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*Annales de la faculté des sciences de Toulouse 6<sup>e</sup> série*, tome 7, n<sup>o</sup> 1  
(1998), p. 51-85

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## Asymptotic expansion of the Bergman kernel for weakly pseudoconvex tube domains in $\mathbb{C}^{2(*)}$

JOE KAMIMOTO<sup>(1)</sup>

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**RÉSUMÉ.** — Dans cet article nous donnons un développement asymptotique du noyau de Bergman pour certains domaines de tube qui sont faiblement pseudo-convexes et de type fini dans  $\mathbb{C}^2$ .

Notre formule asymptotique montre que la singularité du noyau de Bergman à points faiblement pseudo-convexe est essentiellement représentée par deux variables; de plus, un certain éclatement réel est nécessaire pour comprendre cette singularité.

L'expression du développement asymptotique par rapport à chaque variable est analogue à celle de C. Fefferman dans le cas strictement pseudo-convexe.

Nous démontrons aussi un même résultat pour le noyau de Szegő.

**ABSTRACT.** — In this paper we give an asymptotic expansion of the Bergman kernel for certain weakly pseudoconvex tube domains of finite type in  $\mathbb{C}^2$ . Our asymptotic formula asserts that the singularity of the Bergman kernel at weakly pseudoconvex points is essentially expressed by using two variables; moreover certain real blowing-up is necessary to understand its singularity. The form of the asymptotic expansion with respect to each variable is similar to that in the strictly pseudoconvex case due to C. Fefferman. We also give an analogous result in the case of the Szegő kernel.

**AMS Classification :** 32A40, 32F15, 32H10.

**KEY-WORDS :** Bergman kernel, Szegő kernel, weakly pseudoconvex, of finite type, tube, asymptotic expansion, real blowing-up, admissible approach region.

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(\*) Reçu le 3 décembre 1997, accepté le mars 1998

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## 1. Introduction

The purpose of this paper is to give an asymptotic expansion of the Bergman kernel for certain class of weakly pseudoconvex tube domains of finite type in  $\mathbb{C}^2$ . We also give an analogous result for the Szegő kernel for the same class of tube domains.

Let  $\Omega$  be a domain with smooth boundary in  $\mathbb{C}^n$ . The Bergman space  $B(\Omega)$  is the subspace of  $L^2(\Omega)$  consisting of holomorphic  $L^2$ -functions on  $\Omega$ . The Bergman projection is the orthogonal projection  $\mathbb{B} : L^2(\Omega) \rightarrow B(\Omega)$ . We can write  $\mathbb{B}$  as an integral operator

$$\mathbb{B}f(z) = \int_{\Omega} K(z, w)f(w) dV(w) \quad \text{for } f \in L^2(\Omega),$$

where  $K : \Omega \times \Omega \rightarrow \mathbb{C}$  is the *Bergman kernel* of the domain  $\Omega$  and  $dV$  is the Lebesgue measure on  $\Omega$ . In this paper we restrict the Bergman kernel on the diagonal of the domain and study the boundary behavior of  $K(z) = K(z, z)$ .

Although there are many explicit computations for the Bergman kernels of specific domains ([2], [9], [33], [11], [26], [6], [13], [22], [23], [34], [3]), it seems difficult to express the Bergman kernel in closed form in general. Therefore appropriate approximation formulas are necessary to know the boundary behavior of the Bergman kernel. From this viewpoint the following studies have great success in the case of strictly pseudoconvex domains. Assume  $\Omega$  is a bounded strictly pseudoconvex domain. Hörmander [32] showed that the limit of  $K(z) d(z - z^0)^{n+1}$  at  $z^0 \in \partial\Omega$  equals the determinant of the Levi form at  $z^0$  times  $n!/4\pi^n$ , where  $d$  is the Euclidean distance. Moreover Diederich ([14], [15]) obtained analogous results for the first and mixed second derivatives of  $K(z)$ . Fefferman [21] and Boutet de Monvel and Sjöstrand [7] gave the following asymptotic expansion of the Bergman kernel of  $\Omega$ :

$$K(z) = \frac{\varphi(z)}{r(z)^{n+1}} + \psi(z) \log r(z), \quad (1.1)$$

where  $r \in C^\infty(\bar{\Omega})$  is a defining function of  $\Omega$  (i.e.,  $\Omega = \{r > 0\}$  and  $|dr| > 0$  on  $\partial\Omega$ ) and  $\varphi, \psi \in C^\infty(\bar{\Omega})$  can be expanded asymptotically with respect to  $r$ .

On the other hand, there are not so strong results in the weakly pseudoconvex case. Let us recall important studies in this case. Many estimates of the size of the Bergman kernel have been obtained ([29], [52], [19], [8], [45], [30], [16], [31], [46], [10], [53], [25], [47]). In particular Catlin [8] gave a complete estimate from above and below for domains of finite type in  $\mathbb{C}^2$ . Recently Boas, Straube and Yu [4] computed nontangential boundary limit in the sense of Hörmander for bounded domains which are known variously as  $h$ -extendible [56] and semiregular [17]. This class contains many kinds of domains of finite type. Diederich and Herbort [18] gave an alternative proof of the result of [4]. Though the above studies about *estimate* and *boundary limit* are detailed, a clear asymptotic formula like (1.1) is yet to be obtained. In this paper we give an asymptotic expansion of the Bergman kernel for certain class of weakly pseudoconvex tube domains of finite type in  $\mathbb{C}^2$ . Gebelt [24] and Haslinger [28] recently computed certain asymptotic formulas for the special cases, but the method of our expansion is different from theirs.

Our main idea to analyse the Bergman kernel is to introduce certain real blowing-up. Let us briefly indicate how this blowing-up works for the Bergman kernel at a weakly pseudoconvex point  $z^0$ . Since the set of strictly pseudoconvex points are dense on the boundary of the domain of finite type, it is a serious problem to resolve the difficulty caused by the influence of strictly pseudoconvex points near  $z^0$ . This problem can be avoided by restricting the argument on a non-tangential cone in the domain ([29], [19], [30], [16], [4]). We surmount the difficulty in the case of certain class of tube domains in the following. By blowing up at the weakly pseudoconvex point  $z^0$ , we introduce two new variables. The Bergman kernel can be developed asymptotically in terms of these variables in the sense of Sibuya [55] (see also Majima [44]). The expansion, regarded as a function of the first variable, has the form of Fefferman's expansion (1.1), and hence it reflects the strict pseudoconvexity. The characteristic influence of the weak pseudoconvexity appears in the expansion with respect to second variables. Though the form of this expansion is similar to (1.1), we must use  $m$ -th root of the defining function, i.e.  $r^{1/m}$ , as the expansion variable when  $z^0$  is the type  $2m$ . We remark that a similar situation occurs in the case of another class of domains in [24].

Our method of computation is based on the studies [21], [7], [5] and [51]. In [21], Fefferman selected the ball  $\{z \in \mathbb{C}^n \mid \sum_{j=1}^n |z_j|^2 < 1\}$  as a model domain to approximate strictly pseudoconvex domain. We consider  $\{(z_1, z_2) \in \mathbb{C}^2 \mid \Im z_2 > g[\Re z_1]^{2m}\}$  with  $g > 0$ ,  $m = 2, 3, \dots$  as a *model do-*

main. The starting point of our analysis is certain integral representation in [40], [48], [54] and [27]. After introducing the blowing-up to this representation, we compute the asymptotic expansion by using the stationary phase method. For the above computation, it is necessary to localize the Bergman kernel near a weakly pseudoconvex point. This localization can be obtained in a fashion similar to the case of some class of Reinhardt domains ([5], [51]).

This paper is organized as follows. Our main theorem is established in Section 2. The next three sections prepare the proof of the theorem. First an integral representation is introduced, which is a clue to our analysis in Section 3. Second the usefulness of our blowing-up is shown by using a simple tube domain  $\{(z_1, z_2) \in \mathbb{C}^2 \mid \Im z_2 > [\Im z_1]^{2m}\}$ ,  $m = 2, 3, \dots$ , in Section 4. This domain is considered to be a model domain for more general case. Third a localization lemma is established in Section 5, which is necessary to the computation in the proof of our theorem. Our main theorem is proved in Section 6. After an appropriate localization (§ 6.1) and the blowing-up at a weakly pseudoconvex point, an easy computation shows that certain two propositions are sufficient to prove our theorem (§ 6.2). In order to prove these propositions, we compute the asymptotic expansion of two functions by using the stationary phase method (§ 6.3, 6.4). The rest of Section 6 (§ 6.5, 6.6) is devoted to proving two propositions. In Section 7 an analogous theorem about the Szegő kernel is established.

## 2. Statement of main result

Given a function  $f \in C^\infty(\mathbb{R})$  satisfying that:

$$\begin{aligned} f'' &\geq 0 \text{ on } \mathbb{R} \text{ and } f \text{ has the form in some neighborhood of } 0: \\ f(x) &= x^{2m}g(x) \text{ where } m = 2, 3, \dots, g(0) > 0 \text{ and } xg'(x) \leq 0. \end{aligned} \tag{2.1}$$

Let  $\omega_f \subset \mathbb{R}^2$  be a domain defined by  $\omega_f = \{(x, y) \mid y > f(x)\}$ . Let  $\Omega_f \subset \mathbb{C}^2$  be the tube domain over  $\omega_f$ , i.e.,

$$\Omega_f = \mathbb{R}^2 + i\omega_f.$$

Let  $\pi : \mathbb{C}^2 \rightarrow \mathbb{R}^2$  be the projection defined by  $\pi(z_1, z_2) = (\Im z_1, \Im z_2)$ . Set  $O = (0, 0)$ . It is easy to check that  $\Omega_f$  is a pseudoconvex domain; moreover  $z^0 \in \partial\Omega_f$ , with  $\pi(z^0) = O$ , is a weakly pseudoconvex point of type  $2m$  (or  $2m - 1$ ) in the sense of Kohn or D'Angelo and  $\partial\Omega_f \setminus \pi^{-1}(O)$  is strictly pseudoconvex near  $z^0$ .

Asymptotic expansion of the Bergman kernel

Now we introduce the transformation  $\sigma$ , which plays a key role on our analysis. Set  $\Delta = \{(\tau, \varrho) \mid 0 < \tau \leq 1, \varrho > 0\}$ . The transformation  $\sigma: \overline{\omega_f} \rightarrow \overline{\Delta}$  is defined by

$$\sigma : \begin{cases} \tau = \chi\left(1 - \frac{f(x)}{y}\right) \\ \varrho = y, \end{cases} \quad (2.2)$$

where the function  $\chi \in C^\infty([0, 1])$  satisfies the conditions:

$$\chi'(u) \geq \frac{1}{2} \text{ on } [0, 1] \text{ and } \chi(u) = \begin{cases} u & \text{for } 0 \leq u \leq \frac{1}{3} \\ 1 - (1-u)^{1/2m} & \text{for } 1 - \frac{1}{3^{2m}} \leq u \leq 1. \end{cases}$$

Then  $\sigma \circ \pi$  is the transformation from  $\overline{\Omega}$  to  $\overline{\Delta}$ .

The transformation  $\sigma$  induces an isomorphism of  $\omega_f \cap \{x \geq 0\}$  (or  $\omega_f \cap \{x \leq 0\}$ ) on to  $\Delta$ . The boundary of  $\omega_f$  is transferred by  $\sigma$  in the following:

$$\sigma((\partial\omega_f) \setminus \{O\}) = \{(0, \varrho) \mid \varrho > 0\} \quad \text{and} \quad \sigma^{-1}(\{(\tau, 0) \mid 0 \leq \tau \leq 1\}) = \{O\}.$$

This indicates that  $\sigma$  is the real blowing-up of  $\partial\omega_f$  at  $O$ , so we may say that  $\sigma \circ \pi$  is the blowing-up at the weakly pseudoconvex point  $z^0$ . Moreover  $\sigma$  patches the coordinates  $(\tau, \varrho)$  on  $\omega_f$ , which can be considered as the polar coordinates around  $O$ . We call  $\tau$  the angular variable and  $\varrho$  the radial variable, respectively. Note that if  $z$  approaches some strictly (resp. weakly) pseudoconvex points,  $\tau(\pi(z))$  (resp.  $\varrho(\pi(z))$ ) tends to 0 on the coordinates  $(\tau, \varrho)$ .

The following theorem asserts that the singularity of the Bergman kernel of  $\Omega_f$  at  $z^0$ , with  $\pi(z^0) = O$ , can be essentially expressed in terms of the polar coordinates  $(\tau, \varrho)$ .

**THEOREM 2.1.** — *The Bergman kernel of  $\Omega_f$  has the form in some neighborhood of  $z^0$ :*

$$K(z) = \frac{\Phi(\tau, \varrho^{1/m})}{\varrho^{2+1/m}} + \tilde{\Phi}(\tau, \varrho^{1/m}) \log \varrho, \quad (2.3)$$

where  $\Phi \in C^\infty((0, 1] \times [0, \varepsilon])$  and  $\tilde{\Phi} \in C^\infty([0, 1] \times [0, \varepsilon])$  with some  $\varepsilon > 0$ .

Moreover  $\Phi$  is written in the form on the set  $\{\tau > \alpha \varrho^{1/2m}\}$  with some  $\alpha > 0$ : for every nonnegative integer  $\mu_0$

$$\Phi(\tau, \varrho^{1/m}) = \sum_{\pi=0}^{\mu_0} c_\mu(\tau) \varrho^{\mu/m} + R_{\mu_0}(\tau, \varrho^{1/m}) \varrho^{\mu_0/m+1/2m} \quad (2.4)$$

where

$$c_\mu(\tau) = \frac{\varphi_\mu(\tau)}{\tau^{3+2\mu}} + \psi_\mu(\tau) \log \tau, \quad (2.5)$$

for  $\varphi_\mu, \psi_\mu \in C^\infty([0, 1])$ ,  $\varphi_0$  is positive on  $[0, 1]$  and  $R_{\mu_0}$  satisfies

$$|R_{\mu_0}(\tau, \varrho^{1/m})| \leq C_{\mu_0} [\tau - \alpha \varrho^{1/2m}]^{-4-2\mu_0}$$

for some positive constant  $C_{\mu_0}$ .

Let us describe the asymptotic expansion of the Bergman kernel  $K$  in more detail. Considering the meaning of the variables  $\tau, \varrho$ , we may say that each expansion with respect to  $\tau$  or  $\varrho^{1/m}$  is induced by the strict or weak pseudoconvexity, respectively. Actually the expansion (2.5) has the same form as the one of Fefferman (1.1). By (2.4)-(2.5), in order to see the characteristic influence of the weak pseudoconvexity on the singularity of the Bergman kernel  $K$ , it is sufficient to argue about  $K$  on the region

$$\mathcal{U}_\alpha = \{z \in \mathbb{C}^2 \mid \tau \circ \pi(z) > \alpha^{-1}\}, \quad \alpha > 1.$$

This is because  $\mathcal{U}_\alpha$  is the *widest* region where the coefficients  $c_\mu(\tau)$ 's are bounded. We call  $\mathcal{U}_\alpha$  an admissible approach region of the Bergman kernel of  $\Omega_f$  at  $z^0$ . The region  $\mathcal{U}_\alpha$  seems deeply connected with the admissible approach regions studied in [41], [42], [1], [43], etc. We remark that on the region  $\mathcal{U}_\alpha$ , the exchange of the expansion variable  $\varrho^{1/m}$  for  $r^{1/m}$ , where  $r$  is a defining function of  $\Omega_f$  (e.g.,  $r(x, y) = y - f(x)$ ), gives no influence on the form of the expansion on the region  $\mathcal{U}_\alpha$ .

Now let us compare the asymptotic expansion (2.3) on  $\mathcal{U}_\alpha$  with Fefferman's expansion (1.1). The essential difference between them only appears in the expansion variable (i.e.,  $r^{1/m}$  in (2.3) and  $r$  in (1.1)). A similar phenomenon occurs in subelliptic estimates for the  $\bar{\partial}$ -Neumann problem. As is well-known, the finite-type condition is equivalent to the condition that a subelliptic estimate holds, i.e.,

$$\|\phi\|_\varepsilon^2 \leq C \left( \|\bar{\partial}\phi\|^2 + \|\bar{\partial}^* \phi\|^2 + \|\phi\|^2 \right), \quad \varepsilon > 0$$

(refer to [39] for the details). Let  $\varepsilon_0$  be the best possible order of subellipticity. In two dimensional case,  $\varepsilon_0 = 1/2$  in the strictly pseudoconvex case and  $\varepsilon_0 = 1/2m$  in the weakly pseudoconvex case of type  $2m$ . The difference between these two cases only appears in the value of  $\varepsilon$ . From this viewpoint, our expansion (2.3) seems to be a natural generalization of Fefferman's expansion (1.1) in the strictly pseudoconvex case.

*Remarks*

1) From the viewpoint of the studies ([4], [17]), let us consider the limit of  $\varrho^{2+1/m}K(z)$  at  $z^0$ . This limit on a nontangential cone is

$$c_0(1) = \varphi_0(1) = \frac{g(0)^{1/m}}{(4\pi)^2} \Gamma\left(2 + \frac{1}{m}\right) \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{-w^2m + vw} dw \right)^{-1} dv.$$

The above integral seems impossible to be changed into simpler form (see [39]). We remark that if we do not restrict the region for the approach, the above limit is  $c_0(\tau)$ , which depends on the angular variable  $\tau$ .

2) The idea of the blowing-up  $\sigma$  is originally introduced in the study of the Bergman kernel of the domain  $\{z \in \mathbb{C}^n \mid \sum_{j=1}^n |z_j|^{2m_j} < 1\}$  ( $m_j \in \mathbb{N}, m_n \neq 1$ ) in [34]. Since the above domain has high homogeneity, the asymptotic expansion with respect to the radial variable does not appear (see also Sect. 4). Recently the author [37] also gave an analogous result in the case of the decoupled tube domain  $\{z \in \mathbb{C}^{n+1} \mid \Im z_{n+1} > \sum_{j=1}^n [\Im z_j]^{2m_j}\}$  ( $m_j \in \mathbb{N}, m_n \neq 1$ ).

3) If we consider the Bergman kernel on the region  $\mathcal{U}_\alpha$ , then we can remove the condition  $xg'(x) \leq 0$  in (2.1). Namely even if the condition  $xg'(x) \leq 0$  is not satisfied, we can still obtain (2.3)-(2.4) in the theorem where  $c_\mu$ 's are bounded on  $\mathcal{U}_\alpha$ . But, for our method in this paper, the condition  $xg'(x) \leq 0$  is necessary to obtain the asymptotic expansion with respect to  $\tau$ .

4) From the definition of asymptotic expansion of functions of several variables in [55] and [44], the expansion in the theorem is not complete. In order to get a complete asymptotic expansion, we must take a further blowing-up at the point  $(\tau, \varrho) = (0, 0)$ . The real blowing-up  $(\tau, \varrho) \mapsto (\tau, \varrho\tau^{-2m})$  is sufficient for this purpose.

*Notation.* — In this paper, we use  $c, c_j$  or  $C$  for various constants without further comment.



### 3. Integral representation

In this section we give an integral representation of the Bergman kernel, which is a clue to our analysis. Korányi [40], Nagel [48], Saitoh [54] and Haslinger [27] obtain similar representations of Bergman kernels or Szegő kernels for certain tube domains.

In this section, we assume that  $f \in C^\infty(\mathbb{R})$  is a function such that  $f(0) = 0$  and  $f''(x) \geq 0$ . The tube domain  $\Omega_f \subset \mathbb{C}^2$  is defined as in Section 2. Let  $\Lambda, \Lambda^* \subset \mathbb{R}^2$  be the cones defined by

$$\begin{aligned}\Lambda &= \{(x, y) \mid (tx, ty) \in \omega_f \text{ for any } t > 0\} \\ \Lambda^* &= \{(\zeta_1, \zeta_2) \mid x\zeta_1 + y\zeta_2 > 0 \text{ for any } (x, y) \in \Lambda\}\end{aligned}$$

respectively. We call  $\Lambda^*$  the dual cone of  $\omega_f$ . Actually  $\Lambda^*$  can be computed explicitly:

$$\Lambda^* = \{(\zeta_1, \zeta_2) \mid -R^-\zeta_2 < \zeta_1 < R^+\zeta_2\},$$

where  $(R^\pm)^{-1} = \lim_{x \rightarrow \mp\infty} f(x)|x|^{-1} > 0$ , respectively. We allow that  $R^\pm = \infty$ . If  $\lim_{|x| \rightarrow \infty} f(x)|x|^{-1-\varepsilon} > 0$  with some  $\varepsilon > 0$ , then  $R^\pm = \infty$ , i.e.,  $\Lambda^* = \{(\zeta_1, \zeta_2) \mid \zeta_2 > 0\}$ .

The Bergman kernel of  $\Omega_f$  is expressed in the following. Set  $(x, y) = (\Im z_1, \Im z_2)$ ;

$$K(z) = \frac{1}{(4\pi)^2} \iint_{\Lambda^*} e^{-x\zeta_1 - y\zeta_2} \frac{\zeta_2}{D(\zeta_1, \zeta_2)} d\zeta_1 d\zeta_2, \quad (3.1)$$

where

$$D(\zeta_1, \zeta_2) = \int_{-\infty}^{\infty} e^{-\xi\zeta_1 - f(\xi)\zeta_2} d\xi. \quad (3.2)$$

The above representation can be obtained by applying the argument of Korányi [40] and Saitoh [54] to our case, so we omit the proof.

### 4. Analysis on a model domain

Let  $\omega_0 \cup \mathbb{R}^2$  be a domain defined by  $\omega_0 = \{(x, y) \mid y > gx^{2m}\}$ , where  $m = 2, 3, \dots$  and  $g > 0$ . Set  $\Omega_0 = \mathbb{R}^2 + i\omega_0$ . F. Haslinger [28] computes

the asymptotic expansion of the Bergman kernel of  $\Omega_0$  (not only on the diagonal but also off the diagonal). In his result Fefferman's expansion only appears.

In this paper, we consider  $\Omega_0$  as a model domain for the study of singularity of the Bergman kernel for more general domains. The following proposition shows the reason why we take  $\Omega_0$  as a model domain. Set  $(x, y) = (\Im z_1, \Im z_2)$ .

PROPOSITION 4.1. — *The Bergman kernel  $K$  of  $\Omega_0$  has the form:*

$$K(z) = \frac{\Phi(\tau)}{\varrho^{2+1/m}}, \quad (4.1)$$

where  $\tau = \chi(1 - gx^{2m}y^{-1})$ ,  $\varrho = y$  (see (2.2)) and

$$\Phi(\tau) = \frac{\varphi(\tau)}{\tau^3} + \psi(\tau) \log \tau,$$

with  $\varphi, \psi \in C^\infty([0, 1])$  and  $\varphi$  is positive on  $[0, 1]$ .

*Proof.* — Normalizing the integral representation (3.1) and introducing the variables  $t = g^{1/2m}xy^{-1/2m}$ ,  $\varrho = y^{1/2m}$ , we have (4.1) where

$$\Phi(\tau) = \frac{2m}{(4\pi)^2} g^{1/m} \int_0^\infty e^{-s^{2m}} L(ts) s^{4m+1} ds, \quad (4.2)$$

$$L(u) = \int_{-\infty}^\infty e^{uv} \frac{1}{\phi(v)} dv,$$

$$\phi(v) = \int_{-\infty}^\infty e^{-w^{2m}+vw} dw.$$

It turns out from (4.2) and the definition of  $\tau$  that  $\Phi \in C^\infty((0, 1])$ .

Now let  $\widehat{\Phi}$  be defined by

$$\widehat{\Phi}(\tau) = \frac{2m}{(4\pi)^2} \int_1^\infty e^{-s^{2m}} L(ts) s^{4m+1} ds. \quad (4.3)$$

If we admit Lemma 6.2 in Paragraph 6.4 below, we have

$$L(u) = u^{2m-2} e^{u^{2m}} \tilde{L}(u), \quad (4.4)$$

where  $\tilde{L}(u) \sim \sum_{j=0}^{\infty} c_j u^{-2mj}$  as  $u \rightarrow \infty$ . Substituting (4.4) into (4.3), we have

$$\begin{aligned} \widehat{\Phi}(\tau) &= \frac{2m}{(4\pi)^2} t^{2m-2} \int_1^{\infty} e^{-[1-t^{2m}]s^{2m}} \tilde{L}(ts) s^{6m-1} ds \\ &= \frac{1}{(4\pi)^2} \int_1^{\infty} e^{-\chi^{-1}(\tau)\sigma} \widehat{L}(\tau, \sigma) \sigma^2 d\sigma. \end{aligned}$$

Since  $\widehat{L}(\tau, \sigma) \sim \sum_{j=0}^{\infty} c_j(\tau) \sigma^{-j}$  as  $\sigma \rightarrow \infty$  for  $c_j \in C^\infty([0, 1])$ , we have

$$\widehat{\Phi}(\tau) = \frac{\widehat{\varphi}(\tau)}{\tau^3} + \widehat{\psi}(\tau) \log \tau,$$

with  $\widehat{\varphi}, \widehat{\psi} \in C^\infty([0, 1])$  and  $\widehat{\varphi}$  is positive on  $[0, 1]$ .

Finally since the difference between  $\Phi$  and  $\widehat{\Phi}$  is smooth on  $[0, 1]$ , we can obtain Proposition 4.1.  $\square$

## 5. Localization lemma

In this section, we prepare a lemma, which is necessary for the proof of Theorem 2.1. This lemma shows that the singularity of the Bergman kernel for certain class of domains is determined by the local information about the boundary. The method of the proof is similar to the case of some class of Reinhardt domains ([5], [51]). Throughout this section,  $j$  stands for 1 or 2.

Let  $f_1, f_2 \in C^\infty(\mathbb{R})$  be functions such that  $f_j(0) = f'_j(0) = 0$ ,  $f''_j \geq 0$  on  $\mathbb{R}$  and  $f_1(x) = f_2(x)$  on  $|x| < \delta$ . Let  $\omega_j \subset \mathbb{R}^2$  be a domain defined by  $\omega_j = \{(x, y) \mid y > f_j(x)\}$ . Set  $\Omega_j = \mathbb{R}^2 + i\omega_j \subset \mathbb{C}^2$ .

LEMMA 5.1. — *Let  $K_j$  be the Bergman kernels of  $\Omega_j$  for  $j = 1, 2$ , respectively. Then we have*

$$K_1(z) - K_2(z) \in C^\omega(U),$$

where  $U$  is some neighborhood of  $z^0$ .

Asymptotic expansion of the Bergman kernel

*Proof.* — Let  $\Lambda_j^*$  be the dual cone of  $\omega_j$ , i.e.,

$$\Lambda_j^* = \{(\zeta_1, \zeta_2) \mid -R_j^- \zeta_2 < \zeta_1 < R_j^+ \zeta_2\},$$

where  $(R_j^\pm)^{-1} = \lim_{x \rightarrow \mp\infty} f(x)|x|^{-1}$ , respectively (Sect. 3).

Let  $K_j[\Delta](x, y)$  be defined by

$$K_j[\Delta](x, y) = \frac{1}{(4\pi)^2} \iint_{\Delta} e^{-y\zeta_2 - x\zeta_1} \frac{\zeta_2}{D_j(\zeta_1, \zeta_2)} d\zeta_1 d\zeta_2,$$

where  $\Delta \subset \mathbb{R}^2$  and  $D_j(\zeta_1, \zeta_2) = \int_{-\infty}^{\infty} e^{-\zeta_2 f_j(\xi) - \zeta_1 \xi} d\xi$ .

Set  $\Lambda_\varepsilon = \{(\zeta_1, \zeta_2) \mid |\zeta_1| < \varepsilon \zeta_2\}$ , where  $\varepsilon > 0$  is small. Now the following claims (i), (ii) imply Lemma 5.1. Set  $O = (0, 0)$ ;

(i)  $K_j[\Lambda_j^*] \equiv K_j[\Lambda_\varepsilon]$  modulo  $C^\omega(\{O\})$  for any  $\varepsilon > 0$ ,

(ii)  $K_1[\Lambda_{\varepsilon_0}] \equiv K_2[\Lambda_{\varepsilon_0}]$  modulo  $C^\omega(\{O\})$  for some  $\varepsilon_0 > 0$ .

In fact if we substitute  $(x, y) = (\Im z_1, \Im z_2)$ , then  $K_1 = K_1[\Lambda_1^*] \equiv K_1[\Lambda_{\varepsilon_0}] \equiv K_2[\Lambda_{\varepsilon_0}] \equiv K_2[\Lambda_2^*] = K_2$  modulo  $C^\omega(\{z^0\})$ .

Let us show the above claims.

(i) Set  $\Lambda_\varepsilon^\pm = \{(\zeta_1, \zeta_2) \mid 0 < \varepsilon \zeta_2 < \pm \zeta_1 < R_j^\pm \zeta_2\}$ , respectively. Since  $K_j[\Lambda_j^*] - K_j[\Lambda_\varepsilon] = K_j[\Lambda_\varepsilon^+] + K_j[\Lambda_\varepsilon^-]$ , it is sufficient to show  $K_j[\Lambda_\varepsilon^\pm] \in C^\omega(\{O\})$ . We only consider the case of  $K_j[\Lambda_\varepsilon^+]$ . Changing the integral variables, we have

$$K_j[\Lambda_\varepsilon^+](x, y) = \frac{1}{(4\pi)^2} \int_0^\infty \int_\varepsilon^{R_j^+} H_j(\zeta, \eta; x, y) d\zeta d\eta,$$

where

$$H_j(\zeta, \eta; x, y) (= H_j) = e^{-y\eta + x\eta\zeta} \frac{\eta^2}{E_j(\zeta, \eta)},$$

$$E_j(\zeta, \eta) = \int_{-\infty}^{\infty} e^{-\eta(f_j(\xi) - \zeta\xi)} \rightarrow \xi.$$

It is an important remark that  $K_j[\Lambda_\varepsilon^+]$  is real analytic on the region where  $H_j$  is integrable on  $\{(\zeta, \eta) \mid \zeta > \varepsilon, \eta > 0\}$ .

If we take  $\delta_1 > 0$  such that  $|f_j(\xi)\xi^{-1}| < (1/2)\varepsilon$  if  $|\xi| < \delta_1$ , then we have

$$\begin{aligned} E_j(\zeta, \eta) &\geq 2 \int_0^{\delta_1} e^{\eta\xi(\zeta-f(\xi)\xi^{-1})} d\xi \\ &\geq 2 \int_0^{\delta_1} e^{(1/2)\varepsilon\eta\xi} d\xi \geq \frac{4}{\varepsilon\eta} (e^{(1/2)\delta_1\varepsilon\eta} - 1). \end{aligned} \tag{5.1}$$

By (5.1), we have  $H_j(\zeta, \eta; x, y) \leq C\eta e^{-(y-x\zeta+(1/2)\delta_1\varepsilon)\eta}$  for  $\eta \geq 1$ . This inequality implies that if  $x < 0$  and  $y > -(1/2)\delta_1\varepsilon$ , then  $H_j$  is integrable on  $\{(\zeta, \eta) \mid \zeta > \varepsilon, \eta > 0\}$ . Thus  $K_j[\Lambda_\varepsilon^+]$  is real analytic on

$$\omega_j \cup \left\{ (x, y) \mid x < 0, y > -\frac{1}{2}\delta_1\varepsilon \right\} =: \omega_j^+.$$

By regarding  $x, y$  as two complex variables,  $K_j[\Lambda_\varepsilon^+](x, y)$  is holomorphic on  $\omega_j^+ + i\mathbb{R}^2$ , so the shape of  $\omega_j^+$  implies that  $K_j[\Lambda_\varepsilon^+]$  can be extended holomorphically to a region containing some neighborhood of  $\{0\} + i\mathbb{R}^2$ . Consequently we have  $K_j[\Lambda_\varepsilon^+] \in C^\omega(\{0\})$ .

(ii) Changing the integral variables, we have

$$K_1[\Lambda_\varepsilon](x, y) - K_2[\Lambda_\varepsilon](x, y) = \frac{1}{(4\pi)^2} \int_0^\infty \int_{-\varepsilon}^\varepsilon (H_1 - H_2) d\zeta d\eta.$$

We remark that  $K_1[\Lambda_\varepsilon] - K_2[\Lambda_\varepsilon]$  is real analytic on the region where  $H_1 - H_2$  is integrable on  $\{(\zeta, \eta) \mid |\zeta| < \varepsilon, \eta > 0\}$ . To find a positive number  $\varepsilon_0$  satisfying (ii), let us consider the integrability of

$$|H_1 - H_2| = \eta^2 e^{-y\eta+x\eta\zeta} \frac{|E_2(\zeta, \eta) - E_1(\zeta, \eta)|}{|E_1(\zeta, \eta) \cdot E_2(\zeta, \eta)|}. \tag{5.2}$$

First we give an estimate of  $|E_2(\zeta, \eta) - E_1(\zeta, \eta)|$ . Let  $\varepsilon_1 > 0$  be defined by  $|f_j(\xi)\xi^{-1}| \geq \varepsilon_1$  if  $|\xi| \geq \delta$ . If  $|\zeta| \leq \varepsilon_1/2$ , then

$$\int_{|\xi| \geq \delta} e^{-\eta\xi(f_j(\xi)\xi^{-1}-\zeta)} d\xi \leq 2 \int_\delta^\infty e^{-(1/2)\varepsilon_1\eta\xi} d\xi \leq \frac{4}{\varepsilon_1\eta} e^{-(1/2)\delta\varepsilon_1\eta}. \tag{5.3}$$

Therefore (5.3) implies

$$|E_2(\zeta, \eta) - E_1(\zeta, \eta)| \leq \sum_{j=1}^2 \int_{|\xi| \geq \delta} e^{-\eta(f_j(\xi)-\zeta\xi)} d\xi \leq \frac{8}{\varepsilon_1\eta} e^{-(1/2)\delta\varepsilon_1\eta}. \tag{5.4}$$

Second, we give an estimate of  $E_j(\zeta, \eta)$ . By Taylor's formula, we can choose  $\varepsilon_2 > 0$  satisfying the following. If  $|\zeta| < \varepsilon_2$ , then there is a function  $\alpha_j(\zeta) = \alpha_j$  ( $\alpha_j(0) = 0$ ) such that  $f'_j(\alpha_j) = \zeta$  and moreover there is a bounded function  $R_j(\zeta, \xi)$  on  $[-\varepsilon_2, \varepsilon_2] \times [-\xi_0, \xi_0]$ , with some  $\xi_0 > 0$ , such that

$$F_j(\xi) (:= f_j(\xi) - \zeta\xi) = F_j(\alpha_j) + R_j(\zeta, \xi - \alpha_j)(\xi - \alpha_j)^2$$

with  $F_j(\alpha_j) \leq 0$ . Then if  $|\zeta| < \varepsilon_2$ , we have

$$\begin{aligned} E_j(\zeta, \eta) &= \int_{-\infty}^{\infty} e^{-\eta F_j(\xi)} d\xi \geq e^{-\eta F_j(\alpha_j)} \int_{-\xi_0}^{\xi_0} e^{-\eta R_j(\zeta, \xi)\xi^2} d\xi \\ &\geq \frac{C}{\sqrt{\eta}} e^{-\eta F_j(\alpha_j)} \geq \frac{C}{\sqrt{\eta}}. \end{aligned} \tag{5.5}$$

Now we set  $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$ . Then by putting (5.2), (5.4) and (5.5) together, we have

$$|H_1 - H_2| \leq C\eta^2 e^{-\eta(y-x\zeta+(1/2)\delta\varepsilon_1)} \quad \text{for } |\zeta| < \varepsilon_0, \eta > 0.$$

This inequality implies that if  $\varepsilon_0|x| - y - (1/2)\delta\varepsilon_1 > 0$ , then  $|H_1 - H_2|$  is integrable on  $\{(\zeta, \eta) \mid |\zeta| < \varepsilon_0, \eta > 0\}$ . Hence  $K_1[\Lambda_{\varepsilon_0}] - K_2[\Lambda_{\varepsilon_0}]$  is real analytic on the region  $\{(x, y) \mid \varepsilon_0|x| - y - (1/2)\delta\varepsilon_1 > 0\}$ , which contain  $\{O\}$ .

This completes the proof of Lemma 5.1.  $\square$

## 6. Proof of Theorem 2.1

In this section we give a proof of Theorem 2.1. The definitions of  $f$ ,  $\omega_f$  and  $\Omega_f$  are given as in Section 2.

### 6.1 Localization

From the previous section, it turns out that the singularity of the Bergman kernel of  $\Omega_f$  at  $z^0$  is determined by the local information about  $\partial\Omega_f$  near  $z^0$ . Thus we construct an appropriate domain whose boundary coincides  $\partial\Omega_f$  near  $z^0$  for the computation below.

We can easily construct a function  $\tilde{g} \in C^\infty(\mathbb{R})$  such that

$$\tilde{g}(x) = \begin{cases} g(x) & \text{for } |x| \leq \delta \\ \frac{9}{10} g(0) & \text{for } |x| \geq 1 \end{cases} \quad (6.1)$$

and

$$0 \leq -x\tilde{g}'(x), \quad |x^2\tilde{g}''(x)| < \frac{1}{5}g(0) \quad \text{for } x \in \mathbb{R}, \quad (6.2)$$

for some small positive constant  $\delta < 1$ . Note that

$$\frac{9}{10}g(0) \leq \tilde{g}(x) \leq g(0).$$

Set  $\tilde{f}(x) = x^{2m}\tilde{g}(x)$  and  $\omega_{\tilde{f}} = \{(x, y) \in \mathbb{R}^2 \mid y > \tilde{f}(x)\}$ . Let  $\Omega_{\tilde{f}} \subset \mathbb{C}^2$  be the tube domain over  $\omega_{\tilde{f}}$ , i.e.,  $\Omega_{\tilde{f}} = \mathbb{R}^2 + i\omega_{\tilde{f}}$ . Here we remark that the boundary of  $\Omega_{\tilde{f}}$  is strictly pseudoconvex off the set  $\{(z_1, z_2) \mid \Im z_1 = \Im z_2 = 0\}$ . In fact we can easily check that  $\tilde{f}''(x) > 0$  if  $x \neq 0$  by (6.1) and (6.2).

Let  $\tilde{K}$  be the Bergman kernel of  $\Omega_{\tilde{f}}$ . In order to obtain Theorem 2.1, it is sufficient to consider the singularity of the Bergman kernel  $\tilde{K}$  near  $z^0$  by Lemma 5.1.

## 6.2 Two propositions and the proof of Theorem 2.1

A clue to our analysis of the Bergman kernel is the integral representation in Section 3. Normalizing this representation, the Bergman kernel  $\tilde{K}$  of  $\Omega_{\tilde{f}}$  can be expressed in the following:

$$\tilde{K}(z) = \frac{2m}{(4\pi)^2} g(0)^{1/m} \int_0^\infty e^{-yu^{2m}} P(x, u) u^{4m+1} du$$

with

$$P(x, u) = \int_{-\infty}^\infty e^{g(0)^{1/2m}xuv} \frac{1}{\phi(v, u^{-1})} dv,$$

$$\phi(v, X) = \int_{-\infty}^\infty e^{-\widehat{g}(Xw)w^{2m}+vw} dw,$$

where  $\widehat{g}(g(0)^{1/2m}x) = \tilde{g}(x)/g(0)$ . In order to prove Theorem 2.1, it is sufficient to consider the following function  $\bar{K}$  instead of  $\tilde{K}$ :

$$\bar{K}(z) = \frac{2m}{(4\pi)^2} g(0)^{1/m} \int_1^\infty e^{-yu^{2m}} P(x, u) u^{4m+1} du. \quad (6.3)$$

In fact the difference between  $\tilde{K}$  and  $\bar{K}$  is smooth.

Asymptotic expansion of the Bergman kernel

By introducing the variables  $t_0 = g(0)^{1/2m} x y^{-1/2m}$ ,  $\xi = y^{1/2m}$  to the integral representation (6.3), we have

$$\bar{K}(z) = \frac{2m}{(4\pi)^2} \xi^{-4m-2} g(0)^{1/m} \int_{\xi}^{\infty} e^{-s^{2m}} L(t_0, \xi; s) s^{4m+1} ds, \quad (6.4)$$

where

$$L(t_0, \xi; s) = \int_{-\infty}^{\infty} e^{t_0 s v} \frac{1}{\phi(v, \xi s^{-1})} dv. \quad (6.5)$$

We divide the integral in (6.4) into two parts:

$$\bar{K}(z) = \frac{2m}{(4\pi)^2} g(0)^{1/m} \xi^{-4m-2} \{K^{(1)}(z) + K^{(2)}(z)\}, \quad (6.6)$$

where

$$K^{(1)}(z) = \int_1^{\infty} e^{-s^{2m}} L(t_0, \xi; s) s^{4m+1} ds, \quad (6.7)$$

$$K^{(2)}(z) = \int_{\xi}^1 e^{-s^{2m}} L(t_0, \xi; s) s^{4m+1} ds. \quad (6.8)$$

Since the function  $(\phi(v, X))^{-1}$  is smooth function of  $X$  on  $[0, 1]$ , for any positive integer  $\mu_0$

$$\frac{1}{\phi(v, X)} = \sum_{\mu=0}^{\mu_0} a_{\mu}(v) X^{\mu} + r_{\mu_0}(v, X) X^{\mu_0+1}, \quad (6.9)$$

where

$$\begin{aligned} a_{\mu}(v) &= \frac{1}{\mu!} \frac{\partial^{\mu}}{\partial X^{\mu}} \frac{1}{\phi(v, X)} \Big|_{X=0}, \\ r_{\mu_0}(v, X) &= \frac{1}{\mu_0!} \int_0^1 (1-p)^{\mu_0} \frac{\partial^{\mu_0+1}}{\partial Y^{\mu_0+1}} \frac{1}{\phi(v, Y)} \Big|_{Y=Xp} dp. \end{aligned} \quad (6.10)$$

Substituting (6.9) into (6.5), we have

$$L(t_0, \xi; s) = \sum_{\mu=0}^{\mu_0} L_{\mu}(t_0 s) \xi^{\mu} s^{-\mu} + \tilde{L}_{\mu_0}(t_0, \xi; s) \xi^{\mu_0+1} s^{-\mu_0-1}, \quad (6.11)$$



where

$$L_\mu(u) = \int_{-\infty}^{\infty} e^{uv} a_\mu(v) dv,$$

$$\tilde{L}_{\mu_0}(t_0, \xi; s) = \int_{-\infty}^{\infty} e^{t_0sv} r_{\mu_0}(v, \xi s^{-1}) dv.$$

Moreover substituting (6.11) into (6.7)-(6.8), we have

$$K^{(j)}(z) = \sum_{\mu=0}^{\mu_0} K_\mu^{(j)}(\tau, \xi) \xi^\mu + \tilde{K}_{\mu_0}^{(j)}(\tau, \xi) \xi^{\mu_0+1},$$

for  $j = 1, 2$  where

$$K_\mu^{(1)}(\tau, \xi) = \int_1^\infty e^{-s^{2m}} L_\mu(t_0 s) s^{4m+1-\mu} ds,$$

$$\tilde{K}_{\mu_0}^{(1)}(\tau, \xi) = \int_1^\infty e^{-s^{2m}} \tilde{L}_{\mu_0}(t_0, \xi; s) s^{4m-\mu_0} ds,$$

$$K_\mu^{(2)}(\tau, \xi) = \int_\xi^1 e^{-s^{2m}} L_\mu(t_0 s) s^{4m+1-\mu} ds,$$

$$\tilde{K}_{\mu_0}^{(2)}(\tau, \xi) = \int_\xi^1 e^{-s^{2m}} \tilde{L}_{\mu_0}(t_0, \xi; s) s^{4m-\mu_0} ds.$$

The following two propositions are concerned with the singularities of the above functions. Their proofs are given in Paragraphs 6.5 and 6.6.

**PROPOSITION 6.1**

(i) For any nonnegative integer  $k_0$ ,  $K_\mu^{(1)}$  is expressed in the form:

$$K_\mu^{(1)}(\tau, \xi) = \sum_{k=0}^{k_0} c_{\mu,k}(\tau) \xi^k + \tilde{K}_{\mu,k_0}^{(1)}(\tau, \xi) \xi^{k_0+1},$$

where

$$c_{\mu,k}(\tau) = \frac{\varphi_{\mu,k}(\tau)}{\tau^{3+\mu+k}} + \psi_{\mu,k}(\tau) \log \tau,$$

for  $\varphi_{\mu,k}, \psi_{\mu,k} \in C^\infty([0, 1])$  and  $\tilde{K}_{\mu,k_0}^{(1)}$  satisfies  $|\tilde{K}_{\mu,k_0}^{(1)}(\tau, \xi)| < C_{\mu,k_0} (\tau - \alpha\xi)^{-4-\mu-k_0}$  for some positive constants  $C_{\mu,k_0}$  and  $\alpha$ .

(ii)  $\tilde{K}_{\mu_0}^{(1)}$  satisfies  $|\tilde{K}_{\mu_0}^{(1)}(\tau, \xi)| < C_{\mu_0} (\tau - \alpha\xi)^{-4-\mu_0}$  for some positive constants  $C_{\mu_0}$  and  $\alpha$ .

**PROPOSITION 6.2**

(i) (a) For  $0 \leq \mu \leq 4m + 1$ ,  $K_\mu^{(2)} \in C^\infty([0, 1] \times [0, \varepsilon])$ .

(b) For  $\mu \geq 4m + 2$ ,  $K_\mu^{(2)}$  can be expressed in the form:

$$K_\mu^{(2)}(\tau, \xi)\xi^{-4m-2+\mu} = H_\mu(\tau, \xi)\xi^{-4m-2+\mu} \log \xi + \tilde{H}_\mu(\tau, \xi),$$

where  $H_\mu, \tilde{H}_\mu \in C^\infty([0, 1] \times [0, \varepsilon])$ .

(ii) For any positive integer  $r$ , there is a positive integer  $\mu_0$  such that

$$\tilde{K}_{\mu_0}^{(2)}(\tau, \xi)\xi^{-4m-1+\mu_0} \in C^r([0, 1] \times [0, \varepsilon]).$$

First by Proposition 6.1,  $K^{(1)}$  can be expressed in the form:

$$K^{(1)}(z) = \sum_{\mu=0}^{\mu_0} c_\mu(\tau)\xi^\mu + R_{\mu_0}(\tau, \xi)\xi^{\mu_0+1}, \quad (6.12)$$

where  $c_\mu$ 's are expressed as in (2.5) in Theorem 2.1 and  $R_{\mu_0}$  satisfies  $|R_{\mu_0}| < C_{\mu_0}(\tau - \alpha\xi)^{-4-\mu_0}$ .

Next by Proposition 6.2,  $K^{(2)}(z)$  can be expressed in the form: for any positive integer  $r$ ,

$$\xi^{-4m-2}K^{(2)}(z) = H(\tau, \xi)\log \xi + \tilde{H}(\tau, \xi), \quad (6.13)$$

where  $H \in C^\infty([0, 1] \times [0, \varepsilon])$  and  $\tilde{H} \in C^r([0, 1] \times [0, \varepsilon])$ .

Hence putting (6.6), (6.12) and (6.13) together, we can obtain Theorem 2.1. Note that  $K(z)$  is an even function of  $\xi$ .  $\square$

### 6.3 Asymptotic expansion of $a_\mu$

By a direct computation in (6.10),  $a_\mu(v)$  can be expressed in the following form:

$$a_\mu(v) = \sum_{|\alpha|=\mu} C_\alpha \frac{\phi^{[\alpha_1]}(v) \dots \phi^{[\alpha_\mu]}(v)}{\phi(v)^{\mu+1}}, \quad (6.14)$$

where

$$\phi^{[k]}(v) = \left. \frac{\partial^k}{\partial X^k} \phi(v, X) \right|_{X=0},$$

$C_\alpha$ 's are constants depending on  $\alpha = (\alpha_1, \dots, \alpha_\mu) \in \mathbb{Z}_{\geq 0}^\mu$  and  $|\alpha| = \sum_{j=1}^\mu \alpha_j$ . Since

$$\frac{\partial^k}{\partial X^k} \phi(v, X) = \int_{-\infty}^{\infty} \left\{ w^k \sum_{j=1}^k c_{kj}(Xw) w^{2mj} \right\} e^{-\widehat{g}(Xw)w^{2m} + vw} dw,$$

for  $k \geq 1$  where  $c_{kj} \in C^\infty(\mathbb{R})$ , we have

$$\phi^{[k]}(v) = \left. \frac{\partial^k}{\partial X^k} \phi(v, X) \right|_{X=0} = \sum_{j=1}^k c_{kj}(0) \phi_{2mj+k}(v), \quad (6.15)$$

for  $k \geq 1$  where

$$\phi_\ell(v) = \int_{-\infty}^{\infty} w^\ell e^{-w^{2m} + vw} dw.$$

Here the following lemma is concerned with the asymptotic expansion of  $\phi_\ell$  at infinity.

LEMMA 6.1. — Set  $a = [(2m)^{-1}/(2m-1) - (2m)^{-2m/(2m-1)}] > 0$ . Then we have

$$\phi_\ell(v) \sim v^{(1-m+\ell)/(2m-1)} \cdot \exp\{av^{2m/(2m-1)}\} \cdot \sum_{j=0}^{\infty} c_j v^{-2mj/(2m-1)}$$

as  $v \rightarrow +\infty$  for  $\ell \geq 0$ .

The proof of the above lemma will be given soon later.

Lemma 6.1 and (6.15) imply

$$\frac{\phi^{[k]}(v)}{\phi(v)} \sim v^{(2m+1)k/(2m-1)} \cdot \sum_{j=0}^{\infty} c_j v^{-2mj/(2m-1)} \quad \text{as } v \rightarrow \infty \text{ for } k \geq 1.$$

Moreover, we have

$$\frac{\phi^{[\alpha_1]}(v) \cdots \phi^{[\alpha_\mu]}(v)}{\phi(v)^\mu} \sim v^{(2m+1)\mu/(2m-1)} \cdot \sum_{j=0}^{\infty} c_j v^{-2mj/(2m-1)} \quad \text{as } v \rightarrow \infty. \quad (6.16)$$

Therefore (6.14), (6.16) and Lemma 6.1 imply

$$a_\mu(v) \sim v^{(m-1+(2m+1)\mu)/(2m-1)} \cdot \exp\{-av^{2m/(2m-1)}\} \sum_{j=0}^{\infty} c_j v^{-2mj/(2m-1)}$$

as  $v \rightarrow \infty$ .

*Proof of Lemma 6.1.* — Changing the integral variable, we have

$$\phi_\ell(v) = v^{(\ell+1)/(2m-1)} \int_{-\infty}^{\infty} t^\ell e^{-\tilde{v}p(t)} dt, \quad (6.17)$$

where  $\tilde{v} = v^{2m/(2m-1)}$  and  $p(t) = t^{2m} - t$ . We divide (6.17) into two parts:

$$\phi_\ell(v) = v^{(\ell+1)/(2m-1)} \{I_1(\tilde{v}) + I_2(\tilde{v})\}, \quad (6.18)$$

with

$$I_1(\tilde{v}) = \int_{|t-\alpha| \leq \delta} t^\ell e^{-\tilde{v}p(t)} dt \quad \text{and} \quad I_2(\tilde{v}) = \int_{|t-\alpha| > \delta} t^\ell e^{-\tilde{v}p(t)} dt,$$

where  $\delta > 0$  is small and  $\alpha = (2m)^{-1/(2m-1)}$ . Note that  $p'(\alpha) = 0$ .

First we consider the function  $I_1$ . By Taylor's formula, we have

$$\begin{aligned} I_1(\tilde{v}) &= \int_{|t-\alpha| < \delta} t^\ell \exp\left\{-\tilde{v}\left(p(\alpha) + \tilde{p}(t-\alpha)(t-\alpha)^2\right)\right\} dt \\ &= e^{a\tilde{v}} \int_{|t| \leq \delta} (t-\alpha)^\ell e^{-\tilde{v}\tilde{p}(t)t^2} dt, \end{aligned}$$

where  $a = -p(\alpha) = [(2m)^{-1/(2m-1)} - (2m)^{-2m/(2m-1)}] > 0$  and  $\tilde{p}(t) = \int_0^1 (1-u)p''(ut+\alpha) du$ . Set  $s = \tilde{p}(t)^{1/2}t$  for  $|t| \leq \delta$  and  $\delta_\pm = \tilde{p}(\pm\delta)^{1/2}\delta$ , respectively. Then there is a function  $\hat{p} \in C^\infty([-\delta_-, \delta_+])$  such that  $t = \hat{p}(s)$  and  $\hat{p}' > 0$ . Changing the integral variable, we have

$$I_1(\tilde{v}) = \int_{-\delta_-}^{\delta_+} e^{-\tilde{v}s^2} \Psi(s) ds,$$

where  $\Psi(s) = (\hat{p}(s) + \alpha)^\ell \cdot \hat{p}'(s)$ . Since  $\Psi \in C^\infty([-\delta_-, \delta_+])$ , we have

$$\begin{aligned} I_1(v) &= \tilde{v}^{-1/2} \int_{-\delta_- \tilde{v}^{-1/2}}^{\delta_+ \tilde{v}^{1/2}} e^{-u^2} \Psi(u\tilde{v}^{-1/2}) du \\ &\sim \tilde{v}^{-1/2} \sum_{j=0}^{\infty} c_j \tilde{v}^{-j} \quad \text{as } \tilde{v} \rightarrow \infty. \end{aligned} \quad (6.19)$$

We remark that  $\int_{-\infty}^{\infty} e^{-u^2} u^j du = 0$  if  $j \in \mathbb{Z}$  is odd.

Next we consider the function  $I_2$ . Let  $p_d$  be the function defined by  $p_d(t) = d|t - \alpha| - a$  where  $d > 0$ . We can choose  $d > 0$  such that  $p(t) \geq p_d(t)$  for  $|t - \alpha| > \delta$ . Then we have

$$\begin{aligned} |I_2(\tilde{v})| &\leq \int_{|t-\alpha|>\delta} e^{-\tilde{v}p_d(t)} dt \\ &\leq 2C e^{a\tilde{v}} \int_{\delta}^{\infty} e^{-d\tilde{v}t} dt \leq 2C\tilde{v}^{-1} e^{(a-d\delta)\tilde{v}}. \end{aligned} \tag{6.20}$$

Finally putting (6.18), (6.19) and (6.20) together, we have the asymptotic expansion in Lemma 6.1  $\square$

#### 6.4 Asymptotic expansion of $L_\mu$

Let  $A \in C^\infty(\mathbb{R})$  be an even or odd function (i.e.,  $A(-v) = A(v)$  or  $-A(v)$ ) satisfying

$$A(v) \sim v^{n/(2m-1)} \cdot \exp\{-av^{2m/(2m-1)}\} \cdot \sum_{j=0}^{\infty} c_j v^{-2mj/(2m-1)} \quad \text{as } v \rightarrow +\infty,$$

where  $n \in \mathbb{N}$  and the constant  $a$  is as in Lemma 6.1.  $L \in C^\omega(\mathbb{R})$  be the function defined by

$$L(u) = \int_{-\infty}^{\infty} e^{uv} A(v) dv. \tag{6.21}$$

In this section we give the asymptotic expansion of  $L$  at infinity.

LEMMA 6.2.—  $L(u) \sim u^{m-1+n} \cdot e^{u^{2m}} \cdot \sum_{j=0}^{\infty} c_j u^{-2mj}$  as  $u \rightarrow +\infty$ .

*Remark.* — Lemmas 6.1 and 6.2 imply that for  $\mu, \ell \geq 0$ ,

$$\begin{aligned} L_\mu^{(\ell)}(u) &= \frac{d^\ell}{du^\ell} L_\mu(u) \\ &\sim u^{2m-2+(2m+1)\mu+(2m-1)\ell} \cdot e^{u^{2m}} \cdot \sum_{j=0}^{\infty} c_j u^{-2mj} \quad \text{as } u \rightarrow +\infty. \end{aligned} \tag{6.22}$$

Asymptotic expansion of the Bergman kernel

*Proof.* — We only show Lemma 6.2 in the case where  $A$  is an even function. Let  $\widehat{A} \in C^\infty(\mathbb{R} \setminus \{0\})$  be defined by

$$A(v) = v^{n/(2m-1)} \cdot \exp\{-av^{2m/(2m-1)}\} \cdot \widehat{A}(v^{2m/(2m-1)}), \quad (6.23)$$

then  $\widehat{A}(x) \sim \sum_{j=0}^{\infty} c_j x^{-j}$  as  $x \rightarrow \infty$ . Substituting (6.23) into (6.21), we have

$$L(u) = \int_{-\infty}^{\infty} \exp\{-a|v|^{2m/(2m-1)} + uv\} |v|^{n/(2m-1)} \widehat{A}(|v|^{2m/(2m-1)}) dv.$$

Changing the integral variable and setting  $q(t) = at^{2m} - t^{2m-1}$ , we have

$$L(u) = (2m-1)u^{2m+n-1} \int_{-\infty}^{\infty} e^{-u^{2m}q(t)} \widehat{A}(t^{2m}u^{2m}) t^{2m+n-2} dt. \quad (6.24)$$

Now we divide the integral in (6.24) into two parts:

$$L(u) = (2m-1)u^{2m+n-1} \{J_1(\tilde{u}) + J_2(\tilde{u})\}, \quad (6.25)$$

with

$$J_1(\tilde{u}) = \int_{|t-\beta| \leq \delta} e^{-\tilde{u}q(t)} \widehat{A}(\tilde{u}t^{2m}) t^{2m+n-2} dt,$$

$$J_2(\tilde{u}) = \int_{|t-\beta| > \delta} e^{-\tilde{u}q(t)} \widehat{A}(\tilde{u}t^{2m}) t^{2m+n-2} dt,$$

where  $\tilde{u} = u^{2m}$ ,  $\delta > 0$  is small and  $\beta = (2m-1)(2ma)^{-1}$ . Note that  $q'(\beta) = 0$ .

First we consider the function  $J_1$ . By Taylor's formula, we have

$$J_1(\tilde{u}) = e^{\tilde{u}} \int_{|t| \leq \delta} e^{-\tilde{u}\tilde{q}(t)t^2} \widehat{A}(\tilde{u}(t+\beta)^{2m}) (t+\beta)^{2m+n-2} dt,$$

where  $\tilde{q}(t) = \int_0^1 (1-v)q''(vt+\alpha) dv$ . Note that  $q(0) = -1$ . Set  $s = \tilde{q}(t)^{1/2}t$  for  $|t| \leq \delta$  and  $\tilde{\delta}_\pm = \tilde{q}(\pm\delta)^{1/2}\delta$ , respectively. Then there is a function  $\tilde{q} \in C^\infty([-\tilde{\delta}_-, \tilde{\delta}_+])$  such that  $t = \tilde{q}(s)$  and  $\tilde{q}' > 0$ . Changing the integral variable, we have

$$J_1(\tilde{u}) = \int_{-\tilde{\delta}_-}^{\tilde{\delta}_+} e^{-\tilde{u}s^2} \tilde{\Psi}(s, \tilde{u}) ds, \quad (6.26)$$

where

$$\tilde{\Psi}(s, \tilde{u}) = \widehat{A}\left(\tilde{u}(\widehat{q}(s) + \beta)^{2m}\right) (\widehat{q}(s) + \beta)^{2m+n-2} \widehat{q}'(s).$$

Since  $\widehat{A}(x) \sim \sum_{j=0}^{\infty} c_j x^{-j}$  as  $x \rightarrow \infty$ , we have

$$\tilde{\Psi}(s, \tilde{u}) \sim \sum_{j=0}^{\infty} c_j(s) \tilde{u}^{-j} \quad \text{as } \tilde{u} \rightarrow \infty, \quad (6.27)$$

where  $c_j \in C^\infty([-\tilde{\delta}_-, \tilde{\delta}_+])$ . Substituting (6.27) into (6.26), we have

$$J_1(\tilde{u}) \sim \tilde{u}^{-1/2} e^{\tilde{u}} \sum_{j=0}^{\infty} c_j \tilde{u}^{-j} \quad \text{as } \tilde{u} \rightarrow \infty. \quad (6.28)$$

Next we consider the function  $J_2$ . By a similar argument about the estimate of  $I_2(\tilde{v})$  in the proof of Lemma 6.1, we can obtain

$$|J_2(\tilde{u})| \leq C \tilde{u}^{-1} e^{(1-\varepsilon)\tilde{u}}, \quad (2.29)$$

where  $\varepsilon$  is a positive constant.

Finally putting (6.25), (6.28) and (6.29) together, we obtain the asymptotic expansion in Lemma 6.2.  $\square$

### 6.5 Proof of Proposition 6.1

We can construct the function  $h \in C^\infty([0, \infty))$  such that if  $Y = \tilde{f}(X)^{1/2m}$ , then  $X = Yh(Y)$ . In fact

$$\frac{d}{dX} \left[ \tilde{f}(X)^{1/2m} \right] > 0 \quad \text{for } X \geq 0.$$

Set  $t = \tilde{f}(x)^{1/2m} \xi^{-1}$ . Then we can write  $t_0 = th(t\xi)$ . Note that  $\widehat{g}(X)^{1/2m} h(X) = 1$  for  $X, Y \geq 0$  and hence  $h'(Y) \geq 0$  for  $Y \geq 0$ . Let us prepare two lemmas for the proof of Propositions 6.1 and 6.2.

**LEMMA 6.3.** — *Assume that the functions  $a, b$  and  $c$  on  $[0, \varepsilon) \times [0, 1]$  satisfy  $a(\xi, t_0) = b(\xi, t) = c(\xi, \tau)$ . If one of these functions belongs to  $C^\infty([0, \varepsilon) \times [0, 1])$ , then so do the others.*

*Proof.* — This lemma is directly shown by the relation between three variables  $t_0, t$  and  $\tau$ .  $\square$

LEMMA 6.4. — *There is a positive number  $\alpha$  such that  $1 - t_0^{2m} \geq \tau - \alpha\xi$ .*

We remark that the above constant  $\alpha$  is same as that in Proposition 6.2.

*Proof.* — By definition, we have

$$1 - t_0^{2m} = 1 - t^{2m} h(t\xi)^{2m} = (1 - t^{2m}) - (h(t\xi) - 1)t^{2m}.$$

Since  $h(0) = 1$  and  $h'(X) \geq 0$ , we have  $h(t\xi)^{2m} - 1 \leq \alpha t\xi$  for some positive number  $\alpha$ . Therefore we have

$$1 - t_0^{2m} \geq \tau - \alpha t^{2m+1} \xi \geq \tau - \alpha\xi. \quad \square$$

*Proof of Proposition 6.1*

(i) Recall the definition of the function  $K_\mu^{(1)}$ :

$$K_\mu^{(1)}(\tau, \xi) = \int_1^\infty e^{-s^{2m}} L_\mu(t_0 s) s^{4m+1-\mu} ds, \quad (6.30)$$

where

$$L_\mu(u) = \int_{-\infty}^\infty e^{uv} a_\mu(v) dv \quad \text{and} \quad a_\mu(v) = \frac{1}{\mu!} \frac{\partial^\mu}{\partial X^\mu} \frac{1}{\phi(v, X)} \Big|_{X=0}.$$

We obtain the Taylor expansion of  $L_\mu(t_0; s) = L_\mu(tsh(t\xi))$  with respect to  $\xi$ :

$$L_\mu(t_0 s) = \sum_{k=0}^{k_0} L_{\mu,k}(t; s) \xi^k + \tilde{L}_{\mu,k_0}(t, \xi; s) \xi^{k_0+1}, \quad (6.31)$$

where

$$L_{\mu,k}(t; s) = \frac{1}{k!} \frac{\partial^k}{\partial \xi^k} L_\mu(tsh(t\xi)) \Big|_{\xi=0}, \quad (6.32)$$

$$\tilde{L}_{\mu,k_0}(t, \xi; s) = \frac{1}{k_0!} \int_0^1 (1-p)^{k_0} \frac{\partial^{k_0+1}}{\partial X^{k_0+1}} L_\mu(tsh(tX)) \Big|_{X=\xi p} dp. \quad (6.33)$$



Substituting (6.31) into (6.30), we have

$$K_{\mu}^{(1)}(\tau, \xi) = \sum_{k=0}^{k_0} K_{\mu,k}(t) \xi^k + \tilde{K}_{\mu,k_0}(t, \xi) \xi^{k_0+1},$$

where

$$K_{\mu,k}(t) = \int_1^{\infty} e^{-s^{2m}} L_{\mu,k}(t; s) s^{4m+1-\mu} ds, \quad (6.34)$$

$$\tilde{K}_{\mu,k_0}(t, \xi) = \int_1^{\infty} e^{-s^{2m}} \tilde{L}_{\mu,k_0}(t, \xi; s) s^{4m+1-\mu} ds. \quad (6.35)$$

First we consider the singularity of  $K_{\nu,k}$  at  $t = 1$ .

By a direct computation in (6.32), we have

$$L_{\mu,k}(t; s) = \sum_{\ell=1}^k h_{\ell}(t) s^{\ell} L_{\mu}^{(\ell)}(ts),$$

where  $h_{\ell} \in C^{\infty}([0, 1])$  (which depends on  $\mu, k$ ). We define the function  $S_{\mu,k}(t; s^{2m})$  by

$$L_{\mu,k}(t; s) = s^{2m-2+(2m-1)\mu+2mk} e^{t^{2m}s^{2m}} S_{\mu,k}(t; s^{2m}). \quad (6.36)$$

Lemma 6.2 implies

$$S_{\mu,k}(t; s^{2m}) \sim \sum_{j=0}^{\infty} c_j(t) s^{-2mj} \quad \text{as } s \rightarrow \infty, \quad (6.37)$$

where  $c_j \in C^{\infty}([0, 1])$ .

Substituting (6.36) into (6.34), we have

$$\begin{aligned} K_{\mu,k}(t) &= \int_1^{\infty} e^{-(1-t^{2m})s^{2m}} S_{\mu,k}(t; s^{2m}) s^{2m-2+2m\mu+2mk} ds \\ &= \frac{1}{2m} \int_1^{\infty} e^{-\chi^{-1}(\tau)\sigma} S_{\mu,k}(t; \sigma) \sigma^{2+\mu+k} d\sigma. \end{aligned} \quad (6.38)$$

Moreover substituting (6.37) into (6.38), we have

$$K_{\mu,k}(t) = \frac{\varphi_{\mu,k}(\tau)}{\tau^{3+\mu+k}} + \psi_{\mu,k}(\tau) \log \tau,$$

where  $\varphi_{\mu,k}, \psi_{\mu,k} \in C^{\infty}([0, 1])$ .

Next we obtain the inequality  $|\tilde{K}_{\mu,k}^{(1)}(\tau, \xi)| \leq C_{\mu,k_0} [\tau - \alpha\xi]^{-4-\mu-k_0}$  for some positive constant  $C_{\mu,k_0}$ . By a direct computation in (6.33), we have

$$\frac{\partial^{k_0+1}}{\partial X^{k_0+1}} L_\mu(tsh(tX)) = \sum_{\ell=1}^{k_0+1} \tilde{h}_\ell(t, X) s^\ell L_\mu^{(\ell)}(tsh(tX)),$$

where  $\tilde{h}_\ell$  are bounded functions (depending on  $\mu, k_0$ ). Since  $h'(X) \geq 0$ , we can obtain

$$\begin{aligned} |\tilde{L}_{\mu,k_0}(t, \xi; s)| &\leq C \sum_{\ell=1}^{k_0+1} s^\ell L_\mu^{(\ell)}(tsh(ts)) \\ &\leq C s^{k_0+1} L_\mu^{(k_0+1)}(t_0 s) \\ &\leq C s^{2m-2+(2m+1)\mu+2m(k_0+1)} e^{t_0^2 m s^2 m} \end{aligned} \quad (6.39)$$

for  $s \geq 1$ . Substituting (6.39) to (6.35), we obtain

$$\begin{aligned} \tilde{K}_{\mu,k_0}(t, \xi) &\leq C \int_1^\infty e^{-(1-t_0^2 m)s^2 m} s^{6m-2+2m\mu+2m(k_0+1)} ds \\ &\leq C (1-t_0^2 m)^{-4-\mu-k_0} \leq C (\tau - \alpha\xi)^{-4-\mu-k_0} \end{aligned}$$

by Lemma 6.4. This completes the proof of Proposition 6.2(i).

(ii) Recall the definition of the function  $\tilde{K}_\mu^{(1)}$ :

$$\tilde{K}_{\mu_0}^{(1)}(\tau, \xi) = \int_1^\infty e^{-s^2 m} \tilde{L}_{\mu_0}(t_0, \xi; s) s^{4m-\mu_0} ds. \quad (6.40)$$

where

$$\tilde{L}_{\mu_0}(t_0, \xi; s) = \int_{-\infty}^\infty e^{t_0 s v} r_{\mu_0}(v, \xi s^{-1}) dv, \quad (6.41)$$

$$r_{\mu_0}(v, X) = \frac{1}{\mu_0!} \int_0^1 (1-p)^{\mu_0} \frac{\partial^{\mu_0+1}}{\partial Y^{\mu_0+1}} \frac{1}{\phi(v, Y)} \Big|_{Y=X_p} dp. \quad (6.42)$$

The following lemma is necessary to obtain the estimate of  $\tilde{K}_\mu^{(1)}$  in (ii).

LEMMA 6.5. — For  $v \in \mathbb{R}, X \geq 0$ ,

$$|r_{\mu_0}(v, X)| < C |v|^{((2m+1)(\mu_0+1)+m-1)/(2m-1)} e^{-a|v|^{2m/(2m-1)}}.$$

We remark that the constant  $a$  is as in Lemma 6.1. The proof of the above lemma is given soon later.

Applying Lemma 6.5 to (6.41), we have

$$\begin{aligned} |\tilde{L}_{\mu_0}(t_0, \xi, s)| &\leq \\ &\leq C \int_{-\infty}^{\infty} |v|^{(2m+1)(\mu_0+1)+m-1/(2m-1)} e^{t_0 s v - a|v|^{2m/(2m-1)}} dv \quad (6.43) \\ &\leq C s^{2m-2(2m+1)(\mu_0+1)} e^{t_0^{2m} s^{2m}}. \end{aligned}$$

The second inequality is given by Lemma 6.2. Moreover substituting (6.43) into (6.40), we have

$$\begin{aligned} |\tilde{K}_{\mu_0}^{(1)}(t, \xi)| &\leq C \int_1^{\infty} e^{-(1-t_0^{2m})s^{2m}} s^{8m-1+2m\mu_0} ds \\ &\leq C(1-t_0^{2m})^{-4-\mu_0} \leq C(\tau - \alpha\xi)^{-4-\mu_0}, \end{aligned}$$

by Lemma 6.4. Therefore we obtain the estimate of  $\tilde{K}_{\mu_0}^{(1)}$  in Proposition 6.1(ii).  $\square$

*Proof of Lemma 6.5.* — We only consider the case where  $v$  is positive. The proof for the case where  $v$  is negative is given in the same way.

By a direct computation, we have

$$\frac{\partial^{\mu_0+1}}{\partial X^{\mu_0+1}} \frac{1}{\phi(v, X)} = \sum_{|\alpha|=\mu_0+1} C_{\alpha} \frac{\phi^{[\alpha_1]}(v, X) \cdots \phi^{[\alpha_{\mu_0+1}]}(v, X)}{\phi(v, X)^{\mu_0+2}}, \quad (6.44)$$

where

$$\phi^{[k]}(v, X) = \frac{\partial^k}{\partial X^k} \phi(v, X)$$

and  $C_{\alpha}$ 's are constants depending on  $\alpha = (\alpha_1, \dots, \alpha_{\mu_0+1}) \in \mathbb{Z}_{\geq 0}^{\mu_0+1}$ . By a direct computation, the function  $\phi^{[k]}$ 's are expressed in the form:

$$\phi^{[k]}(v, X) = \sum_{|\beta|=k} c_{\beta} F_{\beta}(\tilde{v}, \tilde{X}), \quad (6.45)$$

where  $\tilde{v} = v^{2m/(2m-1)}$ ,  $\tilde{X} = Xv^{-1/2m}$ ,  $c_{\beta}$ 's are constants and

$$\begin{aligned} F_{\beta}(\tilde{v}, \tilde{X}) &= \int_{-\infty}^{\infty} e^{-\tilde{g}(Xw)w^{2m}+vw} w^{\gamma} \prod_{\beta} \hat{g}^{(\beta_k)}(Xw) dw \\ &= \tilde{v}^{\gamma+1/2m} \int_{-\infty}^{\infty} e^{-\tilde{v}p(s, \tilde{X})} s^{\gamma} \prod_{\beta} \hat{g}^{(\beta_k)}(\tilde{X}s) ds, \end{aligned} \quad (6.46)$$

with  $p(s, \tilde{X}) = \hat{g}(\tilde{X}s)s^{2m} - s$ ,  $\beta_k \in \mathbb{N}$  and  $\gamma \in \mathbb{N}$  depending on  $\beta = (\beta_k)_k$ . In order to apply the stationary phase method to the above integral, we must know the location of the critical points of the function  $p(\cdot, \tilde{X})$ . The lemma below gives the information about it.

LEMMA 6.6. — *There exists a function  $\alpha \in C^\infty([0, \infty))$  such that*

$$\frac{\partial}{\partial s} p(\alpha(\tilde{X}), \tilde{X}) = 0 \quad \text{and} \quad \alpha_0 \leq \alpha(\tilde{X}) \leq \alpha_1 \quad \text{for } \tilde{X} \geq 0, \quad (6.47)$$

where  $\alpha_0 = (2m)^{-1/(2m-1)}$  and  $\alpha_1 = (2m \cdot 4/5)^{-1/(2m-1)}$ .

*Proof.* — By a direct computation, we have

$$\begin{aligned} \frac{\partial}{\partial s} p(s, \tilde{X}) &= s^{2m-1} (\hat{g}'(\eta)\eta + 2m\hat{g}(\eta)) - 1 \\ \frac{\partial^2}{\partial s^2} p(s, \tilde{X}) &= s^{2m-2} (\hat{g}''(\eta)\eta^2 + 4m\hat{g}'(\eta)\eta + 2m(2m-1)\hat{g}(\eta)) \end{aligned}$$

where  $\eta = \tilde{X}s$ . It is easy to obtain the following inequalities by using the conditions (6.1)-(6.2):

$$\frac{\partial}{\partial s} p(\alpha_0, \tilde{X}) \leq 0 < \frac{\partial}{\partial s} p(\alpha_1, \tilde{X}) \quad \text{for } \tilde{X} \geq 0 \quad (6.48)$$

$$\frac{\partial^2}{\partial s^2} p(s, \tilde{X}) \geq c > 0 \quad \text{on } [\alpha_0, \alpha_1] \times [0, \infty) \quad (6.49)$$

for some positive constant  $c$ . Then the inequalities (6.48)-(6.49) imply the claim in Lemma 6.6 by the implicit function theorem.  $\square$

Now we divide the integral in (6.46) into two parts.

$$F_\beta(\tilde{v}, \tilde{X}) = \tilde{v}^{\gamma+1/2m} \{I_1(\tilde{v}, \tilde{X}) + I_2(\tilde{v}, \tilde{X})\},$$

where

$$\begin{aligned} I_1(\tilde{v}, \tilde{X}) &= \int_{|t-\alpha(\tilde{X})| \leq \delta} e^{-\tilde{v}p(t, \tilde{X})t^2} t^\gamma \prod_\beta \hat{g}^{(\beta_k)}(\tilde{X}t) dt, \\ I_2(\tilde{v}, \tilde{X}) &= \int_{|t-\alpha(\tilde{X})| > \delta} e^{-\tilde{v}p(t, \tilde{X})t^2} t^\gamma \prod_\beta \hat{g}^{(\beta_k)}(\tilde{X}t) dt, \end{aligned}$$

where  $\delta > 0$  is small.

First we consider the function  $I_1$ . By Lemma 6.6 and Taylor's formula, we have

$$\begin{aligned} I_1(\tilde{v}, \tilde{X}) &= \\ &= e^{a(\tilde{X})\tilde{v}} \int_{|t| \leq \delta} e^{-\tilde{v}\tilde{p}(t, \tilde{X})t^2} (t + \alpha(\tilde{X}))^\gamma \prod_{\beta} \hat{g}^{(\beta_k)}(\tilde{X}(t + \alpha(\tilde{X}))) dt, \end{aligned} \quad (6.50)$$

where  $a(\tilde{X}) = -p(\alpha(\tilde{X}), \tilde{X})$  and

$$\tilde{p}(t, \tilde{X}) = \int_0^1 (1-u) \frac{\partial^2}{\partial s^2} p(ut + \alpha(\tilde{X}), \tilde{X}) du.$$

Set  $s = \tilde{p}(t, \tilde{X})^{1/2}t$  on  $[-\delta, \delta] \times [0, \infty)$  and  $\delta_{\pm}(\tilde{X}) = \tilde{p}(\pm\delta, \tilde{X})^{1/2}\delta$ , respectively. Then there is a function  $\hat{p} \in C^\infty([-\delta_-(\tilde{X}), \delta_+(\tilde{X})] \times [0, \infty))$  such that  $t = \hat{p}(s, \tilde{X})$  and  $(\partial/\partial s)\hat{p}(s, \tilde{X}) > 0$ . Changing the integral variable, we have

$$I_1(\tilde{v}, \tilde{X}) = e^{a(\tilde{X})\tilde{v}} \int_{-\delta_-(\tilde{X})}^{\delta_+(\tilde{X})} e^{-\tilde{v}s^2} \Psi(s, \tilde{X}) ds,$$

where

$$\Psi(s, \tilde{X}) = (\hat{p}(s, \tilde{X}) + \alpha(\tilde{X}))^\gamma \prod_{\beta} \hat{g}^{(\beta_k)}(\tilde{X}(\hat{p}(s, \tilde{X}) + \alpha(\tilde{X}))) \frac{\partial}{\partial s} \hat{p}(s, \tilde{X}).$$

Since  $\Psi \in C^\infty([-\delta_-(\tilde{X}), \delta_+(\tilde{X})] \times [0, \infty))$ , we have

$$\begin{aligned} I_1(\tilde{v}, \tilde{X}) \cdot \{\tilde{v}^{-1/2} e^{a(\tilde{X})\tilde{v}}\}^{-1} &= \\ \int_{-\delta_-(\tilde{X})\tilde{v}^{1/2}}^{\delta_+(\tilde{X})\tilde{v}^{1/2}} e^{-u^2} \Psi(u\tilde{v}^{-1/2}, \tilde{X}) du &\longrightarrow \sqrt{\pi} \Psi(0, \tilde{X}) \quad \text{as } \tilde{v} \rightarrow \infty. \end{aligned} \quad (6.51)$$

Note that

$$\Psi(0, \tilde{X}) = \alpha(\tilde{X})^\gamma \cdot \left\{ \frac{1}{2} \frac{\partial^2}{\partial t^2} p(\alpha(\tilde{X}), \tilde{X}) \right\}^{-1/2} \prod_{\beta} \hat{g}^{(\beta_k)}(\tilde{X}\alpha(\tilde{X})).$$

Next we consider the function  $I_2$ . In a similar argument about the estimate of  $I_2(\tilde{v})$  in the proof in Lemma 6.1, we can obtain

$$|I_2(\tilde{v}, \tilde{X})| \leq C\tilde{v}^{-1} e^{a(\tilde{X})\tilde{v} - \varepsilon\tilde{v}}, \quad (6.52)$$

where  $\varepsilon$  is a positive constant.

Putting (6.51)-(6.52) together, we have

$$\lim_{\tilde{v} \rightarrow \infty} F_{\beta}(\tilde{v}, \tilde{X}) \{ \tilde{v}^{(\gamma+1-m)/2m} e^{-p(\tilde{X})\tilde{v}} \}^{-1} = \sqrt{\pi} \tilde{\Psi}_{\beta}(0, \tilde{X}). \quad (6.53)$$

Now under the condition  $|\beta| = k$ , the number  $\gamma$  in (6.46) attains the maximum value  $(2m+1)k$  when  $\beta = (1, \dots, 1)$ . Therefore (6.45) and (6.53) imply that

$$\left| \frac{\phi^{[k]}(v, X)}{\phi(v, X)} \right| \leq C \tilde{v}^{(2m+1)k/2m}. \quad (6.54)$$

Moreover (6.44) and (6.53) imply that

$$\left| \frac{\partial^{\mu_0+1}}{\partial X^{\mu_0+1}} \frac{1}{\phi(v, X)} \right| \leq C \tilde{v}^{(2m+1)(\mu_0+1)+m-1/2m} e^{-a(\tilde{X})\tilde{v}}.$$

Now we admit the following lemma.

LEMMA 6.7. —  $a(\tilde{X}) \geq a(0) = a$ .

We remark that the constant  $a$  is as in Lemma 6.1. The above lemma implies

$$\left| \frac{\partial^{\mu_0+1}}{\partial X^{\mu_0+1}} \frac{1}{\phi(v, X)} \right| \leq C \tilde{v}^{(2m+1)(\mu_0+1)+m-1/2m} e^{-a\tilde{v}}. \quad (6.55)$$

Finally substituting (6.55) into (6.42), we can obtain the estimate of  $r_{\mu_0}$  in Lemma 6.5.

*Proof of Lemma 6.7.* — The definition of  $\alpha(\tilde{X})$  in (6.47) implies that

$$\begin{aligned} a'(\tilde{X}) &= -\alpha'(\tilde{X}) \cdot \alpha(\tilde{X})^{2m-1} \left( \hat{g}'(\tilde{X}a(\tilde{X})) \tilde{X} \alpha(\tilde{X}) + 2m \hat{g}(\tilde{X}a(\tilde{X})) \right) \\ &\quad + \alpha'(\tilde{X}) - g'(\tilde{X} \alpha(\tilde{X})) \alpha(\tilde{X})^{2m+1} \\ &= -\hat{g}'(\tilde{X} \alpha(\tilde{X})) \alpha(\tilde{X})^{2m+1}. \end{aligned}$$

Since the condition  $xg'(x) \leq 0$  in (2.1) implies

$$\tilde{X} a'(\tilde{X}) = -\tilde{X} \hat{g}'(\tilde{X} \alpha(\tilde{X})) \alpha(\tilde{X})^{2m+1} \geq 0,$$

$a(\tilde{X})$  attains the minimum value when  $\tilde{X} = 0$ . It is easy to check that  $a(0) = a$ .  $\square$

### 6.6 Proof of Proposition 6.2

(i) Recall the definition of the function  $K_\mu^{(2)}$ :

$$K_\mu^{(2)}(\tau, \xi) = \int_\xi^1 e^{-s^{2m}} L_\mu(t_0 s) s^{4m+1-\mu} ds,$$

where

$$L_\mu(u) = \int_{-\infty}^{\infty} e^{uv} a_\mu(v) dv.$$

We remark that  $L_\mu$  extends to an entire function.

By the residue formula, we have

$$K_\mu^{(2)}(\tau, \xi) = h_\mu(t_0) \xi^{-4m-2+\mu} \log \xi + \tilde{h}_\mu(t_0, \xi), \quad (6.56)$$

where

$$h_\mu(t_0) = \frac{1}{2\pi i} \oint_{|\zeta|=\delta} e^{-\zeta^{2m}} L_\mu(t_0 \zeta) \zeta^{4m+1-\mu} d\zeta \quad (6.57)$$

for  $\delta$  is a small positive integer and  $\tilde{h}_\mu \in C^\infty([0, 1] \times [0, \varepsilon])$ . Here (6.57) implies that  $h_\mu \equiv 0$  for  $0 \leq \mu \leq 4m+1$  and  $h_\mu \in C^\infty([0, 1])$  for  $\mu \leq 4m+2$ . By Lemma 6.3, we can obtain (i) in Proposition 6.2.

(ii) Changing the integral variable, we have

$$\begin{aligned} \xi^{-4m-2+\mu_0} \tilde{K}_{\mu_0}^{(2)}(\tau, \xi) &= \int_1^{\xi^{-1}} e^{-\xi^{2m} u^{2m}} \tilde{L}_{\mu_0}(t_0, \xi, \xi u) u^{4m-\mu_0} du, \\ \tilde{L}_{\mu_0}(t_0, \xi, \xi u) &= \int_{-\infty}^{\infty} e^{t_0 \xi uv} r_{\mu_0}(v, u^{-1}) dv. \end{aligned}$$

Note that  $r_{\mu_0}$  satisfies the inequality in Lemma 6.5.

Keeping the above integrals in mind, we define the function  $H$  by

$$H(\alpha, \beta, \gamma, \delta) = t_0^\delta \xi^\alpha \int_1^{\xi^{-1}} e^{-\xi^{2m} u^{2m}} \frac{\partial^\gamma}{\partial X^\gamma} I(t_0 \xi u) u^{-\beta-1} du, \quad (6.58)$$

with

$$I(X) = \int_{-\infty}^{\infty} e^{Xv} r(v) dv, \quad (6.59)$$

where  $\alpha, \beta, \gamma (\geq 0)$ ,  $\delta$  are integers and the function  $r$  satisfies  $|r(v)| \leq C e^{-c|v|^{2m/(2m-1)}}$  for some positive constants  $c, C$ . Note that  $H$  is a function of  $(t_0, \xi)$ .

## Asymptotic expansion of the Bergman kernel

By a direct computation, we have

$$\frac{\partial}{\partial t_0} H(\alpha, \beta, \gamma, \delta) = H(\alpha + 1, \beta - 1, \gamma + 1, \delta), \quad (6.60)$$

$$\begin{aligned} \frac{\partial}{\partial \xi} H(\alpha, \beta, \gamma, \delta) &= -e^{-1} t_0^\delta \xi^{\alpha+\beta-1} \frac{\partial^\gamma}{\partial X^\gamma} I(\xi) + \alpha H(\alpha - 1, \beta, \gamma, \delta) + \\ &\quad - 2m H(\alpha + 2m - 1, \beta - 2m, \gamma, \delta) + \\ &\quad + H(\alpha, \beta - 1, \gamma + 1, \delta + 1). \end{aligned} \quad (6.61)$$

Since  $\partial^\gamma I / \partial X^\gamma$  is bounded on  $[0, 1]$ , we have

$$|H(\alpha, \beta, \gamma, \delta)| \leq C |\xi|^{\alpha+\beta}, \quad (6.62)$$

by (6.58). By induction, we have

$$\left| \frac{\partial^k}{\partial t_0^k} \frac{\partial^\ell}{\partial \xi^\ell} H(\alpha, \beta, \gamma, \delta) \right| \leq C |\xi|^{\alpha+\beta-\ell},$$

by (6.60), (6.61) and (6.62).

Now if replace  $r(v)$  by  $r_{\mu_0}(v, Y)$  in (6.59), then

$$\xi^{-4m-2+\mu_0} \tilde{K}_{\mu_0}^{(2)}(\tau, \xi) = H(0, -4m + \mu_0 - 1, 0, 0).$$

Therefore if  $\mu_0 > 4m + 1 + \ell$ , then

$$\frac{\partial^k}{\partial t_0^k} \frac{\partial^\ell}{\partial \xi^\ell} \tilde{K}_{\mu_0}^{(2)}(\tau, \xi)$$

is a continuous function of  $(t_0, \xi) \in [0, 1] \times [0, \varepsilon]$ . Therefore we can obtain (ii) in Proposition 6.2 by Lemma 6.3.

This completes the proof of Proposition 6.2.  $\square$

## 7. The Szegő kernel

Let  $\Omega_f$  be a tube domain satisfying the condition in Section 2. Let  $H^2(\Omega_f)$  be the subspace of  $L^2(\Omega_f)$  consisting of holomorphic functions  $F$  on  $\Omega_f$  such that

$$\sup_{\varepsilon > 0} \int_{\partial\Omega_f} |F(z_1, z_2 + i\varepsilon)|^2 d\sigma(z) < \infty,$$



where  $d\sigma$  is the measure on  $\partial\Omega_f$  given by Lebesgue measure on  $\mathbb{C} \times \mathbb{R}$  when we identify  $\partial\Omega_f$  with  $\mathbb{C} \times \mathbb{R}$  (by the map  $(z, t + if(\Im z)) \mapsto (z, t)$ ). The Szegő projection is the orthogonal projection  $\mathbb{S} : L^2(\partial\Omega_f) \rightarrow H^2(\Omega_f)$  and we can write

$$\mathbb{S}F(z) = \int_{\partial\Omega_f} S(z, w)F(w) d\sigma(w),$$

where  $\mathbb{S} : \Omega_f \times \Omega_f \rightarrow \mathbb{C}$  is the Szegő kernel of the domain  $\Omega_f$ . We are interested in the restriction of the Szegő kernel on the diagonal, so we write  $S(z) = S(z, z)$ .

The Szegő kernel of  $\Omega_f$  has an integral representation:

$$S(z) = \frac{1}{(4\pi)^2} \iint_{\Lambda^*} e^{-x\zeta_1 - y\zeta_2} \frac{1}{D(\zeta_1, \zeta_2)} d\zeta_1 d\zeta_2,$$

where  $(x, y) = (\Im z_1, \Im z_2)$  and  $D(\zeta_1, \zeta_2)$  is as in Section 3 (3.2).

We also give an asymptotic expansion of the Szegő kernel of  $\Omega_f$ . The theorem below can be obtained in a fashion similar to the case of the Bergman kernel, so we omit the proof.

**THEOREM 7.1.** — *The Szegő kernel of  $\Omega_f$  has the form in some neighborhood of  $z^0$ :*

$$S(z) = \frac{\Phi^S(\tau, \varrho^{1/m})}{\varrho^{1+1/m}} + \tilde{\Phi}^S(\tau, \varrho^{1/m}) \log \varrho,$$

where  $\Phi^S \in C^\infty([0, 1] \times [0, \varepsilon))$  and  $\tilde{\Phi}^S \in C^\infty([0, 1] \times [0, \varepsilon))$  with some  $\varepsilon > 0$ .

Moreover  $\Phi^S$  is written in the form on the set  $\{\tau > \alpha \varrho^{1/2m}\}$  with some  $\alpha > 0$ : for every nonnegative integer  $\mu_0$ ,

$$\Phi^S(\tau, \varrho^{1/m}) = \sum_{\mu=0}^{\mu_0} c_\mu^S(\tau) \varrho^{\mu/m} + R_{\mu_0}^S(\tau, \varrho^{1/m}) \varrho^{\mu_0/m + 1/2m},$$

where

$$c_\mu^S(\tau) = \frac{\varphi_\mu^S(\tau)}{\tau^{2+2\mu}} + \psi_\mu^S(\tau) \log \tau,$$

for  $\varphi_\mu^S, \psi_\mu^S \in C^\infty([0, 1])$ ,  $\varphi_0^S$  is positive on  $[0, 1]$  and  $R_{\mu_0}^S$  satisfies  $|R_{\mu_0}^S(\tau, \varrho^{1/m})| \leq C_{\mu_0}^S (\tau - \alpha \varrho^{1/2m})^{-3-2\mu_0}$  for some positive constant  $C_{\mu_0}^S$ .

## Acknowledgments

I would like to thank Professors Kazuo Okamoto, Takeo Ohsawa and Katsunori Iwasaki for generosity and several valuable suggestions. I would also like to thank Professor Klas Diederich for giving me some important information. Professor Iwasaki carefully read the manuscript and supplied many corrections and Professor Mitsuhiko Kohno translated my abstract in French.

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