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BORIS P. ANDREIANOV

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The Riemann problem for p-systems with continuous flux function [*]

BORIS P. ANDREIANOV [1]

RÉSUMÉ. — On considère les systèmes hyperboliques de la forme $U_t - V_x = 0$, $V_t - f(U)_x = 0$. La solution auto-similaire du problème de Riemann est obtenue comme l'unique limite des solutions bornées auto-similaires des systèmes qui sont régularisés à l'aide d'une viscosité spécifique, qui tend vers zéro. Cette solution est donnée par des formules explicites; on étend ainsi les formules connues au cas d'une fonction de flux $f(\cdot)$ qui n'est pas localement lipschitzienne.

ABSTRACT. — Hyperbolic systems of the form $U_t - V_x = 0$, $V_t - f(U)_x = 0$ are considered. A self-similar solution to the Riemann problem is obtained as the unique limit of bounded self-similar solutions to systems regularized by means of a vanishing viscosity of special form. This solution is given by explicit formulae, which extend the known ones to the case of non-Lipschitz flux function $f(\cdot)$.

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Equipe de Mathématiques UMR CNRS 6623, Université de Franche-Comté, 25030 Besançon cedex, France.
E-mail: borisa@math.univ-fcomte.fr

0. Introduction

Consider the Riemann problem for a so-called p-system, i.e. the initial-value problem

$$\left\{ \begin{array}{l} U_t - V_x = 0 \\ V_t - f(U)_x = 0 \end{array} \right. , \qquad (U,V) : (t,x) \in \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^2; \tag{1}$$

$$U(0,x) = \left\{ \begin{array}{ll} u_+, & x > 0 \\ u_-, & x < 0 \end{array} \right. , \qquad V(0,x) = \left\{ \begin{array}{ll} v_+, & x > 0 \\ v_-, & x < 0 \end{array} \right. \qquad u_\pm, v_\pm \in \mathbb{R}. \tag{2}$$

The flux function $f : \mathbb{R} \mapsto \mathbb{R}$ is assumed to be continuous and strictly increasing.

In the case of piecewise smooth flux function the problem (1),(2) was treated by L.Leibovich, [9] (cf. also [4] and references therein). By analyzing the wave curves on the plane (u,v) it has been shown that a self-similar distribution solution that is consistent with a certain admissibility criterion (cf. B.Wendroff, [12]; also I.Gelfand, [6] and S.Kruzhkov, [8] for the original idea carried out in the case of scalar conservation laws) may be explicitly constructed through convex and concave hulls of the flux function f. It has been noticed by C.Dafermos in [5] that the same solution satisfies the wave fan admissibility criterion, i.e., it can be obtained as limit of self-similar viscous approximations as viscosity goes to 0. Here we follow this last idea.

Let introduce some notation. For given $[a,b] \subset \mathbb{R}$ and $f : u \in [a,b] \mapsto \mathbb{R}$ continuous, the convex hull of f on $[a,b]$ is the function $u \in [a,b] \mapsto \sup\Big\{ \phi(u) \,|\, \phi \text{ is convex and } \phi \leqslant f \text{ on } [a,b] \Big\}$. Respectively, the concave hull of f on $[a,b]$ is the function $u \in [a,b] \mapsto \inf\Big\{ \phi(u) \,|\, \phi \text{ is concave and } \phi \geqslant f \text{ on } [a,b] \Big\}$. Take u_0 in $\mathbb{R}$; by $F_+(\cdot; u_0)$ denote the convex hull of f on $[u_0, u_+]$ if $u_0 \leqslant u_+$, and the concave hull of f on $[u_+, u_0]$ if $u_0 \geqslant u_+$. Replacing u_+ by u_-, define $F_-(\cdot; u_0)$ in the same way. Let shorten $F_\pm(\cdot; u_0)$ to $F_\pm$ when no confusion can arise.

Since f is strictly increasing, the inverse of $\frac{dF_+}{du}$, denoted by $\left[\frac{dF_+}{du}\right]^{-1}$, is well defined in the graph sense as function from $[0,+\infty)$ to $[u_0, u_+]$ if $u_0 < u_+$ (respectively, to $[u_+, u_0]$ if $u_0 > u_+$). In the case $u_0 = u_+$ let $\left[\frac{dF_+}{du}\right]^{-1}$ mean the function on $[0,+\infty)$ identically equal to u_0. With the same notation for F_-, u_- in place of F_+, u_+ and $\dot{F}_\pm$ standing for $\frac{dF_\pm}{du}$, which are non-negative, the self-similar solution of the problem (1),(2) constructed in [9] may be written as

$$U(t,x) = \begin{cases} \left[\dot{F}_+(\cdot;u_0)\right]^{-1}(x^2/t^2), & x \geqslant 0 \\ \left[\dot{F}_-(\cdot;u_0)\right]^{-1}(x^2/t^2), & x \leqslant 0 \end{cases}, \tag{3}$$

$$V(t,x) = v_- - \int_{-\infty}^{x/t} \zeta dU(\zeta), \tag{4}$$

$dU(\zeta)$ being regarded as measure; and, for a bijective flux function f, the value u_0 is uniquely determined by

$$v_- - v_+ = \int_{u_0}^{u_+} \sqrt{\dot{F}_+(u;u_0)}du + \int_{u_0}^{u_-} \sqrt{\dot{F}_-(u;u_0)}du. \tag{5}$$

In the case of bijective locally Lipschitz continuous flux function f, the same formulae (3)-(5) were obtained by P.Krejčí, I.Straškraba ([7]) for the unique solution to satisfy their “maximal dissipation” condition. This solution was also shown to be the unique a.e-limit as $\varepsilon \to 0$ of solutions to Riemann problem for the p-system regularized by means of infinitesimal parameter $\varepsilon > 0$, introduced into the flux function f, and the viscosity $\begin{pmatrix} 0 \\ \varepsilon t V_{xx} \end{pmatrix}$.

In this paper a refinement of these results is presented. The techniques employed are those used by the author while treating the Riemann problem for a scalar conservation law with continuous flux function (cf. [1, 2]). In the general case of continuous strictly increasing flux function f, the Riemann problem (2) for the p-system (1) and the regularized system

$$\begin{cases} U_t - V_x = 0 \\ V_t - f(U)_x = \varepsilon t V_{xx} \end{cases} \tag{6}$$

are treated. The main result is the following theorem:

THEOREM 1.— *Suppose $f : \mathbb{R} \to \mathbb{R}$ is increasing and bijective. Then for all $u_\pm, v_\pm \in \mathbb{R}$, $\varepsilon > 0$ there exists a unique bounded self-similar distribution solution $(U^\varepsilon, V^\varepsilon)$ of the problem (6),(2).*

Besides, as $\varepsilon \downarrow 0$, $(U^\varepsilon, V^\varepsilon)(\xi) \to (U, V)(\xi)$ a.e. on $\mathbb{R}$, where (U, V) is given by the formulae (3)-(5), so that (U, V) is a self-similar distribution solution of the problem (1),(2).

The bijectivity condition is only needed for the existence of solutions and cannot be omitted (see Remark 7.6 in [7]), though it can be relaxed (see Remark 2 in Section 3).

The paper is organized as follows. In the first section the problem (6),(2) is reduced to a pair of boundary-value problems for a second-order ordinary differential equation on the domains $(\min\{u_0, u_\pm\}, \max\{u_0, u_\pm\})$; u_0 is *a priori* unknown and satisfies an additional algebraic equation. In Section 2 existence, uniqueness and convergence (as $\varepsilon \to 0$) results are obtained for the ODE problem stated in Section 1, with $u_0 \in \mathbb{R}$ fixed. Then it is shown in Section 3 that u_0 is in fact uniquely determined by the flux function f, ε, and the Riemann data $u_\pm, v_\pm$; finally, Theorem 1 above is proved.

1. Restatement of the problem

We start by fixing $\varepsilon > 0$. Consider the problem (6),(2) in the class of bounded distribution solutions (U, V) of (6) such that $(U, V)(t, \cdot)$ tends to $(U, V)(0, \cdot)$ in $L^1_{loc}(\mathbb{R}) \times L^1_{loc}(\mathbb{R})$ as t tends to $+0$ essentially. Moreover, since both the initial data (2) and the system (6) are invariant under the transformations $(t, x) \to (kt, kx)$ with k in $\mathbb{R}^+$ (here is the reason to introduce the viscosity with factor t), it is natural to seek for self-similar solutions, i.e. (U, V) depending solely on the ratio x/t. By abuse of notation, let write $(U, V)(t, x) = (U, V)(x/t)$. Let ξ denote x/t and use U', V' for $dU/d\xi, dV/d\xi$ and so on.

LEMMA 1.— *A pair of bounded functions $(U, V) : \xi \in \mathbb{R} \mapsto \mathbb{R}^2$ is a self-similar distribution solution of (6),(2) if and only if $U, V, \xi U'$ and V' are continuous on $\mathbb{R}$, the equations*

$$\varepsilon \xi U'(\xi) = -\int_0^\xi \zeta^2 U'(\zeta) d\zeta + f(U(\xi)) + C \tag{7}$$

$$V(\xi) = -\int_0^\xi \zeta U'(\zeta) d\zeta + K \tag{8}$$

are fulfilled with some constants C,K, and also

$$U(\pm\infty) = u_\pm, \qquad V(\pm\infty) = v_\pm. \tag{9}$$

Besides, there exist $\xi_\pm$ in $\overline{\mathbb{R}^\pm}$, $\xi_- \leqslant \xi_+$, such that U, V are strictly monotone on each of $(-\infty, \xi_-)$, $(\xi_+, +\infty)$, with $U' \neq 0$, and U, V are constant on (ξ_-, ξ_+).

Proof. — Let (U, V) be bounded self-similar distribution solution of the system (6). Then $-\xi U' - V' = 0$ and $-\xi V' - f(U)' = \varepsilon V''$ in $\mathcal{D}'(\mathbb{R})$; therefore $\left[\xi^2 U - f(U) + \varepsilon \xi U'\right]' = 2\xi U$ in $\mathcal{D}'(\mathbb{R})$. Since $U \in L^\infty(\mathbb{R})$, it follows that

$$\xi^2 U - f(U) + \varepsilon \xi U' = \int_0^\xi 2\zeta U(\zeta) d\zeta + C \in C(\mathbb{R}) \tag{10}$$

with some C in $\mathbb{R}$. Hence one deduce consecutively that $\xi U' \in L^\infty_{loc}(\mathbb{R})$, $U \in C(\mathbb{R}\backslash\{0\})$ and finally, $U \in C^1(\mathbb{R}\backslash\{0\})$. Thus for all $\xi \neq 0$ (7) holds.

Now let prove the monotony property stated. For (ξ_-, ξ_+) take the largest interval in $\overline{\mathbb{R}}$ containing $\xi = 0$ such that $U = U(0)$ on (ξ_-, ξ_+). For instance, let ξ_+ be finite and therefore U not constant on $(0, +\infty)$; suppose U is not strictly monotone on $(\xi_+, +\infty)$. Since $U' \in C(\xi_+, +\infty)$, it follows that there exists $c > \xi_+$ such that $U'(c) = 0$ and U' is non-zero in some left neighbourhood of c. For instance, assume $U' > 0$ in this neighbourhood. Clearly, there exists a sequence $\{\xi_n\} \subset \mathbb{R}$ increasing to c such that for all $n \in \mathbb{N}$ the maximum of U' on $[\xi_n, c]$ is attained at the point ξ_n. Since f is increasing, it follows that $f(U(\xi_n)) < f(U(c))$. Take (7) at the points $\xi = \xi_n$ and $\xi = c$; subtraction yields

$$\varepsilon \xi_n U'(\xi_n) - \varepsilon c \cdot 0 \leqslant \int_{\xi_n}^c \zeta^2 U'(\zeta) d\zeta + f(U(\xi_n)) - f(U(c)) \leqslant U'(\xi_n) \int_{\xi_n}^c \zeta^2 d\zeta.$$

As $n \to \infty$, one deduces that $\varepsilon \leqslant 0$, which is impossible.

Thus U, and consequently V, are indeed monotone on $(-\infty, 0)$ and $(0, +\infty)$; therefore there exist $U(\pm 0) = \lim_{\xi \to \pm 0} U(\xi)$. Hence by (10) there exist $\lim_{\xi \to \pm 0} \xi U'(\xi)$, which are necessarily zero since $U \in L^\infty(\mathbb{R})$. Thus (10) yields $f(U(+0)) = f(U(-0))$, so that $U \in C(\mathbb{R})$. Consequently, $\xi U' \in C(\mathbb{R})$, $V' \in C(\mathbb{R})$, and $V \in C(\mathbb{R})$. It follows that (7),(8) hold for all ξ in $\mathbb{R}$.

The converse assertion, i.e. that (7),(8) imply (6) in the distribution sense, is trivial. Finally, since U and V are shown to be monotone on $\mathbb{R}^\pm$ whenever (7),(8) hold, it is evident that (9) is fulfilled if and only if self-similar U, V satisfy (2) in L^1_{loc}-sense as $t \to 0$ essentially. □

Let use this result to obtain another characterisation of self-similar solutions to (6),(2). The idea is to seek for solutions of the same form as in formulae (3)-(5), substituting $F_\pm$ by appropriate functions depending on ε. One thus has to "inverse" (3)-(5).

Set $u_0 := U(0)$ and consider (7) separately on $(-\infty, \xi_-)$, (ξ_-, ξ_+), and $(\xi_+, +\infty)$, where $\xi_\pm$ are defined in Lemma 1. Assume $u_0 \neq u_-$, $u_0 \neq u_+$.

Let introduce the notation $I(a,b)$ for the interval between a and b in $\overline{\mathbb{R}}$. One has $U(\xi)=u_0$ for all $\xi\in(\xi_-,\xi_+)$; besides, the inverse functions $U_+^{-1}: I(u_0,u_+)\mapsto(\xi_+,+\infty)$ and $U_-^{-1}: I(u_0,u_-)\mapsto(-\infty,\xi_-)$ are well defined. For all $u\in I(u_0,u_+)$ (respectively, $u\in I(u_0,u_-)$) set

$$\begin{aligned}\Phi_+^\varepsilon(u;u_0) &:= \int_{u_0}^{u}\left(U_+^{-1}(w)\right)^2 dw - C\\ &\left(\textit{resp.},\ \ \Phi_-^\varepsilon(u;u_0) := \int_{u_0}^{u}\left(U_-^{-1}(w)\right)^2 dw - C\right)\end{aligned} \tag{11}$$

with C taken from (7). The shortened notation $\Phi_\pm(u)$ will be used for $\Phi_\pm^\varepsilon(u;u_0)$ whenever ε,u_0 are fixed. Now (7) can be rewritten as $\varepsilon\xi U'(\xi)=f(U(\xi))-\Phi_\pm(U(\xi))$ for $\xi\in I(\xi_\pm,\pm\infty)$. The reasoning in the proof of Lemma 1 shows that U is not only monotone, but also U' is different from 0 outside of $[\xi_-,\xi_+]$. It follows that for all u in $I(a,b)$, where $a=u_0$, $b=u_+$ (resp., for all u in $I(a,b)$, where $a=u_0$, $b=u_-$), the function Φ_+ (resp., Φ_-) is twice differentiable and satisfies the equation

$$\ddot{\Phi}(u)=\frac{2\varepsilon\dot{\Phi}(u)}{f(u)-\Phi(u)},\quad \textit{with}\ \ \dot{\Phi}(u)>0\ \ \textit{and}\ \ \ddot{\Phi}(u)\cdot(b-a)>0. \tag{12}$$

Hence $\Phi_+<f$ ($\Phi_+>f$) if $u_0<u_+$ (if $u_0>u_+$), and the same for Φ_-,u_- in place of Φ_+,u_+.

Note that one can extend the functions Φ_+,Φ_- to be continuous on $\overline{I(u_0,u_+)}$, $\overline{I(u_0,u_-)}$ respectively, and in this case one has

$$\begin{aligned}&\Phi_+(u_0)=f(u_0),\ \Phi_+(u_+)=f(u_+)\\ &\Big(\ \textit{resp.},\ \Phi_-(u_0)=f(u_0),\ \Phi_-(u_-)=f(u_-)\ \Big).\end{aligned} \tag{13}$$

Indeed, one gets $\Phi_\pm(u_0)=f(u_0)$ directly from (11) and (7). Besides, for $\xi\in\mathbb{R}^\pm$, $\varepsilon\xi U'(\xi)$ is equal to $f(U(\xi))-\Phi_\pm(U(\xi))$, which has finite limits as $\xi\to\pm\infty$ because $U(\pm\infty)=u_\pm$ and $\Phi_\pm$ are convex and bounded on $I(u_0,u_\pm)$. The limits of $\varepsilon\xi U'(\xi)$ cannot be non-zero since U is bounded, thus one naturally assign $\Phi_\pm(u_\pm):=f(u_\pm)$.

Now from (8)-(11) it follows that

$$v_- - v_+ = \int_{u_0}^{u_+}\sqrt{\dot{\Phi}_+^\varepsilon(u;u_0)}du + \int_{u_0}^{u_-}\sqrt{\dot{\Phi}_-^\varepsilon(u;u_0)}du. \tag{14}$$

Note that in the case $u_0=u_+$ ($u_0=u_-$), (12)-(14) formally make sense, with Φ_+ defined at $u=u_0=u_+$ by $f(u_+)$ (resp., with Φ_- defined at $u=u_0=u_-$ by $f(u_-)$).

Finally, the reasoning above is inversible. More presisely, for given $u_0 \in \mathbb{R}$ and $\Phi^\varepsilon_\pm(\cdot;u_0) \in C^2(I(u_0,u_\pm)) \cap C(\overline{I(u_0,u_\pm)})$ such that (12)-(14) hold, define U, V by

$$U(\xi) = \begin{cases} \left[\dot{\Phi}^\varepsilon_+(\cdot;u_0)\right]^{-1}(\xi^2), & \xi \geqslant 0 \\ \left[\dot{\Phi}^\varepsilon_-(\cdot;u_0)\right]^{-1}(\xi^2), & \xi \leqslant 0 \end{cases} \tag{15}$$

$$V(\xi) = v_- - \int_{-\infty}^{\xi} \zeta dU(\zeta), \tag{16}$$

with $[\dot{\Phi}^\varepsilon_+(\cdot;u_0)]^{-1}$ (and $[\dot{\Phi}^\varepsilon_-(\cdot;u_0)]^{-1}$) taken in the graph sense and equal to u_+ (to u_-) identically whenever $u_0 = u_+$ ($u_0 = u_-$). Then (U,V) satisfy (7)-(9). Indeed, U is continuous, $\Phi^\varepsilon_+(u_0;u_0) = \Phi^\varepsilon_-(u_0;u_0)$, and the equation $\varepsilon\xi U'(\xi) = f(U(\xi)) - \Phi^\varepsilon_\pm(U(\xi);u_0)$ holds for all $\xi \in \mathbb{R}^\pm$. Hence $\xi U' \in C(\mathbb{R})$ and (7) is true. Therefore V', V are continuous and (8),(9) are easily checked.

We collect the results obtained above in the following proposition:

PROPOSITION 1.— *Let $\varepsilon, f, u_\pm, v_\pm$ be fixed. Formulae (15),(16) provide a one-to-one correspondence between the sets $\mathcal{A}$ and $\mathcal{B}$ defined by*

$$\begin{aligned} \mathcal{A} := \Big\{ \big(u_0, \Phi_\pm(\cdot)\big) \mid u_0 \in \mathbb{R},\ \Phi_\pm : \overline{I(u_0,u_\pm)} \mapsto \mathbb{R}, \\ \Phi_\pm \in C^2(I(u_0,u_\pm)) \cap C(\overline{I(u_0,u_\pm)}) \ \textit{and}\ (12)-(14)\ \textit{hold} \Big\} \\ \mathcal{B} := \Big\{ (U,V) \mid (U,V)\ \textit{is a bounded self-similar} \\ \textit{distribution solution of}\ (6),(2) \Big\} \end{aligned}$$

In fact, it will be shown in Section 3 that $\mathcal{A}$ and thus $\mathcal{B}$ are one-element or empty sets.

The resemblance of formulae (3),(4),(5) and (15),(16),(14) permits to get the convergence result of Theorem 1 if one has convergence of $\Phi^\varepsilon_\pm$ to $F_\pm$ as $\varepsilon \to 0$.

2. The problem (12),(13) with fixed domain

Let fix $a, b \in \mathbb{R}$ and consider the equation (12) on the interval $I(a,b)$, with the boundary conditions as in (13). For instance, suppose $a \leqslant b$.

PROPOSITION 2.— *For all continuous strictly increasing f, $\varepsilon > 0$, and $a, b \in \mathbb{R}$ there exists a unique Φ in $C^2(I(a,b)) \cap C(\overline{I(a,b)})$ satisfying (12) such that $\Phi(a) = f(a)$ and $\Phi(b) = f(b)$.*

For f and $[a,b]$ fixed, let Φ^ε denote the function Φ from Proposition 2 corresponding to ε, $\varepsilon > 0$.

PROPOSITION 3.— *With the notation above, Φ^ε converge in $C[a,b]$, as $\varepsilon \to 0$, to the convex hull F of the function f on the segment $[a,b]$.*

Remark 1.— In the case $a \geqslant b$, the corresponding limit is the concave hull of f on $[b,a]$.

The following two assertions will be repeatedly used in the proofs in Sections 2,3:

LEMMA 2 [Maximum Principle].— *Let $\Phi, \Psi \in C^2(a,b) \cap C[a,b]$ and satisfy, for all $u \in (a,b)$, the equations $\ddot{\Phi}(u) = G(u, \Phi(u), \dot{\Phi}(u))$ and $\ddot{\Psi}(u) = H(u, \Psi(u), \dot{\Psi}(u))$, respectively, with $G, H : (a,b) \times \mathbb{R} \times (0,+\infty) \mapsto (0,+\infty]$.*

a) Assume that $G(u,z,w) < H(u,\zeta,w)$ for all $u \in (a,b)$ such that $\Phi(u) < \Psi(u)$ and all z,ζ,w such that $z < \zeta$. Then $\Phi \geqslant \Psi$ on $[a,b]$ whenever $\Phi(a) \geqslant \Psi(a)$ and $\Phi(b) \geqslant \Psi(b)$.

b) Assume that $G(u,z,w) \equiv H(u,z,w)$, increases in w and strictly increases in z; let $\Phi(a) = \Psi(a)$ or $\Phi(b) = \Psi(b)$. Then $(\Phi - \Psi)$ is monotone on $[a,b]$.

Proof.— The proof is straightforward. □

LEMMA 3.— *Let functions F, F_n, $n \in \mathbb{N}$, be continuous and convex (or concave) on $[a,b]$. Assume that $F_n(u)$ converge to $F(u)$ for all $u \in [a,b]$. Then this convergence is uniform on all $[c,d] \subset (a,b)$ and*

a) $\dot{F}_n$ converge to $\dot{F}$ a.e. on $[a.b]$;

b) if F_n, F are increasing, then $\int_a^b \sqrt{\dot{F}_n(u)}du$ converge to $\int_a^b \sqrt{\dot{F}(u)}du$;

c) let $\left[\dot{F}\right]^{-1}, \left[\dot{F}_n\right]^{-1}$ denote the graph inverse functions of F, F_n respectively; then $\left[\dot{F}_n\right]^{-1}(\xi)$ tends to $\left[\dot{F}\right]^{-1}(\xi)$ for all ξ such that $\left[\dot{F}\right]^{-1}$ is continuous at the point ξ.

Proof.— An elementary proof of a),c) is given in [2]. Besides, the assumptions of the Lemma imply that for all $\delta > 0$, $\dot{F}_n$ are bounded uniformly in $n \in \mathbb{N}$, for $u \in [a+\delta, b-\delta]$. Since, in addition,

$$\left| \int_a^{a+\delta} \sqrt{\dot{F}_n(u)}du + \int_{b-\delta}^b \sqrt{\dot{F}_n(u)}du \right| \to 0$$

uniformly in $n \in \mathbb{N}$ as $\delta \to 0$, the conclusion b) follows from the Lebesgue Theorem. $\square$

Proof of Proposition 2. — There is nothing to prove if $a = b$; let $a < b$. Consider the penalized problem

$$\ddot{\Phi}(u) = G_n(u, \Phi(u), \dot{\Phi}(u))$$
$$:= \begin{cases} \dfrac{2\varepsilon\dot{\Phi}(u)}{f(u) - \Phi(u)}, & \text{if this value is in } (0, n) \\ n, & \text{otherwise} \end{cases}, \quad \dot{\Phi}(u) > 0 \tag{17}$$

for all $u \in [a, b]$. Since G_n is continuous in all variables and bounded, the existence of solution follows for arbitrary boundary data such that $\Phi(a) < \Phi(b)$; in particular, a solution Φ_n exists such that $\Phi_n(a) = f(a)$, $\Phi_n(b) = f(b)$. The Maximum Principle yields that Φ_n decrease to some convex non-decreasing function Φ on $[a, b]$ as $n \to \infty$.

Further, there exists a solution Ψ of (12) on $[a, b]$ with any assigned value of $\Psi(a)$ less than $f(a)$, or any assigned value of $\Psi(b)$ less than $f(b)$. In fact, in the first case one takes $\Psi(u) \equiv \Psi(a)$; in the second case there exists a solution on the whole of $[a, b]$ to the equation (12) with the Cauchy data $\Psi(b)$ (fixed) and $\dot{\Psi}(b)$ sufficiently large. By the Maximum Principle $\Phi_n \geqslant \Psi$ on $[a, b]$; therefore $\Phi(a + 0) = f(a)$ and $\Phi(b - 0) = f(b)$. Consequently Φ is continuous on $[a, b]$.

Now if for all $[c, d] \subset (a, b)$ there exists $m_0 > 0$ such that $f - \Phi \geqslant m_0$ on $[c, d]$, then the functions $G_n(u, \Phi_n(u), \dot{\Phi}_n(u))$ are bounded uniformly in $n \in \mathbb{N}$ for $u \in [c, d]$; indeed, on $[c, d]$, by convexity, $\dot{\Phi}_n$ are uniformly bounded and Φ_n converge to Φ uniformly, so that $\frac{2\varepsilon\dot{\Phi}_n}{f - \Phi_n} \leqslant M(c, d)$ for all n large enough. Hence it will follow by Lemma 3a) and the Lebesgue Theorem that $\ddot{\Phi}(u) = \frac{2\varepsilon\dot{\Phi}(u)}{f(u) - \Phi(u)}$ for all $u \in [c, d]$, and consequently $\Phi \in \mathbf{C}^2[c, d]$. Thus the existence of solution to problem (12),(13) will be shown.

First let show that $\dot{\Phi}(u \pm 0) > 0$ for all $u > a$. It suffices to prove that $\hat{u} = a$, where $\hat{u} := \sup\Big\{u \in [a, b] \,|\, \Phi(u) = f(a)\Big\}$. Note that $\hat{u} < b$ since $\Phi(b) = f(b) > f(a)$. Assume $\hat{u} > a$; by the Lebesgue Theorem $\ddot{\Phi} = \frac{2\varepsilon\dot{\Phi}}{f - \Phi}$ in some neighbourhood of $\hat{u}$. Since $\dot{\Phi}(\hat{u} - 0) = 0$, by the uniqueness theorem for the Cauchy problem Φ is constant in this neighbourhood. Therefore necessarily $\hat{u} = b$, which is impossible.

Further, by Lemma 3a), (17), and the Fatou Lemma one has $\frac{2\varepsilon\dot{\Phi}}{f - \Phi} \in L^1_{loc}(a, b)$. Hence $\Phi \leqslant f$ and $\frac{2\varepsilon\dot{\Phi}}{f - \Phi} \leqslant \ddot{\Phi}$ on (a, b) in measure sense. Now take $[c, d] \subset (a, b)$ and $\tilde{u} \in [c, d]$; set $m := f(\tilde{u}) - \Phi(\tilde{u}) \geqslant 0$. Set $A := \dot{\Phi}(\frac{a+c}{2} - 0) >$

0, $B := \dot\Phi(d-0) > 0$. For all $u \in [\frac{a+c}{2}, \tilde u]$, $f(u) - \Phi(u) \leqslant m + B(\tilde u - u)$ and $\dot\Phi(u \pm 0) \geqslant A$ since Φ is convex and f increasing. Hence

$$B - A \geqslant \int_{\frac{a+c}{2}}^{\tilde u} \ddot\Phi du \geqslant \int_{\frac{a+c}{2}}^{\tilde u} \frac{2\varepsilon\dot\Phi(u)}{f(u)-\Phi(u)} du$$

$$\geqslant \int_{\frac{a+c}{2}}^{\tilde u} \frac{2\varepsilon A}{m + B(\tilde u - u)} du = K_1 - K_2 \ln m,$$

with some positive constants K_1, K_2 depending only on c, d. Thus $m \geqslant m_0(c,d) > 0$ and the proof of existence is complete.

The uniqueness is clear from the Maximum Principle for solutions of (12). □

Proof of Proposition 3. — Let $a < b$; take $\alpha > 0$ and a barrier function Ψ_α such that $\alpha/2 \leqslant F - \Psi_\alpha \leqslant \alpha$ and $\ddot\Psi_\alpha \geqslant m(\alpha) > 0$ on $[a,b]$. Such a function can be constructed through the Weierstrass Theorem.

By the Maximum Principle Φ^ε increase as ε decrease. Therefore there exists $[c,d]$ inside (a,b) such that for all ε in $(0,1)$, $\Phi^\varepsilon \geqslant \Psi_\alpha$ on $[a,b]\setminus[c,d]$. It follows that $\left\{u \mid \Phi^\varepsilon(u) < \Psi_\alpha(u)\right\} \subset [c,d]$ and thus $\dot\Phi^\varepsilon \leqslant M(\alpha)$ on this set uniformly in ε. Now for all ε less than $\frac{\alpha\cdot m(\alpha)}{2M(\alpha)}$ one may apply the Maximum Principle to Φ^ε and Ψ_α, hence $0 \leqslant F - \Phi^\varepsilon \leqslant \alpha$ for all ε small enough. □

3. Solutions of the problem (6),(2) and the proof of Theorem 1

Proposition 2 above implies that for all $f, \varepsilon, u_\pm$ fixed, for all $u_0 \in \mathbb{R}$ there exist unique $\Phi_+^\varepsilon(\cdot; u_0)$ and $\Phi_-^\varepsilon(\cdot; u_0)$ satisfying (12),(13); thus by Proposition 1, for an arbitrary v_- in $\mathbb{R}$ and v_+ obtained from (14), (U,V) provided by (15),(16) is a self-similar solution to the Riemann problem (6),(2). Now since not u_0 but $v_\pm$ are given by (2), one needs to find u_0 in $\mathbb{R}$ such that (14) holds with these assigned values of $v_\pm$.

PROPOSITION 4. — *a) Assume $f(\pm\infty) = \pm\infty$. Then for all $u_\pm, v_\pm \in \mathbb{R}$, $\varepsilon > 0$ there exists a unique u_0 such that (14) holds, with Φ_+^ε, Φ_-^ε the (unique) solutions to (12),(13).*

b) Assume $f \in W_1^1$ locally in $\mathbb{R}$ and $\displaystyle\int_0^{\pm\infty} \sqrt{\dot f(u)}du = \pm\infty$. Then for all $u_\pm, v_\pm \in \mathbb{R}$ and $\varepsilon < \varepsilon^0 = \varepsilon^0(u_\pm, v_+ - v_-)$ there exists a unique u_0 such that (14) holds, with the same $\Phi_\pm^\varepsilon$.

Let $F_\pm(\cdot; u_0)$ be, as in the Introduction, the convex (concave) hulls of f on $I(u_0, u_\pm)$ according to the sign of $(u_\pm - u_0)$. Set

$$\Delta_\pm^\varepsilon(u_0) := \int_{u_0}^{u_+} \sqrt{\dot\Phi_\pm^\varepsilon(u; u_0)}du, \qquad \Delta_\pm^0(u_0) := \int_{u_0}^{u_+} \sqrt{\dot F_\pm(u; u_0)}du.$$

It will be convenient to extend $\Phi_\pm^\varepsilon(\cdot; u_0)$, $F_\pm(\cdot; u_0)$ to continuous functions on $\mathbb{R}$ by setting each of them constant on $(-\infty, \min\{u_0, u_\pm\}]$ and $[\max\{u_0, u_\pm\}, +\infty)$. In the lemma below a few facts needed for the proofs of Proposition 4 and Theorem 1 are stated.

LEMMA 4.— *With the notation above, and u_0 running through $\mathbb{R}$, the following properties hold.*

a) For all $u \in \mathbb{R}$ and $\varepsilon > 0$, $u_0 \mapsto \Phi_\pm^\varepsilon(u; u_0)$ do not decrease; nor do $u_0 \mapsto F_\pm(u; u_0)$.

b) For all $u \in \mathbb{R}$ and $\varepsilon > 0$, $u_0 \mapsto \operatorname{sign}(u_\pm - u_0)\dot\Phi_\pm^\varepsilon(u; u_0)$ do not increase; nor do $u_0 \mapsto \operatorname{sign}(u_\pm - u_0)\dot F_\pm(u; u_0)$.

c) For all $\varepsilon > 0$ the maps $u_0 \mapsto \Phi_\pm^\varepsilon(\cdot; u_0)$ are continuous for the $L^\infty(\mathbb{R})$ topology; so do $u_0 \mapsto F_\pm(\cdot; u_0)$.

d) For all $\varepsilon \geqslant 0$, $u_0 \mapsto \Delta_\pm^\varepsilon(u_0)$ are continuous and strictly decreasing.

Proof.— Combining the continuity and monotony of f with a),b) of the Maximum Principle for solutions of (12),(13), one gets a)-c) for $\Phi_\pm^\varepsilon$. The same assertions for $F_\pm$ follow now from Proposition 3 and Lemma 3a); they can also be easily derived from the definition of convex hull. Finally, d) results from c), Lemma 3b), b) and the strict monotony of f. □

Proof of Proposition 4.— a) By Lemma 4d), it suffices to prove that $\Delta_\pm^\varepsilon(\pm\infty) = \mp\infty$. Assume the contrary, for instance that $\Delta_+^\varepsilon(-\infty) = M < +\infty$.

Consider $u_0 < u_+$; Φ_+^ε is convex, therefore for all u_0 there exists $c = c(u_0) \in [u_0, u_+]$ such that $\dot\Phi_+^\varepsilon(\cdot; u_0) \geqslant 1$ on $[c, u_+)$ and $\dot\Phi_+^\varepsilon(\cdot; u_0) \leqslant 1$ on $(u_0, c]$. By Lemma 4b) $c(u_0)$ increase with u_0. Obviously, for all u_0, $M > \Delta_+^\varepsilon(u_0) \geqslant [\Phi_+^\varepsilon(c; u_0) - f(u_0)] + [u_+ - c]$. Set $d := u_+ - M$; clearly, $c(u_0) \geqslant d$ for all u_0. Considering the functions $\Phi^\varepsilon(\cdot; u_0)$ with $u_0 \to -\infty$, one obtains a sequence $\{\Psi_n\}$ such that Ψ_n satisfy (12) on $[d, u_+)$, $\dot\Psi_n(d) \leqslant 1$, $\Psi_n(u_+) = f(u_+)$, and finally, $\Psi_n(d) \to -\infty$ (this last holds because $\Psi_n(d) \leqslant f(u_0) + M \to f(-\infty) + M = -\infty$ as $u_0 \to -\infty$). On the other hand, for n large enough, the unique solution Ψ to the equation (12) with the Cauchy data $\Psi(d) = \Psi_n(d)$, $\dot\Psi(d) = 2$ is defined on the whole of $[d, u_+]$, which means that $\Psi(u_+) < f(u_+)$. Now by b) of the Maximum Principle, $(\Psi - \Psi_n)$ is increasing and thus positive. Hence $\Psi_n(u_+) \leqslant \Psi(u_+) < f(u_+)$, which is a contradiction.

b) Take $u_0 < u_+$. First suppose $f \in C^2[u_0, u_+]$ and has a finite number of points of inflexion; denote by F the corresponding convex hull. The segment $[u_0, u_+]$ can be decomposed into the three disjoint sets: $M_1 := \Big\{ u \,|\, \exists\, \delta > 0$ $s.t.\ \dot{F} \equiv const\ on\ (u-\delta, u+\delta) \cap [a,b] \Big\}$, $M_2 := \Big\{ u \,|\, \dot{F}(u) = \dot{f}(u) \Big\} \setminus M_1$, and M_3 finite. Using the Cauchy-Schwarz inequality on every $(c,d) \subset M_1$, one gets $\displaystyle\int_{u_0}^{u_+} \sqrt{\dot{F}(u)}du \equiv \Delta_+^0(u_0) \geqslant \int_{u_0}^{u_+} \sqrt{\dot{f}(u)}du.$

In the general case, let proceed with the density argument, choosing a sequence $\{f_n\}$ such that f_n are increasing and smooth as above, $f_n \to f$ in $C[u_0, u_+]$ with $\sqrt{\dot{f}_n} \to \sqrt{\dot{f}}$ in $L^1[u_0, u_+]$ as $n \to \infty$. Denote the convex hull of f_n on $[u_0, u_+]$ by F_n; it is easy to see that $\|F_n - F\|_{C[u_0,u_+]} \leqslant \|f_n - f\|_{C[u_0,u_+]} \to 0$ as $n \to \infty$. By Lemma 4b), $\Delta_+^0(u_0) = \displaystyle\lim_{n\to\infty} \int_{u_0}^{u_+} \sqrt{\dot{F}_n(u)}du$, so that $\Delta_+^0(u_0) \geqslant \displaystyle\int_{u_0}^{u_+} \sqrt{\dot{f}(u)}du$ in the general case as well. Thus $\Delta_+^0(-\infty) = +\infty$ by the assumption on f.

Now Proposition 3 and Lemma 3b) imply that for given $v_\pm$ in $\mathbb{R}$, there exists $\varepsilon^0 = \varepsilon^0(u_\pm, v_+ - v_-)$ such that one has $\Delta_+^\varepsilon(-L) > |v_- - v_+|$ (and in the same way, $\Delta_+^\varepsilon(L) < -|v_- - v_+|$) for all $\varepsilon < \varepsilon^0$ whenever L is large enough. Lemma 4d) yields now the required fact. □

Finally, here is the proof of the result announced in the Introduction.

Proof of Theorem 1. — The existence and uniqueness of a bounded self-similar distribution solution to the Riemann problem (6),(2) follow immediately from Propositions 1, 2 and 4.

Now let ε decrease to 0. Take $\left(u_0^\varepsilon, \Phi_\pm^\varepsilon(\cdot\,; u_0^\varepsilon)\right)$ corresponding to the unique solution of (6),(2) in the sense of Proposition 1. Take u_0 a limit point in $\overline{\mathbb{R}}$ of $\{u_0^\varepsilon\}_{\varepsilon>0}$. Suppose first $u_0^{\varepsilon_k} \to u_0 \in \mathbb{R}$, $\varepsilon_k \to 0$ as $k \to \infty$; let show that, with the notation as in Lemma 4, $\Phi_+^\varepsilon(\cdot\,; u_0^\varepsilon)$ converge to $F_+(\cdot\,; u_0)$ in $L^\infty(\mathbb{R})$. Indeed, take $\alpha > 0$; $|u_0^{\varepsilon_k} - u_0| < \alpha$ for all k large enough. By Proposition 3 and Lemma 4a), there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon_k < \varepsilon_0$, $F_+(\cdot\,; u_0 - \alpha) - \alpha \leqslant \Phi_+^{\varepsilon_k}(\cdot\,; u_0 - \alpha) \leqslant \Phi_+^{\varepsilon_k}(\cdot\,; u_0^{\varepsilon_k}) \leqslant \Phi_+^{\varepsilon_k}(\cdot\,; u_0 + \alpha) \leqslant F_+(\cdot\,; u_0 + \alpha) + \alpha$. Thus the required result follows from Lemma 4c); clearly, it also holds for $\Phi_-^{\varepsilon_k}, F_-$ in place of $\Phi_+^{\varepsilon_k}, F_+$.

Now by Lemma 3b) $\Delta_+^0(u_0)+\Delta_-^0(u_0)$ is the limit of $\Delta_+^{\varepsilon_k}(u_0^{\varepsilon_k})+\Delta_-^{\varepsilon_k}(u_0^{\varepsilon_k}) \equiv v_- - v_+$; hence by Lemma 4d), u_0 is unique if it is finite. Besides, if for instance $u_0 = -\infty$, then for all $L \in \mathbb{R}$, $v_- - v_+ = \lim_{\varepsilon_k \to 0}[\Delta_+^{\varepsilon_k}(u_0^{\varepsilon_k}) + \Delta_-^{\varepsilon_k}(u_0^{\varepsilon_k})] \geqslant \Delta_+^0(L) + \Delta_-^0(L)$ by Lemma 4d) and Lemma 3b). It is a contradiction; indeed, it is easy to see that $\Delta_\pm^0(L) \to +\infty$ as $L \to -\infty$.

Thus in fact $u_0^\varepsilon \to u_0$ as $\varepsilon \to 0$, $u_0 \in \mathbb{R}$ and (5) holds. Further, let $u_0 < u_\pm$; the other cases are similar and those of $u_0 = u_-$ or $u_0 = u_+$ are trivial. For all $\alpha > 0$ there exists $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ such that for all $\varepsilon < \varepsilon_0$, $[u_0^\varepsilon, u_\pm] \subset [u_0 - \alpha, u_\pm]$. The functions U^ε in the statement of Theorem 1 are given by formula (15), when applied to $\Phi_\pm^\varepsilon(\cdot; u_0)$ with their natural domains $[u_0^\varepsilon, u_\pm]$. Taking for the domains $[u_0 - \alpha, u_\pm]$, one does not change $U^\varepsilon(\xi)$ for $\xi \neq 0$ and $\varepsilon < \varepsilon_0$. The same being valid for U given by (3), one may use the fact, proved above, that $\|\Phi_\pm^\varepsilon(\cdot; u_0^\varepsilon) - F_\pm(\cdot; u_0)\|_{C[u_0-\alpha, u_\pm]} \to 0$ as $\varepsilon \to 0$, and conclude by Lemma 3c) that $U^\varepsilon(\xi) \to U(\xi)$ for a.a. $\xi \in \mathbb{R}$. Hence it follows by (4),(16) that $V^\varepsilon \to V$ a.e., so that (U, V) given by (3)-(5) is the unique a.e.-limit of self-similar bounded distribution solutions of the problem (6),(2). Thus (U, V) is a distribution solution of the Riemann problem (1),(2). □

Remark 2. — Note that using b) of Proposition 4 instead of a), one gets a result similar to the Theorem 1 in the case of $f \in W_1^1$ locally in $\mathbb{R}$, $\int_0^{\pm\infty} \sqrt{\dot{f}(u)}du = \pm\infty$; in fact, the exact condition is the bijectivity of the functions $u_0 \mapsto \Delta_\pm^0(u_0)$ for continuous strictly increasing flux function f. Under each of this conditions the existence of a bounded self-similar solution of (6),(2) is guaranteed for all $\varepsilon < \varepsilon^0 = \varepsilon^0(u_\pm, v_+ - v_-)$.

Note. — After this paper had been completed, the author had an opportunity to meet Prof. A.E.Tzavaras and get acquanted with his papers on viscosity limits for the Riemann problem; in particular, in [10] very close results were obtained for p-systems regularized by viscosity terms of the form $\begin{pmatrix} 0 \\ \varepsilon t(k(U)V_x)_x \end{pmatrix}$, without involving the explicit formulae for the limiting solution.

For results on self-similar viscous limits for general strictly hyperbolic systems of conservation laws, refer to the survey paper [11] and literature cited therein. Let only note that the structure of wave fans in self-similar viscous limits remains the same as in the case of scalar conservation laws ([6, 8]) and in the case of p-systems, where it can be easily observed through the formulae (3),(4).

On the other hand, Prof. B.Piccoli turned my attention to Riemann solvers for hyperbolic-elliptic systems (1) (i.e., the case of non-monotone f). The global explicit Riemann solver extends to this case (see Krejčí, Straškraba, [7]); it can be proved, with the techniques used here and in [1, 2], that this solver is the unique limit of self-similar bounded solutions to the problem (6),(2).

Precise results on hyperbolic-elliptic p-systems and a discussion of other viscosity terms will be given in [3], together with a study of self-similar viscous limits for the corresponding system in Eulerian coordinates.

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Bibliography

[1] ANDREYANOV (B.). — *"Solutions auto-similaires du problème de Riemann pour une loi de conservation scalaire quasilinéaire à fonction de flux continue avec la viscosité $\varepsilon t u_{xx}$"* Publ. Math. Besançon, An. non-linéaire, V.15 (1995/97), pp.127-131.

[2] ANDREYANOV (B.). — *"Vanishing viscosity method and explicit formulae for solutions of the Riemann problem for scalar conservation laws"* Vestn. Mosc. Univ. I (1999), No.1, pp.3-8.

[3] ANDREIANOV (B.). — Ph. D. thesis, Univ. Franche-Comté, 2000.

[4] CHANG (T.), HSIAO (L.) (T. Zhang, L. Xiao). — *"The Riemann Problem and Interaction of Waves in Gas Dynamics"* Pitman Monographs and Surveys in Pure and Appl. Math., 41 (1989).

[5] DAFERMOS (C.M.). — *"Structure of solutions of the Riemann problem for hyperbolic systems of conservation laws"* Arch. Rat. Mech. Anal., V.53 (1974), No.3, pp.203-217.

[6] GELFAND (I.M.). — *"Some problems in the theory of quasilinear equations"* Uspekhi Mat. Nauk, V.14 (1959), No. 2, pp. 87-158; English tr. in Amer. Math. Soc. Transl. Ser. V.29 (1963), No.2, pp.295-381.

[7] KREJČÍ (P.), STRAŠKRABA (I.). — *"A uniqueness criterion for the Riemann problem"* Mat.Ústav AV ČR, V.84 (1993); Hiroshima Math. J. V. 27 (1997), No. 2, pp. 307-346.

[8] KRUZHKOV (S.N.). — *"Nonlinear partial differential equations" Part II. (Lections 3,4)* Mosc. St. Univ. edition, 1970.

[9] LEIBOVICH (L.). — *"Solutions of the Riemann problem for hyperbolic systems of quasilinear equations without convexity conditions"* J. Math. Anal. Appl., V.45 (1974), No. 3, pp. 81-90.

[10] TZAVARAS (A.E.). — *"Elastic as limit of viscoelastic response, in a context of self-similar viscous limits"* J. Diff. Eq., V.123 (1995), No. 1, pp. 305-341.

[11] TZAVARAS (A.E.). — *"Viscosity and relaxation approximations for hyperbolic systems of conservation laws"* in "An introduction to recent developments in theory and numerics of conservation laws", D. Kroener & al., eds.; Lect. Notes in Comp. Sci. and Engin., V.5, Springer, 1998, pp. 73-122.

[12] WENDROFF (B.). — *"The Riemann problem for materials with nonconcave equations of state. I: Isentropic flow"* J. Math. Anal. Appl., V. 38 (1972), pp. 454-466.