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Potentials in pluripotential theory^(*)

MAGNUS CARLEHED⁽¹⁾

RÉSUMÉ. — Nous traitons des potentiels des mesures positives dans un domaine de \mathbb{C}^n , où le noyau est le logarithme ou la fonction de Green pluri-complexe. Leur mesure de Monge-Ampère est calculée, et nous étudions quelles propriétés des potentiels classiques demeurent.

ABSTRACT. — We discuss potentials of positive measures in domains in \mathbb{C}^n , where the kernel is the logarithm or the pluricomplex Green function. Their Monge-Ampère measure is computed, and we investigate which properties of classical potentials remain.

1. Introduction and definition of potentials

In classical potential theory in \mathbb{R}^n , potentials of positive measures play a crucial role. The theory is linear and, due to Riesz decomposition theorem, subharmonic functions are essentially potentials modulo harmonic functions. Furthermore, we can solve Dirichlet problems using convolutions. In the case $n = 2$ we can equally well work in the complex plane, and the Laplacian is invariant under conformal mappings. For a thorough treatment of classical potential theory we refer to the book by Hayman and Kennedy [Ha-Ke]. However, from the point of view of complex analysis classical potential theory is unsatisfactory. The Laplacian is not biholomorphically invariant in higher dimensions.

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Pluripotential theory is the study of plurisubharmonic functions and the complex Monge-Ampère operator. The operator is defined on certain classes of plurisubharmonic functions in domains in \mathbb{C}^n . In the case $n = 1$, it reduces to the Laplacian, but in higher dimension it is non-linear. It is invariant under biholomorphic mappings. We refer to Klimek's monograph [Kl] for an introduction to pluripotential theory and the basic definitions.

Since pluripotential theory is non-linear, we can not expect potentials to be as fruitful as in the classical case. Still, it makes sense to study them, and ask which properties remain, which is the purpose of this article.

We recall the following definition of the multipole Green function.

DEFINITION 1.1 [Lel2]. — Let Ω be a domain in \mathbb{C}^n , and let

$$A = \{(a_k, \nu_k), 1 \leq k \leq p\}$$

be a finite system of points $a_k \in \Omega$ with weights $\nu_k > 0$. Define

$$g(z, A) = \sup \{v(z) : v \in U(A), v < 0\}$$

where

$$U(A) = \{v \in PSH(\Omega) : v(\zeta) \leq \nu_k \log |\zeta - a_k| + O(1), \zeta \rightarrow a_k, 1 \leq k \leq p\}.$$

We call this the *pluricomplex Green function* for Ω , relative to the system A .

In hyperconvex domains the Green function is continuous, and tends to zero at the boundary [Lel2]. In the case $p = 1, a_1 = w, \nu_1 = 1$ we have the standard pluricomplex Green function $g(z, w)$ with one pole at $w \in \Omega$.

The following theorem shows that it makes sense to talk about potentials in pluripotential theory.

THEOREM 1.2 [Lel1]. — *Let Ω be a domain in \mathbb{C}^n , (T, μ) a locally compact measure space, with a positive measure μ . Let $K : \Omega \times T \rightarrow [-\infty, \infty)$ be a function such that*

- 1) $K(ze^{i\theta}, t)$ is measurable with respect to the product measure $d\theta \otimes \mu$,
- 2) $z \mapsto K(z, t)$ is plurisubharmonic for each $t \in T$,
- 3) $t \mapsto K(z, t)$ is measurable for each $z \in \Omega$,
- 4) the function $z \mapsto \sup_{t \in T} K(z, t)$ is locally upper bounded on Ω .

Then $q_\mu^K(z) := \int_T K(z, w) d\mu(w)$ is plurisubharmonic on Ω .

We will mainly study two special cases, so we introduce special notation for these.

DEFINITION 1.3. — Let μ be a finite, positive measure with support in Ω , where Ω is a bounded domain in \mathbb{C}^n . Let $g(z, w)$ be the pluricomplex Green function for Ω with pole in $w \in \Omega$. We define the *pluricomplex potential* of μ as

$$p_\mu(z) = \int_{\Omega} g(z, w) d\mu(w),$$

and the *logarithmic potential* of μ as

$$lp_\mu(z) = \int_{\Omega} \log |z - w| d\mu(w).$$

Clearly both potentials are plurisubharmonic in Ω , and if Ω is hyperconvex, then p_μ vanishes on the boundary. For the logarithmic potential in \mathbb{R}^{2n} , we refer to [Ha-Ke], Chapter 5.

In Section 2 we prove a characterization of logarithmic potentials, which can be viewed as a counterpart to Riesz decomposition theorem.

In Section 3 we give some examples of pluricomplex potentials in the unit ball.

It is possible to define the measures

$$dd^c g(z, v_1) \wedge \dots \wedge dd^c g(z, v_n) \tag{1}$$

and

$$dd^c \log |z - v_1| \wedge \dots \wedge dd^c \log |z - v_n|. \tag{2}$$

This is done in Section 4, and there we also show that the latter is absolutely continuous with respect to the Lebesgue measure unless all v_i coincide. If $g(\cdot, v) \in C^2(\Omega \setminus \{v\})$ for all $v \in \Omega$, the same holds for $dd^c g(z, v_1) \wedge \dots \wedge dd^c g(z, v_n)$.

Then (Section 5) we extend the comparison principle and partial integration formulas to certain classes of unbounded plurisubharmonic functions.

In Section 6, we define so-called integrated measures. Typically we will have a family of positive measures $\nu(v_1, \dots, v_n)$, depending on the parameters v_1, \dots, v_n , such that the total mass is bounded, uniformly in all the parameters. Examples are (1) and (2). We then define measures

$$\int_{\Omega^k} \nu(v_1, \dots, v_n) d\mu(v_1) \cdots d\mu(v_k),$$

where $1 \leq k \leq n$, and μ is a given positive measure.

After that we prove representation formulas for the Monge-Ampère measures of bounded potentials. For the pluricomplex potential, the formula is

$$(dd^c p_\mu(z))^n = \int_{\Omega^n} dd_z^c g(z, v_1) \wedge \dots \wedge dd_z^c g(z, v_n) d\mu(v_1) \cdots d\mu(v_n). \quad (3)$$

This is done in Section 7.

In Section 8, we perform the procedure $\mu \mapsto p_\mu \mapsto (dd^c p_\mu)^n$. It turns out that if $g(\cdot, v) \in C^2$ and p_μ is bounded, then this procedure is “smoothing”, i.e. the resulting measure is absolutely continuous, even if μ is not. We also prove some estimates for this procedure.

Finally, in Section 9, we investigate measures whose pluricomplex potentials in the unit ball have finite energy in the sense of Cegrell [Ce1]. We give a sufficient condition for this, and extend the representation formula to that class. The procedure $\mu \mapsto p_\mu \mapsto (dd^c p_\mu)^n$ is still “smoothing”. We also give an example of a measure such that the potential is unbounded but of finite energy.

We would like to mention two other motivations to study potentials in several complex variables. The first is the concept of pluri-thin sets. For a discussion of these, we refer the reader to [Ca]. The second motivation is the following. We define C to be the cone of non-positive plurisubharmonic functions in a fixed hyperconvex domain. We say that a function φ in C is extremal, if $\varphi = \varphi_1 + \varphi_2$, where each φ_i is in C , implies that there are positive constants α_i such that $\varphi_i = \alpha_i \varphi$, $i = 1, 2$. It is known that the pluricomplex Green function with one pole is always extremal [Ce-T]. From the general Choquet theory of convexity the following theorem is known. Let C be a convex cone. Then for each $q \in C$ there is a positive measure μ_q supported on the extreme elements of C such that $f(q) = \int f(x) d\mu_q(x)$ for all affine functions f . In our situation, taking f as a point evaluation at a point a in the domain, we find that for any function $\varphi \in C$ there is a measure μ_φ such that $\varphi(a) = \int \varphi(x) d\mu_\varphi(x)$ for all a in the domain. Here x varies over all extremal plurisubharmonic functions. A pluricomplex potential is a special case of this, namely where the measure is supported only on the (one-polar) pluricomplex Green functions. We do not carry this discussion any further in this paper, however, the results indicate that the pluricomplex potentials form a very small subset of C (see Theorem 8.1). Therefore there should be plenty of extremal plurisubharmonic functions, in addition to the Green functions. To characterize all of these seems almost impossible, but it would be interesting to have at least a few more examples.

We also make a general remark about dimension. Most of the results of this paper are valid for domains in \mathbb{C}^n , but to avoid too unwieldy expressions

and computations, some results in Sections 8 and 9 are done in \mathbb{C}^2 only. However, we suspect that they are true in higher dimensions too.

Most of the results of this article can be found in Paper I in [Ca], and in some cases more details are given there.

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2. Characterization of logarithmic potentials

For logarithmic potentials, it is possible to give a counterpart to the Riesz decomposition theorem.

PROPOSITION 2.1. — *There is a positive constant C_n , such that $\Delta^n lp_\mu = C_n \mu$, in the distributional sense, for each measure μ . Conversely, if u is a real-valued function, such that $\Delta^n u$ exists in the distributional sense, and equals a positive measure $C_n \mu$, then locally we have*

$$u = lp_\mu + b,$$

where b is a real analytic function, and $\Delta^n b = 0$.

Proof. — The first claim follows from the fact that $\log |x|$ is a fundamental solution for the operator Δ^n in \mathbb{R}^{2n} , see [A-Cr-Li]. To prove the second, note that

$$\Delta^n(u - lp_\mu) = 0$$

in the sense of distributions. Since the operator Δ^n is elliptic it has only classical solutions, and hence $u - lp_\mu$ is an n -harmonic function. These are real analytic. \square

Remark. — In a relatively compact subset of Ω , which is star-shaped with centre a , we can write b as

$$b(z) = \sum_{j=0}^{n-1} |z - a|^{2j} h_j(z),$$

where each h_j is harmonic. This is called the *Almansi expansion* [A-Cr-Li].

3. Some explicit examples of pluricomplex potentials in the unit ball

In [Ca] the pluricomplex potentials of some measures are computed explicitly. Here we only state the results, and refer the reader to [Ca] for the proofs.

PROPOSITION 3.1. — *Let B be the unit ball in \mathbb{C}^2 , and μ be the normalized Lebesgue measure on the disc $D \times \{0\}$. Then*

$$p_\mu(z) = \frac{(|z|^2 - 1) \left(1 - |z_2|^2 + |z_2|^2 \log |z_2|^2\right)}{2 \left(1 - |z_2|^2\right)^2}$$

PROPOSITION 3.2. — *Let $\sigma = \sigma_r$ be the surface measure on the sphere $\partial B(0, r)$ in \mathbb{C}^2 , $r > 0$. Let $t = |z|$. Then*

$$p_\sigma(z) = p_{\sigma_r}(z) = \pi^2 r^3 \cdot \begin{cases} \log r^2 + \frac{1 - r^2}{r^2} + \frac{(1 - r^2)(1 - t^2)}{r^2 t^2} \log(1 - t^2), & \text{if } t \leq r \\ \log t^2 + \frac{1 - t^2}{t^2} + \frac{(1 - r^2)(1 - t^2)}{r^2 t^2} \log(1 - r^2), & \text{if } t \geq r \end{cases}$$

(To get the potential of the normalized surface measure, divide by $2\pi^2 r^3$.)

THEOREM 3.3. — *Let μ be the normalized Lebesgue measure in the unit ball, i.e. λ/b_{2n} , where λ is the Lebesgue measure. Then the potential is*

$$p_\mu(z) = (|z|^2 - 1)/(2n),$$

and hence

$$(dd^c p_\mu(z))^n = \left(\frac{2\pi}{n}\right)^n \mu.$$

4. Definition of the Monge-Ampère operator on certain classes of unbounded functions

It is possible to define measures of the type $dd^c u^1 \wedge \dots \wedge dd^c u^n$, where each u^j is plurisubharmonic and locally bounded outside a finite set A . This can be done by generalizing a method used by Klimek [Kl] in the case where A consists of only one point. We refer the reader to [Kl] and [Ca] for details. The construction is only a special case of the much more general treatment in [S].

Let Ω be a domain in \mathbb{C}^n , and $A = \{a_1, \dots, a_p\}$ a fixed finite subset of Ω . Define

$$PSH(\Omega; A) = PSH(\Omega) \cap L_{loc}^\infty(\Omega \setminus A).$$

PROPOSITION 4.1. — *Let u^1, \dots, u^n be functions in $PSH(\Omega; A)$. Then there exists a positive Borel measure μ on Ω such that if $\{u_i^j\}_{1 \leq i, 1 \leq j \leq n} \subset PSH \cap L_{loc}^\infty(\Omega)$ and $u_i^j \searrow u^j$, when $i \rightarrow \infty$, for all $1 \leq j \leq n$, then $dd^c u_i^1 \wedge \dots \wedge dd^c u_i^n$ is weak*-convergent to μ .*

DEFINITION 4.2. — We define $dd^c u^1 \wedge \dots \wedge dd^c u^n = \mu$, where μ is the measure of the preceding proposition.

The special cases we have in mind are

$$dd^c g(z, v_1) \wedge \dots \wedge dd^c g(z, v_n)$$

and

$$dd^c \log |z - v_1| \wedge \dots \wedge dd^c \log |z - v_n|.$$

In these expressions, we always allow points to be repeated. It is well-known (see e.g. [K1]) that if all the v_j coincide, and are equal to v , then both these measures are equal to $(2\pi)^n \delta_v$. We now want to examine what happens if they do not coincide. We will need the following special case of a theorem by Demailly [D].

THEOREM 4.3. — *Assume that $a \in \Omega$, $u, v \in PSH \cap C(\Omega, [-\infty, \infty])$, $u < v$ in $\Omega \setminus \{a\}$, $u^{-1}(\{-\infty\}) = v^{-1}(\{-\infty\}) = \{a\}$, and*

$$\overline{\lim}_{z \rightarrow a} \frac{u(z)}{v(z)} = 1.$$

Then $(dd^c u)^n(\{a\}) \leq (dd^c v)^n(\{a\})$. If the upper limit is a limit, then equality holds.

PROPOSITION 4.4. — *Assume that $v_j \in \Omega$, for all j , and that $v_i \neq v_j$ for some indices i and j . Then $dd^c \log |z - v_1| \wedge \dots \wedge dd^c \log |z - v_n|$ is an absolutely continuous measure with respect to the Lebesgue measure. If $g(\cdot, v) \in C^2(\Omega \setminus \{v\})$ for all $v \in \Omega$, then $dd^c g(z, v_1) \wedge \dots \wedge dd^c g(z, v_n)$ is an absolutely continuous measure with respect to the Lebesgue measure.*

Proof. — We prove the proposition for g , the proof for the logarithm being essentially the same. Let $1 \leq k \leq n - 1$. It is sufficient to show that if $b_j \neq a$, for all $1 \leq j \leq n - k$, then

$$\int_{\{a\}} (dd^c g(z, a))^k \wedge dd^c g(z, b_1) \wedge \dots \wedge dd^c g(z, b_{n-k}) = 0.$$

Define

$$u(z) = g(z, a) + \sum_{j=1}^{n-k} g(z, b_j).$$

Then using Theorem 4.3 near a , we obtain $\int_{\{a\}} (dd^c u)^n = \int_{\{a\}} (dd^c g(z, a))^n = (2\pi)^n$. On the other hand, expanding the wedge product, we get

$$\begin{aligned} (dd^c u)^n &= (dd^c g(z, a))^n + (dd^c g(z, a))^k \wedge dd^c g(z, b_1) \wedge \dots \wedge dd^c g(z, b_{n-k}) \\ &\quad + \text{other terms} \\ &\geq (2\pi)^n \delta_a + (dd^c g(z, a))^k \wedge dd^c g(z, b_1) \wedge \dots \wedge dd^c g(z, b_{n-k}). \end{aligned}$$

Thus

$$\begin{aligned} \int_{\{a\}} (dd^c g(z, a))^k \wedge dd^c g(z, b_1) \wedge \dots \wedge dd^c g(z, b_{n-k}) \\ \leq \int_{\{a\}} (dd^c u)^n - (2\pi)^n = 0, \end{aligned}$$

and the proof is complete. \square

The hypothesis about g being C^2 is fulfilled in a strictly convex domain with smooth boundary [Lem].

Let A be as in Definition 1.1. Define

$$h(z, A) = \sum_{j=1}^p \nu_j g(z, a_j).$$

As an example, let us assume that $\Omega \subset \mathbb{C}^2$ is such that $g \in C^2$, and compute $(dd^c h(z, A))^2$:

$$\begin{aligned} (dd^c h(z, A))^2 &= \sum_j \nu_j^2 (dd^c g(z, a_j))^2 + \sum_{j \neq k} \nu_j \nu_k dd^c g(z, a_j) \wedge dd^c g(z, a_k) \\ &= (2\pi)^2 \sum_j \nu_j^2 \delta_{a_j} + \sum_{j \neq k} \nu_j \nu_k dd^c g(z, a_j) \wedge dd^c g(z, a_k). \end{aligned}$$

The right hand side is the Lebesgue decomposition of the measure relative to the Lebesgue measure, and the first term of the right hand side is $(dd^c g(z, A))^2$.

5. Comparison principle and partial integration for certain unbounded plurisubharmonic functions

In what follows we will often need to compare certain measures and do partial integration. We therefore need to justify this in a more general setting than the standard one.

DEFINITION 5.1. — For any function u with values in $[-\infty, \infty)$ and any integer L , define $u^L(z) = \max \{u(z), L\}$.

PROPOSITION 5.2. — Let Ω be a bounded domain in \mathbb{C}^n , and suppose that $\varphi_1, \dots, \varphi_n$ are plurisubharmonic functions in Ω . Assume that for some neighbourhood U of $\partial\Omega$, all φ_i are bounded in $U \cap \Omega$. Set

$$L = \min_{i=1, \dots, n} \left(\inf_{z \in U \cap \Omega} \varphi_i(z) \right).$$

Then, for all $M, N \leq L$,

$$\int_{\Omega} dd^c \varphi_1^M \wedge \dots \wedge dd^c \varphi_n^M = \int_{\Omega} dd^c \varphi_1^N \wedge \dots \wedge dd^c \varphi_n^N.$$

We omit the simple proof, which basically depends on the fact that the functions coincide in a neighbourhood of the boundary. See [Ca] for details.

COROLLARY 5.3. — With the same assumptions as in Proposition 5.2, if

$$\int_{\Omega} dd^c \varphi_1 \wedge \dots \wedge dd^c \varphi_n$$

is defined, it equals

$$\int_{\Omega} dd^c \varphi_1^M \wedge \dots \wedge dd^c \varphi_n^M$$

for any M .

COROLLARY 5.4 (Comparison principle). — Let Ω be a hyperconvex, bounded domain in \mathbb{C}^n . Let u and v be plurisubharmonic functions on Ω , and suppose that $u \leq v$. Further assume that u and v are upper bounded exhaustions, that is $\{z \in \Omega : u < c\} \subset\subset \Omega$ for all $c < 0$, and similarly for v . Then

$$\int_{\Omega} (dd^c u)^n \geq \int_{\Omega} (dd^c v)^n.$$

Proof. — For bounded functions, this is Lemma V:3 in [Ce3]. Hence we have

$$\int_{\Omega} (dd^c u^M)^n \geq \int_{\Omega} (dd^c v^M)^n,$$

for all $M \leq -1$. Now Proposition 5.2 applies, and the corollary is proved. \square

THEOREM 5.5 (Partial integration). — Let Ω be a hyperconvex, bounded domain in \mathbb{C}^n . Let $\varphi_0, \varphi_1, \dots, \varphi_n$ be plurisubharmonic functions in Ω , and

assume that for some neighbourhood U of $\partial\Omega$, all φ_i are bounded in $U \cap \Omega$. Further assume that φ_0 and φ_1 are exhaustions, continuous as functions $\Omega \rightarrow [-\infty, 0)$, and that

$$\int dd^c\varphi_1 \wedge \dots \wedge dd^c\varphi_n + \int dd^c\varphi_0 \wedge dd^c\varphi_2 \wedge \dots \wedge dd^c\varphi_n < \infty.$$

Then partial integration is allowed, that is

$$\int_{\Omega} \varphi_0 dd^c\varphi_1 \wedge \dots \wedge dd^c\varphi_n = \int_{\Omega} \varphi_1 dd^c\varphi_0 \wedge dd^c\varphi_2 \wedge \dots \wedge dd^c\varphi_n.$$

We remark that the theorem applies to the class of functions we studied in Section 4, but is actually more general. We refer to [S] for the definition of the Monge-Ampère operator in the more general case.

For the proof of this theorem we refer the reader to the following sources: [B12], [Ce4] (where an even more general result is proved) or [Ca].

6. Definition of integrated measures

Let Ω be a hyperconvex, bounded domain in \mathbb{C}^n , and suppose that A is a finite subset of Ω . We refer to Section 1 for notation and the definition of the multipole pluricomplex Green function. Here we will make the following convention. Let $B = (v_1, \dots, v_n)$ be an n -tuple of points in A , where some v_j might coincide. Let $\tilde{B} = \{v_1, \dots, v_n\} = \{v_{j_1}, \dots, v_{j_p}\}$, where all v_{j_k} are different. Set $D = (v_{j_k}, 1)_{k=1}^p$. Abusing notation, we will write $g(z, B)$ instead of $g(z, D)$.

Note that if $v \in \tilde{B}$, we have

$$0 > g(z, v) \geq g(z, B),$$

for all $z \in \Omega \setminus \{v\}$. Hence

$$\sum_{j=1}^n g(z, v_j) \geq ng(z, B),$$

for all $z \in \Omega \setminus A$. Corollary 5.4 gives

$$\begin{aligned} & \int_{\Omega} dd_z^c g(z, v_1) \wedge \dots \wedge dd_z^c g(z, v_n) \\ & \leq \int_{\Omega} \left(dd_z^c \sum_{j=1}^n g(z, v_j) \right)^n \leq \int_{\Omega} (dd_z^c ng(z, B))^n = p(2\pi n)^n \leq n(2\pi n)^n. \end{aligned}$$

We conclude that the total mass of the measures $dd_z^c g(z, v_1) \wedge \dots \wedge dd_z^c g(z, v_n)$ is bounded, uniformly in $(v_1, \dots, v_n) \in \Omega^n$. Let μ be a finite measure on Ω . Using this fact, we can make the following definition.

DEFINITION 6.1. — For each $1 \leq k \leq n$, we define a new positive measure

$$\int_{\Omega^k} dd_z^c g(z, v_1) \wedge \dots \wedge dd_z^c g(z, v_n) d\mu(v_1) \cdots d\mu(v_k)$$

in the following way:

$$\begin{aligned} & \int_{\Omega} \chi(z) \left(\int_{\Omega^k} dd_z^c g(z, v_1) \wedge \dots \wedge dd_z^c g(z, v_n) d\mu(v_1) \cdots d\mu(v_k) \right) \\ & := \int_{\Omega^k} \left(\int_{z \in \Omega} \chi(z) dd_z^c g(z, v_1) \wedge \dots \wedge dd_z^c g(z, v_n) \right) d\mu(v_1) \cdots d\mu(v_k), \end{aligned}$$

where $\chi \in C_0^\infty(\Omega)$.

Note that the measure is a function in the variables v_{k+1}, \dots, v_n .

Now, let Ω be any bounded domain in \mathbb{C}^n . Let $K(z, w)$ be defined as in Theorem 1.2, and suppose that K is bounded on $\Omega \times T$. In an analogous manner to the above definition, we also define the following measures:

$$\begin{aligned} & \int_{\Omega} \chi(z) \left(\int_{\Omega^k} dd_z^c \log |z - v_1| \wedge \dots \wedge dd_z^c \log |z - v_k| d\mu(v_1) \cdots d\mu(v_k) \right) \\ & := \int_{\Omega^k} \left(\int_{z \in \Omega} \chi(z) dd_z^c \log |z - v_1| \wedge \dots \wedge dd_z^c \log |z - v_k| \right) d\mu(v_1) \cdots d\mu(v_k), \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega} \chi(z) \left(\int_{T^k} dd_z^c K(z, v_1) \wedge \dots \wedge dd_z^c K(z, v_n) d\mu(v_1) \cdots d\mu(v_k) \right) \\ & := \int_{T^k} \left(\int_{z \in \Omega} \chi(z) dd_z^c K(z, v_1) \wedge \dots \wedge dd_z^c K(z, v_n) \right) d\mu(v_1) \cdots d\mu(v_k), \end{aligned}$$

where $\chi \in C_0^\infty(\Omega)$.

7. A formula for the Monge-Ampère measures of potentials

The goal of this section is to prove representation formulas for the Monge-Ampère measure of potentials. In fact the following is true.

PROPOSITION 7.1. — *Let Ω be a bounded domain in \mathbb{C}^n , and μ a finite measure. Let $K(z, w)$ and q_μ^K be defined as in Theorem 1.2, and suppose that K is bounded on $\Omega \times T$. Then we have*

$$(dd^c q_\mu^K(z))^n = \int_{T^n} dd_z^c K(z, v_1) \wedge \dots \wedge dd_z^c K(z, v_n) d\mu(v_1) \cdots d\mu(v_n). \quad (4)$$

Further assume that μ is such that lp_μ is bounded. Then the formula

$$(dd^c lp_\mu(z))^n = \int_{\Omega^n} dd_z^c \log |z - v_1| \wedge \dots \wedge dd_z^c \log |z - v_n| d\mu(v_1) \cdots d\mu(v_n) \quad (5)$$

holds. Finally, if Ω is hyperconvex, and μ is such that p_μ is bounded, we have

$$(dd^c p_\mu(z))^n = \int_{\Omega^n} dd_z^c g(z, v_1) \wedge \dots \wedge dd_z^c g(z, v_n) d\mu(v_1) \cdots d\mu(v_n). \quad (6)$$

The three formulas are equalities between positive measures in the variable z .

To prove these formulas we need the following lemmas.

LEMMA 7.2 (Chern-Levine-Nirenberg’s inequality). — *Let Ω be an open neighbourhood of a compact set $K \subset \mathbb{C}^n$. There exist a constant $C > 0$ and a compact set $L \subset \Omega \setminus K$, which depend on K and Ω , such that for all $u_1, \dots, u_n \in PSH \cap L^\infty(\Omega)$,*

$$\int_K dd^c u_1 \wedge \dots \wedge dd^c u_n \leq C \|u_1\|_L \cdots \|u_n\|_L.$$

Proof. — We refer to [KI], Proposition 3.4.2 and Corollary 3.4.8. □

Remark. — For later reference, we would like to point out two facts about the proof of Lemma 7.2. Firstly, the inequality is proved by a recursive procedure, and if this is stopped after, say, $n - k$ steps we obtain

$$\int_K dd^c u_1 \wedge \dots \wedge dd^c u_n \leq C \prod_{j=k+1}^n \|u_j\|_L \int_K dd^c u_1 \wedge \dots \wedge dd^c u_k \wedge \beta^{n-k}, \quad (7)$$

where

$$\beta = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$$

is the standard Kähler form. Note that $dd^c(|z|^2/4) = \beta$. Secondly, the set L can be chosen “far from” K , i.e. for any compact subset K' of Ω such that $K \subset K'$, the set L can be chosen in the complement of K' . However the constant C may change if L changes.

LEMMA 7.3. — For all C^2 functions u_1, \dots, u_k and all smooth $(n-k, n-k)$ test forms ϕ we have

$$\int_{\Omega} dd^c u_1 \wedge \dots \wedge dd^c u_k \wedge \phi = \int_{\Omega} u_k dd^c u_1 \wedge \dots \wedge dd^c u_{k-1} \wedge dd^c \phi. \quad (8)$$

Proof. — See [Kl], Prop. 3.4.1. □

Proof of Proposition 7.1. — We first prove Formula (6). Since p_{μ} is plurisubharmonic and locally bounded, $(dd^c p_{\mu}(z))^n$ is defined. For each $u \in \Omega$, let $g_{\epsilon}(\cdot, u)$ be the usual regularization of $g(\cdot, u)$, defined on

$$\Omega_{\epsilon} := \{z \in \Omega : \text{dist}(z, \partial\Omega) > \epsilon\}.$$

Then $\{g_{\epsilon}(z, u)\}$ is a family of negative, C^{∞} functions defined on $\Omega_{\epsilon} \times \Omega$, plurisubharmonic in z , decreasing to $g(z, u)$ when $\epsilon \searrow 0$. Define, for $z \in \Omega_{\epsilon}$,

$$p_{\mu, \epsilon}(z) = \int_{\Omega} g_{\epsilon}(z, u) d\mu(u).$$

Let $\chi \geq 0$ be a C^{∞} function with compact support in Ω . Let $\epsilon > 0$ be so small that $\text{supp } \chi \subset \Omega_{\epsilon}$. For brevity, we introduce the notation $MA(u_1, \dots, u_n)$ for $dd^c_z u_1 \wedge \dots \wedge dd^c_z u_n$, where u_1, \dots, u_n are C^2 functions in z . Then we have, using (8) and Fubini's theorem repeatedly:

$$\begin{aligned} & \int_{z \in \Omega} \chi(z) \int_{\Omega^n} MA(g_{\epsilon}(z, v_1), \dots, g_{\epsilon}(z, v_n)) d\mu(v_1) \cdots d\mu(v_n) \\ &= \int_{\Omega^n} \left(\int_{z \in \Omega} \chi(z) MA(g_{\epsilon}(z, v_1), \dots, g_{\epsilon}(z, v_n)) \right) d\mu(v_1) \cdots d\mu(v_n) \\ &= \int_{\Omega^n} \left(\int_{z \in \Omega} g_{\epsilon}(z, v_1) MA(\chi(z), g_{\epsilon}(z, v_2), \dots, g_{\epsilon}(z, v_n)) \right) d\mu(v_1) \cdots d\mu(v_n) \\ &= \int_{z \in \Omega} \left(\int_{\Omega} g_{\epsilon}(z, v_1) d\mu(v_1) \right) \\ & \quad \int_{\Omega^{n-1}} MA(\chi(z), g_{\epsilon}(z, v_2), \dots, g_{\epsilon}(z, v_n)) d\mu(v_2) \cdots d\mu(v_n) \\ &= \int_{z \in \Omega} p_{\mu, \epsilon}(z) \int_{\Omega^{n-1}} MA(\chi(z), g_{\epsilon}(z, v_2), \dots, g_{\epsilon}(z, v_n)) d\mu(v_2) \cdots d\mu(v_n) \\ &= \int_{\Omega^{n-1}} \left(\int_{z \in \Omega} p_{\mu, \epsilon}(z) MA(\chi(z), g_{\epsilon}(z, v_2), \dots, g_{\epsilon}(z, v_n)) \right) d\mu(v_2) \cdots d\mu(v_n) \\ &= \int_{\Omega^{n-1}} \left(\int_{z \in \Omega} g_{\epsilon}(z, v_2) MA(\chi(z), p_{\mu, \epsilon}(z), g_{\epsilon}(z, v_3), \dots, g_{\epsilon}(z, v_n)) \right) \\ & \quad d\mu(v_2) \cdots d\mu(v_n) \\ &= \text{etc.} = \int_{\Omega} \chi(z) (dd^c p_{\mu, \epsilon}(z))^n \rightarrow \int_{\Omega} \chi(z) (dd^c p_{\mu}(z))^n, \end{aligned}$$

when $\epsilon \searrow 0$. In this computation all expressions of the types

$$MA(\varphi_1, \dots, \varphi_n)$$

or

$$\int_{\Omega} MA(\varphi_1, \dots, \varphi_n) d\mu(v_k) \cdots d\mu(v_n)$$

should be interpreted as measures in the variable z .

To complete the proof of (6) we must assure that the sequence of measures

$$\int_{\Omega^n} dd_z^c g_\epsilon(z, v_1) \wedge \dots \wedge dd_z^c g_\epsilon(z, v_n) d\mu(v_1) \cdots d\mu(v_n)$$

tends to

$$\int_{\Omega^n} dd_z^c g(z, v_1) \wedge \dots \wedge dd_z^c g(z, v_n) d\mu(v_1) \cdots d\mu(v_n)$$

weakly when $\epsilon \searrow 0$. We need to justify an application of Lebesgue's dominated convergence theorem in the integral

$$\int_{\Omega^n} \left(\int_{\Omega} \chi(z) dd_z^c g_\epsilon(z, v_1) \wedge \dots \wedge dd_z^c g_\epsilon(z, v_n) \right) d\mu(v_1) \cdots d\mu(v_n),$$

where $\chi \in C_0^\infty(\Omega)$. Since μ is assumed to be finite, it is sufficient to prove that the measures $dd_z^c g_\epsilon(z, v_1) \wedge \dots \wedge dd_z^c g_\epsilon(z, v_n)$ have locally uniformly bounded mass for small ϵ , independent of $(v_1, \dots, v_n) \in \Omega^n$. This is done in the technical Lemma 7.4 below.

We have proved Equation (6). To prove Equation (5), simply repeat the computation with $g(z, u)$ replaced with $\log|z - u|$. Then use Lemma 7.4.

Finally to show that Equation (4) holds, we repeat the same computation again. This time the counterpart to Lemma 7.4 is obvious.

LEMMA 7.4. — *Let Ω be a bounded, hyperconvex domain in \mathbb{C}^n . For each $v \in \Omega$, assume that $g_\epsilon(\cdot, v)$ is the usual regularization of $g(\cdot, v)$ and that $\log_\epsilon(\cdot, v)$ is the regularization of $\log|\cdot - v|$. Then for each compact set K in Ω , there exists a constant $C = C(K) > 0$ and an $\epsilon_0 = \epsilon_0(K) > 0$ such that*

$$\int_K dd_z^c g_\epsilon(z, v_1) \wedge \dots \wedge dd_z^c g_\epsilon(z, v_n) \leq C,$$

and

$$\int_K dd_z^c \log_\epsilon(z, v_1) \wedge \dots \wedge dd_z^c \log_\epsilon(z, v_n) \leq C,$$

for all $v_1, \dots, v_n \in \Omega$ and all $\epsilon \leq \epsilon_0$.

Proof. — Let $\Omega_\epsilon = \{z \in \Omega : \text{dist}(z, \partial\Omega) > \epsilon\}$. Take an open set Ω' such that $K \subset \Omega' \subset \subset \Omega$. For an n -tuple $(v_1, \dots, v_n) \in \Omega^n$, let us assume that exactly k of the v_j are contained in $\overline{\Omega'}$, and that these are v_1, \dots, v_k . We apply Inequality (7) on Ω' to deduce that, for some compact set $L \subset \Omega' \setminus K$, we have

$$\begin{aligned} & \int_K dd_z^c g_\epsilon(z, v_1) \wedge \dots \wedge dd_z^c g_\epsilon(z, v_n) \\ & \leq C_1 \|g_\epsilon(z, v_{k+1})\|_L \cdots \|g_\epsilon(z, v_n)\|_L \\ & \quad \times \int_K dd_z^c g_\epsilon(z, v_1) \wedge \dots \wedge dd_z^c g_\epsilon(z, v_k) \wedge \beta^{n-k}, \end{aligned}$$

for all ϵ such that $\overline{\Omega'} \subset \Omega_\epsilon$. We then apply Lemma 7.2 on Ω , with $\overline{\Omega'}$ as K' (see the remark following Lemma 7.2). This gives us a compact set $L' \subset \Omega \setminus \overline{\Omega'}$, such that

$$\begin{aligned} & \int_K dd_z^c g_\epsilon(z, v_1) \wedge \dots \wedge dd_z^c g_\epsilon(z, v_k) \wedge \beta^{n-k} \\ & \leq C_2 \|g_\epsilon(z, v_1)\|_{L'} \cdots \|g_\epsilon(z, v_k)\|_{L'} \left\| \frac{|z|^2}{4} \right\|_{L'}^{n-k}. \end{aligned}$$

Hence

$$\begin{aligned} & \int_K dd_z^c g_\epsilon(z, v_1) \wedge \dots \wedge dd_z^c g_\epsilon(z, v_n) \\ & \leq C_3 \prod_{j=1}^k \|g_\epsilon(z, v_j)\|_{L'} \prod_{j=k+1}^n \|g_\epsilon(z, v_j)\|_L \\ & \leq C_3 \prod_{j=1}^k \|g(z, v_j)\|_{L'} \prod_{j=k+1}^n \|g(z, v_j)\|_L. \end{aligned}$$

Recall that $g(z, w) \geq \log |z - w| - \log R$, where $R > 0$ is chosen such that $\Omega \subset B(w, R)$ for all $w \in \Omega$. This implies that $\|g(z, v_j)\|_{L'} \leq \log R - \log \text{dist}(L', \partial\Omega')$, $j = 1, \dots, k$, and $\|g(z, v_j)\|_L \leq \log R - \log \text{dist}(L, \partial\Omega')$, $j = k + 1, \dots, n$. Hence

$$\int_K dd_z^c g_\epsilon(z, v_1) \wedge \dots \wedge dd_z^c g_\epsilon(z, v_n) \leq C(k),$$

where $C(k)$ is a constant which depends only on the number of poles inside Ω' . Define $C = \max_{0 \leq k \leq n} C(k)$. Then

$$\int_K dd_z^c g_\epsilon(z, v_1) \wedge \dots \wedge dd_z^c g_\epsilon(z, v_n) \leq C,$$

and we are done. For the logarithm, the proof is similar. \square

For the logarithmic potential, it is possible to get an explicit expression for the Monge-Ampère measure. We will use the notation $\det(a, b) = a_1 b_2 - a_2 b_1$, where a and $b \in \mathbb{C}^2$.

PROPOSITION 7.5. — *If $v \neq w \in \mathbb{C}^2$, we have*

$$dd_z^c \log |z - v| \wedge dd_z^c \log |z - w| = 4 \cdot \frac{|\det(z - v, z - w)|^2}{|z - v|^4 |z - w|^4} d\lambda(z), \quad (9)$$

and this measure is absolutely continuous with respect to the Lebesgue measure. Moreover, if μ is a finite positive measure with compact support in \mathbb{C}^2 , and lp_μ is locally bounded, then

$$(dd^c lp_\mu(z))^2 = 4 \left(\int_{\mathbb{C}^2 \times \mathbb{C}^2} \frac{|\det(z - v, z - w)|^2}{|z - v|^4 |z - w|^4} d\mu(v) d\mu(w) \right) d\lambda(z). \quad (10)$$

Proof. — We compute the first measure using the formula

$$\begin{aligned} dd^c f(z) \wedge dd^c g(z) &= 16 \left(\frac{\partial^2 f}{\partial z_1 \partial \bar{z}_1} \frac{\partial^2 g}{\partial z_2 \partial \bar{z}_2} + \frac{\partial^2 f}{\partial z_2 \partial \bar{z}_2} \frac{\partial^2 g}{\partial z_1 \partial \bar{z}_1} \right) d\lambda(z) \\ &\quad - 16 \left(\frac{\partial^2 f}{\partial z_1 \partial \bar{z}_2} \frac{\partial^2 g}{\partial z_2 \partial \bar{z}_1} + \frac{\partial^2 f}{\partial z_2 \partial \bar{z}_1} \frac{\partial^2 g}{\partial z_1 \partial \bar{z}_2} \right) d\lambda(z) \end{aligned}$$

and the fact that

$$\frac{\partial^2 \log |z|}{\partial z_i \partial \bar{z}_j} = \frac{1}{2} \left(\frac{-\bar{z}_i z_j}{|z|^4} + \frac{\delta_{ij}}{|z|^2} \right).$$

We omit the details. To prove the claim about absolute continuity, note that

$$\frac{|\det(z - v, z - w)|^2}{|z - v|^2 |z - w|^2}$$

is uniformly bounded from above. Since $|z|^{-2} \in L^1_{loc}(\mathbb{C}^2)$, the claim is proved. Finally, inserting Equation (9) into Equation (5) gives Equation (10). □

8. Is there any relation between μ and $(dd^c p_\mu)^n$?

In classical potential theory we have $\Delta U^\mu = c\mu$, where U^μ denotes the classical potential of the measure μ and c is a positive constant. It is natural to ask whether there is any relation between the measures $(dd^c p_\mu)^n$ and μ , such as an inequality. The following results partially answer this question.

THEOREM 8.1. — *Let Ω be a bounded, hyperconvex domain in \mathbb{C}^n , $n \geq 2$. Let μ be a finite measure, such that the logarithmic potential of μ is bounded. Then $(dd^c lp_\mu(z))^n$ is an absolutely continuous measure with respect to the Lebesgue measure. If μ is such that the pluricomplex potential is bounded and $g(\cdot, v) \in C^2(\Omega \setminus \{v\})$ for all $v \in \Omega$, then $(dd^c p_\mu)^n$ is an absolutely continuous measure with respect to the Lebesgue measure.*

Proof. — We prove the theorem for p_μ , the proof for lp_μ being essentially the same. Let K be a set of Lebesgue measure zero. Then using Equation (6) and Fubini's theorem we get

$$\begin{aligned} & \int_K (dd^c p_\mu(z))^n \\ &= \int_K \int_{\Omega^n} dd_z^c g(z, v_1) \wedge \dots \wedge dd_z^c g(z, v_n) d\mu(v_1) \cdots d\mu(v_n) \\ &= \int_{\Omega^n} \left(\int_{z \in K} dd_z^c g(z, v_1) \wedge \dots \wedge dd_z^c g(z, v_n) \right) d\mu(v_1) \cdots d\mu(v_n). \end{aligned}$$

Call the bracketed integral $f(v_1, \dots, v_n)$. If $v_i \neq v_j$, for some indices i and j , we have $f(v_1, \dots, v_n) = 0$, according to Proposition 4.4. If all the variables have the same value v , then $f(v, \dots, v) = (2\pi)^n$ if $v \in K$, with $f(v, \dots, v) = 0$ otherwise. Let $D = \{(v, \dots, v) : v \in \Omega\} \subset \Omega^n$. Then we have

$$\begin{aligned} \int_K (dd^c p(z))^n &= \int_{\Omega^n} f(v_1, \dots, v_n) d\mu(v_1) \cdots d\mu(v_n) \\ &\leq (2\pi)^n \int_D d\mu(v_1) \cdots d\mu(v_n) \\ &= (2\pi)^n \int_{\Omega} \left(\int_{\{w\}} d\mu(v) \right)^{n-1} d\mu(w). \end{aligned}$$

Since p is bounded, μ can not have any mass at points. Hence the inner integral is zero for all w , which implies $\int_K (dd^c p_\mu(z))^n = 0$, and the proposition is proved. \square

EXAMPLE 8.2. — Let $\Omega = B(0, 1) \subseteq \mathbb{C}^2$, and let μ be the normalized Lebesgue measure on the disc $D(0, 1) \times \{0\}$. Then the pluricomplex potential is bounded. Hence it cannot have any mass on the 1-dimensional complex variety $D(0, 1) \times \{0\}$.

Proof. — From Proposition 3.1 we know that

$$p_\mu(z) = \frac{(|z|^2 - 1) \left(1 - |z_2|^2 + |z_2|^2 \log |z_2|^2 \right)}{2 \left(1 - |z_2|^2 \right)^2}.$$

It is easy to see by using routine calculus that $-1/2 = p_\mu(0) \leq p_\mu(z) \leq 0$. The second statement then follows from general properties of plurisubharmonic functions (see [Kl]). \square

The moral here is that, when $n \geq 2$, the procedure $\mu \mapsto p_\mu \mapsto (dd^c p_\mu)^n$ is “smoothing”. The fact that μ is singular is not visible at the end, as long as the singularity is not too strong. This breaks down in dimension 1. We will generalize the result in Section 9.

Also note that Theorem 8.1 says that a necessary condition for a bounded plurisubharmonic function to be a pluricomplex potential is that its Monge-Ampère measure is absolutely continuous with respect to the Lebesgue measure. Since there are many such functions whose Monge-Ampère measure is *not* absolutely continuous, this indicates that the Green functions (with one pole) form a very small subset of all extremal plurisubharmonic functions (see the discussion in Section 1).

THEOREM 8.3. — *Let Ω be a bounded domain in \mathbb{C}^n . Let $q > 1$, $f \in L^q(\Omega)$ with $f \geq 0$, $d\mu = f d\lambda$. Then the pluricomplex and the logarithmic potentials of μ are continuous on $\overline{\Omega}$. Moreover, if $n = 2$, and f is not identically zero, then $(dd^c lp(z))^2 = \varphi(z) d\lambda(z)$, where $\varphi(z) > 0$ throughout $\overline{\Omega}$.*

Proof. — Since for each $z \in \Omega$ and $1 \leq r < \infty$, we have $\log|z - \cdot| \in L^r(\lambda, \Omega)$, it is clear that $lp_\mu(z) > -\infty$ for each z . Let $1/q + 1/r = 1$. We first prove that the logarithmic potential is continuous at every $z \in \Omega$. Without loss of generality, we may assume that $z = 0$. Let $z_j \rightarrow 0$. Then

$$\begin{aligned} & |lp_\mu(z_j) - lp_\mu(0)| \\ &= \left| \int (\log|z_j - u| - \log|u|) f(u) d\lambda(u) \right| \\ &\leq \left(\int |\log|z_j - u| - \log|u||^r d\lambda(u) \right)^{1/r} \left(\int f(u)^q d\lambda(u) \right)^{1/q}, \end{aligned}$$

by Hölder’s inequality. Hence it suffices to show that

$$\int_\Omega |\log|z_j - u| - \log|u||^r d\lambda(u) \rightarrow 0. \tag{11}$$

This, however, follows immediately from Theorem 2.4.2 in [Ku-J-F], and the first statement is proved for lp_μ .

We now turn to p_μ . Again, it is enough to show continuity at $z = 0$. If $\overline{B}(w, R_1) \subset \Omega \subset B(w, R_2)$, then $-\log R_2 \leq g(z, w) - \log|z - w| \leq -\log R_1$, for all $z, w \in \Omega$. Hence, if $\delta > 0$ is small enough, there is a constant A such that $|g(z, w) - \log|z - w|| \leq A$, for all $z, w \in B(0, \delta)$. Let $\epsilon > 0$ be given.

Choose $\delta > 0$, such that $A^r \cdot \text{vol}(B(0, \delta)) < \epsilon^r$. We have by Minkowski's inequality:

$$\begin{aligned} & \left(\int_{B(0, \delta)} |g(z_j, u) - g(0, u)|^r d\lambda(u) \right)^{1/r} \\ & \leq \left(\int_{B(0, \delta)} |g(z_j, u) - \log |z_j - u||^r d\lambda(u) \right)^{1/r} \\ & \quad + \left(\int_{B(0, \delta)} |\log |z_j - u| - \log |u||^r d\lambda(u) \right)^{1/r} \\ & \quad + \left(\int_{B(0, \delta)} |g(0, u) - \log |u||^r d\lambda(u) \right)^{1/r} \\ & \leq \epsilon + \left(\int_{B(0, \delta)} |\log |z_j - u| - \log |u||^r d\lambda(u) \right)^{1/r} + \epsilon. \end{aligned}$$

Using the proof for lp_μ , we conclude that $\int_{B(0, \delta)} |g(z_j, u) - g(0, u)|^r d\lambda(u) < 3\epsilon$, if j is large enough. Outside $B(0, \delta)$, all functions are uniformly bounded if j is large enough. Hence we can use Lebesgue's dominated convergence theorem there, and the first statement is proved for p_μ .

Finally, to prove the statement about φ , we use Equation (10):

$$\varphi(z) = 4 \int_{\Omega \times \Omega} \frac{|\det(z - v, z - w)|^2}{|z - v|^4 |z - w|^4} f(v) f(w) d\lambda(v) d\lambda(w).$$

There is an $\epsilon > 0$ and a set with positive Lebesgue measure where $f \geq \epsilon$. Since the determinant vanishes only in a set in $\Omega \times \Omega$ with Lebesgue measure zero, the integral is greater than zero for all z . \square

THEOREM 8.4. — *Let Ω be a domain in \mathbb{C}^2 . Let $f \geq 0$, $d\mu = f d\lambda$. Then there is a constant $C > 0$ (depending only on Ω), such that*

$$(dd^c lp_\mu(z))^2 \leq C \|f\|_{L^q}^2 (2 - r)^{-2/r} d\lambda(z)$$

for all $q > 2$, $f \in L^q(\Omega)$ and $1/q + 1/r = 1$.

Proof. — It is clear that for all $z, v, w \in \Omega$,

$$\frac{|\det(z - v, z - w)|^2}{|z - v|^2 |z - w|^2}$$

is uniformly bounded. Hence, using Equation (10), we have

$$\begin{aligned} (dd^c l p_\mu(z))^2 &= 4 \left(\int_{\Omega \times \Omega} \frac{|\det(z-v, z-w)|^2}{|z-v|^4 |z-w|^4} d\mu(v) d\mu(w) \right) d\lambda(z) \\ &\leq C_1 \left(\int_{\Omega \times \Omega} \frac{f(v)f(w)}{|z-v|^2 |z-w|^2} d\lambda(v) d\lambda(w) \right) d\lambda(z) \\ &= C_1 \left(\int_{\Omega} \frac{f(v)}{|z-v|^2} d\lambda(v) \right)^2 d\lambda(z). \end{aligned}$$

Since $q > 2$ we have $r < 2$. Hence $|z-v|^{-2} \in L^r$ and the last integral can be estimated by Hölder's inequality:

$$\int_B \frac{f(v)}{|z-v|^2} d\lambda(v) \leq \|f\|_{L^q} \left(\int_{\Omega} |z-v|^{-2r} d\lambda(v) \right)^{1/r}.$$

Take $R > 0$ large enough to ensure that $\Omega \subset B(z, R)$ for all $z \in \Omega$. Then we have

$$\begin{aligned} \int_{\Omega} |z-v|^{-2r} d\lambda(v) &\leq \int_{B(z,R)} |z-v|^{-2r} d\lambda(v) = \{\text{spherical coordinates}\} \\ &= C_2 \int_0^R \rho^{3-2r} d\rho = C_2 \frac{R^{4-2r}}{4-2r} = C_3(2-r)^{-1}. \end{aligned}$$

Putting everything together we obtain the theorem. □

We close this section with a characterization of smooth potentials.

THEOREM 8.5. — *Let Ω be a domain in \mathbb{C}^n . Let μ be a measure (not necessarily bounded) in Ω . Then the logarithmic potential is smooth (i.e. C^∞) in Ω if and only if μ is given by a C^∞ function times the Lebesgue measure. If Ω is such that $g(x, y) - \log|x-y| \in C^\infty(\Omega \times \Omega)$ then the same characterization holds for the pluricomplex potential. In particular, this is true for the unit ball.*

Proof. — We begin with the logarithmic case. As mentioned in Section 2, the logarithm is a fundamental solution of Δ^n in \mathbb{R}^{2n} , and the potential is the convolution of the logarithm and the measure. The first statement of the theorem now follows from distribution theory, see [Hö], Theorem 4.4.1. The second statement is a trivial consequence of the first. For the ball, set $g(x, y) = \log|x-y| + h(x, y)$. We must prove that $h(x, y) \in C^\infty(B \times B)$. We omit the easy computation. □

9. Measures whose potentials have finite energy

Recently, Cegrell [Ce1] introduced the concept *pluricomplex energy*, and defined the Monge-Ampère operator on functions of finite energy.

DEFINITION 9.1 [Ce1]. — *Let Ω be a hyperconvex, bounded domain in \mathbb{C}^n . We define a function u to be in the class E_0 if it is bounded, plurisubharmonic, tends to zero at the boundary, and $\int_{\Omega} (dd^c u)^n < \infty$. We define a function u to be in the class E_p , if there is a sequence $\{u_j\} \subset E_0$, such that $u_j \searrow u$, and*

$$\sup_j \int (-u_j)^p (dd^c u_j)^n < \infty.$$

It is shown that E_p is a convex cone. We remark that functions with logarithmic poles, for instance the logarithm or the pluricomplex Green function, are not contained in E_p for any p . We may say that functions in the class E_p have *finite p -energy*.

THEOREM 9.2 [Ce1]. — *The Monge-Ampère operator is well-defined on E_p .*

The definition of the Monge-Ampère operator is done by means of the defining sequence, and it is shown that this gives a unique measure.

We begin with investigating what happens if we take a function $u \in E_0$, and perform the procedure

$$u \mapsto (dd^c u)^n \mapsto p_{(dd^c u)^n}.$$

PROPOSITION 9.3. — *Let Ω be a hyperconvex domain in \mathbb{C}^n , such that the pluricomplex Green function is symmetric, for example, a convex domain. Let $u \in E_0$ and set $\mu = (dd^c u)^n$. Then $p_{\mu} \in E_0$.*

Proof. — We need to show that p_{μ} is bounded. Fix $z \in \Omega$, and set

$$v^N(w) = \max \{g(w, z), N\}.$$

Set $K = -\inf_{\zeta \in \Omega} u(\zeta)$. We have by Hölder's inequality

$$\begin{aligned} & \int_{\Omega} -\max \{g(z, w), N\} d\mu(w) \\ &= \int_{\Omega} -v^N(w) (dd^c u)^n(w) \\ &\leq \left(\int_{\Omega} (dd^c u)^n \right)^{1-1/n} \left(\int_{\Omega} (-v^N(w))^n (dd^c u)^n(w) \right)^{1/n}, \end{aligned}$$

for $N \leq -1$. We now apply Corollary 2.2 in [B1] to obtain

$$\begin{aligned} \int_{\Omega} -\max\{g(z, w), N\} d\mu(w) &\leq \left(\int_{\Omega} (dd^c u)^n\right)^{1-1/n} (n!K^n \int_{\Omega} (dd^c v^N)^n)^{1/n} \\ &\leq 2\pi K (n!)^{1/n} \left(\int_{\Omega} (dd^c u)^n\right)^{1-1/n}. \end{aligned}$$

Since

$$-p_{\mu}(z) = \lim_{N \rightarrow -\infty} \int_{\Omega} -\max\{g(z, w), N\} d\mu(w),$$

it follows that p_{μ} is bounded. We also need to check that $(dd^c p_{\mu})^n$ is finite. We use Equation (6):

$$\int_{\Omega} (dd^c p_{\mu})^n = \int_{\Omega^n} \left(\int_{z \in \Omega} dd_z^c g(z, v_1) \wedge \dots \wedge dd_z^c g(z, v_n)\right) d\mu(v_1) \cdots d\mu(v_n).$$

Recall that the inner integral is uniformly bounded in v_1, \dots, v_n . Hence

$$\int_{\Omega} (dd^c p_{\mu})^n \leq C \left(\int d\mu\right)^n,$$

and $\int d\mu = \int (dd^c u)^n < \infty$ by assumption. The proof is complete. \square

In the rest of this section we will work in the unit ball in \mathbb{C}^2 and prove that E_1 contains the potential of a measure μ , if μ satisfies a kind of “logarithmic finite energy” condition.

THEOREM 9.4. — *Let $\Omega = B$, the unit ball in \mathbb{C}^2 . If*

$$\int_{B \times B} \frac{-\log|x-y|}{(1-|x|)(1-|y|)} d\mu(x)d\mu(y) < \infty, \tag{12}$$

then $p_{\mu} \in E_1$, and

$$(dd^c p_{\mu})^2(z) = \int_{B \times B} dd_z^c g(z, v) \wedge dd_z^c g(z, w) d\mu(v)d\mu(w). \tag{13}$$

Furthermore, $(dd^c p_{\mu})^2$ is an absolutely continuous measure with respect to the Lebesgue measure.

Proof. — Let

$$g_N(z, w) = \max\{g(z, w), N\},$$

and define, for $N \leq -1$,

$$p_N(z, w) = \int_B g_N(z, w) d\mu(w).$$

Then p_N is a bounded function on $\Omega \times \Omega$, plurisubharmonic in z , so we can apply Equation (4) to obtain

$$(dd^c p_N)^2(z) = \int_{B \times B} dd_z^c g_N(z, v) \wedge dd_z^c g_N(z, w) d\mu(v) d\mu(w).$$

Since, by Corollary 5.4, the total mass of the measures

$$dd_z^c g_N(z, v) \wedge dd_z^c g_N(z, w)$$

is bounded (uniformly in v, w and N), we have

$$\int_B (dd^c p_N)^2 = \int_{B \times B} \left(\int_B dd_z^c g_N(z, v) \wedge dd_z^c g_N(z, w) \right) d\mu(v) d\mu(w) < \infty.$$

It follows that that each p_N is in E_0 . Since also $p_N \searrow p_\mu$ when $N \rightarrow -\infty$, it is sufficient to show that if Inequality (12) holds, then $\sup_N \int_B -p_N (dd^c p_N)^2 < \infty$. We have

$$\begin{aligned} & \int_B -p_N (dd^c p_N)^2 \\ &= \int_{B \times B \times B} \left(\int_{z \in B} -g_N(z, t) dd_z^c g_N(z, v) \wedge dd_z^c g_N(z, w) \right) d\mu(t) d\mu(v) d\mu(w). \end{aligned}$$

Let $I(t, v, w)$ be the inner integral here. We can estimate it using Theorem 5.5 in the following way.

$$\begin{aligned} I(t, v, w) &= \int_{z \in B} -g(z, t) dd_z^c g_N(z, v) \wedge dd_z^c g_N(z, w) \\ &= \int -g_N(z, v) dd_z^c g(z, t) \wedge dd_z^c g_N(z, w) \\ &\leq \int -g(z, v) dd_z^c g(z, t) \wedge dd_z^c g_N(z, w) \\ &= \int -g_N(z, w) dd_z^c g(z, t) \wedge dd_z^c g(z, v) \\ &\leq \int -g(z, w) dd_z^c g(z, t) \wedge dd_z^c g(z, v) \\ &= \int -g(z, t) dd_z^c g(z, v) \wedge dd_z^c g(z, w). \end{aligned}$$

We claim that if $v \neq w$, then

$$dd_z^c g(z, v) \wedge dd_z^c g(z, w) \leq \frac{4}{(1 - |v|)(1 - |w|)} \cdot \frac{1 - \left| \left\langle \frac{z-v}{|z-v|}, \frac{z-w}{|z-w|} \right\rangle \right|^2}{|z-v|^2 |z-w|^2} d\lambda(z).$$

This estimate is contained in Lemma 9.6 below, and combining it with a trivial estimate of g , we conclude that

$$I(t, v, w) \leq \frac{4}{(1 - |v|)(1 - |w|)} \int_B -\log \frac{|z - t|}{4} \cdot \frac{1 - \left| \left\langle \frac{z-v}{|z-v|}, \frac{z-w}{|z-w|} \right\rangle \right|^2}{|z - v|^2 |z - w|^2} d\lambda(z).$$

Since we know that

$$\int_{B \times B} \frac{A - B \log |x - y|}{(1 - |x|)(1 - |y|)} d\mu(x) d\mu(y) < \infty$$

for every $A, B \geq 0$ by assumption, it suffices to prove that there exists $A, B \geq 0$ such that

$$\int_B -\log \frac{|z - t|}{4} \cdot \frac{1 - \left| \left\langle \frac{z-v}{|z-v|}, \frac{z-w}{|z-w|} \right\rangle \right|^2}{|z - v|^2 |z - w|^2} d\lambda(z) \leq A - B \log |v - w|.$$

This is done in the technical Lemma 9.8 below.

Next we prove Formula (13). By Theorem 9.2 and its proof, $(dd^c p_\mu)^2$ is well-defined and equals the weak limit of $(dd^c p_N)^2$. Hence it suffices to show that if $0 \leq \chi(z) \in C_0^\infty(B)$, then

$$\begin{aligned} J &:= \int_{z \in B} \chi(z) \int_{B^2} dd_z^c g_N(z, v) \wedge dd_z^c g_N(z, w) d\mu(v) d\mu(w) \\ &\rightarrow \int_{z \in B} \chi(z) \int_{B^2} dd_z^c g(z, v) \wedge dd_z^c g(z, w) d\mu(v) d\mu(w). \end{aligned}$$

But

$$J = \int_{B^2} \left(\int_{z \in B} \chi(z) dd_z^c g_N(z, v) \wedge dd_z^c g_N(z, w) \right) d\mu(v) d\mu(w),$$

and the inner integral here is bounded by

$$\|\chi\|_{L^\infty} \int_B dd_z^c g_N(z, v) \wedge dd_z^c g_N(z, w),$$

which, again, is uniformly bounded in v, w and N . Hence we can apply Lebesgue's dominated convergence theorem, and Formula (13) is proved.

Finally, to prove the claim about absolute continuity, repeat the proof of Theorem 8.1. This is possible since a measure satisfying Condition (12) can not have any mass at points. \square

It remains to show a number of technical lemmas, used in the preceding proof. The first one is a straight-forward but tedious computation, and we omit the details.

LEMMA 9.5. — We have if $u \neq v$,

$$\begin{aligned} & dd_z^c g(z, u) \wedge dd_z^c g(z, v) \\ &= \frac{4(1 - |u|^2)(1 - |v|^2) |\det(z - u, z - v)|^2}{\left(|z - u|^2 - |\det(z - u, u)|^2\right)^2 \left(|z - v|^2 - |\det(z - v, v)|^2\right)^2} d\lambda(z) \\ &= \frac{1 - \left| \left\langle \frac{z-u}{|z-u|}, \frac{z-v}{|z-v|} \right\rangle \right|^2}{|z - u|^2 |z - v|^2} \\ &\quad \times \frac{4(1 - |u|^2)(1 - |v|^2)}{\left[1 - |u|^2 + \left| \left\langle \frac{z-u}{|z-u|}, u \right\rangle \right|^2\right]^2 \left[1 - |v|^2 + \left| \left\langle \frac{z-v}{|z-v|}, v \right\rangle \right|^2\right]^2} d\lambda(z). \end{aligned}$$

LEMMA 9.6. — We have if $u \neq v$,

$$dd_z^c g(z, u) \wedge dd_z^c g(z, v) \leq \frac{1 - \left| \left\langle \frac{z-u}{|z-u|}, \frac{z-v}{|z-v|} \right\rangle \right|^2}{|z - u|^2 |z - v|^2} \times \frac{4}{(1 - |u|)(1 - |v|)} d\lambda(z).$$

Proof. — Immediate from the previous lemma. □

LEMMA 9.7. — Let $B_2 = B(0, 2) \subset \mathbb{C}^2$. Then the function $\Phi : \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{R}$, defined by

$$\Phi(t, w) = \int_{B_2} \frac{\log \frac{|z||z-w|}{2|z-t|}}{|z|^2 |z-w|^2} d\lambda(z)$$

is bounded from above.

Proof. — Define $f_c(x) = \log(x/c)/x^2$ for $x \geq c > 0$. By differentiation we see that $f_c(x) \leq 1/(2ec^2)$. Hence

$$\begin{aligned} \Phi(t, w) &\leq \int_{B_2 \cap \{|z||z-w| \geq 2|z-t|\}} \frac{\log \frac{|z||z-w|}{2|z-t|}}{|z|^2 |z-w|^2} d\lambda(z) \\ &\leq \frac{1}{8e} \int_{B_2} \frac{d\lambda(z)}{|z-t|^2} \leq C, \end{aligned}$$

which proves the lemma. □

LEMMA 9.8. — Let $B_r = B(0, r) \subset \mathbb{C}^2$, and define a function $\Psi : B_1 \times B_1 \times B_1 \rightarrow \mathbb{R}$ by

$$\Psi(t, u, v) = \int_{B_1} \frac{-\log \frac{|z-t|}{4} \cdot \left(1 - \left| \left\langle \frac{z-u}{|z-u|}, \frac{z-v}{|z-v|} \right\rangle \right|^2\right)}{|z - u|^2 |z - v|^2} d\lambda(z).$$

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Then there are constants A and $B > 0$, such that

$$\Psi(t, u, v) \leq A - B \log |u - v|.$$

Proof. — Let $w = v - u$, and make the change of variables $z' = z - u$, to obtain

$$\Psi(t, u, v) = \int_{B(-u, 1)} \frac{-\log \frac{|z'+u-t|}{4} \cdot \left(1 - \left| \left\langle \frac{z'}{|z'|}, \frac{z'-w}{|z'-w|} \right\rangle \right|^2\right)}{|z'|^2 |z' - w|^2} d\lambda(z').$$

Since $B(-u, 1) \subset B(0, 2)$ and the integrand is non-negative for $|z'| < 2$, we get

$$\Psi(t, u, v) \leq \int_{B_2} \frac{-\log \frac{|z+u-t|}{4} \cdot \left(1 - \left| \left\langle \frac{z}{|z|}, \frac{z-w}{|z-w|} \right\rangle \right|^2\right)}{|z|^2 |z - w|^2} d\lambda(z).$$

We split this into three integrals I_1 , I_2 and I_3 , where

$$I_1 = \int_{B_2} \frac{\log \frac{|z||z-w|}{2|z+u-t|} \cdot \left(1 - \left| \left\langle \frac{z}{|z|}, \frac{z-w}{|z-w|} \right\rangle \right|^2\right)}{|z|^2 |z - w|^2} d\lambda(z),$$

$$I_2 = \int_{B_2} \frac{-\log \frac{|z|}{2} \cdot \left(1 - \left| \left\langle \frac{z}{|z|}, \frac{z-w}{|z-w|} \right\rangle \right|^2\right)}{|z|^2 |z - w|^2} d\lambda(z),$$

$$I_3 = \int_{B_2} \frac{-\log \frac{|z-w|}{4} \cdot \left(1 - \left| \left\langle \frac{z}{|z|}, \frac{z-w}{|z-w|} \right\rangle \right|^2\right)}{|z|^2 |z - w|^2} d\lambda(z).$$

Note that

$$0 \leq 1 - \left| \left\langle \frac{z}{|z|}, \frac{z-w}{|z-w|} \right\rangle \right|^2 \leq 1.$$

Using this and Lemma 9.7 we conclude that I_1 is bounded from above by A_1 , say. To estimate I_2 , note that it is a rotation invariant function of w .

Hence we may assume that $w = (w, 0)$ where $w \in \mathbb{R}^+$. We obtain

$$\begin{aligned} I_2 &= \int_{B_2} \frac{-\log \frac{|z|}{2} \cdot |z|^2 \left(|z|^2 + w^2 - 2w \operatorname{Re} z_1 \right)}{|z|^4 \left[|z|^2 + w^2 - 2w \operatorname{Re} z_1 \right]^2} d\lambda(z) \\ &\quad - \int_{B_2} \frac{-\log \frac{|z|}{2} \cdot \left(|z|^4 + w^2 |z_1|^2 - 2|z|^2 w \operatorname{Re} z_1 \right)}{|z|^4 \left[|z|^2 + w^2 - 2w \operatorname{Re} z_1 \right]^2} d\lambda(z) \\ &= w^2 \int_{B_2} \frac{-|z_2|^2 \log \frac{|z|}{2}}{|z|^4 \left[|z|^2 + w^2 - 2w \operatorname{Re} z_1 \right]^2} d\lambda(z). \end{aligned}$$

We now change to spherical coordinates in \mathbb{R}^4 . This gives

$$I_2 = 2\pi w^2 \int_0^\pi \sin^3 \theta_2 d\theta_2 \cdot \int_0^2 -r \log \frac{r}{2} \left(\int_0^\pi \frac{\sin^4 \theta_1 d\theta_1}{(r^2 + w^2 - 2rw \cos \theta_1)^2} \right) dr.$$

Call the bracketed integral I_4 . To estimate I_4 , we make the substitution $t = \tan \theta_1/2$, and get

$$\begin{aligned} I_4 &= \int_0^\infty \frac{\left(\frac{2t}{1+t^2} \right)^4}{\left(r^2 + w^2 - 2rw \frac{1-t^2}{1+t^2} \right)^2} \cdot \frac{2dt}{1+t^2} \\ &= 32 \int_0^\infty \frac{t^4 dt}{\left((r-w)^2 + (r+w)^2 t^2 \right)^2 (1+t^2)^3} \\ &\leq 32(r+w)^{-4} \int_0^\infty \frac{dt}{(1+t^2)^3} = C(r+w)^{-4}. \end{aligned}$$

Thus

$$I_2 \leq Cw^2 \int_0^2 \frac{-r \log \frac{r}{2}}{(r+w)^4} dr \leq A_2 - B_2 \log w,$$

by direct integration. We conclude that $I_2 \leq A_2 - B_2 \log |v - u|$.

For I_3 , just note that since $|w| < 2$,

$$\begin{aligned} I_3 &= \int_{B(-w,2)} \frac{-\log \frac{|z'|}{4} \cdot \left(1 - \left| \left\langle \frac{z'+w}{|z'+w|}, \frac{z'}{|z'|} \right\rangle \right|^2 \right)}{|z'+w|^2 |z'|^2} d\lambda(z') \\ &\leq \int_{B_4} \frac{-\log \frac{|z|}{4} \cdot \left(1 - \left| \left\langle \frac{z+w}{|z+w|}, \frac{z}{|z|} \right\rangle \right|^2 \right)}{|z+w|^2 |z|^2} d\lambda(z), \end{aligned}$$

and we can estimate it exactly as I_2 . We obtain $I_3 \leq A_3 - B_3 \log |v - u|$. Putting everything together, we obtain $\Psi(t, u, v) \leq A_1 + A_2 - B_2 \log |v - u| + A_3 - B_3 \log |v - u| = A - B \log |v - u|$, as desired. \square

We remark that for measures μ with compact support, Condition (12) reduces to

$$\int_{B \times B} -\log |x - y| d\mu(x) d\mu(y) < \infty. \quad (14)$$

It would be interesting to see whether this condition is also necessary.

EXAMPLE 9.9. — In the unit ball in \mathbb{C}^2 we construct a measure whose potential is unbounded, but still in E_1 .

Proof. — We adopt the convention that C, C_1, C_2 etc. are positive constants but they may have different values in different expressions. For example, we might write $C|f(x)| \leq C$, if we know that $|f(x)|$ is bounded by some other constant (not necessarily 1). Let $\{a_j\}, \{b_j\} \subset (0, 1)$ be sequences which decrease to zero. Further assume that $\sum a_j/b_j < \infty$, and $\sum a_j/b_j^2 = \infty$. We will use radial measures of the type $d\mu = f(|x|)d\lambda(x)$, where $f(r) = \sum a_j r^{b_j - 4}$ if $r < 1/2$, and zero if $r \geq 1/2$. We first note that μ is finite, since

$$\int d\mu = C \sum a_j \int_0^{1/2} r^{b_j - 4} r^3 dr = C \sum a_j/b_j < \infty.$$

Further, p_μ is unbounded, since

$$\begin{aligned} p_\mu(0) &= \int \log |x| d\mu(x) = C \sum a_j \int_0^{1/2} (\log r) r^{b_j - 4} r^3 dr \\ &= -C \sum \frac{a_j 2^{-b_j}}{b_j} - C \sum \frac{a_j 2^{-b_j}}{b_j^2} = -\infty. \end{aligned}$$

It remains to show that μ satisfies Condition (14). First note that since μ is rotation invariant, then so is lp_μ . Let $x = (t, 0)$, $t \in (0, 1/2)$, and change to spherical coordinates. We get

$$-lp_\mu(x) = C \sum a_j \int_0^{1/2} \left(\int_0^\pi -\log(t^2 + r^2 - 2rt \cos \theta) \sin^2 \theta d\theta \right) r^{-1+b_j} dr.$$

Let I be the inner integral here. We use the substitution $x = \tan \theta/2$. This

gives

$$\begin{aligned}
 I &= \int_0^\infty -\log(t^2 + r^2 - 2rt\frac{1-x^2}{1+x^2}) \left(\frac{2x}{1+x^2}\right)^2 \frac{2dx}{1+x^2} \\
 &= -8 \int_0^\infty \log\left(\frac{(t-r)^2 + (t+r)^2 x^2}{1+x^2}\right) \frac{x^2}{(1+x^2)^3} dx \\
 &= 8 \int_0^\infty \log(1+x^2) \frac{x^2}{(1+x^2)^3} dx - 16 \log|t-r| \int_0^\infty \frac{x^2}{(1+x^2)^3} dx \\
 &\quad - 8 \int_0^\infty \log\left(1 + \frac{(t+r)^2 x^2}{(t-r)^2}\right) \frac{x^2}{(1+x^2)^3} dx \\
 &= 8f(1) - 16 \log|t-r| \int_0^\infty \frac{x^2}{(1+x^2)^3} dx - 8f\left(\frac{(t+r)^2}{(t-r)^2}\right),
 \end{aligned}$$

where f is defined by

$$f(a) = \int_0^\infty \log(1+ax^2) \frac{x^2}{(1+x^2)^3} dx.$$

We have $f(0) = 0$ and

$$f'(a) = \int_0^\infty \frac{x^4}{(1+ax^2)(1+x^2)^3} dx = \frac{\pi(3+\sqrt{a})}{16(1+\sqrt{a})^3},$$

by elementary methods from calculus. Hence

$$\begin{aligned}
 f(a) &= f(0) + \int_0^a f'(\alpha) d\alpha = \frac{\pi}{16} \int_0^a \frac{3+\sqrt{\alpha}}{(1+\sqrt{\alpha})^3} d\alpha \\
 &= \frac{\pi}{8} \left(\log(1+\sqrt{a}) - \frac{\sqrt{a}}{(1+\sqrt{a})^2} \right).
 \end{aligned}$$

We also have

$$\int_0^\infty \frac{x^2}{(1+x^2)^3} dx = \frac{\pi}{16}.$$

We conclude that

$$\begin{aligned}
 \frac{I}{\pi} &= \log 2 - \frac{1}{4} - \log|t-r| - \log\left(1 + \frac{r+t}{|t-r|}\right) + \frac{(r+t)/|t-r|}{(1+(r+t)/|t-r|)^2} \\
 &= \log 2 - \frac{1}{4} - \log(|t-r| + r+t) + \frac{|r^2-t^2|}{(|t-r| + r+t)^2} \\
 &= -\frac{1}{4} - \log \max\{r, t\} + \frac{|r^2-t^2|}{4 \max\{r^2, t^2\}}.
 \end{aligned}$$

Thus

$$\begin{aligned} & \frac{1}{\pi} \int_0^{1/2} \left(\int_0^\pi -\log(t^2 + r^2 - 2rt \cos \theta) \sin^2 \theta d\theta \right) r^{-1+b} dr \\ &= \int_0^t \left(-\frac{1}{4} - \log t + \frac{t^2 - r^2}{4t^2} \right) r^{-1+b} dr \\ &+ \int_t^{1/2} \left(-\frac{1}{4} - \log r + \frac{r^2 - t^2}{4r^2} \right) r^{-1+b} dr \\ &= \frac{2^{-b}(2+b)t^2 - t^b}{4-b^2} + \frac{2^{-b} \log 2}{b} + \frac{2^{-b} - t^b}{b^2} \\ &\leq C_1 + \frac{C_2}{b} + C_3 \frac{2^{-b} - t^b}{b^2}. \end{aligned}$$

It follows that

$$-lp_\mu(x) \leq \sum a_j \left(C_1 + \frac{C_2}{b_j} + C_3 \frac{2^{-b_j} - t^{b_j}}{b_j^2} \right).$$

Note that the last sum is convergent for all $0 < t < 1/2$, since $0 < 2^{-b} - t^b \leq -b \log t$, by the Mean Value Theorem applied to the function $b \mapsto 2^{-b} - t^b$.

Now we integrate again, and obtain

$$\begin{aligned} & \int -\log|x-y| d\mu(x)d\mu(y) \\ &= \int -lp_\mu(x) d\mu(x) \\ &\leq C \sum a_j a_k \int_0^{1/2} \left(C_1 + \frac{C_2}{b_j} + C_3 \frac{2^{-b_j} - t^{b_j}}{b_j^2} \right) t^{-1+b_k} dt \\ &= \sum a_j a_k \left(\frac{C_1}{b_k} + \frac{C_2}{b_j b_k} + \frac{C_3}{b_j^2} \left(\frac{1}{b_k} - \frac{1}{b_j + b_k} \right) \right) \\ &= \sum a_j a_k \left(\frac{C_1}{b_k} + \frac{C_2}{b_j b_k} + \frac{C_3}{b_j b_k (b_j + b_k)} \right). \end{aligned}$$

If we now choose, for instance, $a_j = j^{-3}$, $b_j = j^{-1}$, the last series converges, and the corresponding measure has finite energy. \square

Remark. — For the suggested choice of a_j and b_j , the last series becomes $C_1 \zeta(2) \zeta(3) + C_2 \zeta(2)^2 + 2C_3 \zeta(3)$. Here we used the curious fact that

$$\sum_{j,k=1}^{\infty} \frac{1}{jk(j+k)} = 2\zeta(3),$$

which can be proved using an integral representation of the Riemann zeta function.

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