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Entropy of topological directions(*)

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RÉSUMÉ. — Une entropie peut être associée à chaque chemin du graphe de Cayley d'un groupe engendré par un nombre fini de transformations lipschitziennes. La notion de direction topologique introduite dans cet article nous permet d'étendre la définition de l'entropie au cas "directionnel" et d'étudier le comportement ergodique d'un groupe le long d'une direction (et pas seulement de façon globale). L'entropie de la direction topologique est introduite au moyen d'ensembles (n, ε)-séparés et, de manière équivalente, au moyen de recouvrements ouverts finis. Nous démontrons une relation entre l'entropie classique et l'entropie d'une direction topologique.

ABSTRACT. — An entropy can be attached to every path in the Cayley graph of a finitely generated group of Lipschitz transformations. The notion of a topological direction, introduced in this paper, allows us to carry the definition of entropy over the case of a direction and to study the ergodic behaviour of a group along the direction (not only globally). The entropy of the topological direction is introduced by (n, ε)-separated sets and, equivalently, by finite open coverings. A relationship between classical entropy and entropy of a topological direction is shown.

1. Introduction

The notion of an end of a group was introduced in the 40's by H. Freudenthal [Fre] and studied again by J. Stallings [Sta] and D. Cohen [Coh1], [Coh2] in the 70's. D. Cohen showed a new approach to the theory of ends of a group which is purely algebraic in contradistinction to the combinatorical-topological approach of H. Freudenthal and J. Stallings.

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The notion of a direction of a group, introduced in our paper, is a suitable tool allowing us to examine the action of the group on a topological space in a more subtle way than by applying the theory of ends.

Using a properly defined equivalence relation, we obtain that the space of ends of a group splits into equivalence classes-directions of the group.

E. Ghys, R. Langevin and P. Walczak [GLW] introduced the notion of the topological entropy for a finitely generated pseudogroup of local homeomorphisms of a metric space. They defined the topological entropy of a foliation by the topological entropy of the holonomy pseudogroup of this foliation. Making use the same method, one is able to introduce a definition of the topological entropy of a topological direction by (n,ε)-separated sets, (n,ε)-spanning sets and by a finite covering of a space. There is an equivalence between those approaches. The examples included in this paper emphasize the relations between the classical entropy of a homeomorphism, the entropy of a finitely generated pseudogroup, defined in [GLW], and the entropy of the topological direction. We show that the entropy can be attached to every path in the Cayley graph of a finitely generated group.

Let G be a finitely generated group of homeomorphisms acting on a topological space X. We assume that G is generated by a finite set G_1, $G_1^{-1} = G_1$ and $G_0 = \{\mathrm{id}_X\}$. Let $G_m := \{g \in G : g = g_1 \ldots g_m\,,\ g_i \in G_1\}$ and let $\mathrm{Cay}(G, G_1)$ denote the Cayley graph of G generated by G_1.

DEFINITION 1.1.— *Recall that the word distance $d^*(f,g)$ between two elements of G is defined as the smallest m for which $fg^{-1} \in G_m$. Let S be the set of the paths in the Cayley graph* $\mathrm{Cay}(G, G_1)$. *More precisely, S consists of infinite sequences $(f_n)_{n\in\mathbb{N}}$ such that $f_n \in G_{n+1} \backslash G_n$ and $f_{(n+1)}f^{-1} \in G_1$, for each $n \in \mathbb{N}$. We call two paths (f_n), (g_n) a $\sim$-equivalent if the sequence $s(n) := d^*(f_n, g_n)$ is bounded. An algebraic direction is an equivalence class of the relation $\sim$.*

2. Entropy of topological directions

Let (X,d) be a compact metric space with a metric d, G a finitely generated group of Lipschitz mappings defined on X, G_1 a finite set of generators of the group. We assume that $\mathrm{id}_X \in G_1$, $G_1^{-1} = G_1$.

DEFINITION 2.1.— *Let S be, as before, the set of the paths of the Cayley graph of G generated by a finite set G_1. We call the paths (f_n) and (g_n) of S $\approxeq$-equivalent if they are $\sim$-equivalent and there exists a positive constant c such that for all $n \in N$ and for all x, $y \in X$ we have*

$$\frac{1}{c} \leq \frac{\max\Big\{d\big(f_i(x), f_i(y)\big) \mid 0 \leq i \leq n\Big\}}{\max\Big\{d\big(g_i(x), g_i(y)\big) \mid 0 \leq i \leq n\Big\}} \leq c\,.$$

A topological direction is an equivalence class of paths in this relation.

Remark 2.2.— If X is a manifold and G a group with C^1-class generators, acting on X, then each algebraic direction is a topological direction (comp. Proposition 2.4).

DEFINITION 2.3.— *Let*

$$d_n^{(f_p)}(x, y) := \max\Big\{d\big(f_i(x), f_i(y)\big) \mid 0 \leq i \leq n\Big\}\,.$$

PROPOSITION 2.4.— *Let M be a compact Riemannian manifold and G a finitely generated group of Lipschitz transformations defined on M. Denote by G_1 a set of generators of G.*

Then each algebraic direction $K = \big[(f_p)\big]_\sim$ generated by elements of the set G_1 determines exactly one topological direction.

Proof.— At first, we notice that, for any $a_i > 0$, $b_i > 0$, $i = 1, 2, \ldots, n$, we have the inequality

$$\frac{\max\{a_i \mid 1 \leq i \leq n\}}{\max\{b_i \mid 1 \leq i \leq n\}} \leq \max\left\{\frac{a_i}{b_i} \;\Big|\; 1 \leq i \leq n\right\}\,.$$

The sequences (f_p) and (g_p) represent the same algebraic direction. So, for any $i \in \mathbb{N}$ there exist generators $h_{i,1}, \ldots, h_{i,2}, \ldots, h_{i,s(i)} \in G_1$, such that

$$f_i = h_{i,1}\, h_{i,2} \cdots h_{i,s(i)} g_i$$

and the sequence $s(i)$ is bounded

$$\frac{d_n^{(f_p)}(x,y)}{d_n^{(g_p)}(x,y)} = \frac{\max\limits_{1\le i\le n}\left\{d\big(f_i(x),f_i(y)\big)\right\}}{\max\limits_{1\le i\le n}\left\{d\big(g_i(x),g_i(y)\big)\right\}} \le \max_{1\le i\le n}\left\{\frac{d\big(f_i(x),f_i(y)\big)}{d\big(g_i(x),g_i(y)\big)}\right\} =$$

$$= \max_{1\le i\le n}\left\{\frac{d\big(h_{i,1}\,h_{i,2}\,\cdots\,h_{i,s(i)}g_i(x)\,,\,h_{i,1}\,h_{i,2}\,\cdots\,h_{i,s(i)}g_i(y)\big)}{d\big(g_i(x),g_i(y)\big)}\right\}$$

$$\le \max_{\substack{1\le j\le \max\{s(n)\}\\ \in\mathbb{N}}}\left\{\lambda_{h_1}\,\lambda_{h_2}\,\cdots\,\lambda_{h_j}\mid h_1,\,h_2,\,\ldots,\,h_j\in G_1\right\} =$$

$$= c' < \infty$$

where λ_{h_i} is the Lipschitz constant for the Lipschitz mapping $h_i \in G_1$.

In a similar way we obtain that there exists a constant $0 < c'' < \infty$ such that

$$\frac{d_n^{(g_p)}(x,y)}{d_n^{(f_p)}(x,y)} \le c''\,.$$

Putting $c = \max\{c', c''\}$, we obtain the required inequality. □

Consider a topological direction represented by a sequence (f_p).

DEFINITION 2.5.— *Let $n\in\mathbb{N}$ and $\varepsilon>0$. A subset A of the space X is called $\big(n,\varepsilon,(f_p)\big)$-separated if, for any x, $y\in A$ with $x\neq y$, we have*

$$d_n^{(f_p)}(x,y) > \varepsilon\,.$$

Let $s\big(n,\varepsilon,(f_p)\big)$ be the largest cardinality of any $\big(n,\varepsilon,(f_p)\big)$-separated subset of X.

LEMMA 2.6.— *If $0<\varepsilon_1<\varepsilon_2$, then $s\big(n,\varepsilon_1,(f_p)\big) \ge s\big(n,\varepsilon_2,(f_p)\big)$.*

Proof.— Each $\big(n,\varepsilon_2,(f_p)\big)$-separated subset of X is $\big(n,\varepsilon_1,(f_p)\big)$-separated in X. □

The above lemma implies the correctness of the definition below.

DEFINITION 2.7.— *Let*

$$s\big((f_p)\big) := \lim_{\varepsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{n}\log s\big(n,\varepsilon,(f_p)\big)\,.$$

LEMMA 2.8.— *If sequences (f_p) and (g_p) represent the same topological direction, then $s((f_p)) = s((g_p))$.*

Proof.— If $[(f_p)]_{\approx} = [(g_p)]_{\approx}$, there exists a constant $c > 0$, such that for all $n \in \mathbb{N}$ and $x, y \in X$ we get

$$\frac{1}{c} \leq \frac{d_n^{(f_p)}(x,y)}{d_n^{(g_p)}(x,y)} \leq c\,.$$

Let A be an $(n, \varepsilon, (f_p))$-separated subset of X with the largest cardinality. Then, for $x, y \in A$ with $x \neq y$, we obtain

$$\varepsilon < d_n^{(f_p)}(x,y) \leq c \cdot d_n^{(g_p)}(x,y)$$

which yields that the set A is $(n, \varepsilon/c, (g_p))$-separated in X and that the inequality

$$s\Big(n, \frac{\varepsilon}{c}, (g_p)\Big) \geq s(n, \varepsilon, (f_p))\,.$$

holds. So

$$s((g_p)) \geq s((f_p))\,.$$

The inequality $s((g_p)) \leq s((f_p))$ is obtained in a similar way. □

DEFINITION 2.9.— *The number*

$$h\Big([(f_p)]_{\approx}\Big) := s((f_p)) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, (f_p))\,.$$

is called the entropy of the topological direction $[(f_p)]_{\approx}$ in a finitely generated group G with respect to (abrev.: w.r.t.) the generating system G_1.

We can introduce the second (equivalent) definition of the entropy of the topological direction.

DEFINITION 2.10.— *Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. We call a subset B of the space X an $(n, \varepsilon, (f_p))$-spanning set if for any $x \in X$ there exists $b \in B$ such that*

$$d_n^{(f_p)}(x, b) \leq \varepsilon\,.$$

Let $r(n, \varepsilon, (f_p))$ denote the minimal cardinality of an $(n, \varepsilon, (f_p))$-spanning subset of the space X.

DEFINITION 2.11.— *Let*

$$r\big((f_p)\big) := \lim_{\varepsilon \to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log r\big(n, \varepsilon, (f_p)\big) .$$

LEMMA 2.12.— *If a sequence (f_p) determines a topological direction in a finitely generated group G, then $r\big((f_p)\big) = s\big((f_p)\big)$.*

Proof.— Let E be an $\big(n, \varepsilon, (f_p)\big)$-separated set with the largest cardinality. Then E is an $\big(n, \varepsilon, (f_p)\big)$-spanning subset of X. So $s\big(n, \varepsilon, (f_p)\big) \geq r\big(n, \varepsilon, (f_p)\big)$.

Consider an $\big(n, \varepsilon, (f_p)\big)$-separated set A in X and an $\big(n, \varepsilon/2, (f_p)\big)$-spanning set B in X. Fix a mapping $F : A \to B$ such that for any $x \in A$, a point $F(x) \in B$ satisfies the condition

$$d_n^{(f_p)}\big(x, F(x)\big) \leq \frac{\varepsilon}{2} .$$

The mapping $F : A \to B$ is one-to-one. The cardinality of the set A is not greater than the cardinality of the set B, therefore

$$s\big(n, \varepsilon, (f_p)\big) \leq r\left(n, \frac{\varepsilon}{2}, (f_p)\right) .$$

Passing to the limit, we get

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log r\big(n, \varepsilon, (f_p)\big) &\leq \lim_{\varepsilon \to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log s\big(n, \varepsilon, (f_p)\big) \\ &\leq \lim_{\varepsilon \to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log r\left(n, \frac{\varepsilon}{2}, (f_p)\right) \end{aligned}$$

so

$$r\big((f_p)\big) \leq s\big((f_p)\big) \leq r\big((f_p)\big) . \square$$

Example 2.13.— Let G be a finitely generated isometry group acting on a metric space X. Then the entropy of any topological direction in the group G with respect to G_1 is equal to zero.

Indeed, for any $\big(n, \varepsilon, (f_p)\big)$-separated set A with the largest cardinality, we have that for all x, $y \in A$:

$$d_n^{(f_p)}(x, y) = \max_{0 \leq i \leq n} \Big\{ d\big(f_i(x), f_i(y)\big) \Big\} = d(x, y) > \varepsilon$$

so

$$s\big(n,\varepsilon,(f_p)\big)=s\big(0,\varepsilon,(f_p)\big)=\text{constant}$$

and that is why the entropy of this direction is equal to zero.

Remark 2.14.— Let G_1 and H_1 be two sets of generators of G. Let

$$k:=\max_{g\in G_1}\big\{\min\{\ell\in N\mid g=h_{i_1}\cdots h_{i_\ell},\,h_{i_j}\in H_1\}\big\}\,.$$

Fix a sequence (g_p) representing a topological direction in (G,G_1) Then, there exists a sequence (h_m) representing a topological direction in (G,H_1) such that for every $p\in\mathbb{N}$ we get $g_p=h_{m_p}$

Moreover, if $x,\ y\in X$ are $\big(n,\varepsilon,(g_p)\big)$-separated in (G,G_1) then x and y are $\big(k\cdot n,\varepsilon,(h_m)\big)$-separated in (G,H_1). Denoting the entropy of (g_p) with respect to G_1 by $h\big((g_p),G_1\big)$ we get that for certain positive number s:

$$s^{-1}\cdot h\big((g_p),G_1\big)\le h\big((h_m),H_1\big)\le s\cdot h\big((g_p),G_1\big)\,.$$

Thus

$$h\big((h_m),H_1\big)=0\quad\text{iff}\quad h\big((g_p),G_1\big)=0\,.$$

We can introduce an entropy of a topological direction by using only the family of open coverings of the space.

DEFINITION 2.15.— *Let α and β be open coverings of a space X. Then $\alpha\vee\beta$ is an open covering of X which consists of all sets $A\cap B$ where $A\in\alpha$, $B\in\beta$.*

In a similar way we can define a covering $\bigvee_{i=1}^{n}\alpha_i$ for a finite family of coverings of X:

$$\bigvee_{i=1}^{n}\alpha_i:=\bigvee_{i=1}^{n-1}\alpha_i\vee\alpha_n\,.$$

We say that a covering β is subtler than a covering α if any element of β is a subset of a certain element of α. We denote this by $\alpha\prec\beta$.

If α is an open covering of X and $f:X\to X$ is a continuous mapping, then $f^{-1}\alpha$ is an open covering of X which consists of all the sets of the form $f^{-1}A$, $A\in\alpha$.

DEFINITION 2.16.— *Denote by P the family of all open coverings of the space X. For any $U \in P$ we put*

$$N(U) := \min\{\sharp V \mid V \text{ is a finite subcovering of } U\}$$

$$h^*\big((f_p), U\big) := \limsup_{n\to\infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} f_i^{-1} U\right)$$

$$h^*\big((f_p)\big) := \sup_{U \in P} h^*\big((f_p), U\big)\,.$$

PROPOSITION 2.17.— *Let (α_n) be a sequence of open coverings of a compact metric space X. Assume that the diameters of α_n tend to zero. Then there exists $\lim_{n\to\infty} h^*\big((f_n), \alpha_n\big)$; moreover,*

$$\lim_{n\to\infty} h^*\big((f_n), \alpha_n\big) = h^*\big((f_n)\big)\,.$$

Proof.— Let $h^*\big((f_n)\big) < \infty$ and $\varepsilon > 0$. Choose a covering γ such that $h^*\big((f_n), \gamma\big) > h^*\big((f_n)\big) - \varepsilon$. Denote by δ the Lebesgue number for the covering γ. Choose n_0 such that, for $n \geq n_0$ the diameter of the covering α_n is less than δ. Then the covering α_n is subtler than γ, so $\gamma \prec \alpha_n$. For any $j \in \mathbb{N}$, we get

$$\bigvee_{i=1}^{j-1} f_i^{-1}\gamma \prec \bigvee_{i=1}^{j-1} f_i^{-1}\alpha_n\,,$$

thus

$$\begin{aligned} h^*\big((f_n), \gamma\big) &= \limsup_{j\to\infty} \frac{1}{j} \log N\left(\bigvee_{i=0}^{j-1} f_i^{-1}\gamma\right) \\ &\leq \limsup_{j\to\infty} \frac{1}{j} \log N\left(\bigvee_{i=0}^{j-1} f_i^{-1}\alpha_n\right) = h^*\big((f_n), \alpha_n\big)\,. \end{aligned}$$

Finally we obtain

$$h^*\big((f_i)\big) \geq h^*\big((f_i), \alpha_n\big) \geq h^*\big((f_i), \gamma\big) \geq h^*\big((f_i)\big) - \varepsilon$$

which proves that

$$h^*\big((f_i)\big) = \lim_{n\to\infty} h^*\big((f_i), \alpha_n\big)\,.$$

In the case $h^*\big((f_i)\big) = \infty$, the argumentation is similar. □

PROPOSITION 2.18.— *Let a sequence (f_n) of continuous transformations of a compact metric space X represent a topological direction K in a group G w.r.t. G_1, acting on X. Then:*

(1) for any covering α of X with Lebesgue number δ,

$$N\left(\bigvee_{i=0}^{n} f_i^{-1}\alpha\right) \leq r\left(n, \frac{\delta}{2}, (f_i)\right) ;$$

(2) for any $\varepsilon > 0$ and any open covering γ of X, satisfying the condition $\operatorname{diam}(\gamma) \leq \varepsilon$, *we have*

$$s\big(n, \varepsilon, (f_i)\big) \leq N\left(\bigvee_{i=0}^{n} f_i^{-1}\gamma\right) .$$

Proof.— **(1)** Let F be an $\big(n, \delta/2, (f_i)\big)$-spanning set in X of cardinality $r\big(n, \delta/2, (f_i)\big)$. Then

$$X = \bigcup_{x\in F} \bigcap_{i=0}^{n} f_i^{-1}\overline{B}\left(f_i(x), \frac{\delta}{2}\right) .$$

If not, then there would exist some z such that

$$z \in X \setminus \bigcup_{x\in F} \bigcap_{i=0}^{n} f_i^{-1}\overline{B}\left(f_i(x), \frac{\delta}{2}\right) ,$$

i.e. for any $x \in F$ there exists $i \in \{0, \ldots, n\}$ such that

$$d\big(f_i(x)\, , \, f_i(z)\big) > \frac{\delta}{2} ,$$

which yields the inequality

$$d_n^{(f_n)}(x, z) > \frac{\delta}{2} ,$$

a contradiction with the assumption that F is an $\big(n, \delta/2, (f_i)\big)$-spanning set in X.

For each $i \in \mathbb{N}$, the closed ball $\overline{B}(f_i(x), \delta/2)$ is a subset of some set of the covering α, so

$$N\left(\bigvee_{i=0}^{n} f_i^{-1}\alpha\right) \leq r\left(n, \frac{\delta}{2}, (f_i)\right) .$$

(2) Let E be an $(n, \varepsilon, (f_i))$-separated set of cardinality $s(n, \varepsilon, (f_i))$. We claim that no set of the covering $\bigvee_{i=0}^{n} f_i^{-1}\gamma$ includes two distinct elements of E.

Indeed, if there exist distinct x, $y \in E$ such that x, $y \in A \in \bigvee_{i=0}^{n} f_i^{-1}\gamma$, then $d(f_i(x), f_i(y)) \leq \varepsilon$ for any $i \in \{0, \ldots, n\}$, which contradicts the assumption that x and y are $(n, \varepsilon, (f_i))$-separated.

Therefore

$$N\left(\bigvee_{i=0}^{n} f_i^{-1}\gamma\right) \geq s(n, \varepsilon, (f_i)) . \ \square$$

THEOREM 2.19.— *Let G be a finitely generated group of continuous mappings defined on a compact metric space X. Then, for any representations (f_p) and (g_p) defining the same direction K in G w.r.t. G_1, we obtain:*

(1) $h^*((f_p)) = h((f_p))$;

(2) $h^*((f_p)) = h^*((g_p))$.

Proof.— Let $\varepsilon > 0$. Consider the covering α_ε of X which consists of all open balls with radius ε and the open covering γ_ε of X which consists of all open balls with radius $\varepsilon/2$. Applying Proposition 2.18, we get

$$N\left(\bigvee_{i=0}^{n} f_i^{-1}\alpha_\varepsilon\right) \leq r(n, \varepsilon, (f_i)) \leq s(n, \varepsilon, (f_i)) \leq N\left(\bigvee_{i=0}^{n} f_i^{-1}\gamma_\varepsilon\right)$$

so

$$\begin{aligned} h^*((f_i), \alpha_\varepsilon) &\leq \limsup_{n\to\infty} \frac{1}{n} \log r(n, \varepsilon, (f_i)) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log s(n, \varepsilon, (f_i)) \leq h^*((f_i), \gamma_\varepsilon) . \end{aligned}$$

Passing with ε to 0, we get

$$h^*((f_i)) \leq h((f_i)) \leq h^*((f_i)) .$$

We have just proved that

$$h^*((f_i)) = h((f_i)).$$

If the sequences (f_n) and (g_n) represent the same topological direction, then

$$h((f_i)) = h((g_i))$$

therefore,

$$h^*((f_i)) = h^*((g_i)). \square$$

Example 2.20

(a) Let M be a compact metric space and $f : M \to M$ a homeomorphism. Consider a group G generated by f and a topological direction $K = [(g_i)]_{\approxeq}$ defined in the following way: for every $i \in \mathbb{N}$ let $g_i = f^i$. There exists a classical topological entropy $h_{\text{top}}(f)$ of the mapping $f : M \to M$; we also have the entropy $h((g_i))$ of the topological direction K in G w.r.t. $G_1 = \{f, \text{id}_M, f^{-1}\}$. Then the following equality takes place:

$$h_{\text{top}}(f) = h((f^i)).$$

(b) Let M be a compact metric space with metric d and let $f : M \to M$ be a homeomorphism. Fix numbers $c_1,\ c_2 \in \mathbb{N},\ c_1 < c_2$. Let $G = G(f)$ be the cyclic group generated by f. Consider another set of generators of G, the set

$$G_1 := \{f^{c_1}, f^{c_1+1}, \ldots, f^{c_2}, \text{id}_M, f^{-c_1}, f^{-(c_1+1)}, \ldots, f^{-c_2}\}.$$

Fix a topological direction $K = [(g_i)]_{\approxeq}$ in G w.r.t. G_1, defined by $g_i = f^{n_i}$ where (n_i) is an increasing sequence of positive integers such that

$$c_2 = \sup_{i \in \mathbb{N}} \{n_{i+1} - n_i\} \quad \text{and} \quad c_1 = \inf_{i \in \mathbb{N}} \{n_{i+1} - n_i\}.$$

We shall prove that the entropy of the topological direction $K = [(g_i)]_{\approxeq}$ in G w.r.t. G_1 satisfies the inequalities

$$c_1 \cdot h_{\text{top}}(f) \leq h((g_i)) \leq c_2 \cdot h_{\text{top}}(f)$$

where $h_{\text{top}}(f)$ is the classically defined entropy of $f : M \to M$.

Let $\varepsilon > 0$ and $m \in \mathbb{N}$. Take a minimal $(mc_2, \varepsilon, (f^i))$-spanning set A. Then for any $x \in M$ there exists $a \in A$ such that

$$\max_{0 \le i \le mc_2} \left\{ d\big(f^i(x)\,,\, f^i(a)\big) \right\} \le \varepsilon$$

and, the more so

$$\max_{0 \le i \le m} \left\{ d\big(f^{n_i}(x)\,,\, f^{n_i}(a)\big) \right\} \le \varepsilon\,;$$

thus the set A is $(m, \varepsilon, (f^{n_i}))$–spanning and, consequently,

$$r\big(m, \varepsilon, (f^{n_i})\big) \le r\big(mc_2, \varepsilon, (f^i)\big)$$

$$\frac{1}{m} \log r\big(m, \varepsilon, (f^{n_i})\big) \le \frac{c_2}{c_2 m} \log r\big(mc_2, \varepsilon, (f^i)\big)$$

$$h\big((f^{n_i})\big) \le c_2 h\big((f^i)\big) = c_2 h_{\mathrm{top}}(f)\,.$$

The mapping f is uniformly continuous, so, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(x,y) < \delta$, then

$$\max_{0 \le j \le c_2} \left\{ d\big(f^j(x)\,,\, f^j(y)\big) \right\} < \varepsilon\,.$$

For an $(m, \delta, (f^{n_i}))$-spanning set B of minimal cardinality, we have that for any $x \in M$ there exists $b \in B$ such that

$$\max_{0 \le i \le m} \left\{ d\big(f^{n_i}(x)\,,\, f^{n_i}(b)\big) \right\} \le \delta\,.$$

that is why

$$\max_{0 \le j \le mc_1} \left\{ d\big(f^j(x)\,,\, f^j(b)\big) \right\} \le \varepsilon\,;$$

therefore the set B is $(c_1 m, \varepsilon, (f^i))$-spanning and

$$r\big(m, \delta, (f^{n_i})\big) \ge r\big(mc_1, \varepsilon, (f^i)\big)$$

$$h\big((f^{n_i})\big) \ge c_1 h\big((f^i)\big) = c_1 h_{\mathrm{top}}(f)\,.$$

Example 2.21

Let M be a compact manifold and G a finitely generated group of transformations of M. As usual, G_1 is a set of generators of G such that $\mathrm{id}_M \in G_1$. Choose a topological direction $K = [(g_i)]_{\approx}$ in G w.r.t. G_1.

In [GLW, p. 106], E. Ghys, R. Langevin and P. Walczak defined the entropy $h_{\mathrm{GLW}}(G, G_1)$ of G with respect to G_1. Let

$$G_m = \{g_1 \cdots g_m \mid g_j \in G_1\}.$$

Two points x, $y \in M$ are $(n, \varepsilon, (g_i))$-separated if and only if

$$\max_{0 \le i \le n} \{d(g_i(x), g_i(y))\} > \varepsilon,$$

and this occurs if there exists $g_{i_0} \in G_{i_0}$, $i_0 \in \{0, \ldots, n\}$, such that

$$d(g_{i_0}(x), g_{i_0}(y)) > \varepsilon.$$

Thus the points, x, y are (n, ε)-separated in the sense of the definition given in paper [GLW, p. 106]. So, the largest cardinality of an $(n, \varepsilon, (g_i))$-separated set is less than or equal to the cardinality of an (n, ε)-separated set in the sense of the definition from paper [GLW]. Besides,

$$h((g_i)) \le h_{\mathrm{GLW}}(G, G_1).$$

Generally h_{GLW} is not a supremum of all $h(K)$.

PROPOSITION 2.22.— *Let S^1 be the unit circle with Riemannian metric d and G a finitely generated homeomorphism group of S^1 generated by G_1. Let $K = [(g_i)]_{\approx}$ be a topological direction in G w.r.t. G_1. Then the entropy of K is equal to zero.*

Proof.— Using the continuity of elements of G, we get that there exists $\varepsilon > 0$ such that for all x, $y \in S^1$ and $g \in G_1$:

$$d(x, y) \le \varepsilon \Longrightarrow d(g(x), g(y)) < 1. \qquad (*)$$

Choose ε sufficiently small to satisfy condition $(*)$. Then

$$r(0, \varepsilon, (f_i)) \le \left[\frac{2\pi}{\varepsilon}\right] + 1.$$

Let A be an $(n-1, \varepsilon, (f_i))$-spanning set on the circle S^1 with the minimal cardinality. Denote by B a subset of S^1 with minimal cardinality, for which the distance between the closest points of S^1 is less than or equal to ε. Then

$$\operatorname{card} B \leq \left[\frac{2\pi}{\varepsilon}\right] + 1 .$$

Put

$$C := A \cup f_n^{-1} B .$$

Fix a point $x \in S^1$. Then there exists some $y \in A$ such that

$$\max_{0 \leq i \leq n-1} \{d(f_i(x), f_i(y))\} \leq \varepsilon .$$

Define sets $I_0, I_1, \ldots, I_{n-1}, I_n$ in the following way:

(1) I_j is an arc of S^1 with end points $f_j(x)$ and $f_j(y)$;

(2) the homeomorphism $f_{j+1} f_j^{-1}$ transforms the arc I_j onto the arc I_{j+1};

(3) the length of the arc I_{n-1} is less than or equal to ε.

There exists $z \in C$ such that $f_n(z) \in I_n$ and $d(f_n(x), f_n(z)) \leq \varepsilon$. The mapping $(f_n f_{n-1}^{-1})^{-1}$ tranforms homeomorphically the arc I_n onto the arc I_{n-1}. So $f_{n-1}(z) \in I_{n-1}$; that is why

$$d(f_{n-1}(x), f_{n-1}(z)) \leq \varepsilon .$$

Similarly,

$$(f_{n-1} f_{n-2}^{-1})^{-1} : I_{n-1} \xrightarrow{\text{homeo}} I_{n-2} ,$$

so $f_{n-2}(z) \in I_{n-2}$.

It remains to show that $d(f_{n-2}(x), f_{n-2}(z)) \leq \varepsilon$. The arc I_{n-2} is the homeomorphic image of I_{n-1} in the mapping $(f_{n-1} f_{n-2}^{-1})^{-1}$. The distance between end points of I_{n-2} is less than or equal to ε. Condition $(*)$ states that the length of I_{n-2} is less than or equal to 1, so the shortest way between $f_{n-2}(x)$ and $f_{n-2}(y)$ is included in the arc I_{n-2}; that is why the length of the arc I_{n-2} is less than or equal to ε. In consequence, we obtain

$$d(f_{n-2}(x), f_{n-2}(y)) \leq \varepsilon .$$

Repeating the above argumentation, we get the inequality

$$\max_{0\le j\le n} \{d(f_j(x), f_j(z))\} \le \varepsilon$$

which proves that the set C is $(n, \varepsilon, (f_i))$-spanning. Notice that

$$\begin{aligned} r(n, \varepsilon, (f_i)) &\le \operatorname{card} A + \operatorname{card} B \\ &\le r(n-1, \varepsilon, (f_i)) + \left[\frac{2\pi}{\varepsilon}\right] + 1 \le (n+1)\left(\left[\frac{2\pi}{\varepsilon}\right] + 1\right) ; \end{aligned}$$

therefore the entropy of the direction K is equal to zero. □

3. Expansive topological direction and entropy

DEFINITION 3.1.— *Let K be a topological direction in a finitely generated group G of transformations of a compact metric space (X, d). We say that K is an expansive direction with a constant $\delta > 0$ if there exists a sequence (f_n) determining the direction K such that for every distinct x, $y \in X$, there exists $m \in \mathbb{N}$ with the following property:*

$$d_m^{(f_n)}(x, y) \ge \delta .$$

PROPOSITION 3.2.— *Let G be a finitely generated group of Lipschitz homeomorphisms of a compact Riemannian manifold M. Then the entropy of every topological direction in G w.r.t. G_1 is finite.*

Proof.— Consider a sequence (f_p) such that $[(f_p)]_{\approx} = K$. By induction we obtain that for any $p \in \mathbb{N}$ there exists $\lambda_p > 0$ such that for any x, $y \in M$, we get

$$d(f_p(x), f_p(y)) \le \lambda_p d(x, y) .$$

Assume that λ_p is the smallest number with this property. Let $\varepsilon > 0$ and A be the largest $(n, \varepsilon, (f_p))$-separated set in M. Then, for any x, $y \in A$, we have

$$\begin{aligned} \varepsilon < d_n^{(f_p)}(x, y) &= \max\{d(f_i(x), f_i(y)) \mid i \in \{0, \dots, n\}\} \le \\ &\le \max\{\lambda_i \mid i \in \{0, \dots, n\}\} d(x, y) \le \\ &\le \max\{\{\lambda_i \mid i \in \{0, \dots, n\}\} \cup \{1\}\} d(x, y) . \end{aligned}$$

Put $\max\{\{\lambda_i \mid i \in \{0, \dots, n\}\} \cup \{1\}\} := \Theta_n$. Then for all x, $y \in A$,

$$\frac{\varepsilon}{\Theta_n} \leq d(x, y)$$

that is why A is $(0, \varepsilon/\Theta_n, (f_p))$-separated and we have the inequality

$$s\left(0, \frac{\varepsilon}{\Theta_n}, (f_p)\right) \geq s\big(n, \varepsilon, (f_p)\big) ;$$

hence

$$\begin{aligned} h\big((f_p)\big) &= \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log s\big(n, \varepsilon, (f_p)\big) \leq \\ &\leq \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log s\left(0, \frac{\varepsilon}{\Theta_n}, (f_p)\right) \leq \\ &\leq \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log \frac{\operatorname{vol} M}{\max_{x \in M} \operatorname{vol} B\big(x, \varepsilon/2\Theta_n\big)} . \end{aligned}$$

Having regard to the fact that the volume of the ball $B(r)$ with radius r satisfies the condition $\operatorname{vol} B(r) \geq c r^m$ where $m = \dim M$, $r \in (0\,,\ \operatorname{diam} M\,]$, c some positive number dependent on the curvature of M, we obtain

$$\begin{aligned} h\big((f_p)\big) &\leq \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log \frac{\operatorname{vol} M}{c\big(\varepsilon/2\Theta_n\big)^m} \\ &\leq \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log 2^m \operatorname{vol} M - \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log c\varepsilon^m + \\ &\quad + \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log \Theta_n^m \\ &\leq \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{m}{n} \log \Theta_n = \limsup_{n \to \infty} \frac{m}{n} \log \Theta_n . \end{aligned}$$

Notice that $\lambda_n \leq \big(\max\{\lambda_f \mid f \in G_1\}\big)^n$ so

$$\Theta_n \leq \max\left\{1\,,\ \big(\max\{\lambda_f \mid f \in G_1\}\big)^n\right\} .$$

That is why the sequence $a_n := 1/n \log \Theta_n$ is bounded and

$$h\big((f_p)\big) < \infty . \;\square$$

COROLLARY 3.3.— *Let M be a compact connected Riemannian manifold with metric d and let G be a finitely generated group of C^1-class transformations of M. Then:*

(1) any generator f of the group G satisfies the condition: there exists $\lambda_f > 0$ such that for all x, $y \in M$,

$$d\big(f(x), f(y)\big) \leq \lambda_f \, d(x,y) \ ;$$

(2) the entropy of any topological direction K in G is finite.

Example 3.4

There are examples of an expansive direction generated by non-expansive generators. Let $\Gamma = \mathbb{Z}^2$ operate on $\mathbb{R}^2$ through translations. Endow $\mathbb{R}^2$ with the standard metric d and $T^2 = \mathbb{R}^2 \setminus \mathbb{Z}^2$ with quotient metric d_1

Define the mappings ϕ, $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ in the following way:

$$\phi(x_1, x_2) = (x_1 + x_2 \, , \, x_2) \, ,$$

$$\psi(x_1, x_2) = (x_1 \, , \, x_1 + x_2) \, .$$

These mappings induce bijective mappings ϕ_1, $\psi_1 \ : T^2 \to T^2$. The generators ϕ_1 and ψ_1 are not expansive but the topological direction K determined by the sequence (f_n) defined by:

$$f_1 = \psi_1 \, , \quad f_2 = \psi_1 \psi_1 \, , \quad f_3 = \psi_1 \psi_1 \psi_1$$

$$f_{n+1} = \begin{cases} \psi_1 f_n & \text{if } n \in [\,2^{2k} - 1 \, , \, 2^{2k+1} - 2\,], \, k \in \mathbb{N} \\ \phi_1 f_n & \text{if } n \in [\,2^{2k+1} - 1 \, , \, 2^{2k+2} - 2\,], \, k \in \mathbb{N} \end{cases}$$

is expansive.

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