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Special Directions on Contact Metric Manifolds of Negative ξ-sectional Curvature(*)

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RÉSUMÉ. — Dans cet article nous introduisons des vecteurs particuliers dans le sous-fibré de contact d'une variété de contact à courbure sectionnelle négative, pour les plans contenant la direction caractéristique. Sur une variété métrique de contact en dimension 3, si le champ caractéristique est de type Anosov et si les vecteurs particuliers coïncident avec les directions stable et instable, alors la variété métrique de contact est une 3-τ-métrique. De plus, si la variété est supposée compacte, alors c'est un quotient de $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Dans le cas du fibré unitaire tangent d'une surface à courbure négative, équipé de la métrique de contact canonique, les vecteurs particuliers ne coïncident cependant jamais avec les directions stable et instable.

ABSTRACT. — In this paper we introduce special directions in the contact subbundle on a contact metric manifold with negative sectional curvature for plane sections containing the characteristic vector field. If on a 3-dimensional contact metric manifold whose characteristic vector field is Anosov, the special directions agree with the stable and unstable directions of the Anosov flow, then the contact metric structure is a 3-τ-structure. Moreover if the manifold is compact, then it is a compact quotient of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. In the case of the tangent sphere bundle of a negatively curved surface with its standard contact metric structure, the special directions never agree with the stable and unstable directions.

1. Introduction

The purpose of this paper is to introduce special directions belonging to the contact subbundle on a contact metric manifold with negative sectional curvature for plane sections containing the characteristic vector field ξ or more generally when the operator h (see below) admits an eigenfunction

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everywhere greater than 1. As an application we turn to 3-dimensional contact manifolds whose characteristic vector field is Anosov and compare the special directions with the stable and unstable directions (Anosov directions) of the Anosov flow. We show (Theorem 4.2) that if on a 3-dimensional contact metric manifold with negative ξ-sectional curvature, ξ is Anosov and the special directions agree with the Anosov directions, then the contact metric structure is a 3-τ-structure in the sense of Gouli-Andreou and Xenos [10]. Moreover if the manifold is compact we will see (Theorem 4.3) that it is a compact quotient of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Since the special directions are smooth, this is a consequence of a result of E. Ghys [9] that if ξ is Anosov on a compact 3-dimensional contact manifold M and the Anosov directions are smooth, then M is a compact quotient of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. We will, however, give a proof using properties of a 3-τ-structure. There has been recent interest in questions related to the Anosovicity of the characteristic vector field of a contact structure and the reader may want to look at the 3-dimensional contact manifolds constructed by Y. Mitsumatsu in [12]. For results in higher dimensions see the paper of Benoist, Foulon and Labourie [3].

The most notable example of a contact manifold for which the characteristic vector field is Anosov is the tangent sphere bundle of a negatively curved manifold; here the characteristic vector field is the geodesic flow. In the case of the tangent sphere bundle of a surface, this is closely related to the structure on $\mathrm{SL}(2,\mathbb{R})$ from both the topological and Anosov points of view. If we set

$$Z_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

then $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/Z_2$ is homeomorphic to the tangent sphere bundle of the hyperbolic plane. Moreover the geodesic flow on a compact surface of constant negative curvature may be realized on $\mathrm{PSL}(2,\mathbb{R})/\Gamma$ by

$$\left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right\},$$

where Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ for which $\mathrm{SL}(2,\mathbb{R})/\Gamma$ is compact (see e.g., [2, pp. 26-27]). However from the Riemannian point of view these examples are quite different as we shall see. In fact in the case of the tangent sphere bundle of a negatively curved surface, the special directions never agree with the Anosov directions (Theorem 4.4).

Finally for the Lie group $\mathrm{SL}(2,\mathbb{R})$ we exhibit the special directions which do agree with the Anosov directions.

2. Contact manifolds

By a *real contact manifold* we mean a C^∞ manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (\mathrm{d}\eta)^n \neq 0$. It is well known that given η there exists a unique vector field ξ such that $\mathrm{d}\eta(\xi, X) = 0$ and $\eta(\xi) = 1$ called the *characteristic vector field* or *Reeb vector field* of the contact structure η. A classical theorem of Darboux states that on a contact manifold there exist local coordinates with respect to which $\eta = \mathrm{d}z - \sum_{i=1}^n y^i\,\mathrm{d}x^i$. We denote the *contact subbundle* or *contact distribution* defined by the subspaces $\{X \in T_m M \mid \eta(X) = 0\}$ by $\mathcal{D}$. Roughly speaking the meaning of the contact condition, $\eta \wedge (\mathrm{d}\eta)^n \neq 0$, is that the contact subbundle is as far from being integrable as possible. In fact for a contact manifold the maximum dimension of an integral submanifold of $\mathcal{D}$ is only n; whereas a subbundle defined by a 1-form η is integrable if and only if $\eta \wedge \mathrm{d}\eta \equiv 0$.

A Riemannian metric g is an *associated metric* for a contact form η if there exists a tensor field ϕ of type $(1,1)$ such

$$\phi^2 = -I + \eta \otimes \xi\,, \quad \eta(X) = g(X, \xi)\,, \quad \mathrm{d}\eta(X, Y) = g(X, \phi Y)\,;$$

We refer to (η, g) or (ϕ, ξ, η, g) as a *contact metric structure.* All associated metrics have the same volume element, viz.,

$$\frac{(-1)^n}{2^n n!}\,\eta \wedge (\mathrm{d}\eta)^n\,.$$

Since $\mathrm{d}\eta(\xi, X) = 0$ and $\eta(\xi) = 1$, computing Lie derivatives, we have $\mathcal{L}_\xi \eta = 0$ and $\mathcal{L}_\xi\,\mathrm{d}\eta = 0$. Thus the flow generated by ξ is volume preserving with respect to any associated metric.

In the theory of contact metric manifolds there is another tensor field that plays a fundamental role, viz. $h = (1/2)\mathcal{L}_\xi \phi$. h is a symmetric operator which anti-commutes with ϕ, $h\xi = 0$ and h vanishes if and only if ξ is Killing. We denote by ∇ the Levi-Civita connection of g and by R its curvature tensor. On a contact metric manifold we have the following further important relations involving h,

$$\nabla_X \xi = -\phi X - \phi h X\,, \tag{2.1}$$

$$\frac{1}{2}\left(R_{\xi X}\xi - \phi R_{\xi\,\phi X}\xi\right) = h^2 X + \phi^2 X\,, \tag{2.2}$$

$$(\nabla_\xi h)X = \phi(X - h^2X - R_{X\xi}\xi)\,. \tag{2.3}$$

As a corollary we see from equation (2.2) that on a contact metric manifold M^{2n+1} the Ricci curvature in the direction ξ is given by

$$\mathrm{Ric}(\xi) = 2n - \operatorname{tr} h^2\,. \tag{2.4}$$

Since $h\phi + \phi h = 0$, if λ is an eigenvalue of h with eigenvector X, then $-\lambda$ is also an eigenvalue with eigenvector ϕX. Thus, since $h\xi = 0$, in dimension 3 we have only one eigenfunction λ on the manifold to be concerned with.

The sectional curvature of a plane section containing ξ is called a *ξ-sectional curvature*. In this paper, except for the result from [7] described in the next paragraph, we do not need the notion of a Sasakian manifold, though it may be worth pointing out that the ξ-sectional curvature of a Sasakian manifold is $+1$. For a general reference to the ideas so far in this section see [4].

In [7] it was shown that a 3-dimensional contact metric manifold M^3 whose Ricci operator Q commutes with the tensor field ϕ is either Sasakian, flat or locally isometric to a left-invariant metric on the Lie group SU(2) or $\mathrm{SL}(2,\mathbb{R})$. In the latter cases M^3 has constant ξ-sectional curvature $k = 1 - \lambda^2 < 1$ and the sectional curvature of a plane section orthogonal to ξ is $-k$ (see also [8]); the structure occurs on these Lie groups with $k > 0$ for SU(2) and $k < 0$ for $\mathrm{SL}(2,\mathbb{R})$. It was also shown in [8] (see Lemma 3.1) that on a 3-dimensional contact metric manifold satisfying $Q\phi = \phi Q$, the eigenfunction λ is a constant.

In [10] Gouli-Andreou and Xenos introduced the notion of a *3-τ-manifold*, namely a 3-dimensional contact metric manifold on which

$$\nabla_\xi h = 0\,.$$

The name comes from the equivalent condition $\nabla_\xi \tau = 0$ where $\tau = \mathcal{L}_\xi g$; in particular τ and h are related by $\tau(X,Y) = 2g(h\phi X, Y)$. The following lemma shows that a 3-dimensional contact metric manifold on which $Q\phi = \phi Q$ is a 3-τ-manifold and gives a partial converse. In general the converse is not true and an example is given in [6].

LEMMA 2.1.— *A 3-dimensional contact metric manifold on which $Q\phi = \phi Q$ is a 3-τ-manifold. A 3-τ-manifold on which $Q\xi$ is collinear with ξ satisfies $Q\phi = \phi Q$.*

Proof.— If $Q\phi = \phi Q$, then $\phi\xi = 0$ gives $\phi Q\xi = 0$ and hence that $Q\xi$ is collinear with ξ. In [13] (Proposition 3.1) Perrone proved that on a 3-dimensional contact metric manifold

$$(\nabla_\xi \tau)(X,Y) = g(Q\phi X, \phi Y) - g(QX,Y) + \eta(X)g(Q\xi,Y) + \\ + \eta(Y)g(Q\xi,X) - \eta(X)\eta(Y)g(Q\xi,\xi).$$

Thus if $Q\phi = \phi Q$, $\nabla_\xi \tau = 0$ giving the first statement.

If $(\nabla_\xi \tau)(X,Y) = 0$ and $Q\xi = f\xi$, Perrone's formula yields

$$g(Q\phi X, \phi Y) - g(QX,Y) + f\eta(X)\eta(Y) = 0$$

or

$$-\phi Q\phi X - QX + f\eta(X)\xi = 0.$$

Applying ϕ and noting that $\eta(Q\phi X) = g(\xi, Q\phi X) = g(Q\xi, \phi X) = 0$, we have $Q\phi = \phi Q$ as desired. □

LEMMA 2.2.— *On a 3-τ-manifold the eigenfunction λ is constant along integral curves of ξ.*

Proof.— Let X be a unit eigenvector field corresponding to λ. Then

$$0 = (\nabla_\xi h)X = (\xi\lambda)X + \lambda\nabla_\xi X - h\nabla_\xi X.$$

Since X is unit and h symmetric, taking the inner product with X yields $\xi\lambda = 0$. □

Let $\{X, \phi X, \xi\}$ be an eigenvector basis of h with $hX = \lambda X$, $\lambda \neq 0$; in [10] the following were obtained on a 3-τ-manifold.

$$\nabla_\xi X = \nabla_\xi \phi X = 0, \quad \nabla_X \xi = -(\lambda+1)\phi X, \quad \nabla_{\phi X}\xi = (1-\lambda)X$$

$$\nabla_X(\phi X) = -\frac{1}{2\lambda}\left(\phi X\lambda + \eta(QX)\right)X + (\lambda+1)\xi,$$

$$\nabla_{\phi X}X = -\frac{1}{2\lambda}\left(X\lambda + \eta(Q\phi X)\right)\phi X + (\lambda-1)\xi.$$

Other derivatives are easily obtained from these.

LEMMA 2.3.— *If on a 3-τ-manifold $hX = \lambda X$ and λ is a non-zero constant, then*

$$\xi\xi\eta(QX) = (\lambda^2 - 1)\eta(QX).$$

Proof.— Using the constancy of λ in the covariant derivatives above and computing Lie brackets as differences of covariant derivatives, the Jacobi identity yields

$$\begin{aligned} 0 &= [[X, \phi X], \xi] + [[\phi X, \xi], X] + [[\xi, X], \phi X] \\ &= \left[-\frac{1}{2\lambda}\eta(QX)X + \frac{1}{2\lambda}\eta(Q\phi X)\phi X, \xi\right]. \end{aligned}$$

Therefore

$$\begin{aligned} 0 &= -[\eta(QX)X, \xi] + [\eta(Q\phi X)\phi X, \xi] \\ &= (\xi\eta(QX))X + \eta(QX)(\lambda+1)\phi X + \\ &\quad - (\xi\eta(Q\phi X))\phi X + \eta(Q\phi X)(1-\lambda)X. \end{aligned}$$

Taking X and ϕX components we have

$$\xi\eta(QX) = (\lambda - 1)\eta(Q\phi X), \quad \xi\eta(Q\phi X) = (\lambda+1)\eta(QX)$$

from which the result easily follows. □

3. Anosov flows

Classically an Anosov flow is defined as follows [1, pp. 6-7]. Let M be a compact differentiable manifold, ξ a non-vanishing vector field and $\{\psi_t\}$ its 1-parameter group of diffeomorphisms. $\{\psi_t\}$ is said to be an *Anosov flow* (or ξ to be *Anosov*) if there exist subbundles E^s and E^u which are invariant along the flow and such that $TM = E^s \oplus E^u \oplus \{\xi\}$ and there exists a Riemannian metric such that

$$|\psi_{t*}Y| \leq \begin{cases} a\, e^{-ct}|Y| & \text{for } t \geq 0 \text{ and } Y \in E^s_p, \\ a\, e^{ct}|Y| & \text{for } t \leq 0 \text{ and } Y \in E^u_p \end{cases}$$

where a and c are positive constants independent of $p \in M$ and Y in E^s_p or E^u_p. E^s and E^u are called the *stable* and *unstable* subbundles or the *contracting* and *expanding* subbundles .

When M is compact the notion is independent of the Riemannian metric. If M is not compact the notion is metric dependent; the example of a 3-τ-manifold on which the Ricci operator does not commute with ϕ given in [6] is a metric on $\mathbb{R}^3$ with respect to which the coordinate field $\partial/\partial z$ is Anosov, even though $\partial/\partial z$ is clearly not Anosov with respect to the Euclidean metric on $\mathbb{R}^3$. Since we are dealing with Riemannian metrics associated to a contact structure, when we speak of the characteristic vector field being Anosov, we will mean that it is Anosov with respect to an associated metric of the contact structure.

The following properties of Anosov flows will be of importance here. The subbundles $E^s \oplus \{\xi\}$ and $E^u \oplus \{\xi\}$ are integrable, [1, Theorem 8]. Let μ denote the measure induced on M by the Riemannian metric. Recall that a flow is *ergodic* if for every measurable set S, $\psi_t(S) = S$ for all t implies $\mu(S)\mu(M - S) = 0$. If on a compact manifold an Anosov flow admits an integral invariant, in particular if it is volume preserving, then it is ergodic [1, Theorem 4] and in turn by the Ergodic Theorem almost all orbits are dense (see e.g, [15, pp. 29-30]).

As an aside we note that on a compact manifold, an Anosov flow has a countable number of periodic orbits [1, Theorem 2] and if the flow admits an integral invariant, then the set of periodic orbits is dense in M [1, Theorem 3]. This in itself has some implications for contact geometry. An important conjecture of Weinstein [16] is that on a simply connected compact contact manifold ξ must have a closed orbit, so in particular the Weinstein conjecture holds for a compact contact manifold on which ξ is Anosov.

4. Special directions

We may regard equation (2.1) as indicating how ξ or, by orthogonality, the contact subbundle, rotates as one moves around on the manifold. For example when $h = 0$, as we move in a direction X orthogonal to ξ, ξ is always "turning" or "falling" toward $-\phi X$. If $hX = \lambda X$, then $\nabla_X \xi = -(1+\lambda)\phi X$ and again ξ is turning toward $-\phi X$ if $\lambda > -1$ or toward ϕX if $\lambda < -1$. Recall, as we noted above, that if λ is an eigenvalue of h with eigenvector X, then $-\lambda$ is also an eigenvalue with eigenvector ϕX.

Now one can ask if there can ever be directions, say Y orthogonal to ξ, along which ξ "falls" forward or backward in the direction of Y itself.

THEOREM 4.1.— *Let M^{2n+1} be a contact metric manifold. If the tensor field h admits an eigenvalue $\lambda > 1$ at a point P, then there exists a vector Y orthogonal to ξ at P such that $\nabla_Y \xi$ is collinear with Y. In particular if M^{2n+1} has negative ξ-sectional curvature, such directions Y exist.*

Proof.— As stated in the Theorem we will let λ denote a positive eigenvalue of h and let X be a corresponding unit eigenvector. Then from equation (2.1)

$$\nabla_X \xi = -(1+\lambda)\phi X\,, \quad \nabla_{\phi X}\xi = (1-\lambda)X\,.$$

Now let $Y = aX + b\phi X$ with $a > 0$, $b > 0$, $a^2 + b^2 = 1$ and suppose that $\nabla_Y \xi = \alpha Y$. Then

$$\alpha(aX + b\phi X) = \nabla_Y \xi = -(1+\lambda)a\phi X + (1-\lambda)bX$$

from which $\alpha a = (1-\lambda)b$, $\alpha b = -(1+\lambda)a$ and hence

$$a^2 = \frac{\lambda - 1}{2\lambda}\,, \quad b^2 = \frac{\lambda+1}{2\lambda}\,, \quad \alpha = -\sqrt{\lambda^2 - 1}\,.$$

Thus we see that directions along which $\nabla_Y \xi$ is collinear with Y exist whenever h admits an eigenvalue greater than 1. From equation (2.4) we see that if M^{2n+1} has negative ξ-sectional curvature, at least one of the eigenvalues of h must exceed 1. □

Note that when there exists a direction Y along which $\nabla_Y \xi$ is collinear with Y as above, there is also a second such direction, namely $Z = aX - b\phi X$. For Z we have $\nabla_Z \xi = -\alpha Z$; thus we think of ξ as falling backward as we move in the direction Y and falling forward as we move in the direction Z.

Next let us note that

$$g(Y,Z) = a^2 - b^2 = -\frac{1}{\lambda}$$

and hence that such directions Y and Z are never orthogonal. Also if λ has multiplicity $m > 1$, then there are m-dimensional subbundles $\mathcal{Y}$ and $\mathcal{Z}$ such that $\nabla_Y \xi = \alpha Y$ for any $Y \in \mathcal{Y}$ and $\nabla_Z \xi = -\alpha Z$ for any $Z \in \mathcal{Z}$.

We now turn to 3-dimensional contact metric manifolds where the dimension of $\mathcal{Y}$ and $\mathcal{Z}$ will be 1 and ask what happens if these special directions or subbundles agree with the stable and unstable bundles (Anosov directions) of the Anosov flow generated by ξ.

THEOREM 4.2.— *Let M be a 3-dimensional contact metric manifold with negative ξ-sectional curvature. If the characteristic vector field ξ generates an Anosov flow and the special directions agree with the Anosov directions, then the contact metric structure is a 3-τ-structure.*

Proof.— Suppose that Y is a local unit vector field such that $\nabla_Y \xi = \alpha Y$, $\alpha = -\sqrt{\lambda^2 - 1}$. Since ξ is Anosov and $\mathcal{Y}$ agrees with the stable Anosov subbundle, the subbundle $\mathcal{Y} \oplus \{\xi\}$ is integrable. Thus from $[\xi, Y] = \nabla_\xi Y - \alpha Y$, $\nabla_\xi Y$ belongs to $\mathcal{Y} \oplus \{\xi\}$; but $g(\nabla_\xi Y, \xi) = 0$ and Y is unit, so $g(\nabla_\xi Y, Y) = 0$. Thus $\nabla_\xi Y = 0$. Similarly $\nabla_\xi Z = 0$. For simplicity define an operator ℓ by $\ell X = R_{X\xi}\xi$ for any X; clearly ℓ is a symmetric operator. Computing $R_{Y\xi}\xi$ and $R_{Z\xi}\xi$ we have

$$\ell Y = -(\xi\alpha + \alpha^2)Y\,, \quad \ell Z = (\xi\alpha - \alpha^2)Z\,;$$

but Y and Z are not orthogonal, so $\xi\alpha = 0$ and $\ell\big|_{\mathcal{D}} = -\alpha^2 I\big|_{\mathcal{D}}$. Now compute $\nabla_\xi h$ acting on each vector of the eigenvector basis $\{X, \phi X, \xi\}$ using equation (2.3).

$$(\nabla_\xi h)X = \phi(X - h^2 X - \ell X) = \phi(X - \lambda^2 X + \alpha^2 X) = 0$$

and similarly $(\nabla_\xi h)\phi X = 0$. $(\nabla_\xi h)\xi = 0$ is immediate. Thus $\nabla_\xi h = 0$ and M is a 3-τ-manifold. □

If M is compact in Theorem 4.2, then M is a compact quotient of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. This follows from a result of E. Ghys [9] that if ξ is Anosov on a compact 3-dimensional contact manifold M and the Anosov directions are smooth, then M is a compact quotient of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Here we present this as a compact version of Theorem 4.2 and give a proof using properties of a 3-τ-structure.

THEOREM 4.3.— *Let M be a compact 3-dimensional contact metric manifold with negative ξ-sectional curvature. If the characteristic vector field ξ generates an Anosov flow and the special directions agree with the Anosov directions, then M is a compact quotient of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.*

Proof.— By Theorem 4.2 the contact metric structure is a 3-τ-structure. Since M is compact and ξ is a volume preserving Anosov flow, ξ has a dense orbit by the ergodicity. Thus the eigenfunction λ, which is invariant along the flow by Lemma 2.2, is constant on M. Again by the density of the orbit, Lemma 2.3 implies that $\eta(QX) = 0$ and hence by Lemma 2.1, $Q\phi = \phi Q$. Finally, as we noted in Section 2, the result of [7] for an eigenvalue $\lambda > 1$, $k = 1 - \lambda^2 < 0$, is that the universal covering of M is the universal covering of $\mathrm{SL}(2, \mathbb{R})$. □

THEOREM 4.4.— *With respect to the standard contact metric structure on the tangent sphere bundle of a negatively curved surface, the characteristic vector field is Anosov, but the special directions never agree with the stable and unstable directions.*

Proof.— It is well known that for the standard contact metric structure on the tangent sphere bundle of a Riemannian manifold, the characteristic vector field ξ is (twice) the geodesic flow (cf. [4]), which is, as is also well known, an Anosov vector field when the base manifold is negatively curved (see e.g., [1]). By Theorem 4.2 if the special directions of the contact metric structure agree with the Anosov directions, then $\nabla_\xi h = 0$. Now Perrone [14] showed that the standard contact metric structure of the tangent sphere bundle of any Riemannian manifold satisfies $\nabla_\xi h = 0$ if and only if the base manifold is of constant curvature 0 or $+1$. □

5. Contact metric structures on $\mathrm{SL}(2, \mathbb{R})$

In this section we briefly exhibit a family of contact metric structures on the Lie group $\mathrm{SL}(2, \mathbb{R})$, show that the characteristic vector field is Anosov and show that the special directions agree with the Anosov directions. For further discussion of these contact metric structures on $\mathrm{SL}(2, \mathbb{R})$ see [5].

On a 3-dimensional unimodular Lie group we have a Lie algebra structure of the form

$$[e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \quad [e_1, e_2] = c_3 e_3 .$$

In [11], J. Milnor gave a complete classification of 3-dimensional Lie groups and their left invariant metrics. If one c_i is non-zero, the dual 1-form ω_i

is a contact form and e_i is the characteristic vector field. However for the Riemannian metric defined by $g(e_i, e_j) = \delta_{ij}$ at the identity and extended by left translation to be an associated metric for ω_i, we must have $c_i = 2$ [7]. For $\mathrm{SL}(2,\mathbb{R})$ two of the c_i's are positive and one negative in the Milnor classification, so taking ω_1 as the contact form, we write the Lie algebra structure as

$$[e_2, e_3] = 2e_1, \quad [e_3, e_1] = (1-\lambda)e_2, \quad [e_1, e_2] = (1+\lambda)e_3 \tag{5.1}$$

where $\lambda > 1$. Further by way of notation, we set

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ u & v \end{pmatrix} \;\middle|\; xv - yu = 1 \right\}.$$

Now consider the matrices

$$\begin{pmatrix} \frac{1}{2}\sqrt{\lambda^2-1} & 0 \\ 0 & -\frac{1}{2}\sqrt{\lambda^2-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & -\sqrt{\frac{\lambda+1}{2}} \\ \sqrt{\frac{\lambda+1}{2}} & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & -\sqrt{\frac{\lambda-1}{2}} \\ -\sqrt{\frac{\lambda-1}{2}} & 0 \end{pmatrix}$$

in the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ which we regard as the tangent space of $\mathrm{SL}(2,\mathbb{R})$ at the identity. Applying the differential of left translation by

$$\begin{pmatrix} x & y \\ u & v \end{pmatrix}$$

to these matrices gives the vector fields

$$\zeta_1 = \frac{1}{2}\sqrt{\lambda^2-1}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right),$$
$$\zeta_2 = \sqrt{\frac{\lambda+1}{2}}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right),$$
$$\zeta_3 = -\sqrt{\frac{\lambda-1}{2}}\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\right)$$

whose Lie brackets satisfy (5.1). Using these matrices again, define a left invariant metric g; then $\{\zeta_1, \zeta_2, \zeta_3\}$ is an orthonormal basis. The contact form ω_1 we denote by η and it is given by

$$\eta = \frac{2}{\sqrt{\lambda^2 - 1}}(v\,dx - y\,du)\,.$$

The characteristic vector field ξ is ζ_1. g is an associated metric and ϕ as a skew-symmetric operator is given by $\phi\xi = 0$ and $\phi\zeta_2 = \zeta_3$. The symmetric operator h is given by $h\xi = 0$, $h\zeta_2 = \lambda\zeta_2$, $h\zeta_3 = -\lambda\zeta_3$. The special directions are

$$Y = \sqrt{\frac{\lambda-1}{2\lambda}}\zeta_2 + \sqrt{\frac{\lambda+1}{2\lambda}}\zeta_3 = -\frac{\sqrt{\lambda^2-1}}{\sqrt{\lambda}}\left(x\frac{\partial}{\partial y} + u\frac{\partial}{\partial v}\right)$$

and

$$Z = \sqrt{\frac{\lambda-1}{2\lambda}}\zeta_2 - \sqrt{\frac{\lambda+1}{2\lambda}}\zeta_3 = \frac{\sqrt{\lambda^2-1}}{\sqrt{\lambda}}\left(y\frac{\partial}{\partial x} + v\frac{\partial}{\partial u}\right).$$

The 1-parameter group $\{\psi_t\}$ of ξ is given by

$$\psi_t\begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} xe^{\frac{1}{2}\sqrt{\lambda^2-1}\,t} & ye^{-\frac{1}{2}\sqrt{\lambda^2-1}\,t} \\ ue^{\frac{1}{2}\sqrt{\lambda^2-1}\,t} & ve^{-\frac{1}{2}\sqrt{\lambda^2-1}\,t} \end{pmatrix}.$$

Then

$$\psi_{t*}Y = -\frac{\sqrt{\lambda^2-1}}{\sqrt{\lambda}}\,e^{-\frac{1}{2}\sqrt{\lambda^2-1}\,t}\left(x\frac{\partial}{\partial y} + u\frac{\partial}{\partial v}\right) = (e^{-\frac{1}{2}\sqrt{\lambda^2-1}\,t})Y\,,$$

$$\psi_{t*}Z = \frac{\sqrt{\lambda^2-1}}{\sqrt{\lambda}}\,e^{\frac{1}{2}\sqrt{\lambda^2-1}\,t}\left(y\frac{\partial}{\partial x} + v\frac{\partial}{\partial u}\right) = (e^{\frac{1}{2}\sqrt{\lambda^2-1}\,t})Z\,.$$

Thus the corresponding subbundles $\mathcal{Y}$ and $\mathcal{Z}$ are invariant under the flow. Finally since $\{\zeta_1, \zeta_2, \zeta_3\}$ is orthonormal,

$$|Y|^2 = \frac{\lambda-1}{2\lambda} + \frac{\lambda+1}{2\lambda} = 1$$

and hence

$$|\psi_{t*}Y| = (e^{-\frac{1}{2}\sqrt{\lambda^2-1}\,t})|Y|\,;$$

similarly

$$|\psi_{t*}Z| = (e^{\frac{1}{2}\sqrt{\lambda^2-1}\,t})|Z|\,.$$

Thus ξ is an Anosov vector field and the special directions Y and Z agree with the Anosov directions.

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