M_{i,j} x_j, \dots) \quad \forall M \in Sl_r(\mathbb{Z}) \quad (3.1)$$

A *perinomial system* is a system of equations of the form :

$$\{ \underline{M}^1 f = \overset{1}{\varphi}(x, f), \dots, \underline{M}^s f = \overset{s}{\varphi}(x, f) \} \quad (f \text{ unknown}, \overset{i}{M} \in Sl_r(\mathbb{Z})) \quad (3.2)$$

with data $\overset{i}{\varphi}(x, f)$ usually linear or affine in f and ‘elementary’ in x .

A *perinomial function* is a solution of such a system.

A function is said to have finite *perinomial degrees* $d_{i,j}$ if $f(x_1, \dots, x_i + k x_j, \dots, x_r)$ is *polynomial* in k of degree $d_{i,j}$.

A *perinomial number* is a number attached to a perinomial function – by integration, summation, or taking its Taylor coefficients at the origin, etc.

Perinomial systems are a cross between *difference* and *q-difference* systems, but they also commend themselves to our attention for a number of more specific reasons :

1. *Finiteness properties* : Important spaces of perinomial functions admit a natural gradation by a global degree d , with a finite basis for any given d . The subject is of course closely tied up with the theory of finite linear representations of $Sl_r(\mathbb{Z})$.

2. *Closure properties* : Perinomal functions tend to be stable under partial differentiation, multiplication, various types of convolution, etc

3. *Self-duality under Fourier/Borel/Laplace* : This tends to be the case whenever any of these transforms applies. For instance, we have a correspondence between *homogeneous linear perinomal systems* of the form :

$$\{e^{x_i \partial_{x_j}}, f(x) = L_{i,j}(f), \forall i, j\} \xleftrightarrow{\mathcal{F}/\mathcal{B}/\mathcal{L}} \{e^{-\partial_{\xi_i} \xi_j} \hat{f}(\xi) = L_{i,j}(\hat{f}), \forall i, j\} \quad (3.3)$$

which is reminiscent of the self-duality properties for *homogeneous linear differential* equations with polynomial coefficients :

$$\left(\sum a_{m,n} x^m \partial_x^n \right) \cdot f(x) \xleftrightarrow{\mathcal{B}/\mathcal{L}} \left(\sum (-1)^n a_{m,n} \partial_\xi^n \xi^n \right) \cdot \hat{f}(\xi) \quad (3.4)$$

4. *Duality between perinomal meromorphic functions and their residues* :

This applies in particular to *eupolar* meromorphic functions f with multipoles of maximal order, located ‘at’ multi-integers \mathbf{n} and carrying multiresidues ρ . Thus, for a rather trivial type of eupolarity :¹²

$$f(x) \stackrel{ess}{:=} \sum_{n_i \in \mathbb{Z}} \frac{\rho(n_1, \dots, n_r)}{(n_1 - x_1) \dots (n_r - x_r)} : \{f \text{ perinomal}\} \Leftrightarrow \{\rho \text{ perinomal}\} \quad (3.5)$$

5. *Link with multizeta arithmetic* : Multizeta arithmetic makes extensive use of *perinomal numbers* $\rho^\#$ attached to discrete perinomal functions ρ via the series :

$$\rho^\#(s_1, \dots, s_r) \stackrel{ess}{:=} \sum_{n_i \in \mathbb{N}^*} \rho(n_1, \dots, n_r) n_1^{-s_1} \dots n_r^{-s_r} \quad (3.6)$$

or, equivalently, attached to meromorphic perinomal functions f under the taking of Taylor coefficients.

To put some flesh on these definitions, let us give two simple examples :

Example 3.1. — Fix (k_1, \dots, k_r) in \mathbb{N}^r . The perinomal function f :

$$f(x) \stackrel{ess}{:=} \sum_{n_i \in \mathbb{Z}^*} \frac{\rho(n_1, \dots, n_r)}{(n_1 - x_1) \dots (n_r - x_r)} \quad (3.7)$$

$$\begin{aligned} \text{with } \rho(n_1, \dots, n_r) &:= |n_1|^{k_1} \dots |n_r|^{k_r} \text{ if } (n_1, \dots, n_r) \text{ coprime} \\ &:= 0 \text{ otherwise} \end{aligned}$$

⁽¹²⁾ in (3.5) and throughout the sequel, the warning “*ess(entially)*” shall mean : *up to the addition of simple (usually constant) corrective terms that ensure absolute convergence, or after suitable regroupings that ensure semi-convergence.*

has Taylor coefficients $\rho^\#$ of the form¹³ :

$$\begin{aligned} \rho^\#(s_1, \dots, s_r) &= 2^r \frac{\zeta(s_1 - k_1) \dots \zeta(s_r - k_r)}{\zeta(\sum s_i - \sum k_i)} \quad \text{if all } s_i \text{ are even} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Despite being given as infinite sums, these $\rho^\#$ are clearly *rational* whenever all data k_i are even integers. This phenomenon shall be pivotal to the construction of the rational-coefficiented bimould $loma^\bullet/lomi^\bullet$.

Example 3.2. — Consider now the less simplistic perinomial function :

$$f(x) := \sum_{n_i \in \mathbb{Z}^\bullet} \frac{\rho(n_1, n_2)}{(n_1 - x_1)(n_2 - x_2)} \quad (3.8)$$

$$\{ \rho(n_1 + n_2, n_2) = \rho(n_1, n_2) + 1, \rho(n_1, n_1 + n_2) = \rho(n_1, n_2) - 1 \} \quad (3.9)$$

$$\rho(n_1, n_2) = \text{sign}(n_1) \text{sign}(n_2) (c_1 - c_2 + c_3 - c_4 + \dots) \quad (3.10)$$

$$\text{if } \left| \frac{n_1}{n_2} \right| = [c_1, c_2, c_3, c_4, \dots] = \text{continued fraction} \quad (3.11)$$

with residues ρ defined by the perinomial system (3.9). What about the arithmetical nature of the corresponding *perinomial numbers*?

$$\rho^\#(s_1, s_2) \stackrel{ess}{:=} \sum_{n_i \in \mathbb{N}^\bullet} \rho(s_1, s_2) n_1^{-s_1} n_2^{-s_2} \quad (3.12)$$

Whether we start from (3.9) or (3.10), that nature is far from clear. But the moment we form the functions fa and fi :

$$\begin{array}{ccc} & f(x_1, x_2) & \\ & \swarrow \quad \searrow & \\ fa(u_1, u_2) := & & fi(v_1, v_2) := \\ +f(u_1, u_2) - f(u_2, u_1) & \parallel & +f(v_1, v_2) - f(v_2, v_1) \\ +f(u_1 + u_2, u_1) - f(u_1, u_1 + u_2) & \parallel & +f(v_1 - v_2, v_2) - f(v_2, v_1 - v_2) \\ -f(u_1 + u_2, u_2) + f(u_2, u_1 + u_2) & \parallel & -f(v_1, v_2 - v_1) + f(v_2 - v_1, v_1) \end{array}$$

we see that the multiresidues simplify dramatically, and that the $\rho^\#(s_1, s_2)$ are in fact simple rational combinations of multizetas. Furthermore, fa links the $\rho^\#(s_1, s_2)$ to the Wa^\bullet -basis, while fi links them to the Ze^\bullet -basis. This example is but the tip of a mighty iceberg – namely the *direct-impartial* expression of the multizeta irreducibles.

⁽¹³⁾ for homogeneity reasons, $\rho^\#(s_1, \dots, s_r)$ always denotes the coefficient of $x_1^{s_1-1} \dots x_r^{s_r-1}$.

4. The adequate structure: ARI/GARI and AXI/GAXI

The starting point is the algebra BIMU. Its elements are bimoulds, ie moulds $A^\bullet = \{A^{w_1, \dots, w_r}\} = \{A^{(u_1, \dots, u_r)}_{(v_1, \dots, v_r)}\}$ with double-storeyed indices $w_i = (u_i, v_i)$. BIMU is endowed with the ordinary mould product \times , which is often noted mu to avoid confusion with a host of other operations on bimoulds. All these operations involve simultaneous additions of the u_i -variables and subtractions of the v_i -variable, which makes it expedient to systematically use the abbreviations (0.5) for sums and differences.

There is on BIMU a basic involution, the *swap*, which exchanges both sets of variables :

$$B^\bullet = \text{swap}(A^\bullet) \iff B^{(u_1, \dots, u_r)}_{(v_1, \dots, v_r)} = A^{(u_1, v_r, \dots, v_{r-1}, r, \dots, v_{2:3}, v_{1:2})}_{(u_1, r, \dots, u_1, r-1, \dots, u_{1:2}, u_1)} \quad (4.1)$$

and a basic shift operator¹⁴, the *push*, which acts as follows :

$$C^\bullet = \text{push}(A^\bullet) \iff C^{(u_1, \dots, u_r)}_{(v_1, \dots, v_r)} = A^{(-u_1, r, \dots, u_1, u_2, \dots, u_{r-1})}_{(v_r, r, \dots, v_1, r, v_2, r, \dots, v_{r-1}, r)} \quad (4.2)$$

All further operations involve ‘*sequence flexions*’ $\mathbf{a} \cdot \mathbf{b} \mapsto \mathbf{a}] [\mathbf{b}$ or $\mathbf{a}] [\mathbf{b}$. Thus, relative to the factorisation $\mathbf{w} = \dots \mathbf{a} \cdot \mathbf{b} \dots = \dots (u_3, u_4, u_5)_{(v_3, v_4, v_5)} (u_6, u_7, u_8, u_9)_{(v_6, v_7, v_8, v_9)} \dots$ the flexion markers $] , [,] , [$ should be interpreted as follows :

$$\mathbf{a}] := (u_3, u_4, u_{5,6,7,8,9})_{(v_3, v_4, v_5)} \quad] \mathbf{b} := (u_6, u_7, u_8, u_9)_{(v_6, v_7, v_8, v_9)} \quad (4.3)$$

$$\mathbf{a}] := (u_3, u_4, u_5)_{(v_{3:6}, v_{4:6}, v_{5:6})} \quad] \mathbf{b} := (u_{3,4,5,6}, u_7, u_8, u_9)_{(v_6, v_7, v_8, v_9)} \quad (4.4)$$

The binary operation *ari* defined by the flexions¹⁵ :

$$C^\bullet = \text{ari}(A^\bullet, B^\bullet) \iff C^\mathbf{w} = \sum_{\mathbf{w}=\mathbf{b} \cdot \mathbf{c}} (A^\mathbf{b} B^\mathbf{c} - B^\mathbf{b} A^\mathbf{c}) + \sum_{\mathbf{w}=\mathbf{b} \cdot \mathbf{c} \cdot \mathbf{d}} (A^{[\mathbf{c} B^{\mathbf{b}]} \mathbf{d}} - B^{[\mathbf{c} A^{\mathbf{b}]} \mathbf{d}}) + \sum_{\mathbf{w}=\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}} (A^{\mathbf{a} [\mathbf{c} B^{\mathbf{b}]}]} - B^{\mathbf{a} [\mathbf{c} A^{\mathbf{b}]}]} \quad (4.5)$$

turns BIMU_{*}¹⁶ into a Lie algebra known as ARI.

(14) of order $r+1$ when restricted to components of length r .

(15) with $\mathbf{b} \neq \emptyset, \mathbf{c} \neq \emptyset$ in all three sums; but \mathbf{a} and \mathbf{d} may be empty.

(16) i.e. the set of all A^\bullet with vanishing length-0 component ($A^\emptyset = 0$).

Likewise, the binary operation *gari* defined by the flexions ¹⁷ :

$$C^\bullet = \text{gari}(A^\bullet, B^\bullet) \iff C^{\mathbf{w}} = \sum_{\mathbf{w}=\mathbf{a}^1.\mathbf{b}^1.\mathbf{c}^1\dots\mathbf{a}^s.\mathbf{b}^s.\mathbf{c}^s.\mathbf{a}^{s+1}} A^{[\mathbf{b}^1]\dots[\mathbf{b}^s]B^{\mathbf{a}^1}} \dots B^{\mathbf{a}^s}]B^{\mathbf{a}^{s+1}}]B_\star^{[\mathbf{c}^1} \dots B_\star^{[\mathbf{c}^s} \quad (4.6)$$

turns $\text{BIMU}^{\ast 18}$ into a Lie group GARI, with ARI as its Lie algebra.

Central bimoulds, by definition, *gari*-commute with, and *ari*-annihilate, everyone else. They are of the form :

$$C^{w_1, \dots, w_r} := c(r) \in \mathbb{C} \quad \text{if } (v_1, \dots, v_r) = (0, \dots, 0) \quad (\forall (u_1, \dots, u_r)) \quad (4.7)$$

$$:= 0 \quad \text{if } (v_1, \dots, v_r) \neq (0, \dots, 0) \quad (\forall (u_1, \dots, u_r)) \quad (4.8)$$

The following are important subalgebras/subgroups of GARI/ARI :

$$\begin{aligned} \text{ARI}_{\text{push}} &:= \{A^\bullet ; A^\bullet \text{ push-invariant} \} \\ \text{ARI}_{\text{al}} &:= \{A^\bullet ; A^\bullet \text{ alternal} \} \\ \text{ARI}_{\text{al/al}} &:= \{A^\bullet ; A^\bullet \text{ alternal} \quad , \text{swap}(A^\bullet) \text{ alternal} \quad , IP \} \\ \text{ARI}_{\text{al/il}} &:= \{A^\bullet ; A^\bullet \text{ alternal} \quad , \text{swap}(A^\bullet) \text{ alternil} \quad , IP \} \\ \text{GARI}_{\text{spush}} &:= \{A^\bullet ; A^\bullet \text{ spush-invariant} \} \\ \text{GARI}_{\text{as}} &:= \{A^\bullet ; A^\bullet \text{ symmetrical} \} \\ \text{GARI}_{\text{as/as}} &:= \{A^\bullet ; A^\bullet \text{ symmetrical} \quad , \text{swap}(A^\bullet) \text{ symmetrical} \quad , IP \} \\ \text{GARI}_{\text{as/is}} &:= \{A^\bullet ; A^\bullet \text{ symmetrical} \quad , \text{swap}(A^\bullet) \text{ symmetril} \quad , IP \} \end{aligned}$$

Initial parity clause (IP) : in the above definitions, we demand that bimoulds in $\text{ARI}_{\text{al/al}}$ or $\text{ARI}_{\text{al/il}}$ (resp. $\text{GARI}_{\text{as/as}}$ or $\text{GARI}_{\text{as/is}}$) should have as their length-1 component an *even* function of w_1 , but we allow for the addition of (resp. multiplication by) a *central bimould* C^\bullet before taking the *swap*.

We have the non-trivial inclusions and isomorphisms :

$$\begin{array}{ccc} \text{ARI}_{\text{push}} & \supset & \text{ARI}_{\text{al/al}} & \xleftrightarrow{\text{algebra isom.}} & \text{ARI}_{\text{al/il}} \\ \downarrow \text{expari} & & \downarrow \text{expari} & & \downarrow \text{expari} \\ \text{GARI}_{\text{spush}} & \supset & \text{GARI}_{\text{al/as}} & \xleftrightarrow{\text{group isom.}} & \text{GARI}_{\text{as/is}} \end{array}$$

and a non-trivial action *arit/garit* of ARI/GARI on the *mu*-algebra BIMU.

(17) with $s \geq 2$ and all factor sequences $\mathbf{b}^i \neq \emptyset$ and $\mathbf{c}^i.\mathbf{a}^{i+1} \neq \emptyset$. The factors \mathbf{c}^i et \mathbf{a}^{i+1} may turn empty but *separately so* and the extreme factors $\mathbf{a}^1, \mathbf{c}^s, \mathbf{a}^{s+1}$ may also turn empty, *separately or jointly*. As for B_\star^\bullet , it denotes the inverse *invmu*(B^\bullet) of B^\bullet with respect to ordinary mould multiplication *mu* (same as \times).

(18) i.e. the set of all A^\bullet with unit length-0 component ($A^0 = 1$).

Though ARI/GARI traces its origins to *singularity theory*, its double series of variables u_i and v_i as well as its property of *accommodating* and *reproducing* double symmetries, makes it an ideal tool for investigating *arithmetical dimorphy*. ARI/GARI is actually part of a larger umbrella structure, AXI/GAXI, which regroups all *flexion* derivations/automorphisms¹⁹ of BIMU.

5. Multizeta arithmetic: the main steps

R1. Formalisation: from numbers to symbols.

Formalising the scalar multizetas means replacing the familiar systems of numbers Wa^\bullet/Ze^\bullet by symbols wa^\bullet/ze^\bullet subject to the same quadratic relations, conversion rule, and self-consistency constraints. In terms of generating series, it means replacing Zag^\bullet/Zig^\bullet by the most general pair zag^\bullet/zig^\bullet of symmetral/symmetril bimoulds connected under the *swap*:²⁰

$$\text{swap}(\text{zig}^\bullet) = \text{gari}(\text{zag}^\bullet, \text{mana}^\bullet) \quad (\text{with } \text{mana}^\bullet \text{ central}) \quad (5.1)$$

and subject to the old self-consistency constraints (2.19) but with components that are arbitrary power series instead of well-defined meromorphic functions.

R2. Free generation.

It says that the \mathbb{Q} -rings of (scalar) formal multizetas are *polynomial rings* in countably many indeterminates – the so-called irreducibles.

R3. From the atomic to the subatomic level.

It means re-interpreting the apparently *unbreakable* irreducibles as elements of a Lie algebra $\text{ARI}_{\text{al,il}}^{\text{reg,disc}}$ ²¹, which opens the way to a finer analysis.

R4. Free subgeneration.

It says that the finer and truly ultimate building blocks²², or *subgenerators*, are either *free* (for small values of the root order p) or “very nearly free” (for larger p). The various dimensions (additive/multiplicative) that can be

⁽¹⁹⁾ i.e. of all those derivations or automorphisms of the *mu*-algebra BIMU that can be expressed in terms of the *flexions* (4.3),(4.4). Elements of AXI/GAXI are determined not by single bimoulds A^\bullet , but by *pairs* $(A_L^\bullet, A_R^\bullet)$ consisting of a left and a right bimould.

⁽²⁰⁾ due to *mana* $^\bullet$ being a *central bimould*, we have in fact :

$$\text{gari}(\text{mana}^\bullet, \text{zag}^\bullet) = \text{gari}(\text{zag}^\bullet, \text{mana}^\bullet) = \text{mu}(\text{zag}^\bullet, \text{mana}^\bullet) \left(\text{but } \neq \text{mu}(\text{mana}^\bullet, \text{zag}^\bullet) \right)$$

⁽²¹⁾ the upper index *reg* means *regular in u at the origin*, i.e. without Laurent terms etc; the upper index *disc* for *discrete* means *with v-variables in \mathbb{Q}/\mathbb{Z}* ; the underlining of both upper indices means *subject to the self-consistency constraints (2.19)*.

⁽²²⁾ in ARI.

attached to subspaces/subrings of multizetas and that depend on the various filtrations/gradations (by $s, r, d, p...$ etc) mostly follow from that.

R5. Decomposition into irreducibles: constructive.

It should be fully constructive, i.e. amenable to effective computation.

R6. Decomposition into irreducibles: explicit.

In a context such as this, if we are to maintain a meaningful distinction between *constructive* and *explicit*, the latter can mean only one thing, namely: given by direct formulas which, though inevitably complex, are nonetheless perspicuous enough²³ and reasonably compact²⁴. Above all, *explicit* means that we are not required to solve larger and larger linear systems as the natural filtration/gradation parameters (s, r, d, p etc) increase.

R7. Decomposition into irreducibles: canonical.

Though redolent of subjectivity, the notion of *canonicity* matters immensely. Here, we are fortunate in being able to construct, among all possible, more or less natural systems of irreducibles, one that is indisputably canonical.

R8. Direct and ‘impartial’ expression of the irreducibles.

It goes way beyond the mere reversing of the canonical-explicit decomposition of multizetas into irreducibles; rather, it asks for a direct and ‘impartial’ (i.e. ‘equidistant’ from the two competing bases wa^\bullet and ze^\bullet) expression of the irreducibles. This is where perinomial algebra comes in.

R9. Materialisation: from symbols to numbers.

That would mean: showing that the \mathbb{Q} -ring of ‘actual’ or ‘genuine’ multizetas is actually isomorphic to its formalisation. This is the one great challenge that still lies ahead. It apparently defies extant mathematical tools, but the advent of *direct*²⁵ *numerical derivations*²⁶ might change that.

6. The general scheme

To simplify, we set out the general procedure for *ordinary multizetas*²⁷. The \mathbb{Q} -ring of multizetas splits into two/three factors rings:

$$\text{Zeta} = \text{Zeta}_{\text{I+II}} \otimes \text{Zeta}_{\text{III}} \tag{6.1}$$

$$\text{Zeta} = \text{Zeta}_{\text{I}} \otimes \text{Zeta}_{\text{II+III}} \tag{6.2}$$

$$\text{Zeta} = \text{Zeta}_{\text{I}} \otimes \text{Zeta}_{\text{II}} \otimes \text{Zeta}_{\text{III}} \quad \text{with} \quad \text{Zeta}_{\text{I}} = \mathbb{Q}[\pi^2] \tag{6.3}$$

(23) to make the main features, symmetries etc of the objects at hand easily detectable.

(24) both as mathematical text or as computation programmes.

(25) all the emphasis here is on *direct*; other derivations are useless chaff.

(26) that annihilate \mathbb{Q} but act non-trivially on \mathbb{Q} -rings of transcendental numbers

(27) i.e. multizetas without unit roots.

The factor-ring $\mathbb{Z}\eta_{\mathbf{I}}$ is generated by π^2 .

The factor-ring $\mathbb{Z}\eta_{\mathbf{II}}$ contains all irreducibles of *even* weight and ‘length’.

The factor-ring $\mathbb{Z}\eta_{\mathbf{III}}$ contains all irreducibles of *odd* weight and ‘length’.

This splitting of the ring $\mathbb{Z}\eta$ stems from a corresponding factorisation of the generating functions :

$$\text{zag}^\bullet = \text{gari}(\text{zag}_{\mathbf{I+II}}^\bullet, \text{zag}_{\mathbf{III}}^\bullet) \quad (\text{zag}_{\mathbf{III}}^\bullet \in \text{GARI}_{\underline{\text{as}}/\underline{\text{is}}}^{\text{o.l.}}) \quad (6.4)$$

$$\text{zag}^\bullet = \text{gari}(\text{zag}_{\mathbf{I}}^\bullet, \text{zag}_{\mathbf{II+III}}^\bullet) \quad (\text{zag}_{\mathbf{II+III}}^\bullet \in \text{GARI}_{\underline{\text{as}}/\underline{\text{is}}}) \quad (6.5)$$

$$\text{zag}^\bullet = \text{gari}(\text{zag}_{\mathbf{I}}^\bullet, \text{zag}_{\mathbf{II}}^\bullet, \text{zag}_{\mathbf{III}}^\bullet) \quad (\text{zag}_{\mathbf{II}}^\bullet \in \text{GARI}_{\underline{\text{as}}/\underline{\text{is}}}^{\text{e.l.}}) \quad (6.6)$$

The factors $\text{zag}_{\mathbf{I}}^\bullet$ and $\text{zag}_{\mathbf{II}}^\bullet$ carry only terms of *even* weight. As a consequence, their components of *even/odd* length are *even/odd* functions of \mathbf{u} .

The factor $\text{zag}_{\mathbf{III}}^\bullet$ carries only terms of *odd* weight. As a consequence, its components of *even/odd* length are *odd/even* functions of \mathbf{u} .

The factor $\text{zag}_{\mathbf{I}}^\bullet$ is symmetral/il²⁸ but doesn’t verify the initial parity condition *IP* (see §4). Therefore, its *gari*-logarithm is *not* alternal/il²⁹. The factors $\text{zag}_{\mathbf{II}}^\bullet$ and $\text{zag}_{\mathbf{III}}^\bullet$, on the other hand, do verify that condition and so belong to the proper symmetral/il group $\text{GARI}_{\underline{\text{as}}/\underline{\text{is}}}$. Consequently, their *gari*-logarithms $\text{lozag}_{\mathbf{II}}^\bullet$ and $\text{lozag}_{\mathbf{III}}^\bullet$ belong to the proper alternal/il algebra $\text{ARI}_{\underline{\text{al}}/\underline{\text{il}}}$ and can be *further analysed* therein. Actually, it turns out that $\text{lozag}_{\mathbf{II}}^\bullet$ and $\text{lozag}_{\mathbf{III}}^\bullet$ can be uniquely generated by the \mathbf{u} -homogeneous parts of a crucial bimould, $\text{loma}^\bullet \in \text{ARI}_{\underline{\text{al}}/\underline{\text{il}}}$, with well-defined coefficients in front of the multibrackets. These coefficients are none other than the *formal irreducibles*.

Isolating the factor $\text{zag}_{\mathbf{III}}^\bullet$ from the other two is quite easy. Indeed :³⁰

$$\text{gari}(\text{zag}_{\mathbf{III}}^\bullet, \text{zag}_{\mathbf{III}}^\bullet) = \text{gari}(\text{nepar}(\text{invgari}(\text{zag}^\bullet)), \text{zag}^\bullet) \quad (6.7)$$

But separating $\text{zag}_{\mathbf{I}}^\bullet$ from $\text{zag}_{\mathbf{II}}^\bullet$ is a far more arduous undertaking, especially if we insist, as we do, on getting a ‘canonical’ separation. This requires an elaborate construction, with the introduction of three special bimoulds, leading to a subfactorisation of $\text{zag}_{\mathbf{I}}^\bullet$:

$$\text{zag}_{\mathbf{I}}^\bullet = \text{gari}^\bullet(\text{tal}^\bullet, \text{invgari}(\text{pal}^\bullet), \text{expari}(\text{roma}^\bullet)) \quad (6.8)$$

$$= \text{gari}^\bullet(\text{tal}^\bullet, \text{expari}(\text{viroma}^\bullet), \text{invgari}(\text{pal}^\bullet)) \quad (6.9)$$

(28) i.e. symmetral and with a symmetral *swappee*.

(29) i.e. more exactly, it is alternal all right, but its *swappee* is not.

(30) *invgari* denotes the *gari*-inversion; and *nepar* multiplies each length- r component by $(-1)^r$ while changing the signs of all u_i ’s and v_i ’s.

- with a ‘eupolar’ factor $pal^\bullet \in \text{GARI}_{\text{as/as}}$ but $\notin \text{GARI}_{\underline{\text{as/as}}}$ (see §7);
- with a ‘eutrigonometric’ factor $tal^\bullet \in \text{GARI}_{\text{as/as}}$ but $\notin \text{GARI}_{\underline{\text{as/as}}}$ (see §7);
- with a ‘corrective factor’ $roma^\bullet \in \text{ARI}_{\underline{\text{al/il}}}$ or its variant $viroma^\bullet \in \text{ARI}_{\underline{\text{al/al}}}$.

In sum, everything begins with the construction of two special bimoulds³¹ :

- pal^\bullet/pil^\bullet and tal^\bullet/til^\bullet
- both *symmetral/symmetral* (as bimoulds)
- both relatively elementary (as functions of \mathbf{u})

but the really sensitive part consists in constructing and understanding two further, even more crucial, and highly non-elementary, bimoulds :

- $loma^\bullet/lomi^\bullet$ and $roma^\bullet/romi^\bullet$
- both *alternal/alternil* (as bimoulds)
- both with rational coefficients³² (as formal series in \mathbf{u})
- both strongly transcendental (as meromorphic functions of \mathbf{u}) and actually of *perinomial* and *eupolar* type.

7. The bisymmetral bimoulds pal^\bullet/pil^\bullet and tal^\bullet/til^\bullet

The two semi-elementary factors in the decomposition of Zag_i^\bullet are built from the following simple ingredients:

$$P(t) := 1/t \tag{7.1}$$

$$Q(t) := \pi/\tan(\pi t) \tag{7.2}$$

$$Qa^{w_0} = Qa^{(u_0)} := \frac{1}{p} \sum_{1 \leq k \leq p} e^{2\pi i k v_0} Q(u_0 + \frac{k}{p}) \quad \text{if } v_0 \in \frac{1}{p}\mathbb{Z}/\mathbb{Z} \tag{7.3}$$

$$Ea^{w_0} = Ea^{(u_0)} := \pi \text{ if } v_0 = 0_{\mathbb{Q}/\mathbb{Z}} \quad (\text{resp. } 0 \text{ if } v_0 \neq 0_{\mathbb{Q}/\mathbb{Z}}) \tag{7.4}$$

The simpler bimould, pal^\bullet , depends only on the \mathbf{u} -variables. It is called *eupolar* because of the very specific form of its poles. Its length- r component is a homogeneous polynomial of total degree r in simple P -expressions:

$$pal^{w_1, \dots, w_r} \in \mathbb{Q}[P(u_1), P(u_2), \dots, P(u_r), P(u_{1,2}), P(u_{1,2,3}), \dots, P(u_{1, \dots, r})]$$

The second bimould, tal^\bullet , is called *eutrigonometric*. For vanishing \mathbf{v} -variables, it closely resembles the eupolar bimould since its length- r component

(31) as usual, the *swappee* of a bimould bears the same name, with i instead of a .

(32) in the case of $roma^\bullet/romi^\bullet$, the coefficients become rational after an elementary rescaling $\pi^2 \mapsto 1$.

is again a homogeneous polynomial³³ of degree r in Q -expressions and π :
 $\text{tal}^{w_1, \dots, w_r} \in \mathbb{Q}[\pi, Q(u_1), Q(u_2), \dots, Q(u_r), Q(u_{1,2}), Q(u_{1,2,3}), \dots, Q(u_{1, \dots, r})]$

For general \mathbf{v} -variables, $\text{tal}^{w_1, \dots, w_r}$ is still a homogeneous polynomial of total degree r , but in the variables $Ea^{w_i^*}$ and $Qa^{w_i^*}$, with double indices $w_i^* = \binom{u_i^*}{v_i^*}$ subject to $\sum u_i^* v_i^* = \sum u_i v_i$ and of the form $\binom{u_j}{v_j}$ or $\binom{u_j}{v_{j:k}}$ or $\binom{u_1, \dots, j}{v_k}$.

Both bimoulds verify the *self-consistency constraints* and both are *bissymmetrical*³⁴ with, in the case of tal^\bullet , an elementary connection factor mana_I^\bullet much like the ‘global’ mana^\bullet in (2.15) but carrying only even powers of π :

$$\text{swap}(\text{pil}^\bullet) = \text{pal}^\bullet = \text{symmetr}al \tag{7.5}$$

$$\text{swap}(\text{til}^\bullet) = \text{gari}(\text{tal}^\bullet, \text{mana}_I^\bullet) = \text{gari}(\text{mana}_I^\bullet, \text{tal}^\bullet) = \text{symmetr}al \tag{7.6}$$

These properties³⁵ completely determine $\text{pal}^\bullet/\text{pil}^\bullet$ and $\text{tal}^\bullet/\text{til}^\bullet$. These bimoulds admit fully explicit expressions and enjoy an incredible number of properties. As far as multizeta algebra is concerned, they intervene at three critical junctures :

- tal^\bullet describes ‘almost all’ poles of Zag_I^\bullet . It is therefore the mainstay of the *canonical-rational Drinfel’d associator*.
- pal^\bullet provides us with a canonical-explicit isomorphism between the two ‘doubly symmetric’ or ‘dimorphic’ algebras $\text{ARI}_{\text{al}/\text{al}}$ and $\text{ARI}_{\text{al}/\text{il}}$.
- pal^\bullet enables us to construct the so-called “singulators” and these in turn make it possible to remove all the unwanted singularities at $\mathbf{u} = \mathbf{0}$ which appear during the inductive construction of the moulds loma^\bullet and roma^\bullet .

8. From the atomic to subatomic level. Free generation and subgeneration

$\text{ARI}_{\text{al}/\text{al}}$ is easily shown to be closed under the *ari* bracket. Under the adjoint action of pal^\bullet in ARI , this closure property carries over to $\text{ARI}_{\text{al}/\text{il}}$:

$$\text{adari}(\text{pal}^\bullet) : \text{ARI}_{\text{al}/\text{al}} \xrightarrow{\text{algebra isom.}} \text{ARI}_{\text{al}/\text{il}} \tag{8.1}$$

Further, using the factorisation (or stability) property :

$$\text{GARI}_{\text{as}/\text{is}} = \text{GARI}_{\text{as}/\text{is}} \cdot \text{GARI}_{\text{as}/\text{is}} \tag{8.2}$$

$$\text{zag}^\bullet = \text{gari}(\text{zag}_0^\bullet, \text{expari}(\mathbf{A}^\bullet)) \quad \text{with} \quad \mathbf{A}^\bullet \in \text{ARI}_{\text{al}/\text{il}}^{\text{ent}/\text{disc}} \tag{8.3}$$

(33) but with only *even* powers of π in it.

(34) however, $\text{pal}^\bullet, \text{tal}^\bullet \notin \text{GARI}_{\text{as}, \text{as}}$ since pal^{w_1} and tal^{w_1} are *odd* functions of w_1 .

(35) together with the initial conditions $\text{pal}^{w_1} = -\frac{1}{2u_1}$ and $\text{tal}^{w_1} = -\frac{1}{2}Qa^{w_1}$.

we get the general zag^\bullet verifying all the constraints of *R1* in §5 by postcomposing any particular solution³⁶ zag_0^\bullet by the *ari*-exponential of the generic element A^\bullet of $\text{ARI}_{\text{al/il}}^{\text{ent/disc}}$. Expanding $A^\bullet := \sum c_J A_J^\bullet$ along any rational linear basis $\{A_J\}$ of $\text{ARI}_{\text{al/il}}^{\text{ent/disc}}$, we get one degree of freedom per basis element. The corresponding scalar coefficients c_J are none other than the sought-after *irreducibles*. Thus, the \mathbb{Q} -ring $\text{Zeta}^{\text{form}}$ is seen to be isomorphic to the polynomial ring $\mathbb{Q}[\pi^2] \otimes \mathbb{Q}[c_{J_1}, c_{J_2}, \dots]$: this is the *free generation theorem*.

Now, as *scalars*, the irreducibles a_J are ‘*atoms*’, i.e. incapable of further analysis. But they correspond one-to-one to dual objects A_J^\bullet which, being elements of the Lie algebra $\text{ARI}_{\text{al/il}}^{\text{ent/disc}}$, may be broken down as Lie brackets of still ‘*simpler*’ and much less numerous ‘*subatoms*’ B_J^\bullet . Moreover, for small values of the root-order p , in particular for $p = 1, 2, 3$, the subatoms in question *freely generate* $\text{ARI}_{\text{al/il}}^{\text{ent/disc}}$ as an algebra. This is the so-called *free subgeneration theorem*. And even for general values of p , the relations between the subatoms B_J^\bullet are *few* in number and *easy to describe*, so that we may speak of a *nearly free subgeneration*. In all instances, knowledge of the subatoms implies knowledge of all the relevant *dimensions*: additive, multiplicative, etc (see §5).

We already saw how the change *scalars* \rightarrow *generating series/functions* entails a far-going *restoration of symmetry* between the two fundamental encodings, and a welcome ‘compactification’ of the *conversion rule* and *quadratic constraints*. We now register another, more decisive gain: the possibility of moving from the *atomic* to the *subatomic level*, leading to a complete understanding of the irreducibles. As for the further shift, from *generating series* to *generating functions*, the specific reward it brings is the notion of *perinomial function*, with the attendant direct-impartial description of the irreducibles.

9. Constructing $loma^\bullet/lomi^\bullet$ and $roma^\bullet/romi^\bullet$: the easy steps

This leaves us with two main tasks:

- constructing in $\text{ARI}_{\text{al/il}}^{\text{ent/disc}}$ a mould $loma^\bullet := \sum_s loma_s^\bullet$, regular in \mathbf{u} at the origin, and with weight-homogeneous summands³⁷ $loma_s^\bullet$ that shall

(36) For example the ready-made Zag^\bullet constructed from the genuine multizetas, or the factor zag_1^\bullet in (6.5).

(37) the summand $loma_s^\bullet$ of weight s has s non-vanishing components, and its component of length r is a polynomial $loma_s^{w_1, \dots, w_r} \in \mathbb{Q}[u_1, \dots, u_r]$ of total degree $d := s - r$.

yield the afore-mentioned sub-atoms B_J and support the description of all irreducibles but π^2 .

– constructing in $\text{ARI}_{\underline{al}/\underline{il}}^{\text{sing}/\text{disc}}$ a mould roma^\bullet , singular in \mathbf{u} at the origin, but with singularities that exactly compensate those of the factors pal^\bullet and tal^\bullet in (6.8),(6.9) so as to produce a *regular* factor zag_I^\bullet , leading to an automatic *separation* of the ‘rogue’ irreducible π^2 from all others.

Both constructions rely on an induction on the component length r ³⁸ but there is a sharp dichotomy between *easy steps*:

- for loma^\bullet : going from length r odd to length $r+1$ even
- for roma^\bullet : going from length r even to length $r+1$ odd

which are automatic under the ARI/GARI machinery, and *tricky steps* (the alternate ones!) which involve a complex compensation mechanism.

Here is a table setting forth the *easy* induction steps for loma^\bullet :

$$\begin{array}{ll}
 \text{loma}^\bullet \parallel_r & \in \text{ARI}_{\underline{al}/\underline{il}} \quad \text{and regular at } 0. \\
 \downarrow \text{adari}(\text{pal}^\bullet)^{-1} & \\
 \text{viloma}^\bullet \parallel_r & \in \text{ARI}_{\underline{al}/\underline{al}} \quad \text{and singular at } 0. \\
 \downarrow \text{trivial extension} & \\
 \text{viloma}^\bullet \parallel_{r+1} & \in \text{ARI}_{\underline{al}/\underline{al}} \quad \text{and singular at } 0. \\
 \downarrow \text{adari}(\text{pal}^\bullet) & \\
 \text{loma}^\bullet \parallel_{r+1} & \in \text{ARI}_{\underline{al}/\underline{il}} \quad \text{and regular at } 0.
 \end{array}$$

The trivial extension $\text{viloma}^\bullet \parallel_r \mapsto \text{viloma}^\bullet \parallel_{r+1}$ consists of course in setting the $(r+1)^{\text{th}}$ component of $\text{viloma}^\bullet \parallel_{r+1}$ equal to 0.

The same basic procedure holds for roma^\bullet , but with *even/odd* exchanged and with the *regular/singular* dichotomy applying not to roma^\bullet itself, but to the product zag_I^\bullet .

(38) in the sequel, $M^\bullet \parallel_r$ denotes M^\bullet defined *modulo* all components of length $> r$.

10. Singulators and the removal of singularities: the tricky steps

If we apply the same procedure for the ‘*tricky steps*’, the machinery will work all right and still produce an extension $loma\|_{r+1}^\bullet$ with the proper symmetries, but with unwanted singularities at the origin. To removed these, some finely honed terms have to be added. The key notion here is that of *singulator*. These are operators $slank_{r_0}$ and $slang_{r_0}$ that turn elementary bimoulds (regular in \mathbf{u} at 0, and with a single non-zero component, for $r = 1$) into bimoulds with the proper symmetries (either $\underline{al}/\underline{al}$ or $\underline{al}/\underline{il}$) and the proper singularities (‘eupolar’) at the component of length r_0 .

Here is how they are defined :

$$slank_{r_0}^\bullet H^\bullet := \text{neginvar} \cdot \text{leng}_{r_0} \cdot \text{adari}(\text{pal}^\bullet)^{-1} \cdot \text{mu}(\text{mupaj}^\bullet, \text{leng}_1 \cdot H^\bullet, \text{paj}^\bullet) \quad (10.1)$$

$$\equiv \text{pushinvar} \cdot \text{leng}_r \cdot \text{mu}(\text{anti} \cdot \text{pal}^\bullet, \text{garit}(\text{pal}^\bullet) \cdot \text{leng}_1 \cdot H^\bullet, \text{pari} \cdot \text{pal}^\bullet) \quad (10.2)$$

$$slang_{r_0}^\bullet H^\bullet := \text{adari}(\text{pal}^\bullet) \cdot slank_{r_0}^\bullet H^\bullet \quad (10.3)$$

$$slank_{r_0}^\bullet H^\bullet \in \text{ARI}_{\underline{al}/\underline{al}} \quad ; \quad slang_{r_0}^\bullet H^\bullet \in \text{ARI}_{\underline{al}/\underline{il}} \quad (r_0 \in \mathbb{N}^*) \quad (10.4)$$

and here is how they affect their argument H^\bullet :

$$\begin{array}{l} H^\bullet \in \text{ARI}_{\underline{al}/\underline{al}}^{\text{reg}/\text{disc}} \quad \parallel \quad \text{sole non-zero component for } r=1, \\ \quad \quad \quad \downarrow \quad \quad \quad \parallel \quad \text{regular in } \mathbf{u} \text{ at } \mathbf{0}. \\ slank_{r_0} H^\bullet \in \text{ARI}_{\underline{al}/\underline{al}}^{\text{sing}/\text{disc}} \quad \parallel \quad \text{sole non-zero component for } r=r_0, \\ \quad \quad \quad \downarrow \quad \quad \quad \parallel \quad \text{polar part in } \mathbf{u} \text{ of order } r_0-1 \text{ at } \mathbf{0}. \\ slang_{r_0} H^\bullet \in \text{ARI}_{\underline{al}/\underline{il}}^{\text{sing}/\text{disc}} \quad \parallel \quad \text{sole non-zero components for } r \geq r_0, \\ \quad \quad \quad \parallel \quad \text{polar part in } \mathbf{u} \text{ of order } r-1 \text{ at } \mathbf{0}. \end{array}$$

In the above relations leng_r , neginvar , pushinvar denote projectors on $BIMU$:

- leng_r retains the component of length r and annihilates all others.
 - neginvar turns any bimould into one that is an *even* function of \mathbf{w} .
 - pushinvar turns any bimould into one that is *push*-invariant (see (4.2)).
- The elementary symmetral bimoulds paj^\bullet , mupaj^\bullet are mutually inverse under the ordinary mould product mu and depend only on the \mathbf{u} -variables :

$$\text{paj}^{w_1, \dots, w_r} = P(u_1)P(u_{1,2}) \dots P(u_{1, \dots, r}) \quad (10.5)$$

$$\text{mupaj}^{w_1, \dots, w_r} = (-1)^r P(u_{1, \dots, r})P(u_{2, \dots, r}) \dots P(u_r) \quad (10.6)$$

The operator *pari* multiplies components of length r by $(-1)^r$ and *anti* reverses the order in index sequences. Lastly, *garit* denotes the natural action of GARI on BIMU.³⁹

11. Explicit formulae for $\text{loma}^\bullet/\text{lomi}^\bullet$

Starting from the two elementary bimoulds H_s^\bullet and K_n^\bullet with length-1 components of the form $H_s^{w_1} := u_1^{s-1}$ and $K_n^{w_1} := P(n-u_1)$ and applying slang_r , we get two series of bimoulds :

$$H_{[r]}^\bullet := \text{slang}_r H_s^\bullet \in \text{ARI}_{\underline{al}/\underline{il}}^{\text{sing}/\text{disc}}, \quad K_{[r]}^\bullet := \text{slang}_r K_n^\bullet \in \text{ARI}_{\underline{al}/\underline{il}}^{\text{sing}/\text{disc}} \quad (11.1)$$

that make it possible to compensate the unwanted singularities produced at each *tricky step* of the induction. This leads to two parallel expansions.

First expansion of loma^\bullet and each loma_s^\bullet : as power series in \mathbf{u} .

$$\begin{aligned} \text{act}(\text{loma}_s^\bullet) &:= +\text{act}(H_{[1]}^\bullet) \\ (r=3) &\parallel + \sum_{\substack{[s_1, s_2] \\ s_1+s_2=s \\ 1+2=3}} \beta^{[s_1, s_2]} \text{act}(H_{[s_1]}^\bullet) \text{act}(H_{[s_2]}^\bullet) \\ &\parallel \dots\dots\dots \\ (r \text{ odd} \geq 5) &\parallel + \sum_{\substack{[s_1, \dots, s_q] \\ s_1+\dots+s_q=s \\ r_1+\dots+r_q=r}} \beta^{[s_1, \dots, s_q]} \text{act}(H_{[s_1]}^\bullet) \dots \text{act}(H_{[s_q]}^\bullet) \\ &\parallel \dots\dots\dots \end{aligned}$$

Second expansion of loma^\bullet : as meromorphic-perinomial functions in \mathbf{u} .

$$\begin{aligned} \text{act}(\text{loma}^\bullet) &\stackrel{ess}{:=} +\text{act}(K_{[1]}^\bullet) \\ (r=3) &\parallel + \sum_{\substack{[n_1, n_2] \\ n_1+n_2=3 \\ \text{coprime}}} \rho^{[n_1, n_2]} \text{act}(K_{[n_1]}^\bullet) \text{act}(K_{[n_2]}^\bullet) \\ &\parallel \dots\dots\dots \\ (r \text{ odd} \geq 5) &\parallel + \sum_{\substack{[n_1, \dots, n_q] \\ n_1+\dots+n_q=r \\ \text{coprime}}} \rho^{[n_1, \dots, n_q]} \text{act}(K_{[n_1]}^\bullet) \dots \text{act}(K_{[n_q]}^\bullet) \\ &\parallel \dots\dots\dots \end{aligned}$$

The second expansion is unique. It involves well-defined, rational multiresidues ρ^\bullet that are discrete perinomial functions of the integers n_i .

⁽³⁹⁾ which of course is distinct from the adjoint action of GARI in ARI.

The first expansion is not unique, but becomes so if we want it to coincide with the second one. It then involves well-defined coefficients β^\bullet which are, unexpectedly but crucially, rational numbers. Each one of them is expressible as a ratio of finitely many *hyper-bernoullian numbers*. In fact, for $r = 3$, they are quotients $\beta' \beta'' / \beta'''$ of just three *ordinary Bernoulli numbers*.⁴⁰

As moulds, both ρ^\bullet and β^\bullet are alternal.

We have similar expansions for the bimould *roma*[•].

12. Explicit-canonical decomposition of multizetas into irreducibles

Let τ^s be the projector on $\text{ARI}_{\text{al/il}}^{\text{reg/disc}}$ which, when applied to a bimould M^\bullet , retains only the part of weight s . For a component of length r , this means retaining only the part of degree $d = s - r$ in the \mathbf{u} -variables.

$$\tau^s M^{\binom{u_1 \dots u_r}{v_1 \dots v_r}} := M^{\binom{u_1 \dots u_r}{v_1 \dots v_r}} \Big|_{\mathbf{u}\text{-part of degree } s - r} \quad (12.1)$$

For the generating series of the multizetas, whether ‘genuine’⁴¹ or formal⁴², this leads to unique decompositions, with a *loma*[•]-part that carries only rational coefficients, and coefficients $\text{Irr}^\bullet / \text{irr}^\bullet$ that absorb all the transcendence:

$$\text{act}(\text{Zag}_{\text{II}}^\bullet) = 1 + \sum \text{Irr}_{\text{II}}^{s_1, \dots, s_r} \text{act}(\tau^{s_1} \text{loma}^\bullet) \dots \text{act}(\tau^{s_r} \text{loma}^\bullet) \quad (12.2)$$

$$\text{act}(\text{Zag}_{\text{III}}^\bullet) = 1 + \sum \text{Irr}_{\text{III}}^{s_1, \dots, s_r} \text{act}(\tau^{s_1} \text{loma}^\bullet) \dots \text{act}(\tau^{s_r} \text{loma}^\bullet) \quad (12.3)$$

$$\text{act}(\text{zag}_{\text{II}}^\bullet) = 1 + \sum \text{irr}_{\text{II}}^{s_1, \dots, s_r} \text{act}(\tau^{s_1} \text{loma}^\bullet) \dots \text{act}(\tau^{s_r} \text{loma}^\bullet) \quad (12.4)$$

$$\text{act}(\text{zag}_{\text{III}}^\bullet) = 1 + \sum \text{irr}_{\text{III}}^{s_1, \dots, s_r} \text{act}(\tau^{s_1} \text{loma}^\bullet) \dots \text{act}(\tau^{s_r} \text{loma}^\bullet) \quad (12.5)$$

In all four sums, the indices s_i are odd integers ≥ 3 and “act” denotes *any* transitive action of ARI/GARI in BIMU – it doesn’t matter which. The *gari*-factorisations between symmetral bimoulds:

$$\text{Zag}_{\text{II+III}}^\bullet := \text{gari}^\bullet(\text{Zag}_{\text{II}}^\bullet, \text{Zag}_{\text{III}}^\bullet) \quad (12.6)$$

$$\text{zag}_{\text{II+III}}^\bullet := \text{gari}^\bullet(\text{zag}_{\text{II}}^\bullet, \text{zag}_{\text{III}}^\bullet) \quad (12.7)$$

(40) the step 3 \mapsto 4 requiring no corrections, these harmless quotients $\beta' \beta'' / \beta'''$ already yield the explicit-canonical decomposition of all multizetas of length $r \leq 4$ and of any weight s , up to infinity!

(41) with upper-case initials.

(42) with lower-case initials.

induce corresponding *mu*-factorisations for the symmetral, scalar moulds :

$$\text{Irr}_{\text{II+III}}^\bullet := \text{Irr}_{\text{II}}^\bullet \times \text{Irr}_{\text{III}}^\bullet \tag{12.8}$$

$$\text{irr}_{\text{II+III}}^\bullet := \text{irr}_{\text{II}}^\bullet \times \text{irr}_{\text{III}}^\bullet \tag{12.9}$$

Moreover, due to the *parity* which governs everything here, the only non-zero components in the mould logarithms $\text{logmu}(\text{Irr}_{\text{II}}^\bullet)$ and $\text{logmu}(\text{irr}_{\text{II}}^\bullet)$ (resp. $\text{logmu}(\text{Irr}_{\text{III}}^\bullet)$ and $\text{logmu}(\text{irr}_{\text{III}}^\bullet)$) are those of *even* (resp. *odd*) length r . As an easy consequence, the mould $\text{irr}_{\text{II+III}}^\bullet$, or irr^\bullet for short, actually *determines* its two factors $\text{irr}_{\text{II}}^\bullet$ and $\text{irr}_{\text{III}}^\bullet$. Summing up, we may say :

Together with the symbol $\text{irr}_I^2 \sim \pi^2$, the symmetral mould ⁴³:

$$\text{irr}_{\text{II+III}}^\bullet = \text{irr}^\bullet = \{ \text{irr}^{s_1, s_2, \dots, s_r} \in \mathbb{C} \}, \quad r = 1, 2, 3 \dots, \quad s_i \in \{3, 5, 7, 9 \dots\} \tag{12.10}$$

constitutes a system, both complete and free ⁴⁴, *of canonical irreducibles for the (ordinary or ‘rootless’) formal multizetas. More precisely, any such multizeta may be uniquely linearised as a sum :*

$$\sum_{r \geq 0} \sum_{\substack{s_0 \text{ even} \geq 0 \\ s_1, \dots, s_r \text{ odd} \geq 3}} \gamma^{s_0; s_1, \dots, s_r} \pi^{s_0} \text{irr}^{s_1, \dots, s_r} \quad (\gamma^\bullet \in \mathbb{Q}). \tag{12.11}$$

Analogous results hold for the *rooted* multizetas.

13. ‘Impartial’ expression of the irreducibles as perinomial numbers

Plugging (6.8),(12.4),(12.5) into the factorisation (6.6) and then picking the Taylor coefficients of either zag^\bullet or zig^\bullet , we get the formal multizetas, in both encodings wa^\bullet and ze^\bullet , automatically ⁴⁵, uniquely, and explicitly expanded as finite sums of irreducibles irr^\bullet . The process of course may be reversed, yielding expressions of irr^\bullet in terms of either wa^\bullet and ze^\bullet , but

⁽⁴³⁾ i.e. subject to no other constraints than symmetry.

⁽⁴⁴⁾ that is, free up to the *symmetry constraints*. These constraints could easily be removed by working with the alternal mould $\text{logmu}(\text{irr}^\bullet)$ and picking some Lyndon basis in the corresponding Lie algebra, but that would entail a slight loss of canonicity. The truth of the matter is that the irreducibles *spontaneously* present themselves in the shape of a symmetral mould – and there is no going against that. The whole point of the *reduction*, of course, lies in the change from a mould ze^\bullet with a double symmetry and indices s_i running through $\{1, 2, 3, 4 \dots\}$, to a mould irr^\bullet with a single symmetry and indices s_i running through $\{3, 5, 7, 9 \dots\}$.

⁽⁴⁵⁾ via the ARI/GARI machinery

these reverse formulae are trebly defective: they are not particularly explicit; they are decidedly non-unique; and they are ‘partial’, in the sense of *leaning* towards one or the other of the two natural encodings. To remove these blemishes, we require expansions which, like (12.2)-(12.3), express Zag_{II}^\bullet , Zag_{III}^\bullet in terms of $loma^\bullet$, but treating these bimoulds as perinomial-meromorphic *functions* and no longer as *power series*. So, instead of breaking up $loma^\bullet$ under the projectors τ^s , we now apply the following dilation automorphisms⁴⁶ δ^n :

$$\delta^n M \binom{u_1 \dots u_r}{v_1 \dots v_r} := n^{-r} M \binom{u_1/n \dots u_r/n}{v_1 \dots v_r} \quad (13.1)$$

The new expansions read (with all n_i running through \mathbb{N}^*):

$$act(Zag_{II}^\bullet) \stackrel{ess}{=} 1 + \sum Urr_{II}^{n_1, \dots, n_r} act(\delta^{n_1} loma^\bullet) \dots act(\delta^{n_r} loma^\bullet) \quad (13.2)$$

$$act(Zag_{III}^\bullet) \stackrel{ess}{=} 1 + \sum Urr_{III}^{n_1, \dots, n_r} act(\delta^{n_1} loma^\bullet) \dots act(\delta^{n_r} loma^\bullet) \quad (13.3)$$

with symmetral moulds $Urr_{II}^\bullet, Urr_{III}^\bullet$ that are rational-valued perinomial functions of the integers n_i .

A first fallout (from inverting (??)) is yet another expansion for $loma^\bullet$:

$$act(loma^\bullet) \stackrel{ess}{=} \sum Orr_{III}^{n_1, \dots, n_r} act(\delta^{n_1} Zag_{III}^\bullet - 1^\bullet) \dots act(\delta^{n_r} Zag_{III}^\bullet - 1^\bullet) \quad (13.4)$$

which is often referred to as “*wasteful-useful*”⁴⁷. But the main consequence is a *direct-impartial expression for the irreducibles*. Indeed, if for any index X of the form II , III or $II+III$ we set:

$$Yrr_X^{s_1, \dots, s_r} \stackrel{ess}{:=} \sum_{n_i \geq 1} Urr_X^{n_1, \dots, n_r} n_1^{-s_1} \dots n_r^{-s_r} \quad (13.5)$$

we get three parallel identities between *symmetral* moulds:

$$Zag_{II+III}^\bullet = gari^\bullet(Zag_{II}^\bullet, Zag_{III}^\bullet) \quad (13.6)$$

$$Urr_{II+III}^\bullet := Urr_{II}^\bullet \times Urr_{III}^\bullet \quad (13.7)$$

$$Yrr_{II+III}^\bullet := Yrr_{II}^\bullet \times Yrr_{III}^\bullet \quad (13.8)$$

and we find that the moulds Irr_X^\bullet and Yrr_X^\bullet actually *coincide*, though they vastly differ as to the way they are defined.

⁽⁴⁶⁾ they are automorphisms of $ARI_{\underline{al}/\underline{al}}$ and, thanks to the factor n^{-r} , of $ARI_{\underline{al}/\underline{il}}$ as well.

⁽⁴⁷⁾ “*wasteful*”, because it derives an object with sparse poles and rational Taylor coefficients from one with “dense” poles and transcendental Taylor coefficients; “*useful*”, because it automatically transports important properties (like invariance under the *digonal involution*: see [E2], Appendix) from upper-case Zag^\bullet and Zag_{III}^\bullet over to $loma^\bullet$ and then, via (6.8), (12.4), (12.5), back to lower-case zag^\bullet , thus proving that these properties (digonal invariance etc) are algebraically implied by the quadratic relations.

Remark 13.1. — There is also a notion of *formal perinomial numbers*, parallel to that of *formal multizetas* (but relative to constraints altogether different from the *quadratic relations*) and the above relations translate into a direct-impartial expression of the irreducibles irr_X^\bullet attached to the *formal multizetas*.

The whole theory also carries over to the case of *rooted multizetas*, but with the new phenomenon of *retroaction*.⁴⁸ Results are particularly simple for the root orders $p = 2$ and $p = 3$. If anything, the case of *Eulerian multizetas* ($p = 2$) is even simpler than that of *ordinary multizetas* ($p = 1$). Perinomial functions still rule the roost. They fall into six main classes:⁴⁹

<i>root order</i>	<i>Bimoulds in</i> ARI ^{reg/disc} al/il	<i>Eupolar functions</i> (meromorphic-perinomial)	<i>Residues ρ</i> (discrete-perinomial)	<i>Taylor coefficients $\rho^\#$</i> (perinomial numbers)
$p = 1$	loma [•] roma [•]	“sparse” multipoles	well-defined perinomial degrees $d_{i,j}$	rational and “Bernoullian”
$p = 2$	loma [•] roma [•]	“dense” multipoles	well-defined perinomial degrees $d_{i,j}$	rational and “Bernoullian”
$p \geq 1$	logari(Zag [•] _{II}) logari(Zag [•] _{III})	“dense” multipoles	well-defined perinomial degrees $d_{i,j}$	multizetaic and “impartial”

14. Conclusion. Looking back/ahead/sideways

Looking back: Despite interesting work by M. Petitot, J. van der Hoeven, Minh, etc, and some vigorous numerical exploration by Broadhurst, Borwein, Kreimer etc, plus some inspired conjectures based on these numerical data, the main problems in multizeta arithmetic were still open by the end of the 90s. The intervention in the last months of 2000 of the (pre-existing) ARI/GARI apparatus unfroze the situation. It yielded at once the *free generation theorem* and the basic dimorphic *special bimoulds*, leading to the *canonical-explicit decomposition* into irreducibles. Later still, in 2002, the recourse to *perinomial objects* opened the way for the *direct-impartial expansion* of irreducibles.

Looking ahead: Here are four possible avenues for further research :
 – going through the list of known multizetaic constraints and showing that each of them is algebraically derivable from the *quadratic relations*. This

(48) which means that, for a fixed weight s , some of the constraints binding the multizetas of a given length $r_0 < s$ do not result from the double symmetries written down for $r = 1, \dots, r_0$: the full sequence $r = 1, \dots, s$ must be taken into account.

(49) we say that a meromorphic-eupolar functions on \mathbb{C}^r with multipoles “at” the multiintegers $\in \mathbb{Z}^r$ has “dense” (resp. “sparse”) multipoles if the latter’s number inside a ball of radius l grows like $\mathcal{O}(l^r)$ (resp. $o(l^r)$).

has already been done in special instances. The trouble with that process is of course its open-endedness. . .

– attempting to establish once and for all the rigorous arithmetical isomorphism of *formal* and *genuine* multizetas. The most promising approach here is that of *direct numerical derivations* patterned on the *alien derivations* of analysis, which have largely systematised the proving of transcendence results for resurgent functions.

– pursuing the investigation of dimorphy beyond multizetas, in the next main dimorphic \mathbb{Q} -rings $\subset \mathbb{C}$ that have already been identified, beginning of course with suitable *formalisations* of these rings.

– exploring the trigeбра $\mathcal{N}a$ of *natural analysable germs*⁵⁰ and its numerical accompaniment, the ring $\mathcal{N}a$ of *natural numbers*, as the (probably) broadest setting for the understanding of numerical dimorphy.

Looking sideways: strangely, multizeta theory is rife with mistakes and misconceptions, some of which persist long after exposure. Here is a sample :

– **the conversion rule (2.15)-(2.17):** though long known and immediate to derive (see §2), it sometimes receives needlessly convoluted proofs.

– **the meromorphic continuation of Ze^\bullet in the s-variables:** this almost self-evident fact (two lines of proof!) has been questioned, even denied, and then given clumsy, roundabout proofs⁵¹. Nor is there any awareness of the existence of another s-continuable bimould Za^\bullet , analogous to Ze^\bullet but linked to the Wa^\bullet encoding.⁵² The only difficult, and still open, question in this context pertains to the irreducibles Irr^\bullet : these are easily definable as holomorphic functions for large positive s-variables, but do they admit a meromorphic continuation on the negative side?

– **status of the self-consistency constraints (2.19):** it has been variously stated or suggested that they don't exhaust the *additional constraints* for rooted multizetas⁵³. In fact the *exact opposite is the case*: they are always *sufficient* and for small values of the root order p they are even *redundant*.

– **status of ARI/GARI:** it has been likened to various constructs, in particular the Ihara algebra, which is downright absurd, if only because :
 (i) the Ihara algebra lacks the dual set of variables \mathbf{u}/\mathbf{v} which is indispensable for a 'symmetrical' treatment of dimorphy⁵⁴
 (ii) it cannot accommodate the *singular functions*⁵⁵ which fit effortlessly into the ARI/GARI framework and on which everything revolves

(50) with its two sets of exotic derivations: *alien* and *foreign*.

(51) which fall well short of a full description of the multipoles and their residues.

(52) it is closely connected with the Taylor coefficients of the *gari*-inverse of Zag^\bullet .

(53) i.e. the constraints that must be added to the *quadratic relations* and the *conversion rule* to get an (empirically) complete system of constraints.

(54) as a consequence, even the *bracket* of ARI and that of the Ihara algebra are irredeemably non-isomorphic.

(55) i.e. those bimoulds of ARI/GARI, which, as functions of \mathbf{u} or \mathbf{v} , have poles at the origin, or away from it.

(iii) it has no place for *any* of the sixty-odd *special moulds*⁵⁶ which are essential to the construction and description of irreducibles.

– ***status of numerical dimorphy***: when not ignored purely and simply, this central fact about transcendental numbers is often discussed within the quite uncongenial framework of “period theory” which, due to its partiality for *integrals* over *series*, distorts at the outset all the symmetries that underpin dimorphy.

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[E3],[E4] shall be available as Orsay preprints in February 2005, and [E5],[E6] shall appear later on in the course of that same year. All four texts [E3,4,5,6] are extremely lengthy, but abridged versions shall be submitted to ordinary mathematical journals. Electronic files of the unabridged versions shall be put on the WEB, along with an assortment of Tables and Maple programmes.

⁽⁵⁶⁾ the infatuation of algebraists with the Ihara apparatus must be the reason why, after more than a decade of fudging with multizetas, such conspicuous objects as pal^\bullet/pil^\bullet and tal^\bullet/til^\bullet had still escaped discovery, although these moulds were standing *right there*, tall and erect, and almost impossible to miss, at the very entrance of multizeta territory, as its guardians!