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Critical exponent and minimization problem in $\mathbb{R}^N$

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RÉSUMÉ. — L'objet de cet article est d'obtenir une solution au problème suivant :

$$\inf\left\{\int_{\mathbb{R}^N}|\nabla u|^2\exp\left(\frac{|x|^2}{4}\right)-\lambda\int_{\mathbb{R}^N}u^2\exp\left(\frac{|x|^2}{4}\right);\right.$$
$$\left.\int_{\mathbb{R}^N}|u+\varphi|^{q_c}\exp\left(\frac{|x|^2}{4}\right)=1\right\},$$

où $\varphi\in C\left(\mathbb{R}^N\right)\cap\left\{u:\mathbb{R}^N\to\mathbb{R};u\exp\left(\frac{|x|^2}{4}\right)\in L^\infty\left(\mathbb{R}^N\right)\right\}, q_c=\dfrac{2N}{N-2}$, $N\geqslant 3$, est l'exposant critique de Sobolev et $\lambda\in\mathbb{R}$. On montre lorsque $\varphi\neq 0$ et sous certaines conditions sur λ, que le problème admet au moins une solution.

ABSTRACT. — Let $K(x)=\exp\left(\dfrac{|x|^2}{4}\right)$,
for $x\in\mathbb{R}^N$, $L^q(K)=\left\{u:\mathbb{R}^N\to\mathbb{R};\int_{\mathbb{R}^N}|u|^qK<\infty\right\}$ and $H^1(K)=\left\{u\in L^2(K);|\nabla u|\in L^2(K)\right\}$. We are concerned with the following minimization problem

$$\inf\left\{\int_{\mathbb{R}^N}|\nabla u|^2K-\lambda\int_{\mathbb{R}^N}u^2K;u\in H^1(K),\int_{\mathbb{R}^N}|u+\varphi|^{q_c}K=1\right\},$$

where $q_c=\dfrac{2N}{N-2}$, $N\geqslant 3$, $\lambda\in\mathbb{R}$ and $\varphi\in C\left(\mathbb{R}^N\right)$ is such that $K\varphi\in L^\infty\left(\mathbb{R}^N\right)$. We show that for $\varphi\neq 0$, the infimum is achieved under some condition on λ.

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1. Introduction and main result

Let $K(x) = \exp\left(\frac{|x|^2}{4}\right)$, for $x \in \mathbb{R}^N$, $N \geqslant 3$ and

$$L^q(K) = \left\{u : \mathbb{R}^N \to \mathbb{R}; \int_{\mathbb{R}^N} |u|^q K < \infty\right\}$$

$$H^1(K) = \left\{u : \mathbb{R}^N \to \mathbb{R}; u, |\nabla u| \in L^2(K)\right\}.$$

Let us consider the minimization problem

$$S_\lambda(K) = \inf\left\{\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \lambda u^2\right) K; u \in H^1(K), \int_{\mathbb{R}^N} |u|^{q_c} K = 1\right\}, \tag{1.1}$$

where $q_c = \frac{2N}{N-2}$ and λ is a real parameter.

It is well known [5] that the infimum $S_\lambda(K)$ is never achieved for $\lambda \leqslant \frac{N}{4}, N \geqslant 3$. Moreover it is shown that the problem

$$-\Delta u - \frac{x.\nabla u}{2} = |u|^{q_c-2} + \lambda u, u > 0, u \in H^1(K),$$

has no solution if $\lambda \notin \left(\frac{N}{4}, \frac{N}{2}\right), N \geqslant 4$.

In this paper we are interested in the perturbed minimization of (1.1) :

$$S_{\varphi,\lambda}(K) = \inf\left\{\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \lambda u^2\right) K; u \in H^1(K), \|u + \varphi\|_{q_c} = 1\right\}, \tag{1.2}$$

where

$$\|v\|_p^p := \int_{\mathbb{R}^N} |v|^p K,$$

$\lambda \in \mathbb{R}$ and where the function $\varphi \in C\left(\mathbb{R}^N\right)$ satisfying

$$|\varphi(x)| \leqslant C \exp\left(-\frac{|x|^2}{4}\right); C > 0. \tag{1.3}$$

We prove that if $\varphi \not\equiv 0$ the infimum (1.2) is achieved. Note that if $\|\varphi\|_{q_c} = 1$, and $\lambda \leqslant \lambda_1$, where $\lambda_1 = \frac{N}{2}$ is the least eigenvalue of $Lv := -\Delta v - x\frac{\nabla v}{2}$ in $H^1(K)$, we get $S_{\varphi,\lambda}(K) = 0$ and the infimum is achieved by 0. Our main result is the following.

THEOREM 1.1.— *Let* $q_c = \frac{2N}{N-2}$ $N \geqslant 3$. $\varphi \in C(\mathbb{R}^N)$ *is not identically zero and satisfies* (1.3).

1. *If* $N \leqslant 6$, $S_{\varphi,\lambda}$ *is achieved for any* λ.
2. *If* $N \geqslant 7$, $S_{\varphi,\lambda}$ *is achieved for any* $\lambda \in \left(\frac{N}{4}, +\infty\right)$.

This result is similar to those proved by Brezis-Nirenberg [1] and by [9] for the minimization problem

$$\inf\left\{\int_\Omega |\Delta u|^2 ; u \in H^2_\theta(\Omega), \int_\Omega |u+\varphi|^{\frac{2N}{(N-4)}} = 1\right\},$$

where Ω is a bounded domain of $\mathbb{R}^N, N > 4$, $H^2_\theta(\Omega) = H^1_0(\Omega) \cap H^2(\Omega)$.

2. Preliminary results

Before going to the proof of the Theorem 1.1 we denote, for $N \geqslant 3$, by

$$S(K) = S_0(K) = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2 K; u \in H^1(K), \int_{\mathbb{R}^N} |u|^{q_c} K = 1\right\}, \tag{2.1}$$

and

$$S = S(1) = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2 ; u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^{q_c} = 1\right\}. \tag{2.2}$$

It is known that the best Sobolev constant S is approached by the test functions $\left(\varepsilon + |x-a|^2\right)^{-\frac{N-2}{2}}$, and $S \leqslant S(K)$ [4, 5]. In fact we can see, by an easy argument, that $S = S(K)$.

LEMMA 2.1.— *We have*

$$S = S(K).$$

Proof. — Let $\zeta \in C^\infty(\mathbb{R}^N)$, $0 \leqslant \zeta \leqslant 1$ and $\zeta(x) = 1$ if $|x| \leqslant 1$ and $\zeta(x) = 0$ if $|x| \geqslant 2$. For $\varepsilon > 0$, let

$$v_\varepsilon(x) = \frac{K^{-\frac{1}{2}}\zeta}{\left(\varepsilon + |x|^2\right)^{\frac{N-2}{2}}},$$

and

$$v^t_\varepsilon(x) = v_\varepsilon(tx)$$

By the definition of $S(K)$ we obtain

$$S(K) \leqslant \frac{\int_{\mathbb{R}^N} |\nabla v_\varepsilon^t(x)|^2 K(x)\, dx}{\left(\int_{\mathbb{R}^N} |v_\varepsilon^t(x)|^{q_c} K(x)\, dx\right)^{\frac{2}{q_c}}}.$$

Thus

$$S(K) \leqslant \frac{\int_{\mathbb{R}^N} |\nabla v_\varepsilon(x)|^2 K\left(\frac{x}{t}\right) dx}{\left(\int_{\mathbb{R}^N} |v_\varepsilon(x)|^{q_c} K\left(\frac{x}{t}\right) dx\right)^{\frac{2}{q_c}}}.$$

Letting $t \to \infty$ one has

$$S(K) \leqslant \frac{\int_{\mathbb{R}^N} |\nabla v_\varepsilon(x)|^2 dx}{\left(\int_{\mathbb{R}^N} |v_\varepsilon(x)|^{q_c} dx\right)^{\frac{2}{q_c}}}.$$

Then

$$S(K) \leqslant S + o(1),$$

thanks to [5]. This implies that $S = S(K)$.

In the sequel we shall use the following estimates from [5].

Let

$$A_0 = \int_{\mathbb{R}^N} \frac{dx}{\left(1+|x|^2\right)^N}; N \geqslant 3,$$

$$A_1 = (N-2)^2 \int_{\mathbb{R}^N} \frac{|x|^2\, dx}{\left(1+|x|^2\right)^N}; N \geqslant 3,$$

$$A_2 = \tfrac{N-2}{2} \int_{\mathbb{R}^N} \frac{|x|^2\, dx}{\left(1+|x|^2\right)^{N-1}}; N \geqslant 5,$$

$$A_3 = \tfrac{1}{16} \int_{\mathbb{R}^N} \frac{|x|^2\, dx}{\left(1+|x|^2\right)^{N-2}}; N \geqslant 7,$$

$$A_4 = \int_{\mathbb{R}^N} \frac{dx}{\left(1+|x|^2\right)^{N-2}}; N \geqslant 5,$$

$$A_5 = \int_{\mathbb{R}^N} \frac{\exp\left(-\frac{|x|^2}{4}\right) dx}{|x|^2}.$$

PROPOSITION 2.2. — *Let*

$$u_\varepsilon = \varepsilon^{\frac{(N-2)}{4}} v_\varepsilon.$$

Set

$$F(w) = \|\nabla w\|_2^2 - \lambda \|w\|_2^2,$$

hence

$$F(u_\varepsilon) = \begin{cases} A_1 + \varepsilon (A_2 - \lambda A_4) + \varepsilon^2 A_3 + O(\varepsilon^2), for N \geqslant 7, \\ A_1 + \varepsilon (A_2 - \lambda A_4) + \frac{1}{32}\omega_5 \varepsilon^2 |\log \varepsilon| + O(\varepsilon^2), for N = 6, \\ A_1 + \varepsilon (A_2 - \lambda A_4) + O\left(\varepsilon^{\frac{3}{2}}\right), for N = 5, \\ A_1 + \frac{1}{2}\omega_3 (1-\lambda) \varepsilon |\log \varepsilon| + O(\varepsilon), for N = 4, \\ A_1 + \varepsilon^{\frac{1}{2}} A_5 (1-\lambda) + O(\varepsilon), for N = 3, \end{cases} \tag{2.3}$$

and

$$\|u_\varepsilon\|_{q_c}^{q_c} = A_0 + O\left(\varepsilon^{\frac{N}{2}}\right), \tag{2.4}$$

where

$$S = \frac{A_1}{A_0^{\frac{2}{q_c}}}, \frac{A_2}{A_4} = \frac{N}{4},$$

and w_{N-1} denotes the measure of the $N-1$ dimensional unit sphere.

To prove Theorem 1.1, we follow an idea intoduced by Brezis-Nirenberg [1] wich involves a careful analysis of a minimizing sequence $\{u_j\} \subset H^1(K)$ for $S_{\varphi,\lambda}$; that is

$$\|u_j + \varphi\|_{q_c} = 1, \tag{2.5}$$

and

$$\|\nabla u_j\|_2^2 - \lambda \|u_j\|_2^2 = S_{\varphi,\lambda} + o(1), \tag{2.6}$$

It is clear that if $\lambda \leqslant 0$, $\{u_j\}$ is bounded. Now assume that $\lambda > 0$. (2.5) implies that $\{u_j\}$ is $L^{q_c}(K)$-bounded, in particular $\{u_j\} \in H^1(K) \cap L^{q_c}_{Loc}(\mathbb{R}^N)$. Thanks to corollary 1.11 in [5] $\forall \varepsilon > 0$ there exists constants $c = c(\lambda, q) > 0$, $R > 0$ such that

$$\int_{\mathbb{R}^N} |u_j|^2 K \leqslant \varepsilon \int_{\mathbb{R}^N} |\nabla u_j|^2 K + c \|u_j\|^2_{L^q(B(0,R))},$$

we deduce from (2.6) that $\{\nabla u_j\}$and $\{u_j\}$ are $L^2(K)$-bounded. Hence there exists a subsequence, still denoted by $\{u_j\}$, and a function $u \in H^1(K)$ such that

$$\begin{array}{llllll} u_j & \rightharpoonup & u & weakly & in & H^1(K), \\ u_j & \rightarrow & u & strongly & in & L^2(K) \quad (H^1(K) \hookrightarrow L^2(K) \text{ is compact } [5]), \\ u_j & \rightarrow & u & a.e. & on & \mathbb{R}^N, \end{array}$$

$$\int_{\mathbb{R}^N} |\nabla u|^2 K - \lambda \int_{\mathbb{R}^N} |u|^2 K \leqslant S_{\varphi,\lambda},$$

and

$$\|u+\varphi\|_{q_c} \leqslant 1.$$

We shall establish that $\|u+\varphi\|_{q_c} = 1$ to deduce that $S_{\varphi,\lambda}(K)$ is achieved by u. Actually, we shall prove that the assumption

$$\|u+\varphi\|_{q_c} < 1 \tag{2.7}$$

leads to a contradiction. This will be a consequence of the following lemmas.

LEMMA 2.3. — *We have*

$$S_{\varphi,\lambda} - \int_{\mathbb{R}^N} |\nabla u|^2 K + \lambda \int_{\mathbb{R}^N} |u|^2 K \geqslant S \left[1 - \int_{\mathbb{R}^N} |u+\varphi|^{q_c} K\right]^{\frac{2}{q_c}}. \tag{2.8}$$

Proof. — Let $w_j = u_j - u$, then

$$\begin{array}{llllll} w_j & \rightharpoonup & 0 & \textit{weakly} & \textit{in} & H^1(K), \\ w_j & \to & 0 & \textit{strongly} & \textit{in} & L^2(K), \\ w_j & \to & 0 & \textit{a.e.} & \textit{on} & \mathbb{R}^N. \end{array}$$

By the definition of $S = S(K)$ and (2.6) we deduce

$$\|\nabla w_j\|_2^2 \geqslant S \|w_j\|_{q_c}^2, \tag{2.9}$$

$$\|\nabla w_j\|_2^2 - \lambda \|w_j\|_2^2 = S_{\varphi,\lambda} - \|\nabla u\|_2^2 + \lambda \|u\|_2^2 + o(1), \tag{2.10}$$

and, by (2.5) ,

$$1 = \|u+\varphi\|_{q_c}^{q_c} + \|w_j\|_{q_c}^{q_c} + o(1), \tag{2.11}$$

thanks to Brezis-Lieb Lemma [3] . Combining (2.9) – (2.11) leads to (2.8) .

LEMMA 2.4. — *For any $v \in H^1(K)$ such that $\|v+\varphi\|_{q_c} < 1$ we have*

$$S_{\varphi,\lambda} - \int_{\mathbb{R}^N} |\nabla v|^2 K + \lambda \int_{\mathbb{R}^N} |v|^2 K \leqslant S \left[1 - \int_{\mathbb{R}^N} |v+\varphi|^{q_c} K\right]^{\frac{2}{q_c}}, \tag{2.12}$$

and therefore

$$S_{\varphi,\lambda} - \int_{\mathbb{R}^N} |\nabla u|^2 K + \lambda \int_{\mathbb{R}^N} |u|^2 K = S \left[1 - \int_{\mathbb{R}^N} |u+\varphi|^{q_c} K\right]^{\frac{2}{q_c}}. \tag{2.13}$$

Proof. — Since $\|v+\varphi\|_{q_c} < 1$, there exists $c_\varepsilon > 0$ such that

$$\|v+\varphi+c_\varepsilon u_\varepsilon\|_{q_c} = 1.$$

Using again Brezis-Lieb Lemma one sees

$$c_\varepsilon^{q_c} \|u_\varepsilon\|_{q_c}^{q_c} = 1 - \|v+\varphi\|_{q_c}^{q_c} + o(1),$$

thus

$$c_\varepsilon^2 = \frac{S}{A_1}\left[1 - \|u+\varphi\|_{q_c}^{q_c}\right]^{\frac{2}{q_c}} + o(1) \tag{2.14}$$

On the other hand we have

$$\|v+c_\varepsilon u_\varepsilon\|_2^2 = \|v\|_2^2 + c_\varepsilon^2 \|u_\varepsilon\|_2^2 + o(1)$$

and

$$\|\nabla(v+c_\varepsilon u_\varepsilon)\|_2^2 = \|\nabla v\|_2^2 + c_\varepsilon^2 \|\nabla u_\varepsilon\|_2^2 + o(1).$$

As

$$S_{\varphi,\lambda} \leqslant \|\nabla(v+c_\varepsilon u_\varepsilon)\|_2^2 - \lambda\|v+c_\varepsilon u_\varepsilon\|_2^2,$$

then

$$S_{\varphi,\lambda} \leqslant F(v) + c_\varepsilon^2 F(u_\varepsilon) + o(1). \tag{2.15}$$

Inequality (2.12) follows directly by substituing (2.14) and estimates (2.3) in (2.15), and the proof of Lemma 2.4 is completed.

As consequence of Lemmas 2.3 and 2.4 we have

LEMMA 2.5. — *Suppose that assumption* (2.7) *holds, then the limit function* $u \in H^1(K)$ *satisfies the following*

$$-\Delta u - \frac{x.\nabla u}{2} = \nu|u+\varphi|^{q_c-2}(u+\varphi) + \lambda u, \text{ on } \mathbb{R}^N \tag{2.16}$$

where

$$\nu = S\left(1 - \|u+\varphi\|_{q_c}^{q_c}\right)^{\frac{2}{q_c}-1}.$$

Proof. — Let $v \in H^1(K)$. Since $\|u+\varphi\|_{q_c} < 1$, there exists $t_0 > 0$ such that

$$\|u+\varphi+tv\|_{q_c} < 1,$$

for all $|t| < t_0$. We deduce from Lemma 2.3

$$S_{\varphi,\lambda} - \int_{\mathbb{R}^N} (\nabla(u+tv))^2 K + \lambda\int_{\mathbb{R}^N} (u+tv) K \leqslant S\left[1 - \int_{\mathbb{R}^N} |u+tv+\varphi|^{q_c} K\right]^{\frac{2}{q_c}}.$$

Thus

$$S_{\varphi,\lambda}-F(u)-2t\int_{\mathbb{R}^N}\nabla u\nabla vK+2t\lambda\int_{\mathbb{R}^N}uvK+o(t)$$
$$\leqslant S\left[1-\int_{\mathbb{R}^N}|u+tv+\varphi|^{q_c}K\right]^{\frac{2}{q_c}},$$

$$S_{\varphi,\lambda}-F(u)-2t\int_{\mathbb{R}^N}\nabla u\nabla vK+2t\lambda\int_{\mathbb{R}^N}uvK+o(t)$$
$$\leqslant S\left[1-\int_{\mathbb{R}^N}|u+\varphi|^{q_c}K\right]^{\frac{2}{q_c}}\times$$
$$\left[1-2t\left(1-\|u+\varphi\|_{q_c}^{q_c}\right)^{-1}\int_{\mathbb{R}^N}|u+\varphi|^{q_c-2}(u+\varphi)vK+o(t)\right].$$

Using (2.13), we get

$$-2t\int_{\mathbb{R}^N}\nabla u\nabla vK+2t\lambda\int_{\mathbb{R}^N}uvK+o(t)$$
$$\leqslant -2tS\left(1-\|u+\varphi\|_{q_c}^{q_c}\right)^{\frac{2}{q_c}-1}\int_{\mathbb{R}^N}|u+\varphi|^{q_c-2}(u+\varphi)vK,$$

Letting $t\to 0^{\pm}$ we deduce Lemma 2.5.

LEMMA 2.6.— *Suppose that* (2.7) *holds, then the limit function* u *satisfies*

$$u+\varphi\neq 0,\ \textit{and}\ u\neq 0.$$

Proof. — On the contrary, suppose that $u+\varphi\equiv 0$. Then $u=-\varphi\neq 0$ and satisfies

$$-\Delta u-\frac{x.\nabla u}{2}=\lambda u. \tag{2.17}$$

Next since $\|u+\varphi\|_{q_c}=0$, there exists $t_0>0$ such that

$$\|t_0u+\varphi\|_{q_c}=1.$$

Using Lemma 2.4 and (2.17) we deduce that $S_{\varphi,\lambda}\leqslant 0$.

By the fact that $u+\varphi=0$ equality (2.11) shows that $S=S_{\varphi,\lambda}\leqslant 0$, which is impossible. Now assume that $u\equiv 0$. Using again equation (2.16) we infer that $v=0$, since $u+\varphi\neq 0$. Therefore $1=\|u+\varphi\|_{q_c}$, a contradiction.

Remark 2.7. — Arguing as in [5, p.1121] one sees that the limit function u satisfying (2.16) is in $L^\infty(\mathbb{R}^N)$. Now $w = u \exp\left(\frac{|x|^2}{8}\right)$ satisfies

$$-\Delta w + \left(\frac{N}{4} - \lambda + \frac{|x|^2}{16} - \nu |u+\varphi|^{q_c-2}\right) w = \nu \varphi K^{\frac{1}{2}} |u+\varphi|^{q_c-2}.$$

The last equation can be written as

$$-\Delta w + V(x) w = f,$$

where $V^-(x) = \max(-V(x), 0) \in L^\infty(\mathbb{R}^N) \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ and $f \in L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. We conclude as in [5, p.1119] that $w \in C^2(\mathbb{R}^N)$ and then $u \in C^2(\mathbb{R}^N)$.

3. Existence of a minimizer for $S_{\varphi,\lambda}$

As consequence of Lemma 2.6 we shall prove that assumption (2.7) leads to a contradiction. Suppose λ and N as in Theorem 1.1, then we have

LEMMA 3.1.— *Assumption* (2.7) *implies*

$$S_{\varphi,\lambda} - \int_{\mathbb{R}^N} |\nabla u|^2 K + \lambda \int_{\mathbb{R}^N} u^2 K < S\left[1 - \|u+\varphi\|_{q_c}^{q_c}\right]^{\frac{2}{q_c}}. \tag{3.1}$$

This lemma contradicts (2.13), that means that hypothesis (2.7) is not true, hence

$$\|u+\varphi\|_{q_c} = 1,$$

and then $S_{\varphi,\lambda}$ is achieved.This ends the proof of Theorem 1.1. Now we return to the proof of Lemma 3.1.

Proof of Lemma 3.1. — Since $u + \varphi \neq 0$, we may assume that $(u+\varphi)(0) > 0$. Then there exists a ball $B(0,r), r > 0$ such that $u + \varphi > 0$ on $B(0,r)$. Choose the function ζ in u_ε such that $\zeta \in C_0^\infty(B(0,r))$.

As in the proof of Lemma 2.4, there exists $c_\varepsilon > 0$ such that

$$\|u + \varphi + c_\varepsilon u_\varepsilon\|_{q_c} = 1,$$

where

$$c_\varepsilon^2 = \frac{S}{A_1}\left[1 - \|u+\varphi\|_{q_c}^{q_c}\right]^{\frac{2}{q_c}} + o(1). \tag{3.2}$$

Put

$$R_\varepsilon := 1 - \int_{\mathbb{R}^N} |u+\varphi|^{q_c} K - c_\varepsilon^{q_c} \int_{\mathbb{R}^N} |u_\varepsilon|^{q_c} K - q_c c_\varepsilon^{q_c-1} \int_{\mathbb{R}^N} |u_\varepsilon|^{q_c-1} (u+\varphi) K$$
$$-q_c c_\varepsilon \int_{\mathbb{R}^N} |u+\varphi|^{q_c-2} (u+\varphi) u_\varepsilon K. \tag{3.3}$$

Arguing as in [1], [9] one sees that

$$R_\varepsilon = o\left(\varepsilon^{\frac{N-2}{4}}\right), \tag{3.4}$$

for ε small enough.

We have from [4] and [5],

$$\int_{\mathbb{R}^N} |u_\varepsilon|^{q_c} K = \left(\frac{A_1}{S}\right)^{\frac{q_c}{2}} + O\left(\varepsilon^{\frac{N}{2}}\right), \tag{3.5}$$

on the other hand

$$\int_{\mathbb{R}^N} |u_\varepsilon|^{q_c-1} (u+\varphi) K dx = \int_{\mathbb{R}^N} (u+\varphi) K^{\frac{1-q_c}{2}} \zeta^{q_c-1} \frac{\varepsilon^{\frac{N+2}{4}}}{\left(\varepsilon+|x|^2\right)^{\frac{N+2}{2}}} dx$$
$$= \varepsilon^{\frac{N-2}{4}} \int_{\mathbb{R}^N} (u+\varphi) \zeta^{q_c-1} K^{\frac{N-6}{2(N-2)}} \frac{1}{\varepsilon^{\frac{N}{2}}} \Psi_1\left(\frac{x}{\varepsilon^{\frac{1}{2}}}\right) dx,$$

where

$$\Psi_1(x) = \frac{1}{\left(1+|x|^2\right)^{\frac{N+2}{2}}} \in L^1\left(\mathbb{R}^N\right).$$

Therefore, Hölder's inequality implies that $(u+\varphi)\zeta^{q_c-1} K^{\frac{N-6}{2(N-2)}} \in L^1\left(\mathbb{R}^N\right)$.

Setting

$$B = \int_{\mathbb{R}^N} \Psi_1(x)\, dx,$$

we derive (see [6], Theorem. 8.15, p. 235)

$$\int_{\mathbb{R}^N} (u+\varphi)(x) K^{\frac{N-6}{2(N-2)}}(x) \zeta^{q_c-1}(x) \frac{1}{\varepsilon^{\frac{N}{2}}} \Psi_1\left(\frac{x}{\varepsilon^{\frac{1}{2}}}\right) dx \to (u+\varphi)(0) B.$$

Thus

$$\int_{\mathbb{R}^N} |u_\varepsilon|^{q_c-1} (u+\varphi) K = \varepsilon^{\frac{N-2}{4}} B (u+\varphi)(0) + o\left(\varepsilon^{\frac{N-2}{4}}\right), \tag{3.6}$$

and a simple computation yields that

$$\int_{\mathbb{R}^N} |u+\varphi|^{q_c-2}(u+\varphi)\,u_\varepsilon K = C\varepsilon^{\frac{N-2}{4}} + o\left(\varepsilon^{\frac{N-2}{4}}\right), C>0. \tag{3.7}$$

Let δ_ε and c_0 given by

$$c_\varepsilon = c_0\left(1-\delta_\varepsilon\right), c_0^2 = \frac{S}{A_1}\left[1-\|u+\varphi\|_{q_c}^{q_c}\right]^{\frac{2}{q_c}}.$$

In view of $(3.2), (3.3)$ one sees

$$\begin{aligned}\delta_\varepsilon c_0^{q_c}\left(\frac{A_1}{S}\right)^{\frac{q_c}{2}} &= q_c c_0^{q_c-1}\int_{\mathbb{R}^N} |u_\varepsilon|^{q_c-1}(u+\varphi)\,K \\ &\quad + q_c c_\varepsilon \int_{\mathbb{R}^N} |u+\varphi|^{q_c-2}(u+\varphi)\,u_\varepsilon K + o\left(\delta_\varepsilon\right) + o\left(\varepsilon^{\frac{N-2}{4}}\right),\end{aligned}$$

therefore $\delta_\varepsilon = O\left(\varepsilon^{\frac{N-2}{4}}\right)$.

Now, using $u + c_\varepsilon u_\varepsilon$ as a test function in problem (1.2) we get

$$S_{\varphi,\lambda} \leqslant F(u) + c_\varepsilon^2 F\left(u_\varepsilon\right) + 2c_\varepsilon \int_{\mathbb{R}^N} \nabla u_\varepsilon \nabla u K - 2c_\varepsilon\lambda \int_{\mathbb{R}^N} u u_\varepsilon K, \tag{3.8}$$

and then

$$S_{\varphi,\lambda} \leqslant F(u) + c_\varepsilon^2 F\left(u_\varepsilon\right) + 2\nu c_\varepsilon \int_{\mathbb{R}^N} |u+\varphi|^{q_c-2}(u+\varphi)\,u_\varepsilon K, \tag{3.9}$$

by Lemma 2.3.

First assume that $N \geqslant 7$. Using (3.7) and (2.3), we deduce

$$\begin{aligned}S_{\varphi,\lambda} - F(u) \leqslant{}& c_0^2\left(1-2\delta_\varepsilon+\delta_\varepsilon^2\right)\left(A_1 + \varepsilon\left(A_2-\lambda A_4\right) + \varepsilon^2 A_3\right. \\ &\left. + O\left(\varepsilon^2\right)\right) + 2\nu c_0 C\varepsilon^{\frac{N-2}{4}} + o\left(\varepsilon^{\frac{N-2}{4}}\right).\end{aligned} \tag{3.10}$$

It follows from this that

$$\begin{aligned}S_{\varphi,\lambda} - \int_{\mathbb{R}^N} |\nabla u|^2 K + \lambda\int_{\mathbb{R}^N} |u^2| K \leqslant{}& c_0^2 A_1 - \varepsilon(c_0^2(\lambda A_4 - A_2) \\ & -2\nu c_0 C\varepsilon^{\frac{N-6}{4}} + o(\varepsilon^{\frac{N-6}{4}})),\end{aligned}$$

and the last expression leads to (3.1) when $\lambda A_4 - A_2 > 0$.

Now if $N \leqslant 6$, we have $q_c \geqslant 3$ and then we use the elementary inequality [9]

$$(x+y)^p - x^p - y^p - px^{p-1}y - pxy^{p-1} \geqslant 0, x, y \geqslant 0, p \geqslant 3.$$

Thus

$$\begin{aligned}
q_c c_\varepsilon \int_{\mathbb{R}^N} |u+\varphi|^{q_c-2}(u+\varphi) u_\varepsilon K &\leqslant 1 - \|u+\varphi\|_{q_c}^{q_c} - c_\varepsilon^{q_c} \|u_\varepsilon\|_{q_c}^{q_c} \\
&\quad - q_c c_\varepsilon^{q_c-1} \int_{\mathbb{R}^N} |u_\varepsilon|^{q_c-1}(u+\varphi) K, \\
&\leqslant 1 - \|u+\varphi\|_{q_c}^{q_c} - \varepsilon^{\frac{N-2}{4}} q_c c_0^{q_c-1} B(u+\varphi)(0) + o\left(\varepsilon^{\frac{N-2}{4}}\right) \\
&\quad - c_0^{q_c}\left(1 - q_c\delta_\varepsilon + \frac{q_c(q_c-1)}{2}\delta_\varepsilon^2 + o\left(\delta_\varepsilon^2\right)\right) \\
&\quad \times \left(\left(\frac{A_1}{S}\right)^{\frac{q_c}{2}} + O\left(\varepsilon^{\frac{N}{2}}\right)\right), \\
&\leqslant q_c c_0^{q_c}\left(\frac{A_1}{S}\right)^{\frac{q_c}{2}}\delta_\varepsilon - \frac{q_c(q_c-1)}{2} c_0^{q_c}\left(\frac{A_1}{S}\right)^{\frac{q_c}{2}}\delta_\varepsilon^2 + o\left(\delta_\varepsilon^2\right) \\
&\quad - \varepsilon^{\frac{N-2}{4}} q_c c_0^{q_c-1} B(u+\varphi)(0) + o\left(\varepsilon^{\frac{N-2}{4}}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
2\nu c_\varepsilon \int_{\mathbb{R}^N} |u+\varphi|^{q_c-2}(u+\varphi) u_\varepsilon K &\leqslant 2\nu c_0^{q_c}\left(\frac{A_1}{S}\right)^{\frac{q_c}{2}}\delta_\varepsilon - \nu c_0^{q_c}(q_c-1)\left(\frac{A_1}{S}\right)^{\frac{q_c}{2}}\delta_\varepsilon^2 \\
&\quad + o\left(\delta_\varepsilon^2\right) - \varepsilon^{\frac{N-2}{4}} \nu c_0^{q_c-1} B(u+\varphi)(0) + o\left(\varepsilon^{\frac{N-2}{4}}\right), \\
&\leqslant \delta_\varepsilon A_1 c_0^2 (2 - (q_c-1)\delta_\varepsilon) - \varepsilon^{\frac{N-2}{4}} \nu c_0^{q_c-1} B(u+\varphi)(0) \\
&\quad + o\left(\varepsilon^{\frac{N-2}{4}}\right) + o\left(\delta_\varepsilon^2\right).
\end{aligned}$$

Using this we obtain for $N = 6$,

$$\begin{aligned}
S_{\varphi,\lambda} - \int_{\mathbb{R}^6} |\nabla u|^2 K + \lambda \int_{\mathbb{R}^6} \left|u^2\right| K &\leqslant c_0^2\left(1 - 2\delta_\varepsilon + \delta_\varepsilon^2\right) \times \\
&\quad \left(A_1 + \varepsilon(A_2 - \lambda A_4) - \frac{1}{32}\omega_5 \varepsilon^2 |\log\varepsilon| + O\left(\varepsilon^2\right)\right) \\
&\quad + \delta_\varepsilon A_1 c_0^2 (2 - (q_c-1)\delta_\varepsilon) - \varepsilon\nu c_0^{q_c-1} B(u+\varphi)(0) \\
&\quad + o(\varepsilon) + o\left(\delta_\varepsilon^2\right),
\end{aligned}$$

and then

$$S_{\varphi,\lambda} - \int_{\mathbb{R}^6} |\nabla u|^2 K + \lambda \int_{\mathbb{R}^6} |u^2| K \leqslant c_0^2 A_1 - \varepsilon\nu c_0^{q_c-1} B(u+\varphi)(0)$$
$$+c_0^2\delta_\varepsilon^2\left[(2-q_c)A_1 + \varepsilon(A_2 - \lambda A_4)\right]$$
$$+o(\varepsilon) + o\left(\delta_\varepsilon^2\right).$$

Therefore

$$S_{\varphi,\lambda} - \int_{\mathbb{R}^6} |\nabla u|^2 K + \lambda \int_{\mathbb{R}^6} |u^2| K < c_0^2 A_1, \tag{3.11}$$

for any λ. A similar argument can be used to extend (3.11) to the case $N \leqslant 5$. The proof is left to the reader. This completes the proof of Lemma 3.1.

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