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Scattering theory with two L^1 spaces: application to transport equations with obstacles[(*)]

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RÉSUMÉ. — Nous donnons une théorie des opérateurs d'ondes pour les groupes positifs agissant dans deux espaces L^1 et relions leur existence à des principes d'absorption limite dans l'esprit de [2]. Une application à des équations de transport dans des domaines extérieurs [1] est donnée.

ABSTRACT. — We give a theory of wave operators for positive groups acting on two L^1 spaces by relating them to limiting absorption principles in the spirit of [2]. An application to transport equations in exterior domains [1] is given.

1. Introduction

This paper is motivated by existence of wave operators for neutron like transport equations in *exterior domains*. The existence of such operators will imply that the time evolution of the transport solution is asymptotically ($|t| \to \infty$) equivalent to that of a free solution (i.e. with no collision nor obstacle) on the whole space. This problem was already analyzed directly by Stefanov [1]. Our aim here is to tackle it by means of the abstract ideas introduced by Mokhtar-Kharroubi [2] to deal with scattering problems on L^1 spaces. Such ideas rely on limiting absorption principles and are intimately

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connected to positivity and to the L^1 structure of the underlying space (i.e. the *additivity* of the norm on the positive cone). We note that the abstract scattering theory in [2] is concerned with two groups acting on the *same* L^1 space and differing only by a (relatively) bounded perturbation of the generator. Our purpose here is to extend this formalism to groups acting on *different* L^1 spaces. This is clearly necessary in view of the comparison of *exterior Cauchy problems* to free dynamics on the whole space. Of course some compatibility conditions relating the two spaces are to be imposed. In our concrete problem the two spaces are related by the restriction operator (to the exterior domain) and the (trivial) extension operator to $\mathbb{R}^N$.

Actually in the abstract framework we give in the first part of this paper, the two L^1 spaces are connected by abstract operators under a *structure* assumption.

We point out that scattering problems with two spaces were analyzed by several authors (e.g. Kato [3], Birman ([4], [5]), Schechter [6]). However they deal with *unitary groups on Hilbert spaces* while we are concerned with *positive groups* (in the lattice sense) *on L^1 spaces*. Actually the problems are different as well as the mathematical tools to deal with them. In our analysis the *positivity* and the *additivity* of the L^1 norm on the positive cone play a key role. More precisely our framework is the following.

Let $X = L^1(\mu_1)$ and $Y = L^1(\mu_2)$ be two L^1 spaces and let $\{U_0(t) \mid t \in \mathbb{R}\}$ be a positive and bounded group on X with generator T. Let $B \in \mathcal{L}(D(T), X)$ be a positive operator and $\{U(t) \mid t \in \mathbb{R}\}$ the group generated by $T + B$. Finally let $\{W_0(t) \mid t \in \mathbb{R}\}$ be a positive and bounded group on Y with generator G.

We are concerned with the existence of the wave operators

$$\begin{cases} W_-(T+B\,,\,G) = \operatorname*{s-lim}_{t\to-\infty} U(-t)RW_0(t) \\ W_+(G\,,\,T+B) = \operatorname*{s-lim}_{t\to+\infty} W_0(-t)JU(t) \end{cases}$$

where $R \in \mathcal{L}(Y;X)$ and $J \in \mathcal{L}(X;Y)$. Our basic assumption is

$$\begin{cases} W_-(T,G) = \operatorname*{s-lim}_{t\to-\infty} U_0(-t)RW_0(t) \quad \text{exists} \\ W_+(G,T) = \operatorname*{s-lim}_{t\to+\infty} W_0(-t)JU_0(t) \quad \text{exists.} \end{cases} \tag{H}$$

Then taking advantage of some results of [2] one proves that the existence of $W_-(T+B\,,\,G)$ and $W_+(G\,,\,T+B)$ is closely related to the existence of the strong limits s-$\lim_{\lambda \to 0_+} B(\lambda - T)^{-1}$ and s-$\lim_{\lambda \to 0_-} B(\lambda - T)^{-1}$ and to the size of their spectral radii. We also give converse results which show the optimality of our assumptions.

In the second part of our paper, we consider neutron transport equations in exterior domains where T stands for the advection operator with reflecting boundary condition, and B for the collision (scattering) operator, while G is the advection operator on the whole space. Then we show how our abstract formalism applies to this problem. In particular we show how Assumption (H) may be verified using some technical results by Stefanov [1].

2. Existence of the wave operators

In this section we shall recall some results of [2] and give sufficient conditions for the existence of wave operators defined in section 1.

Let X, Y, $U_0(t)$, $W_0(t)$ and B be given as in section 1 then we have the following results.

THEOREM 1

1) If $B(0_+ - T)^{-1} =$ s-$\lim_{\lambda \to 0_+} B(\lambda - T)^{-1}$ exists and $r_\sigma[B(0_+ - T)^{-1}] < 1$, then $T + B$ generates a positive and bounded (c_0) semigroup $\{U(t) \mid t \geq 0\}$ and s-$\lim_{t \to +\infty} U_0(-t)U(t)$ exists.

2) If, in addition, s-$\lim_{t \to +\infty} W_0(-t)JU_0(t)$ exists, then $W_+(G, T + B) =$ s-$\lim_{t \to +\infty} W_0(-t)JU(t)$ exists.

Proof.— See [2, Th. 1 and Th. 4] for the first part. For the second part, we consider the decomposition

$$W_0(-t)JU(t) = W_0(-t)JU_0(t)U_0(-t)U(t)\,.$$

Since

$$\operatorname*{s-lim}_{t \to +\infty} U_0(-t)U(t) \quad \text{and} \quad \operatorname*{s-lim}_{t \to +\infty} W_0(-t)JU_0(t)$$

exist, we have the result. □

THEOREM 2

1) Let $B(0_+ - T)^{-1}$ and $B(0_- - T)^{-1}$ exist and let $r_\sigma[B(0_+ - T)^{-}] < 1$; then $T + B$ generates a bounded (c_0) semigroup $\{U(t) \mid t \geq 0\}$ and $W_-(T + B\,, T) =$ s-$\lim_{t \to -\infty} U(-t)U_0(t)$ exists.

2) If, in addition, s-$\lim_{t\to-\infty} U_0(-t)RW_0(t)$ *exists, then* $W_-(T+B\,,\,G)$ $=$ s-$\lim_{t\to-\infty} U(-t)RW_0(t)$ *exists.*

Proof.— See [2, Th. 1 and Th. 2] for the first part. For the second part, we consider the decomposition

$$U(-t)RW_0(t) = U(-t)U_0(t)U_0(-t)RW_0(t)\,.$$

Since s-$\lim_{t\to-\infty} U_0(-t)RW_0(t)$ and s-$\lim_{t\to-\infty} U(-t)U_0(t)$ exist, the result follows. □

3. Converse results

We prove converse theorems to show that our assumptions are necessary in some sense.

THEOREM 3.— *Let* $B\big(0_\pm - T\big)^{-1}$ *exist and* $r_\sigma\big[B\big(0_\pm - T\big)^{-1}\big] < 1$*; if*

$$W_+(G\,,\,T+B) = \operatorname*{s-lim}_{t\to+\infty} W_0(-t)JU(t)$$

exists then $W_+(G,T) =$ s-$\lim_{t\to+\infty} W_0(-t)JU_0(t)$ *exists.*

Proof.— In view of [2, Th. 1)], $T+B$ generates a (c_0) group $U(\,\cdot\,)$ and then by [2, Th. 3], the strong limit s-$\lim_{t\to+\infty} U(-t)U_0(t)$ exists and the result follows from the decomposition

$$W_0(-t)JU_0(t) = W_0(-t)JU(t)U(-t)U_0(t)\,. \ \square$$

In the following, another consequence of the existence of the wave operators is given.

Let $W_0(\,\cdot\,)$, $U_0(\,\cdot\,)$ J, R and B be as in section 1 such that $T+B$ generates a (c_0) group (not necessarily bounded).

We introduce X^* the dual space of X and R^* the dual of R defined on X^*, then we have the following theorem.

THEOREM 4

1) Let there exist $\alpha_1 > 0$ such that

$$\|Jx\| \geq \alpha_1 \|x\|, \quad \forall\, x \in X,$$

and let $W_+(G, T+B)$ exist; then $\{U(t) \mid t \geq 0\}$ is bounded, $B(0_+ - T)^{-1}$ exists and $r_\sigma[B(0_+ - T)^{-1}] < 1$.

2) Let there exist $\alpha_2 > 0$ such that

$$\|R^* x\| \geq \alpha_2 \|x\|, \quad \forall\, x \in X^*,$$

and let $W_-(T+B, G)$ exist; then $\{U(t) \mid t \geq 0\}$ is bounded, $B(0_+ - T)^{-1}$ exists and $r_\sigma[B(0_+ - T)^{-1}] < 1$.

Proof

1) Suppose that $W_+(G, T+B)$ exists then, by the uniform boundedness theorem, there exists $M > 0$ such that

$$\|W_0(-t) J U(t)\| \leq M, \quad \forall\, t \geq 0,$$

and

$$\begin{aligned}\|U(t)x\| &\leq \frac{1}{\alpha_1} \|JU(t)x\| = \frac{1}{\alpha_1} \|W_0(t) W_0(-t) J U(t) x\| \\ &\leq \frac{M}{\alpha_1} \left(\sup_{t \geq 0} \|W_0(t)\| \right) \|x\|, \quad \forall\, t \geq 0,\end{aligned}$$

i.e. $U(\cdot)$ is bounded.

Finally using [2, Th. 6 and Corol. 2] yields the result.

2) Suppose that $W_-(T+B, G)$ exists then, by the uniform boundedness theorem, there exists $M > 0$ such that

$$\|U(-t) R W_0(t)\| \leq M, \quad \forall\, t \leq 0,$$

or equivalently $\|W_0^*(t) R^* U^*(-t)\| \leq M$ and

$$\begin{aligned}\|U^*(-t)x\| &\leq \frac{1}{\alpha_2} \|R^* U^*(-t) x\| = \frac{1}{\alpha_2} \|W_0^*(-t) W_0^*(t) R^* U^*(-t) x\| \\ &\leq \frac{M}{\alpha_2} \left(\sup_{t \leq 0} \|W_0^*(-t)\| \right) \|x\|, \quad \forall\, t \leq 0,\end{aligned}$$

i.e. $\{U^*(t) \mid t \geq 0\}$ is bounded, consequently $\{U(t) \mid t \geq 0\}$ is bounded and we argue as in the first part. □

4. Application to transport equations in exterior domains

We consider a problem studied by Stefanov [1]. First of all, we introduce the relevant operators.

Let Ω be the exterior of a compact obstacle $\Theta \subset \mathbb{R}^N$ with twice continuously differentiable boundary $\partial\Omega$ and let $V = \mathbb{R}^N$. We define the operator T_0 by

$$\begin{cases} T_0\psi(x,v) = -v\dfrac{\partial\psi}{\partial x}(x,v) & \text{for } (x,v) \in \Omega \times V \\ \psi(x,v) = \psi(x,\omega) & \text{for } x \in \partial\Omega \end{cases}$$

where ω denotes the velocity obtained by reflecting v at the point x with the explicit expression

$$\omega = v - 2\langle v, n\rangle n$$

where n denotes the inner normal to $\partial\Omega$ at the point $x \in \partial\Omega$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the scalar product in $\mathbb{R}^N$.

It is known (see [1]) that, with a suitable domain $D(T_0)$, T_0 generates the following (c_0) group on $L^1(\Omega \times V)$

$$S_0(t)\psi(x,v) = \psi\big(\tau_{-t}(x,v)\big)\,, \quad t \in \mathbb{R}\,,$$

where

$$\tau_t(x,v) = \big(\tau_t^x(x,v)\,,\,\tau_t^v(x,v)\big)$$

denotes the flow in the space of a particle starting from the point x with velocity v. Let $\sigma_a(\,\cdot\,,\,\cdot\,) \in L_+^\infty(\Omega \times V)$ be the collision frequency and let T be the operator defined on $D(T) = D(T_0)$ by

$$\begin{cases} T\psi(x,v) = -v\dfrac{\partial\psi}{\partial x}(x,v) - \sigma_a(x,v)\psi(x,v) & \text{for } (x,v) \in \Omega \times V \\ \psi(x,v) = \psi(x,\omega) & \text{for } x \in \partial\Omega \end{cases}$$

then (see [1]) T generates the following (c_0) group:

$$U_0(t)\psi(x,v) = e^{-\int_0^t \sigma_a(\tau_{-s}(x,v))\,\mathrm{d}s}\psi\big(\tau_{-t}(x,v)\big)\,, \quad t \in \mathbb{R}\,.$$

The collision operator is defined as

$$B : \psi \in L^1(\Omega \times V) \longrightarrow \int_V k(x,v,v')\psi(x,v')\,\mathrm{d}v'$$

where $k(x,v,v') \geq 0$ a.e.

On the other hand, let G_0 be the advection operator on $L^1(\mathbb{R}^N \times V)$

$$\begin{cases} G_0\psi(x,v) = -v\dfrac{\partial\psi}{\partial x}(x,v) \\ D(G_0) = \left\{\psi \in L^1(\mathbb{R}^N \times V) \;\middle|\; v\dfrac{\partial\psi}{\partial x} \in L^1(\mathbb{R}^N \times V)\right\} \end{cases}$$

then G_0 generates the (c_0) group (of isometries) [7]

$$W_0(t)\psi(x,v) = \psi(x - tv\,,\, v)\,, \quad t \in \mathbb{R}\,.$$

Let $\sigma_p(\,\cdot\,,\,\cdot\,) = \int k(\,\cdot\,, v, \,\cdot\,)\,\mathrm{d}v$. We introduce some assumptions on Θ, σ_a and σ_p.

DEFINITION .— *We say that the obstacle Θ is non-trapping if for every compact $K \subset \overline{\Omega}$, there exists a constant $\ell(K)$ such that*

$$\int |v| \chi_K\big(\tau_t^x(x,v)\big)\,\mathrm{d}t < \ell(K)\,, \quad \forall\,(x,v) \in \Omega \times V$$

(where χ_K is the caracteristic function of K).

Remark 1.— Θ is non-trapping means that for every compact $K \subset \overline{\Omega}$, there are no trajectories with lengths greater than $\ell(K)$ having their initial and final points in K.

We introduce the assumptions

$$\operatorname*{ess\,sup}_{(x,v)} \int_0^\infty \sigma_a(x - tv\,,\, v)\,\mathrm{d}t = F_-(\sigma_a) < \infty \tag{F0}$$

$$\operatorname*{ess\,sup}_{(x,v)} \int_0^\infty \sigma_a\big(\tau_t(x,v)\,\mathrm{d}t = F_+(\sigma_a) < \infty \tag{F1}$$

$$\operatorname*{ess\,sup}_{(x,v)} \int_0^\infty \sigma_p\big(\tau_t(x,v)\,\mathrm{d}t = F_+(\sigma_p) < \infty \tag{F2}$$

$$\operatorname*{ess\,sup}_{(x,v)} \int_0^\infty \sigma_p\big(\tau_{-t}(x,v)\,\mathrm{d}t = F_-(\sigma_p) < \infty\,. \tag{F3}$$

Remark 2.— Since $\tau_t(x,v)$ is measure preserving, $S_0(\,\cdot\,)$ is a (c_0) group of isometries. On the other hand, (F1) is necessary and sufficient for

$\{U_0(t) \mid t \leq 0\}$ to be bounded, while $\{U_0(t) \mid t \geq 0\}$ is always bounded when σ_a is positive.

Let $\widetilde{\mathcal{C}}_0^\infty(\Omega \times V)$ (resp. $\widetilde{\mathcal{C}}_0^\infty(\mathbb{R}^N \times V)$) denote the subspace of $\mathcal{C}_0^\infty$-function f on $\Omega \times V$ (resp. $\mathbb{R}^N \times V$) such that $f(x, v) = 0$ for $|v| < v_0$, where v_0 is a constant depending on f, then $\widetilde{\mathcal{C}}_0^\infty(\Omega \times V)$ (resp. $\widetilde{\mathcal{C}}_0^\infty(\mathbb{R}^N \times V)$) is dense in $L^1(\Omega \times V)$ (resp. $L^1(\mathbb{R}^L \times V)$) [7] and we have the following results.

LEMMA 1.— *For any $K \subset \mathbb{R}^N$, we denote $\|f\|_K = \iint_K |f(x,v)| \, dx \, dv$. Then*

a1) $\int_{-\infty}^0 \|W_0(t)f\|_K \, dt < \infty$ *for all f in $\widetilde{\mathcal{C}}_0^\infty(\mathbb{R}^N \times V)$ and for all compact $K \subset \mathbb{R}^N$;*

a2) $\|W_0(t)f\|_K \to 0$ *as $t \to -\infty$ for all f in $L^1(\mathbb{R}^N \times V)$ and for all compact $K \subset \mathbb{R}^N$;*

b1) let Θ be non-trapping, then

$$\int_0^{+\infty} \|U_0(t)f\|_K \, dt \leq \ell(K) \| |v|^{-1} f \|$$

for all f in $\widetilde{\mathcal{C}}_0^\infty(\Omega \times V)$ and for all compact $K \subset \overline{\Omega}$;

b2) Let Θ be non-trapping, then

$$\|U_0(t)f\|_K \to 0 \quad \text{as } t \to \infty$$

for all f in $L^1(\Omega \times V)$ and for all compact $K \subset \overline{\Omega}$.

Proof.— Let $f \in \widetilde{\mathcal{C}}_0^\infty(\mathbb{R}^N \times V)$ and let $K \subset \mathbb{R}^N$ be compact; then for $t \to -\infty$,

$$\chi_{K \times V} \cdot W_0(t)f = 0$$

so that

$$\int_{-\infty}^0 \|W_0(t)f\|_K \, dt < \infty .$$

On the other hand, $\|\chi_{K \times V} \cdot W_0(t)f\| \leq 1$ implies

$$\chi_{K \times V} \cdot W_0(t)f \longrightarrow 0 \quad \text{as } t \to -\infty , \quad \forall\, f \in L^1(\mathbb{R}^N \times V)$$

by density arguments. This ends the proof of a1) and a2).

b1) and b2) were proved in [1] when $\sigma_a = 0$, we use the same arguments. Let $K \subset \overline{\Omega}$ be compact and let $f \in \widetilde{\mathcal{C}}_0^\infty(\Omega \times V)$.

Denote by $\chi(x)$ the caracteristic function of the set K and let $f \in \widetilde{\mathcal{C}}_0^\infty(\Omega \times V)$, then

$$\begin{aligned}
&\int_0^\infty \left\|U_0(t)f\right\|_K \mathrm{d}t = \\
&\quad = \int_0^{+\infty} \iint \chi(x) \left| e^{-\int_0^t \sigma_a(\tau_{-s}(x,v))\,\mathrm{d}s} f\big(\tau_{-t}(x,v)\big) \right| \mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\
&\quad \leq \int_0^{+\infty} \iint \chi(x) \left| f\big(\tau_{-t}(x,v)\big) \right| \mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t
\end{aligned}$$

since $\tau_t(x,v)$ is measure preserving, using a change of variables $(y,w) = \tau_{-t}(x,v)$, we get

$$\begin{aligned}
\int_0^\infty \left\|U_0(t)f\right\|_K \mathrm{d}t &\leq \int_0^{+\infty} \iint \chi\big(\tau_t^x(y,w)\big) \left|f(y,w)\right| \mathrm{d}y\,\mathrm{d}w\,\mathrm{d}t = \\
&= \iint \left(\int_0^{+\infty} \chi\big(\tau_t^x(y,w)\big) |w|\,\mathrm{d}t \right) |w|^{-1} \left|f(y,w)\right| \mathrm{d}y\,\mathrm{d}w \\
&\leq \ell(K) \left\| |v|^{-1} f \right\| ;
\end{aligned}$$

this proves b1).

Finally, since the function $t \longrightarrow \left\|U_0(t)\right\|_K$ is uniformly continuous, we derive b2) for $f \in \widetilde{\mathcal{C}}_0^\infty(\Omega \times V)$ and we end the proof by density arguments. □

We introduce the inclusion operator $J : L^1(\Omega \times V) \longrightarrow L^1(\mathbb{R}^N \times V)$ and the restriction map $R : L^1(\mathbb{R}^N \times V) \longrightarrow L^1(\Omega \times V)$, then we have the following result relating $U_0(t)$ to $W_0(t)$.

THEOREM 5

1) Let (F0) hold, then the wave operator

$$W_-(T, G_0) = \operatorname*{s-lim}_{t \to -\infty} U_0(-t) R W_0(t) \quad \textit{exists}.$$

2) Let Θ be non-trapping and let (F1) hold, then the wave operator

$$W_+(G_0, T) = \operatorname*{s-lim}_{t \to +\infty} W_0(-t) J U_0(t) \quad \textit{exists}.$$

Proof.— Let $d > 0$ be such that $\Theta \subset \{x \in \mathbb{R}^N \mid |x| < d\}$ and let $\varphi(x)$ be a smooth function such that

$$\varphi(x) = \begin{cases} 0 & \text{for } |x| \leq d \\ 1 & \text{for } |x| \geq 2d \end{cases}$$

and denote by φ the multiplication operator by the function $\varphi(x)$. Then for $f \in L^1(\mathbb{R}^N \times V)$, we have

$$U_0(-t)RW_0(t)f = U_0(-t)\varphi RW_0(t)f + U_0(-t)(I - \varphi)RW_0(t)f\,.$$

By Lemma 1 a2),

$$\left\|U_0(-t)(I - \varphi)RW_0(t)f\right\| \longrightarrow 0 \quad \text{as } t \to -\infty\,.$$

On the other hand,

$$\begin{aligned} &U_0(-t)\varphi RW_0(t)f = \\ &= \varphi Rf + \int_0^t \left(U_0(-s)v\frac{\partial\varphi}{\partial x}\, RW_0(s)f + U_0(-s)\sigma_a\varphi RW_0(s)f\right)\, \mathrm{d}s\ ; \end{aligned}$$

let $f \in \widetilde{C}_0^\infty(\mathbb{R}^N \times V)$ then, by using Lemma 1 a1),

$$\int_{-\infty}^0 \left\|U_0(-s)v\frac{\partial\varphi}{\partial x}\, RW_0(s)f\right\|\, \mathrm{d}s < \infty\,.$$

By changing variables and using assumption (F0), we get

$$\int_{-\infty}^0 \left\|U_0(-s)\sigma_a\varphi RW_0(s)f\right\|\, \mathrm{d}s < \infty$$

so that the wave operator $W_-(T, G_0) = \text{s-lim}_{t\to-\infty} U_0(-t)RW_0(t)$ exists by density arguments. For 2), the same arguments give for $f \in L^1(\Omega \times V)$

$$W_0(-t)JU_0(t)f = W_0(-t)\varphi JU_0(t)f + W_0(-t)(I - \varphi)JU_0(t)f\,.$$

By Lemma 1 b2),

$$\left\|W_0(-t)(I - \varphi)U_0(t)f\right\| \longrightarrow 0 \quad \text{as } t \to +\infty\,.$$

On the other hand,

$$\begin{aligned} &W_0(-t)\varphi JU_0(t)f = \\ &= \varphi Jf + \int_0^t \left(W_0(-s)v\frac{\partial\varphi}{\partial x}\, JU_0(s)f + W_0(-s)\varphi J\sigma_a U_0(s)f\right)\, \mathrm{d}s\,. \end{aligned}$$

Let $f \in \widetilde{C}_0^\infty(\Omega \times V)$, then by using Lemma 1 b1),

$$\int_0^\infty \left\| W_0(-s) v \frac{\partial \varphi}{\partial x} \, J U_0(s) f \right\| \, \mathrm{d}s < \infty .$$

By changing variables and using assumption (F1), we get

$$\int_0^\infty \left\| W_0(-s) \varphi J \sigma_a U_0(s) f \right\| \, \mathrm{d}s < \infty$$

so that the wave operator $W_+(G_0, T) = \text{s-lim}_{t \to -\infty} U_0(-t) R W_0(t)$ exists by density arguments. □

Finally we consider the perturbation B and begin with a result which shows that the above assumptions are sufficient for the existence of the absorption limits.

LEMMA 2

1) Let (F2) hold, then $B(0_+ - T)^{-1} = \text{s-lim}_{\lambda \to 0_+} B(\lambda - T)^{-1}$ *exists and* $\|B(0_+ - T)^{-1}\| \le F_+(\sigma_p)$.

2) Let (F1) and (F3) hold, then $B(0_- - T)^{-1} = \text{s-lim}_{\lambda \to 0_-} B(\lambda - T)^{-1}$ *exists and* $\|B(0_- - T)^{-1}\| \le F_-(\sigma_p) \, e^{F_+(\sigma_a)}$.

Proof

1) Let $f \in L^1(\Omega \times V)$, we have

$$B(\lambda - T)^{-1} f =$$

$$= \int_V k(x, v, v') \, \mathrm{d}v' \int_0^\infty e^{-\lambda t} e^{-\int_0^t \sigma_a(\tau_{-s}(x, v')) \, \mathrm{d}s} f(\tau_{-t}(x, v')) \, \mathrm{d}t;$$

the monotone convergence theorem yields for $f \ge 0$ and $\lambda \to 0_+$

$$\lim_{\lambda \to 0_+} B(\lambda - T)^{-1} f =$$

$$= \int_V k(x, v, v') \, \mathrm{d}v' \int_0^\infty e^{-\int_0^t \sigma_a(\tau_{-s}(x, v')) \, \mathrm{d}s} f(\tau_{-t}(x, v')) \, \mathrm{d}t.$$

On the other hand

$$\int_{\Omega \times V} \mathrm{d}x \, \mathrm{d}v \int_V k(x, v, v') \, \mathrm{d}v' \int_0^\infty e^{-\int_0^t \sigma_a(\tau_{-s}(x, v')) \, \mathrm{d}s} f(\tau_{-t}(x, v')) \, \mathrm{d}t =$$

$$= \int_V \mathrm{d}v' \int_0^\infty \mathrm{d}t \int_\Omega \sigma_p(x, v') \, e^{-\int_0^t \sigma_a(\tau_{-s}(x, v')) \, \mathrm{d}s} f(\tau_{-t}(x, v')) \, \mathrm{d}x \, ;$$

since $\sigma_a(\cdot,\cdot) > 0$ and $\tau_t(x,v')$ is measure preserving on $\Omega \times V$ (see [1, Th. 2.1]) and using Fubini theorem, we get

$$\iint B(0_+ - T)^{-1} f(x,v) \le \int_{\Omega\times V} \left(\int_0^\infty \sigma_p\big(\tau_t(x,v')\big)\, \mathrm{d}t \right) f(x,v')\, \mathrm{d}x\, \mathrm{d}v' .$$

Now let (F2) hold then

$$\left\| B(0_+ - T)^{-1} f \right\| \le F_+(\sigma_p) \|f\| ,$$

i.e. $B(0_+ - T)^{-1}$ exists and $\left\| B(0_+ - T)^{-1} \right\| \le F_+(\sigma_p)$.

2) For $f \in L^1(\Omega \times V)$ and $\lambda < 0$, we have

$$(\lambda - T)^{-1} f(x,v) = - \int_0^\infty e^{\lambda t} e^{\int_0^t \sigma_a(\tau_s(x,v'))\, \mathrm{d}s} f\big(\tau_t(x,v')\big)\, \mathrm{d}t$$

then $-B(\lambda - T)^{-1}$ is a positive operator for $\lambda < 0$, as in 1), $B(0_- - T)^{-1}$ exists if (F3) holds and $\left\| B(0_- - T)^{-1} \right\| \le F_-(\sigma_p)\, e^{F_+(\sigma_a)}$. □

The application of Theorems 1 and 2 gives the following results.

Theorem 6

1) Let (F0)-(F3) hold and let $F_+(\sigma_p) < 1$ then the wave operator

$$W_-(T + B\,,\, G_0) = \operatorname*{s-lim}_{t\to -\infty} U(-t) R W_0(t) \quad \textit{exists}.$$

2) Let Θ be non-trapping and let (F1) and (F2) hold and let $F_+(\sigma_p) < 1$, then the wave operator

$$W_+(G_0\,,\, T + B) = \operatorname*{s-lim}_{t\to +\infty} W_0(-t) J U(t) \quad \textit{exists}.$$

Proof

1) In view of Theorem 5 1), $\text{s-lim}_{t\to-\infty} U_0(-t) R W_0(t)$ exists and by Lemma 2 2), $B(0_+ - T)^{-1}$ and $B(0_- - T)^{-1}$ exist and $\left\| B(0_+ - T)^{-1} \right\| \le F_+(\sigma_p) < 1$, then the result follows from Theorem 2.

2) In view of Theorem 5 2), $\text{s-lim}_{t\to-\infty} W_0(-t) J U_0(t)$ exists and by Lemma 2 1), $B(0_+ - T)^{-1}$ exists and $\left\| B(0_+ - T)^{-1} \right\| \le F_+(\sigma_p) < 1$, then the result follows from Theorem 1. □

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