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Trace imbeddings for T-sets and application to Neumann–Dirichlet problems with measures included in the boundary data[*]

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RÉSUMÉ. — Dans cet article, on étudie l'existence de solutions pour un problème quasi-linéraire elliptique à données mesures y compris sur la frontière et on montre l'unicité de solutions lorsque ces données sont dans L^1.

ABSTRACT. — In this paper, we investigate in the existence and uniqueness of a quasilinear elliptic problem with measure data included in the free terms both of equation and boundary. We study first some Sobolev-type imbeddings for T-sets and we use the notion of renormalized solution for having uniqueness.

0. Introduction

Dirichlet problems with measures as data have been widely studied recently by various authors, for instance [BG], [BGV], [BS], [BCP], [GV], [Ra1]-[Ra6], [Lia], [LM], [At]. Most of them concern the homogeneous boundary Dirichlet condition except in [GV], where a semilinear equation is studied with a measure as a boundary condition. In this paper, we will discuss a kind of more general mixed boundary condition.

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More precisely, consider the following problem:

$$\begin{cases} -\operatorname{div}(\widehat{a}(x,u,\mathrm{D}u)) + \varepsilon_a |u|^{p-2} u = \mu \in M(\Omega) \\ Bu = \nu \in M(\Gamma) \end{cases} \tag{P}$$

where $\widehat{a}$ is a Caratheodory function defined in $\Omega \times \mathbb{R} \times \mathbb{R}^N$, Ω is a smooth bounded open set of $\mathbb{R}^N$, B is defined on $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$ formally by $Bu = u$ on Γ_0 and $Bu = \widehat{a} \cdot \overrightarrow{n} + \varepsilon_b |u|^{s-1} u$ on Γ_1, ε_a and ε_b equal to 0 or 1, $p \in \,]1, \infty[$, s will be precised later, $\overrightarrow{n}$ is the outer normal to Γ_1, $M(\Omega)$ (resp. $M(\Gamma)$) is the set of bounded Radon measures on Ω (resp. Γ). We assume $\nu = 0$ on Γ_0.

The function $\widehat{a}$ is required to satisfy the standard Leray–Lions assumptions:

(A1) for a.e. $x \in \Omega$, for all $u \in \mathbb{R}$, all $\xi \in \mathbb{R}^N$

$$\widehat{a}(x,u,\xi)\xi \geq \alpha |\xi|^p \quad \text{for some } \alpha > 0 \,;$$

(A2) for a.e. $x \in \Omega$, for all $u \in \mathbb{R}$, all $\xi \in \mathbb{R}^N$

$$|\widehat{a}(x,u,\xi)| \leq C\Big(|u|^\sigma + |\xi|^{p-1} + a_0(x)\Big)$$

where C is a positive constant, $\sigma < N(p-1)^2/\big(p(N-p)\big)$ and

$$a_0 \in L^{p'}(\Omega)\,, \quad \frac{1}{p} + \frac{1}{p'} = 1 \,;$$

(A3) for a.e. $x \in \Omega$, for all $u \in \mathbb{R}$, all $\xi \neq \xi' \in \mathbb{R}^N$

$$\big(\widehat{a}(x,u,\xi) - \widehat{a}(x,u,\xi')\big)(\xi - \xi') > 0 \,.$$

As a model, we can condiser

$$Au = -\operatorname{div}\Big(|\mathrm{D}u|^{p-2}\mathrm{D}u\Big) + \varepsilon_a |u|^{p-2} u$$

or

$$Au = -\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{i,j}(x)\frac{\partial}{\partial x_i}\right).$$

If $\varepsilon_a = 0$, there is generally no solution for problem (P); for this reason, we need a compatibility condition involving the measures μ and ν, that is $\mu(\Omega) + \nu(\Gamma_1) = 0$.

The problem (P) does not possess solutions in the usual Sobolev spaces: one can verify that fact on the fundamental equation.

$$-\operatorname{div}\left(\left|\mathrm{D}u\right|^{p-2}\mathrm{D}u\right) = \delta_{x_0}\,, \quad u = 0 \text{ on } \partial\Omega\,,\ x_0 \in \Omega\,,\ p \leq 2 - \frac{1}{N}\,.$$

For this reason, as in [BBGGVP], [RA2] and [RA3], we introduce some convenient functional sets in which we search weak solutions (sects 1 and 2). In the first time, we interpret in the distribution sense, but as pointed out by a counter-example given by Serrin [Se], the weak solution given in that sense is not unique. This lack of uniqueness is widely explained in [Ra6]. Another notion of weak solution is then useful to ensure the uniqueness of solutions. When the data μ and ν are in L^1, we can borrow the notion of renormalized solution of Di Perna–Lions and adapt it to our case. As it is shown in [Ra6], when μ and ν are smooth, say in $L^{p'}(\Omega)$, then this notion of renormalized solution is completely equivalent to the notion of classic weak solution.

For proving the existence of a weak solution, we consider a family of approximating problems, and, by compacity arguments, we construct a solution u. This function u is also a renormalized solution (when μ and ν are L^1), and we prove uniqueness result in this case by comparing an arbitrary renormalized solution w with the solution u mentioned above (see [Ra6] for Dirichlet equations).

An uniqueness result is also given in [BBGGVP] under less general conditions than ours and technics are completely different, for instance they compare directly two arbitrary solutions.

Recently, Xu [X] borrows the same ideas as in [BBGGVP], [Ral] and [Ra2] to show the existence of solution for a multivoque problem. There is no uniqueness result in this paper and the data are in L^1.

We will distinguish in the proofs the mixed problems corresponding to the case of $H_{N-1}(\Gamma_0) > 0$ ($H_{N-1}(\Gamma_0)$ is the $N-1$ dimensional Hausdorf measure of Γ_0) and the Neumann problems corresponding to the case of $H_{N-1}(\Gamma_0) = 0$. We give a sense for the traces of the founded weak solution by using truncations.

1. Functional sets; T-sets

Let Ω be a smooth bounded open connected set of $\mathbb{R}^N$, $N \geq 2$, $p \in \,]1,\infty[$. $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$, Γ_0 and Γ_1 disjoints. We introduce the following sets:

$$\mathrm{Lip}_p(\mathbb{R}) = \left\{\Phi \in W^{1,\infty}(\mathbb{R}) \mid \text{ such that } \Phi' \in L^p(\mathbb{R}) \text{ and } \Phi(0) = 0\right\},$$

$$C^\infty_{\Gamma_0}(\Omega) = \left\{u \in C^\infty(\overline{\Omega}) \mid u\restriction_{\Gamma_0} = 0\right\},$$

$$W^{1,p}_{\Gamma_0}(\Omega) = \left\{u \in W^{1,p}(\Omega) \mid u\restriction_{\Gamma_0} = 0\right\},$$

$$L^{1,p}_{\Gamma_0}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable} \;\middle|\; \forall\, \Phi \in \mathrm{Lip}_p(\mathbb{R}),\ \Phi(u) \in W^{1,p}_{\Gamma_0}(\Omega) \right.$$

$$\left. \text{and } \sup_{k>0} \int_\Omega \frac{|\mathrm{D}u^k|^p}{\left(1+|u^k|\right)^{1+\delta}}\,\mathrm{d}x \leq C(\delta), \text{ for all } \delta > 0\right\},$$

where $v^k = T_k(v)$, $T_k(\sigma) = \left(k - \left(k - |\sigma|\right)_+\right)\mathrm{sign}(\sigma)$ for $\sigma \in \mathbb{R}$,

$$\Lambda^{1,p}_{\Gamma_0}(\Omega) = \left\{u \in L^{1,p}_{\Gamma_0}(\Omega) \cap L^{p-1}(\Omega) \;\middle|\; \exists\, C > 0,\ \forall\, \Phi \in \mathrm{Lip}_p(\mathbb{R}), \right.$$

$$\left. \left\|\mathrm{D}(\Phi(u))\right\|_{L^p(\Omega)} \leq C\left\|\Phi'\right\|_{L^p(\mathbb{R})}\right\},$$

$$\Pi^{1,p}_{\Gamma_0}(\Omega) = \left\{u \in \Lambda^{1,p}_{\Gamma_0}(\Omega) \;\middle|\; \lim_{m\to\infty} \int_{m \leq |u| \leq m+1} |\mathrm{D}u|^p\,\mathrm{d}x = 0\right\}.$$

Here $L^{1,p}_{\Gamma_0}(\Omega)$ (resp. $\Lambda^{1,p}_{\Gamma_0}(\Omega)$ and $\Pi^{1,p}_{\Gamma_0}(\Omega)$) is called T-set (resp. T-subset).

We note that $C^\infty_{\Gamma_0}(\Omega) \subset \Pi^{1,p}_{\Gamma_0}(\Omega)$ and, in the case of $H_{N-1}(\Gamma_0) = 0$, the spaces $C^\infty_{\Gamma_0}(\Omega)$ and $W^{1,p}_{\Gamma_0}(\Omega)$ are the classical spaces $C^\infty(\overline{\Omega})$ and $W^{1,p}(\Omega)$ and the T-set and T-subset are simply denoted by $L^{1,p}$, $\Lambda^{1,p}$ and $\Pi^{1,p}$.

These spaces and sets possess similar properties as Sobolev spaces. The first lemma concerns the derivability result.

Lemma 1.1.— *If $v \in L^{1,p}_{\Gamma_0}(\Omega)$, $\mathrm{D}v(x)$ exists almost everywhere and for $f \in C^1(\mathbb{R})$, $k \in \mathbb{N}$.*

$$\mathrm{D}(f \circ v)(x) = (f' \circ v)\mathrm{D}v(x)$$

$$\mathrm{D}v^k(x) = \begin{cases} \mathrm{D}v(x) & \text{if } |v(x)| \leq k \\ 0 & \text{otherwise} \end{cases} \qquad a.e.\ x \in \Omega.$$

For the proof see [Ra3] or [At].

LEMMA 1.2.— $W^{1,p}_{\Gamma_0}(\Omega) \subset L^{1,p}_{\Gamma_0}(\Omega)$.

The proof is easy.

The two lemmas below are fundamental: they concern trace results in the T-spaces and provide inequalities for estimations in section 3 and convergence results in section 4. We introduce four constants defined with $p \in]1, N]$:

$$r_c = \frac{N}{N-p}(p-1) \qquad \text{and} \qquad r'_c = \frac{N-1}{N-p}(p-1) \text{ eventually infinite},$$

$$\widetilde{p}\begin{cases} = \dfrac{Np}{N-p} & \text{if } 1 \le p < N \\ < +\infty & \text{if } p = N \end{cases} \qquad \text{and} \qquad \widetilde{p}'\begin{cases} = \dfrac{(N-1)p}{N-p} & \text{if } 1 \le p < N \\ < +\infty & \text{if } p = N. \end{cases}$$

LEMMA 1.3.— *For $r \in]0, r_c[$ and $r' \in]0, r'_c[$ satisfying $r(N-1) = r'N$, there exists a constant C, depending only on Ω and r, such that for all $u \in L^{1,p}(\Omega) \cap L^{p-1}(\Omega)$, $u(x)$ makes sense for a.e. $x \in \Gamma$, and we have with $\tau = (\widetilde{p} - r)/\widetilde{p} = (\widetilde{p}' - r')/\widetilde{p}'$:*

$$\begin{aligned} &\left(\int_\Omega |u|^r \,\mathrm{d}x\right)^{1/\widetilde{p}} + \left(\int_\Gamma |u|^{r'} \,\mathrm{d}H_{N-1}\right)^{1/\widetilde{p}'} \le \\ &\qquad \le C\left(\left(\int_\Omega \frac{|\mathrm{D}u|^p}{(1+|u|)^{p\tau}} \,\mathrm{d}x\right)^{1/p} + \left(\int_\Gamma |u|^{p-1} \,\mathrm{d}x\right)^{1/p}\right) + C. \end{aligned} \tag{1}$$

Moreover, if $u \in L^{1,p}(\Omega)$, and for any $\tau \in]1/p, 1[$,

$$\int_0^u \frac{\mathrm{d}\sigma}{(1+|\sigma|)^\tau} = \Phi_\tau(u) \in L^1(\Omega)$$

with $\int_\Omega \Phi_\tau(u)\,\mathrm{d}x = 0$, we have for $\widetilde{p}$, $\widetilde{p}'$ defined as before and for $r = (1-\tau)\widetilde{p}$, $r' = (1-\tau)\widetilde{p}'$:

$$\left(\int_\Omega |u|^r \,\mathrm{d}x\right)^{1/\widetilde{p}} + \left(\int_\Gamma |u|^{r'} \,\mathrm{d}H_{N-1}\right)^{1/\widetilde{p}'} \le C\left(\int_\Omega \frac{|\mathrm{D}u|^p}{(1+|u|)^{p\tau}} \,\mathrm{d}x\right)^{1/p} + C. \tag{2}$$

Proof.— Consider $\Phi(t) = \int_0^t \mathrm{d}\sigma/(1+|\sigma|)^\tau$. Let $u \in L^{1,p}_{\Gamma_0}(\Omega) \cap L^{p-1}(\Omega)$, by definition $\Phi(u^k) \in W^{1,p}_{\Gamma_0}(\Omega) \subset L^{\widetilde{p}}(\Omega)$, for all $k > 0$ we write,

$$(1+|u^k|)^{r/\widetilde{p}} = 1 + \frac{r}{\widetilde{p}} \int_0^{|u^k|} \frac{\mathrm{d}\sigma}{(1+|\sigma|)^\tau} = 1 + \frac{r}{\widetilde{p}} |\Phi(u^k)|.$$

We arise this equality to the power $\widetilde{p}$ and integrate over Ω, we get

$$\int_\Omega (1+|u^k|)^r \,\mathrm{d}x = \int_\Omega \left(1+\frac{r}{\widetilde{p}}\left|\Phi(u^k)\right|\right)^{\widetilde{p}} \mathrm{d}x\,.$$

Thus we obtain:

$$\begin{aligned}\left(\int_\Omega (1+|u|^k)^r \,\mathrm{d}x\right)^{1/\widetilde{p}} &\le \left\|\left(1+\frac{r}{\widetilde{p}}\left|\Phi(u^k)\right|\right)\right\|_{L^{\widetilde{p}}}\\ &\le \|1\|_{L^{\widetilde{p}}}+\frac{r}{\widetilde{p}}\|\Phi(u^k)\|_{L^{\widetilde{p}}}\,;\end{aligned}$$

by the usual Sobolev inequality applied to $\Phi(u^k)$, we deduce:

$$\left(\int_\Omega (1+|u|^k)^r \,\mathrm{d}x\right)^{1/\widetilde{p}} \le \|1\|_{L^{\widetilde{p}}}+C\|\Phi(u^k)\|_{W^{1,p}}\,.$$

Since $|\Phi(u^k)|^p \le C|u^k|^{(1-\tau)p} \le C|u^k|^{p-1}$, we then have

$$\left(\int_\Omega |u^k|^r \,\mathrm{d}x\right)^{1/\widetilde{p}} \le C+C\left(\left(\int_\Omega \frac{|\mathrm{D}u^k|^p}{(1+|u^k|)^{p\tau}}\,\mathrm{d}x\right)^{1/p}+\left(\int_\Omega |u^k|^{p-1}\,\mathrm{d}x\right)^{1/p}\right).$$

Similarly, by the trace imbedding theorem on usual Sobolev spaces, applied to the function $u^k = T_k(u)$, we have:

$$\begin{aligned}&\left(\int_\Gamma |u^k|^{r'} \,\mathrm{d}H_{N-1}\right)^{1/\widetilde{p}'} \le\\ &\quad\le \left(\int_\Gamma (1+|u^k|)^{r'} \,\mathrm{d}H_{N-1}\right)^{1/\widetilde{p}'}\\ &\quad\le \|1\|_{L^{\widetilde{p}'}}+\frac{r'}{\widetilde{p}'}\|\Phi(u^k)\|_{L^{\widetilde{p}'}(\Gamma)}\\ &\quad\le C\left(\left(\int_\Omega |\mathrm{D}\Phi(u^k)|^p\,\mathrm{d}x\right)^{1/p}+\left(\int_\Omega |\Phi^p(u^k)|\,\mathrm{d}x\right)^{1/p}+1\right)\\ &\quad\le C\left(1+\left(\int_\Omega \frac{|\mathrm{D}u^k|^p}{(1+|u^k|)^{p\tau}}\,\mathrm{d}x\right)^{1/p}+\left(\int_\Omega |u^k|^{p-1}\,\mathrm{d}x\right)^{1/p}\right).\end{aligned}$$

(C is independent of k). For $x \in \Gamma$, $u^k(x) = u(x)_+^k - u(x)_-^k$: thus, it has a limit in $[-\infty, +\infty]$, when k go to infinity; we denote this limit by $u(x)$. By Fatou's lemma and Lebesgue dominated convergence, when letting k go to infinity, we obtain the inequality (1).

For the last inequality, let $u \in L^{1,p}(\Omega)$ with the condition

$$\int_\Omega \left(\int_0^u \frac{\mathrm{d}\sigma}{(1+|\sigma|)^\tau} \right) \mathrm{d}x = 0 .$$

From usual Sobolev imbedding theorem, applied to $\Phi(u^k)$, we can obtain:

$$\begin{aligned} \|\Phi(u^k)\|_{L^{\widetilde{p}}} &\le C\left(\|\mathrm{D}\Phi(u^k)\|_{L^p} + \left| \int_\Omega \Phi(u^k)\,\mathrm{d}x \right| \right) \\ &= C\left(\|\mathrm{D}\Phi(u^k)\|_{L^p} + \left| \int_\Omega \left(\int_0^{u^k} \frac{\mathrm{d}\sigma}{(1+|\sigma|)^\tau} \right) \mathrm{d}x \right| \right) \end{aligned}$$

(C is independent of k) and noticing that

$$\lim_{k\to\infty} \int_\Omega \left(\int_0^{u^k} \frac{\mathrm{d}\sigma}{(1+|\sigma|)^\tau} \right) \mathrm{d}x = \int_\Omega \left(\int_0^{u} \frac{\mathrm{d}\sigma}{(1+|\sigma|)^\tau} \right) \mathrm{d}x = 0 ,$$

thus arguing as for the inequality (1) and using the above inequality, we derive the inequality (2). □

Remark.— The existence of a trace function can also be derived by showing that u is p-quasicontinuous in $\overline{\Omega}$.

LEMMA 1.4.— *For all $q \in [1, N/(N-1)[$, there exists $r \in]0, r_c[$, there exists a constant C such that for all $u \in L^{1,p}(\Omega)$:*

$$\int_\Omega |\mathrm{D}u|^{q(p-1)}\,\mathrm{d}x \le C\left(1 + \left(\int_\Omega |u|^r\,\mathrm{d}x \right)^{1-q/p'} \right), \quad \frac{1}{p} + \frac{1}{p'} = 1 . \qquad (3)$$

Proof.— Choose $t = q(p-1)$ and $\delta > 0$ small enough, then

$$\begin{aligned}
&\int_\Omega |\mathrm{D}u|^t \, \mathrm{d}x = \\
&\quad = \int_\Omega \frac{|\mathrm{D}u|^t}{(1+|u|)^{t(1+\delta)/p}} (1+|u|)^{t(1+\delta)/p} \, \mathrm{d}x \\
&\quad \leq \left(\int_\Omega \frac{|\mathrm{D}u|^p}{(1+|u|)^{t(1+\delta)}} \, \mathrm{d}x \right)^{t/p} \left(\int_\Omega (1+|u|)^{t(1+\delta)/(p-t)} \, \mathrm{d}x \right)^{1-t/p} \\
&\quad \leq C \left(1 + \left(\int_\Omega |u|^r \, \mathrm{d}x \right)^{1-t/p} \right)
\end{aligned}$$

with $r = t(1+\delta)/(p-t) < r_c$. □

LEMMA 1.5.— *If $p > 2 - 1/N$, then $L^{1,p}_{\Gamma_0}(\Omega) \cap L^{p-1}(\Omega) \subset W^{1,q}_{\Gamma_0}(\Omega)$ for all $q \in \left[1\,, N(p-1)/(N-1)\right[$.*

The proof is the same as in [Ra3].

2. Weak solutions and renormalized solutions. Main results

For $p \in \left]1\,, \infty\right[$ we set $s_c = (N-1)(p-1)/(N-p)$ if $1 < p < N$. We want to solve the following problem: find u in an appropriate T-set satisfying:

$$\begin{cases} -\operatorname{div}\left(\widehat{a}(x,u,\mathrm{D}u)\right) + \varepsilon_a |u|^{p-2} u = \mu \in M(\Omega) \\ Bu = \nu \in M(\Gamma)\,. \end{cases} \tag{P}$$

B is defined on $\Gamma = \Gamma_0 \cup \Gamma_1$ formally by $Bu = u$ on Γ_0, $Bu = \widehat{a} \cdot \overrightarrow{n} + \varepsilon_b |u|^{s-1} u$ on Γ_1, $\nu = 0$ on Γ_0, ε_a and ε_b equal to 0 or 1, $s < s_c$.

The above problem has to be understood in a precise sense, this is why we introduce a few definitions (see below). The first one is available for general measure and the second one is true for integrable function. We will see that one can get an uniqueness result for renormalized solution.

DEFINITION 2.1.— *We will say u is a weak solution of the problem (P) if $u \in \Lambda^{1,p}_{\Gamma_0}(\Omega)$ and for any $v \in C^{\infty}_{\Gamma_0}(\Omega)$, there holds:*

$$\begin{aligned}\int_\Omega \widehat{a}(x,u,\mathrm{D}u)\mathrm{D}v\,\mathrm{d}x + \varepsilon_a \int_\Omega |u|^{p-2}uv\,\mathrm{d}x + \varepsilon_b \int_{\Gamma_1} |u|^{s-1}uv\,\mathrm{d}H_{N-1} = \\ = \int_\Gamma v\,\mathrm{d}\nu + \int_\Omega v\,\mathrm{d}\mu\,.\end{aligned} \tag{4}$$

DEFINITION 2.2.— *Let $\mu \in L^1(\Omega)$ and $\nu \in L^1(\Gamma)$, we will say that $u \in \Pi^{1,p}_{\Gamma_0}$ is a renormalized solution if: for any $v \in W^{1,p}_{\Gamma_0}(\Omega) \cap L^\infty(\Omega)$, for any $T \in W^{1,\infty}_{\text{compact}}(\mathbb{R})$, there holds:*

$$\begin{aligned}\int_\Omega \widehat{a}(x,u,\mathrm{D}u)\mathrm{D}\big(T(u)v\big)\,\mathrm{d}x + \varepsilon_a \int_\Omega |u|^{p-2}uT(u)v\,\mathrm{d}x + \\ + \varepsilon_b \int_{\Gamma_1} |u|^{s-1}uT(u)v\,\mathrm{d}H_{N-1} = \\ = \int_\Gamma T(u)v\nu\,\mathrm{d}H_{N-1} + \int_\Omega T(u)v\mu\,\mathrm{d}\mu\,.\end{aligned} \tag{5}$$

The main results in this paper are stated in Theorem 1 below and Theorems 2 and 3 in section 7: Theorem 1 concerns an existence result and the others concern uniqueness results.

THEOREM 1.— *Let $\mu \in M(\Omega)$ and $\nu \in M(\Gamma)$. If $p \in \,]1, N]$ and $s \in \,]0, s_c[$ if $p < N$, under the assumptions (A1) to (A3), there exists at least a weak solution u of problem (P). ($H_{N-1}(\Gamma_0) = 0$ in the case $\varepsilon_a = \varepsilon_b = 0$, and $\nu = 0$ on Γ_0 if $H_{N-1}(\Gamma_0) > 0$).*

Futhermore, if $u \in L^1(\Omega)$ and $\nu \in L^1(\Gamma)$ this weak solution is a renormalized solution.

The proof will be divided in two cases according to the values of ε_a and ε_b: The first one is the case when ε_a or $\varepsilon_b = 0$ which will include three subcases (that is $\varepsilon_a = \varepsilon_b = 1$, $\varepsilon_a = 1 - \varepsilon_b = 1$, $\varepsilon_a = 1 - \varepsilon_b = 0$). But since the discussions of them are similar, we just consider the case of $\varepsilon_a = \varepsilon_b = 1$. And the other case that we will discuss is $\varepsilon_a = \varepsilon_b = 0$.

The first step consists in considerating an approximating problem. We will make some uniform a priori estimates compatible to the structure of T-set that we consider. Using some compactness results similar to those produced in [Ra1]-[Ra3], we pass to the limit.

3. An Approximating problem of problem in the case $\varepsilon_a = \varepsilon_b = 1$

We will consider an approximating problem of problem (P) in the following manner.

Let $\mu_n \in \mathcal{D}(\Omega)$, $\nu_n \in L^1(\Gamma) \cap W^{-1/p',p}(\Gamma)$, $\nu_n = 0$ on Γ_0 satisfying: for any $\Phi \in C(\overline{\Omega})$ (set of continuous functions)

$$\int_\Omega \Phi \mu_n \, \mathrm{d}x \longrightarrow \int_\Omega \Phi \, \mathrm{d}\mu \,, \quad \int_\Gamma \Phi \nu_n \, \mathrm{d}H_{N-1} \longrightarrow \int_\Gamma \Phi \, \mathrm{d}\nu$$

and for all n,

$$\|\mu_n\|_{L^1} \leq \mu(\Omega) \,, \quad \|\nu_n\|_{L^1} \leq \nu(\Gamma_1) \,.$$

With the assumptions of Theorem 1, there exists a weak solution $u_n \in W^{1,p}_{\Gamma_0}(\Omega)$ of problem (P_n) below and the proof can be found in [LL].

Find $u_n \in W^{1,p}_{\Gamma_0}(\Omega)$ that satisfies for any $v \in W^{1,p}_{\Gamma_0}(\Omega)$: $\qquad (\mathrm{P}_n)$

$$\int_\Omega \widehat{a}(x, u_n, \mathrm{D}u_n)\mathrm{D}v \, \mathrm{d}x + \int_\Omega |u_n|^{p-2} u_n v \, \mathrm{d}x + \int_{\Gamma_1} |u_n|^{s-1} u_n v \, \mathrm{d}H_{N-1} =$$
$$= \int_\Gamma v\nu_n \, \mathrm{d}H_{N-1} + \int_\Omega v\mu_n \, \mathrm{d}x \,. \tag{6}$$

Uniform estimates for the sequence u_n

(i) We prove first that the sequence u_n lies in a bounded subset of $L^{p-1}(\Omega)$ and a bounded subset of $L^s(\Gamma)$.

We take $v = \mathrm{sign}_\eta(u_n)$, $\eta > 0$ in (6) where

$$\mathrm{sign}_\eta(\sigma) = \begin{cases} \dfrac{\sigma}{\eta} & \text{if } |\sigma| \leq \eta \\ \mathrm{sign}(\sigma) & \text{if } |\sigma| > \eta, \end{cases}$$

after dropping the positive terms in the first member and letting $\eta \longrightarrow 0$, we obtain:

$$\int_{\Gamma_1} |u_n|^s \, \mathrm{d}H_{N-1} + \int_\Omega |u_n|^{p-1} \, \mathrm{d}x \leq \|\nu_n\|_{L^1(\Gamma_1)} + \|\mu_n\|_{L^1(\Omega)}$$
$$\leq \mu(\Omega) + \nu(\Gamma_1) \,.$$

(ii) Let $T \in \mathrm{Lip}_p(\mathbb{R})$, we shall prove that the sequence $\mathrm{D}(T(u_n))$ lies in a bounded subset of $L^p(\Omega)$ (as $n \to \infty$): for this, we take $v_* = \int_0^{u_n} |T'(\sigma)|^p \,\mathrm{d}\sigma$ in (6) and we obtain

$$\begin{aligned}
\alpha \int_\Omega &\left|\mathrm{D}(T(u_n))\right|^p \,\mathrm{d}x \leq \\
&\leq \|u_n\|_{L^{p-1}(\Omega)} \|v_*\|_{L^\infty(\Omega)} + \|u_n\|_{L^s(\Gamma)} \|v_*\|_{L^\infty(\Gamma)} \\
&\quad + \|\nu_n\|_{L^1(\Gamma)} \|v_*\|_{L^\infty(\Gamma)} + \|\mu_n\|_{L^1(\Omega)} \|v_*\|_{L^\infty(\Omega)} \\
&\leq C \|T'\|_{L^p}^p
\end{aligned}$$

(because $\|v_*\|_{L^\infty(\Omega)} \leq \|T'\|_{L^p}^p$) where C is independent of n.

Thus $T(u_n)$ remains in a bounded set of $W_{\Gamma_0}^{1,p}(\Omega)$.

(iii) Now take

$$v = \int_0^{u_n} \frac{\mathrm{d}\sigma}{\left(1 + |\sigma|\right)^{1+\delta}}, \quad \delta > 0,$$

as a test function in (6), we obtain:

$$\int_\Omega \frac{|\mathrm{D}u_n|^p}{\left(1 + |u_n|\right)^{1+\delta}} \,\mathrm{d}x \leq C \tag{7}$$

where C is independent of n and k.

(iv) We apply Lemma 1.3 to u_n and we have for any r and r' verifying $0 < r < r_c$, $0 < r' < r'_c$, $r(N-1) = r'N$ and for $\tau = 1 - r/\widetilde{p}$:

$$\begin{aligned}
&\left(\int_\Omega |u_n|^r \,\mathrm{d}x\right)^{1/\widetilde{p}} + \left(\int_\Gamma |u_n|^{r'} \,\mathrm{d}H_{N-1}\right)^{1/\widetilde{p}'} \leq \\
&\leq C\left(\left(\int_\Omega \frac{|\mathrm{D}u_n|^p}{\left(1 + |u_n|\right)^{p\tau}} \,\mathrm{d}x\right)^{1/p} + \left(\int_\Omega |u_n|^{p-1} \,\mathrm{d}x\right)^{1/p}\right) + C \qquad (8) \\
&\leq C
\end{aligned}$$

(using the previous estimate and noticing that $p\tau > 1$), where C is independent of n.

Moreover, using Lemma 1.4 and the above estimate for $q \in [1, N/(N-1)[$,

$$\int_\Omega |\mathrm{D}u_n|^{q(p-1)} \,\mathrm{d}x \leq C \tag{9}$$

where C is independent of n.

4. The limit process. Existence of a weak solution in the case $\varepsilon_a = \varepsilon_b = 1$

Let us choose $\Phi(t) = \arctan(t) \in \mathrm{Lip}_p(\mathbb{R})$. Set $w_n = \Phi(u_n)$, $\Phi(u_n)$ remains in a bounded set of $W^{1,p}_{\Gamma_0}(\Omega)$. By (ii) in section 3, and the usual compactness on Sobolev Spaces there exists a subsequence still denoted w_n which verifies:

$$\begin{aligned} w_n &\longrightarrow w \quad W^{1,p}_{\Gamma_0}(\Omega) \text{ weakly} \\ &\longrightarrow w \quad L^p(\Omega) \text{ strongly} \\ &\longrightarrow w \quad \text{a.e. in } \Omega\,. \end{aligned}$$

We introduce $u = \tan(w)$, then we have

$$u_n \longrightarrow u \quad \text{a.e. in } \Omega\,.$$

For $T \in \mathrm{Lip}_p(\mathbb{R})$, the sequence $T(u_n)$ lies in a bounded subset of $W^{1,p}(\Omega)$, thus we have $T(u) \in W^{1,p}(\Omega)$. From the previous estimates on u_n, we see that u belongs to $\Lambda^{1,p}_{\Gamma_0}(\Omega)$.

We can use $v = S_\eta(u_n - u^k) \in W^{1,p}_{\Gamma_0}(\Omega)$ as a test function in (6), where

$$S_\eta(\sigma) = \begin{cases} \sigma & \text{if } |\sigma| \le \eta \\ \eta \,\mathrm{sign}(\sigma) & \text{if } |\sigma| > \eta \end{cases}$$

to obtain, as in [Ra2]-[Ra3],

$$\mathrm{D}u_n \longrightarrow \mathrm{D}u \text{ a.e. in } \Omega\,, \quad u_n \longrightarrow u \text{ a.e. in } \Omega\,,$$

and then

$$\widehat{a}(x, u_n, \mathrm{D}u_n) \longrightarrow \widehat{a}(x, u, \mathrm{D}u) \text{ a.e. in } \Omega\,.$$

We want to show the following result on the trace of u on Γ: u exists a.e. on Γ and $u_n \to u$ H_{N-1}-a.e. on Γ when $n \to \infty$.

For any integer $k > 0$, we have $u_n^k \in W^{1,p}(\Omega)$ uniformly with respect to n. So $u_n^k \to u^k$, in $W^{1,p}(\Omega)$ weakly. Thus by trace lemma $\left|u_n^k - u^k\right|_{L^1(\Gamma)} \to 0$ (as $n \to \infty$). Then there exists E_k such that $H_{N-1}(E_k) = 0$ and $u^k(x)$ exists on $\Gamma \setminus E_k$. Now, for any $x \in \Gamma \setminus \bigcup_k E_k$, $u^k(x)$ exists for all k, then letting $k \to \infty$ as in Lemma 1.3, we have that $u^k(x)$ has a limit a.e. on Γ denoted by $u(x) \in [-\infty, +\infty]$. To show that $u(x)$ is finite, we remark that

$$\int_\Gamma \left|u_n^k(x)\right|^s \mathrm{d}H_{N-1} \le \int_\Gamma \left|u_n(x)\right|^s \mathrm{d}H_{N-1} \le C$$

with C independent on n and k, so that $\int_\Gamma \left|u(x)\right|^s \mathrm{d}H_{N-1} \le C$, and consequently $\left|u(x)\right| < +\infty$. Futhermore, we have

$$\begin{aligned} H_{N-1}\left\{x \in \Gamma \mid \left|u_n(x)\right| > k\right\} &\le \frac{1}{k^s} \int_{\{|u_n|>k\}} \left|u_n(x)\right|^s \mathrm{d}H_{N-1} \\ &\le \frac{1}{k^s} \int_\Gamma \left|u_n(x)\right|^s \mathrm{d}H_{N-1} \le \frac{C}{k^s} \\ H_{N-1}\left\{x \in \Gamma \mid \left|u(x)\right| > k\right\} &\le \frac{1}{k^s} \int_{\{|u|>k\}} \left|u(x)\right|^s \mathrm{d}H_{N-1} \\ &\le \frac{1}{k^s} \int_\Gamma \left|u(x)\right|^s \mathrm{d}H_{N-1} \le \frac{C}{k^s}\,. \end{aligned}$$

We decompose Γ in three subsets: $\Gamma = \Gamma_{\alpha_n} \cup \Gamma_{\beta_n} \cup \Gamma_{\gamma_n}$, with

$$\begin{aligned} \Gamma_{\alpha_n} &= \left\{x \in \Gamma \mid \left|u_n(x)\right| > k\right\} \\ \Gamma_{\beta_n} &= \left\{x \in \Gamma \mid \left|u_n(x)\right| \le k\right\} \cap \left\{x \in \Gamma \mid \left|u(x)\right| > k\right\} \\ \Gamma_{\gamma_n} &= \left\{x \in \Gamma \mid \left|u_n(x)\right| \le k\right\} \cap \left\{x \in \Gamma \mid \left|u(x)\right| \le k\right\} \end{aligned}$$

and, for $\eta \in \,]0, s[$, we consider $\int_\Gamma \left|u_n(x) - u(x)\right|^\eta \mathrm{d}H_{N-1}$:

$$\begin{aligned} \int_{\Gamma_{\alpha_n}} \left|u_n(x) - u(x)\right|^\eta \mathrm{d}H_{N-1} &\le C\big(H_{N-1}(\Gamma_{\alpha_n})\big)^{1-\eta/s} \le C\frac{1}{k^{s-\eta}} \\ \int_{\Gamma_{\beta_n}} \left|u_n(x) - u(x)\right|^\eta \mathrm{d}H_{N-1} &\le C\big(H_{N-1}(\Gamma_{\beta_n})\big)^{1-\eta/s} \le C\frac{1}{k^{s-\eta}} \end{aligned}$$

and

$$\int_{\Gamma_{\gamma_n}} \left|u_n(x) - u(x)\right|^\eta \mathrm{d}H_{N-1} = \int_{\Gamma_{\gamma_n}} \left|u_n^k(x) - u^k(x)\right|^\eta \mathrm{d}H_{N-1}\ ;$$

so,

$$\lim_{n\to\infty}\int_\Gamma \big|u_n(x)-u(x)\big|^\eta \,\mathrm{d}H_{N-1}=0\,,$$

and we have proved that

$$u_n \longrightarrow u \quad H_{N-1}\text{-a.e. in } \Gamma \text{ when } n\to\infty\,.$$

The estimates (7), (8) and (9) show that there exist r and r', $p-1<r<r_c$, $s<r'<r'_c=s_c$ so that $u\in L^r(\Omega)$, $\mathrm{D}u\in L^r(\Omega)$, $u\in L^{r'}(\Gamma)$ and

$$\sup_{k>0}\int_\Omega \frac{|\mathrm{D}u^k|^p}{\big(1+|u^k|\big)^{1+\delta}}\,\mathrm{d}x\le C\,,$$

then with Vitali's lemma we deduce from the previous pointwise convergence that

$$\begin{aligned} u_n &\longrightarrow u \quad \text{in } L^{p-1}(\Omega)\\ &\longrightarrow u \quad \text{in } L^s(\Gamma)\\ \widehat{a}(x,u_n,\mathrm{D}u_n) &\longrightarrow \widehat{a}(x,u,\mathrm{D}u) \quad \text{in } L^1(\Omega)\,. \end{aligned}$$

We finally can pass to the limit in (6) and obtain that u is a weak solution of problem (P). □

5. An approximating problem in the case $\varepsilon_a=\varepsilon_b=0$

In this section $H_{N-1}(\Gamma_0)=0$, we consider the problem (P) when $\varepsilon_a=\varepsilon_b=0$ under the supplementary following compatibility condition:

$$\int_\Omega \mathrm{d}\mu=\mu(\Omega)=-\nu(\Gamma_1)=-\int_{\Gamma_1}\mathrm{d}\nu=-\int_\Gamma \mathrm{d}\nu\,.$$

We define the sequences μ_n and ν_n as in section 3, but μ_n and ν_n must satisfy an additional condition:

$$\int_\Omega \mu_n\,\mathrm{d}x+\int_\Gamma \nu_n\,\mathrm{d}H_{N-1}=0\,.$$

We will distinguish the cases $p>2N/(N+2)$ and $p\le 2N/(N+2)$.

First case, $p > 2N/(N+2)$

We consider the approximating problem below.

Find $u_n \in W^{1,p}(\Omega)$ verifying $\int u_n \,\mathrm{d}x = 0$, and for all $v \in W^{1,p}(\Omega)$

$$\int_\Omega \widehat{a}(x, u_n, \mathrm{D}u_n)\mathrm{D}v \,\mathrm{d}x = \int_\Gamma v\nu_n \,\mathrm{d}H_{N-1} + \int_\Omega v\mu_n \,\mathrm{d}x \,. \tag{10}$$

There is a solution $u_n \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ for this problem (see [LL] for the existence and [RT] for L^∞-estimates). We have for u_n the following estimates obtained with the same test functions as in section 3, that is:

(i) $\forall\, \delta > 0$,

$$\int_\Omega \frac{|\mathrm{D}u_n|^p}{(1+|u_n|)^{1+\delta}} \,\mathrm{d}x \le C(\delta)$$

where $C(\delta)$ is independent of n;

(ii) $\forall\, T \in \mathrm{Lip}_p(\mathbb{R})$,

$$\left\| \mathrm{D}(T(u_n)) \right\|_{L^p(\Omega)} \le C \left\| T' \right\|_{L^p(\mathbb{R})}$$

with C independent of n.

But we cannot use Lemma 1.3 for u_n; we introduce another function $\overline{u}_n$ by noticing that, for all $\tau < 1$ there exists $c_n^\tau \in \mathbb{R}$ such that

$$\int_\Omega \Phi_\tau(u_n + c_n^\tau) \,\mathrm{d}x = 0 \,, \quad \overline{u}_n = u_n + c_n^\tau \,.$$

We fix

$$\tau \in \left] \frac{1}{p} \,, \min \left\{ \left(1 - \frac{N-p}{Np}\right), \left(1 - \left(\frac{p-1}{p}\right)^2\right) \right\} \right[\,.$$

Note that

$$|\Omega|\, |c_n^\tau| = \left| \int_\Omega c_n^\tau \,\mathrm{d}x \right| = \left| \int_\Omega (\overline{u}_n - u_n) \,\mathrm{d}x \right| \le \int_\Omega |\overline{u}_n| \,\mathrm{d}x \,,$$

and Lemma 1.3 can be applied to $\overline{u}_n$ with $r = (1-\tau)Np/(N-p)$ (> 1). Exactly:

$$\int_\Omega |\overline{u}_n|^r \,\mathrm{d}x \le C \int_\Omega \frac{|\mathrm{D}\overline{u}_n|^p}{(1+|\overline{u}_n|)^{p\tau}} \,\mathrm{d}x \le C$$

with C independent of n (choose $v = \Phi_{p\tau}(\overline{u}_n)$ in (10)). So $|c_n^\tau| \le C$ with C independent of n and then $\int_\Omega |u_n|^r \,\mathrm{d}x \le C$.

Using Lemma 1.4 for u_n we have the following inequality with

$$r = (1-\tau)\frac{Np}{N-p}\ (> p-1) \quad \text{and} \quad q = \frac{r}{r+1}\,\frac{p}{p-1}\ (> 1)$$

$$\int_\Omega |\mathrm{D}u_n|^{q(p-1)}\,\mathrm{d}x \le C$$

with C independent of n.

We introduce u as in section 4. We can apply Vitali's lemma when $n \to \infty$ (note that $r > \sigma$) to prove that $\widehat{a}(x, u_n, \mathrm{D}u_n) \to \widehat{a}(x, u, \mathrm{D}u)$ and that u is a weak solution of problem (P).

Second case, $p \le 2N/(N+2)$

We assume here that $\widehat{a}$ does no depend explicitely on u_n; we note $\widehat{a}(x, \mathrm{D}u_n)$ and we consider the following problem, after fixing a real

$$\tau \in \left]\frac{1}{p}\,, 1-(p-1)\frac{N-p}{Np}\right[\,.$$

Find $u_n \in W^{1,p}(\Omega)$ verifying $\int_\Omega \Phi_\tau(u_n)\,\mathrm{d}x = 0$, and for all $v \in W^{1,p}(\Omega)$:

$$\int_\Omega \widehat{a}(x, \mathrm{D}u_n)\mathrm{D}v\,\mathrm{d}x = \int_\Gamma v\nu_n\,\mathrm{d}H_{N-1} + \int_\Omega v\mu_n\,\mathrm{d}x\,. \tag{10bis}$$

There exists a solution $u_n \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ verifying (10bis): for proving this we apply the Leray–Lions theorem [LL] and L^∞-estimates results [RT]; we find a function $w_n \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that $\int_\Omega w_n\,\mathrm{d}x = 0$, and for all $v \in W^{1,p}(\Omega)$:

$$\int_\Omega \widehat{a}(x, \mathrm{D}w_n)\mathrm{D}v\,\mathrm{d}x = \int_\Gamma v\nu_n\,\mathrm{d}H_{N-1} + \int_\Omega v\mu_n\,\mathrm{d}x\,.$$

As in the first case, there exists $c_n^\tau \in \mathbb{R}$ such that $\int_\Omega \Phi_\tau(w_n + c_n^\tau)\,\mathrm{d}x = 0$. So, $u_n = w_n + c_n^\tau$ verifies (10bis), and we derive the estimates below:

(i) $\forall\, \delta > 0$,

$$\int_\Omega \frac{|\mathrm{D}u_n|^p}{(1+|u_n|)^{1+\delta}}\,\mathrm{d}x \le C(\delta)$$

where $C(\delta)$ is independent of n;

(ii) $\forall\, T \in \mathrm{Lip}_p(\mathbb{R})$,

$$\left\| \mathrm{D}(T(u_n)) \right\|_{L^p(\Omega)} \leq C \left\| T' \right\|_{L^p(\mathbb{R})}$$

with C independent of n;

(iii) for $r = (1-\tau)Np/(N-p)$ $(> p-1)$ and $q = \big(r/(r+1)\big)p/(p-1)$ (> 1):

$$\int_\Omega \left| u_n \right|^r \mathrm{d}x \leq C\,,$$

$$\int_\Omega \left| \mathrm{D}u_n \right|^{q(p-1)} \mathrm{d}x \leq C$$

with C independent of n.

We conclude as in the first case and find a weak solution u for the problem (P). □

6. End of the proof of Theorem 1

u is also a renormalized solution when μ and ν are in L^1.

Here μ and ν are respectively in $L^1(\Omega)$ and $L^1(\Gamma)$. We use the sequences u_n defined in sections 3 and 5 and u the corresponding weak solution. We give additional properties to prove (5) for $v \in W^{1,p}_{\Gamma_0}(\Omega) \cap L^\infty(\Omega)$ and $T \in W^{1,\infty}_{\mathrm{compact}}(\mathbb{R})$.

LEMMA 6.1.— *The sequences u_n introduced in (6) and (10) verify:*

$$\limsup_{n\to\infty} \int_{m \leq |u_n| \leq m+1} \left| \mathrm{D}u_n \right|^p \mathrm{d}x = \mathrm{o}(1) \quad \textit{when } m \to \infty\,. \tag{11}$$

Proof.— Case $\varepsilon_a = \varepsilon_b = 1$. First we take $v = u_n^{m+1} - u_n^m$ as a test function in (6) and we obtain:

$$\alpha \int_{m \leq |u_n| \leq m+1} \left| \mathrm{D}u_n \right|^p \mathrm{d}x \leq$$

$$\leq \int_\Omega \left| u_n \right|^{p-1} \left| u_n^{m+1} - u_n^m \right| \mathrm{d}x + \int_{\Gamma_1} \left| u_n \right|^s \left| u_n^{m+1} - u_n^m \right| \mathrm{d}H_{N-1} +$$

$$+ \int_{\Gamma_1} \left| u_n^{m+1} - u_n^m \right| \nu_n \,\mathrm{d}H_{N-1} + \int_\Omega \left| u_n^{m+1} - u_n^m \right| \mu_n \,\mathrm{d}x\,,$$

and consequently, using the properties of the sequence u_n and Fatous's lemma,

$$\alpha \limsup_{n\to\infty} \int_{m\le|u_n|\le m+1} |\mathrm{D}u_n|^p \,\mathrm{d}x \le$$
$$\le \int_\Omega |u|^{p-1}|u^{m+1}-u^m|\,\mathrm{d}x + \int_{\Gamma_1} |u|^s|u^{m+1}-u^m|\,\mathrm{d}H_{N-1} +$$
$$+ \int_{\Gamma_1} |u^{m+1}-u^m|\nu\,\mathrm{d}H_{N-1} + \int_\Omega |u^{m+1}-u^m|\mu\,\mathrm{d}x\,.$$

Then let m tend to infinity, using the dominated convergence theorem we get (11).

Case $\varepsilon_a = \varepsilon_b = 0$. For the sequence defined in section 5, taking $v = u_n^{m+1} - u_n^m$ as a test function in (10) or (10bis), we obtain:

$$\alpha \int_{m\le|u_n|\le m+1} |\mathrm{D}u_n|^p \,\mathrm{d}x \le$$
$$\le \int_\Gamma |u_n^{m+1}-u_n^m|\nu_n\,\mathrm{d}H_{N-1} + \int_\Omega |u_n^{m+1}-u_n^m|\mu_n\,\mathrm{d}x$$

$$\alpha \limsup_{n\to\infty} \int_{m\le|u_n|\le m+1} |\mathrm{D}u_n|^p \,\mathrm{d}x \le$$
$$\le \int_\Gamma |u^{m+1}-u^m|\nu\,\mathrm{d}H_{N-1} + \int_\Omega |u^{m+1}-u^m|\mu\,\mathrm{d}x$$

and for $m\to\infty$ we get (11). □

Corollary of Lemma 6.1.— *Let h_m be the continuous function defined on $\mathbb{R}$ by:*

$$h_m(\sigma)\begin{cases} = 1 & \textit{if } |\sigma|\le m \\ = 0 & \textit{if } |\sigma|\ge m+1 \\ \textit{affine on } [m\,,\,m+1] \textit{ and on } [-m-1\,,\,-m]; \end{cases}$$

then for any $v\in W^{1,p}_{\Gamma_0}(\Omega)\cap L^\infty(\Omega)$, for any $T\in W^{1,\infty}_{\mathrm{compact}}(\mathbb{R})$,

$$\limsup_{n\to\infty}\int_\Omega \widehat{a}(x,u_n,\mathrm{D}u_n)\mathrm{D}u_n T(u)vh'_m(u_n)\,\mathrm{d}x = \mathrm{o}(1) \quad \textit{when } m\to\infty\,.$$

Proof

$$\left|\int_\Omega \widehat{a}(x, u_n, \mathrm{D}u_n)\mathrm{D}u_n T(u) v h'_m(u_n)\,\mathrm{d}x\right| \leq$$
$$\leq \int_{m\leq|u_n|\leq m+1} \widehat{a}(x, u_n, \mathrm{D}u_n)\mathrm{D}u_n \big|T(u)v\big|\,\mathrm{d}x$$
$$\leq C\int_{m\leq|u_n|\leq m+1} \big|u_n\big|^\sigma \big|\mathrm{D}u_n\big|\,\big|T(u)v\big|\,\mathrm{d}x +$$
$$+ C\int_{m\leq|u_n|\leq m+1} \big|\mathrm{D}u_n\big|^p \big|T(u)v\big|\,\mathrm{d}x +$$
$$+ C\int_{m\leq|u_n|\leq m+1} a_0(x)\big|\mathrm{D}u_n\big|\,\big|T(u)v\big|\,\mathrm{d}x$$

with hypothesis (A2); futhermore $\big|T(u)v\big|$ is bounded with M, the support of T is compact, a_0 is in $L^{p'}(\Omega)$, then

$$\limsup_{n\to\infty} \int_{m\leq|u_n|\leq m+1} \big|\mathrm{D}u_n\big|^p\big|T(u)v\big|\,\mathrm{d}x \leq$$
$$\leq M \limsup_{n\to\infty} \int_{m\leq|u_n|\leq m+1} \big|\mathrm{D}u_n\big|^p\,\mathrm{d}x$$

$$\limsup_{n\to\infty} \int_{m\leq|u_n|\leq m+1} \big|u_n\big|^\sigma \big|\mathrm{D}u_n\big|\,\big|T(u)v\big|\,\mathrm{d}x \leq$$
$$\leq \limsup_{n\to\infty} \left[\left(\int_{m\leq|u_n|\leq m+1} \big|u_n\big|^{\sigma p/(p-1)}\big|T(u)v\big|\,\mathrm{d}x\right)^{1/p'} \times\right.$$
$$\left. \times \left(\int_{m\leq|u_n|\leq m+1} \big|\mathrm{D}u_n\big|^p\big|T(u)v\big|\,\mathrm{d}x\right)^{1/p}\right]$$
$$\leq MC \limsup_{n\to\infty} \left(\int_{m\leq|u_n|\leq m+1} \big|\mathrm{D}u_n\big|^p\,\mathrm{d}x\right)^{1/p},$$

where C is independent of n,

$$\limsup_{n\to\infty} \int_{m\le|u_n|\le m+1} a_0(x)\big|\mathrm{D}u_n\big|\,\big|T(u)v\big|\,\mathrm{d}x \le$$

$$\le \limsup_{n\to\infty} \left[\left(\int_{m\le|u_n|\le m+1} a_0(x)^{p'}\big|T(u)v\big|\,\mathrm{d}x\right)^{1/p'} \times\right.$$

$$\left.\times \left(\int_{m\le|u_n|\le m+1} \big|\mathrm{D}u_n\big|^p\big|T(u)v\big|\,\mathrm{d}x\right)^{1/p}\right]$$

$$\le M\big\|a_0\big\|_{L^{p'}} \limsup_{n\to\infty} \left(\int_{m\le|u_n|\le m+1} \big|\mathrm{D}u_n\big|^p\,\mathrm{d}x\right)^{1/p}.$$

We conclude with Lemma 6.1. □

We now prove that u is a renormalized solution of problem (P).

• First we consider the case $\varepsilon_a = \varepsilon_b = 1$. For $v \in W^{1,p}_{\Gamma_0}(\Omega) \cap L^\infty(\Omega)$, $T \in W^{1,\infty}_{\mathrm{compact}}(\mathbb{R})$ and $m \in \mathbb{N}$, we take $h_m(u_m)T(u)v$ as a test function in (6):

$$\int_\Omega \widehat{a}(x,u_n,\mathrm{D}u_n)T(u)v\mathrm{D}u_n h'_m(u_n)\,\mathrm{d}x =$$

$$= -\int_\Omega \widehat{a}(x,u_n,\mathrm{D}u_n)\mathrm{D}\big(T(u)v\big)\,h'_m(u_n)\,\mathrm{d}x +$$

$$-\int_\Omega \big|u_n\big|^{p-2}u_nT(u)vh_m(u_n)\,\mathrm{d}x +$$

$$-\int_{\Gamma_1} \big|u_n\big|^{s-1}u_nT(u)vh_m(u_n)\,\mathrm{d}H_{N-1} +$$

$$+\int_\Gamma T(u)v\nu_nh_m(u_n)\,\mathrm{d}H_{N-1} + \int_\Omega T(u)v\mu_nh_m(u_n)\,\mathrm{d}x\,.$$

First, when $n\to\infty$

The five terms in the second member have a limit and then:

$$\lim_{n\to\infty}\int_\Omega \widehat{a}(x,u_n,\mathrm{D}u_n)T(u)v\mathrm{D}u_nh'_m(u_n)\,\mathrm{d}x =$$

$$= -\int_\Omega \widehat{a}(x,u,\mathrm{D}u)\mathrm{D}\big(T(u)v\big)h_m(u)\,\mathrm{d}x - \int_\Omega |u|^{p-2}uT(u)vh_m(u)\,\mathrm{d}x +$$

$$-\int_{\Gamma_1}|u|^{s-1}uT(u)vh_m(u)\,\mathrm{d}H_{N-1} + \int_\Gamma T(u)v\nu h_m(u)\,\mathrm{d}H_{N-1} +$$

$$+\int_\Omega T(u)v\mu h_m(u)\,\mathrm{d}x\,.$$

Secondly, when $m \to \infty$

For m large enough (say $\operatorname{supp} T \subset [-m, m]$), we can write

$$\widehat{a}(x, u, \mathrm{D}u)\mathrm{D}\big(T(u)v\big) = \widehat{a}(x, u^m, \mathrm{D}u^m)\mathrm{D}\big(T(u)v\big).$$

The corollary of Lemma 6.1 and the dominated convergence theorem permit to conclude that u verifies (5).

• In the case of $\varepsilon_0 = \varepsilon_b = 0$ and u_n Is the sequence defined in section 5, the proof is similar.

7. Uniqueness result

We have seen that there exists at least one function $u \in \Pi^{1,p}_{\Gamma_0}(\Omega)$ being a renormalized solution in the sense of definition 2.2. Futhermore, this solution is a limit of a smooth sequence u_n being a classical solution of an approximate problem.

We want to show some uniqueness result; we begin by a "simple" case.

THEOREM 2.— *Assume that* $\widehat{a}(x, \eta, \xi)$ *is independent of* η *(we note* $\widehat{a}(x, \xi)$*), then there exists a unique renormalized solution* $w \in \Pi^{1,p}_{\Gamma_0}(\Omega)$ *up to a constant if* $\varepsilon_a = \varepsilon_b = 0$ *verifying for any* $v \in W^{1,p}_{\Gamma_0}(\Omega) \cap L^\infty(\Omega)$, *for any* $T \in W^{1,\infty}_{\text{compact}}(\mathbb{R})$*:*

$$\begin{aligned}\int_\Omega \widehat{a}(x, \mathrm{D}w)\mathrm{D}\big(T(w)v\big)\,\mathrm{d}x + \varepsilon_a \int_\Omega |w|^{p-2} wT(w)v\,\mathrm{d}x +{}& \\ + \varepsilon_b \int_{\Gamma_1} |w|^{s-1} wT(w)v\,\mathrm{d}H_{N-1} ={}& \\ = \int_\Gamma T(w)v\nu\,\mathrm{d}H_{N-1} + \int_\Omega T(w)v\mu\,\mathrm{d}x\,.&\end{aligned}$$

Proof.— It remains to show the uniqueness of the solution; let us call w an arbitrary solution of the preceding problem; we keep the notation u for the particular solution being a limit of sequence u_n (that we found before). The aim is to show that $w = u$.

Let $m \in \mathbb{N}$, $k \in \mathbb{N}$ and consider $h_m(\sigma)$ as in section 6; the function $h_m(w)\big(w^{m+1} - u_n\big)^k$ can be used as a test function in (6) or (10) or (10bis), that is

$$\begin{aligned}
&\int_\Omega \widehat{a}(x, \mathrm{D}u_n)\mathrm{D}\left(h_m(w)\big(w^{m+1} - u_n\big)^k\right) \mathrm{d}x + \\
&\qquad + \varepsilon \int_\Omega |u_n|^{p-2} u_n h_m(w)\big(w^{m+1} - u_n\big)^k \,\mathrm{d}x + \\
&\qquad + \varepsilon \int_\Gamma |u_n|^{s-1} u_n h_m(w)\big(w^{m+1} - u_n\big)^k \,\mathrm{d}H_{N-1} = \\
&\quad = \int_\Gamma h_m(w)\big(w^{m+1} - u_n\big)^k \nu_n \,\mathrm{d}H_{N-1} + \int_\Omega h_m(w)\big(w^{m+1} - u_n\big)^k \mu_n \,\mathrm{d}x
\end{aligned}$$

($\varepsilon = 1$ for (6) and $\varepsilon = 0$ for (10) or (10bis)).

But we have also (5) for $u = w$ with $T = h_m$ and $v = \big(w^{m+1} - u_n\big)^k$, that is

$$\begin{aligned}
&\int_\Omega \widehat{a}(x, \mathrm{D}w)\mathrm{D}\left(h_m(w)\big(w^{m+1} - u_n\big)^k\right) \mathrm{d}x + \\
&\qquad + \varepsilon \int_\Omega |w|^{p-2} w h_m(w)\big(w^{m+1} - u_n\big)^k \,\mathrm{d}x + \\
&\qquad + \varepsilon \int_\Gamma |w|^{s-1} w h_m(w)\big(w^{m+1} - u_n\big)^k \,\mathrm{d}H_{N-1} = \\
&\quad = \int_\Gamma h_m(w)\big(w^{m+1} - u_n\big)^k \nu \,\mathrm{d}H_{N-1} + \int_\Omega h_m(w)\big(w^{m+1} - u_n\big)^k \mu \,\mathrm{d}x\,.
\end{aligned}$$

Let us make the difference between the two last equations; we get:

$$\begin{aligned}
&\int_\Omega \big(\widehat{a}(x, \mathrm{D}w) - \widehat{a}(x, \mathrm{D}u_n)\big) h_m(w)\mathrm{D}\left(\big(w^{m+1} - u_n\big)^k\right) \mathrm{d}x + \\
&\qquad + \int_\Omega \big(\widehat{a}(x, \mathrm{D}w) - \widehat{a}(x, \mathrm{D}u_n)\big)(\mathrm{D}w) h_m'(w)\big(w^{m+1} - u_n\big)^k \,\mathrm{d}x + \\
&\qquad + \varepsilon \int_\Omega \left(|w|^{p-2} w - |u_n|^{p-2} u_n\right) h_m(w)\big(w^{m+1} - u_n\big)^k \,\mathrm{d}x + \\
&\qquad + \varepsilon \int_\Gamma \left(|w|^{s-1} w - |u_n|^{s-1} u_n\right) h_m(w)\big(w^{m+1} - u_n\big)^k \,\mathrm{d}H_{N-1} = \\
&\quad = \int_\Gamma h_m(w)\big(w^{m+1} - u_n\big)^k (\nu - \nu_n) \,\mathrm{d}H_{N-1} + \\
&\qquad + \int_\Omega h_m(w)\big(w^{m+1} - u_n\big)^k (\mu - \mu_n) \,\mathrm{d}x\,. \qquad (12)
\end{aligned}$$

The third ans fourth terms of the left hand-side are non-negative; when dropping those terms, we get:

$$\int_\Omega \big(\widehat{a}(x,\mathrm{D}w)-\widehat{a}(x,\mathrm{D}u_n)\big)h_m(w)\mathrm{D}\Big(\big(w^{m+1}-u_n\big)^k\Big)\,\mathrm{d}x \le$$

$$\le k\int_{m\le|w|\le m+1}\Big(C\big|\mathrm{D}u_n\big|^{p-1}+C\big|\mathrm{D}w\big|^{p-1}+2Ca_0(x)\Big)\,|\mathrm{D}w|\,\mathrm{d}x+$$

$$+\,k\int_\Gamma|\nu-\nu_n|\,\mathrm{d}H_{N-1}+k\int_\Omega|\mu-\mu_n|\,\mathrm{d}x\,.$$

• In a first time, letting m go to infinity (n and k being fixed), we have the two inequalities:

$$\int_{m\le|w|\le m+1}\big|\mathrm{D}u_n\big|^{p-1}\big|\mathrm{D}w\big|\,\mathrm{d}x\le$$

$$\le\left(\int_{m\le|w|\le m+1}\big|\mathrm{D}u_n\big|^{p}\,\mathrm{d}x\right)^{1/p'}\left(\int_{m\le|w|\le m+1}\big|\mathrm{D}w\big|^{p}\,\mathrm{d}x\right)^{1/p}$$

$$\int_{m\le|w|\le m+1}a_0(x)\big|\mathrm{D}w\big|\,\mathrm{d}x\le$$

$$\le\left(\int_{m\le|u_n|\le m+1}a_0(x)^{p'}\,\mathrm{d}x\right)^{1/p'}\left(\int_{m\le|u_p|\le m+1}\big|\mathrm{D}w\big|^{p}\,\mathrm{d}x\right)^{1/p}$$

and, with the fact $w\in\Pi^{1,p}_{\Gamma_0}(\Omega)$, it comes:

$$\int_{|w-u_n|\le k}\big(\widehat{a}(x,\mathrm{D}w)-\widehat{a}(x,\mathrm{D}u_n)\big)\mathrm{D}\big(w-u_n\big)\,\mathrm{d}x\le$$

$$\le k\int_\Gamma|\nu-\nu_n|\,\mathrm{d}H_{N-1}+k\int_\Omega|\mu-\mu_n|\,\mathrm{d}x\,.$$

• Now, k being fixed, we let $n\to\infty$; Fatous's lemma and the monotony of $\widehat{a}$ (condition A3) lead to:

$$\int_{|w-u|\le k}\big(\widehat{a}(x,\mathrm{D}w)-\widehat{a}(x,\mathrm{D}u)\big)\mathrm{D}\big(w-u\big)\,\mathrm{d}x=0\,.$$

• Finally, we let $k\to\infty$ and we have, using Beppo Lévi theorem,

$$\int_\Omega\big(\widehat{a}(x,\mathrm{D}w)-\widehat{a}(x,\mathrm{D}u)\big)\mathrm{D}\big(w-u\big)\,\mathrm{d}x=0\,.$$

Consequently $\mathrm{D}w = \mathrm{D}u$, then $\mathrm{D}w^k = \mathrm{D}u^k$ for all $k \in \mathbb{N}$, that is w^k and u^k differ from a constant C_k; these constants are uniformly bounded because u^k and w^k are in a bounded subset of L^{p-1}, so a subsequence of (C_k) converges to a constant C, and we conclude $w = u + C$.

We prove that $C = 0$ if $\varepsilon \neq 0$: if $\varepsilon_a \neq 0$, without dropping the positive terms in (12) and using Fatou's lemma, we deduce:

$$\int_\Omega \left(|w|^{p-2}w - |u|^{p-2}u\right)(w-u)^k \,\mathrm{d}x \leq 0$$

and, when $k \to \infty$,

$$\int_\Omega \left(|w|^{p-2}w - |u|^{p-2}u\right)(w-u)\,\mathrm{d}x \leq 0\,,$$

so $u = w$ that is $C = 0$; if $\varepsilon_b \neq 0$,

$$\int_\Gamma \left(|w|^{s-1}w - |u|^{s-1}u\right)(w-u)\,\mathrm{d}H_{N-1} \leq 0$$

for the same reason, and then $C = 0$. □

Now consider the general case for $\widehat{a}$, that is $\widehat{a}(x,\eta,\xi)$ depends of η, for problem (P) with $\varepsilon_a = \varepsilon_b = 1$.

If $\widehat{a}$ satisfies the supplementary condition:

(A4) for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^N$, for all η_1 and η_2 in $[-h\,,+h] \subset \mathbb{R}$:

$$\left|\widehat{a}(x,\eta_1,\xi) - \widehat{a}(x,\eta_2,\xi)\right| \leq C(h)|\eta_1 - \eta_2|\left(b_0(x) + |\xi|^{p-1}\right)$$

where $C(h)$ is a constant depending only of h, and $b_0 \in L^{p'}(\Omega)$,

then we have an uniqueness result for the mixed problem.

THEOREM 3.— *Assume $\widehat{a}$ satisfies the hypothesis (A1) to (A4). There exists a unique renormalized solution $w \in \Pi^{1,p}_{\Gamma_0}(\Omega)$ verifying for any $v \in W^{1,p}_{\Gamma_0}(\Omega) \cap L^\infty(\Omega)$, for any $T \in W^{1,\infty}_{\text{compact}}(\mathbb{R})$:*

$$\begin{aligned}\int_\Omega \widehat{a}(x,w,\mathrm{D}w)\mathrm{D}(T(w)v)\,\mathrm{d}x &+ \int_\Omega |w|^{p-2}wT(w)v\,\mathrm{d}x + \\ &+ \int_{\Gamma_1}|w|^{s-1}wT(w)v\,\mathrm{d}H_{N-1} = \\ &= \int_\Gamma T(w)v\nu\,\mathrm{d}H_{N-1} + \int_\Omega T(w)v\mu\,\mathrm{d}x\,.\end{aligned} \tag{13}$$

Proof. — As in the proof of Theorem 2 we call w an abritrary solution of (13) and compare w and u (the solution founded in section 4).

We introduce, for $\eta \in \,]0\,,1[$ the function F_η defined from $\mathbb{R}$ to $\mathbb{R}$ by:

$$F_\eta(\sigma) = \begin{cases} 1 - \dfrac{\ln(\sigma)}{\ln(\eta)} & \text{if } \sigma \geq \eta \\ 0 & \text{if } \sigma < \eta. \end{cases}$$

Then, for $m \in \mathbb{N}$, $k \in \mathbb{N}$, $\eta \in \,]0\,,1[$, we take $v = h_m(w)F_\eta\big((w^{m+1} - u_n)^k\big)$ as a test function in (6); we have also (13) with $T = h_m$ and $v = F_\eta\big((w^{m+1} - u_n)^k\big)$; we take the difference between the two corresponding equations and we obtain:

$$\begin{aligned}
&\int_\Omega (\widehat{a}(x,w,\mathrm{D}w) - \widehat{a}(x,u_n,\mathrm{D}u_n))\, h_m(w) \mathrm{D}\Big((w^{m+1} - u_n)^k\Big)\, F'_\eta\big((w^{m+1} - u_n)^k\big)\,\mathrm{d}x \,+ \\
&\quad + \int_\Omega (\widehat{a}(x,w,\mathrm{D}w) - \widehat{a}(x,u_n,\mathrm{D}u_n))\, \mathrm{D}w\, h'_m(w) F_\eta\Big((w^{m+1} - u_n)^k\Big)\,\mathrm{d}x \,+ \\
&\quad + \int_\Omega \Big(|w|^{p-2}w - |u_n|^{p-2}u_n\Big)\, h_m(w) F_\eta\Big((w^{m+1} - u_n)^k\Big)\,\mathrm{d}x \,+ \\
&\quad + \int_\Gamma \Big(|w|^{s-1}w - |u_n|^{s-1}u_n\Big)\, h_m(w) F_\eta\Big((w^{m+1} - u_n)^k\Big)\,\mathrm{d}H_{N-1} = \\
&\quad = \int_\Gamma h_m(w) F_\eta\Big((w^{m+1} - u_n)^k\Big)\,(\nu - \nu_n)\,\mathrm{d}H_{N-1} \,+ \\
&\qquad + \int_\Omega h_m(w) F_\eta\Big((w^{m+1} - u_n)^k\Big)\,(\mu - \mu_n)\,\mathrm{d}x\,.
\end{aligned} \tag{14}$$

We consider separately all these terms (four in the first left hand-side and two in the right hand-side) and let successively $m \to \infty$, $\eta \to 0$, $n \to \infty$, $k \to \infty$.

The fourth term of the left hand-side of (14) is non-negative; we drop it.

The two terms in the second member are respectively bounded by

$$\left(1 - \frac{\ln k}{\ln \eta}\right) \int_\Gamma |\nu - \nu_n|\,\mathrm{d}H_{N-1} \quad \text{and} \quad \left(1 - \frac{\ln k}{\ln \eta}\right) \int_\Omega |\mu - \mu_n|\,\mathrm{d}x\,.$$

The first term is decomposed in two parts, using

$$\begin{aligned}
&(\widehat{a}(x,w,\mathrm{D}w) - \widehat{a}(x,u_n,\mathrm{D}u_n)) = \\
&\quad = \big(\widehat{a}(x,w,\mathrm{D}w) - \widehat{a}(x,w,\mathrm{D}u_n)\big) + \big(\widehat{a}(x,w,\mathrm{D}u_n) - \widehat{a}(x,u_n,\mathrm{D}u_n)\big)\,,
\end{aligned}$$

we use the hypothesis (A2) and (A4) on $\widehat{a}$, it comes

$$\int_{|w^{m+1}-u_n|\le k} (\widehat{a}(x,w,\mathrm{D}w)-\widehat{a}(x,w,\mathrm{D}u_n))h_m(w)\mathrm{D}(w^{m+1}-u_n)\times \\ \times F'_\eta\Big((w^{m+1}-u_n)^k\Big)\,\mathrm{d}x + \\ + \int_\Omega \Big(|w|^{p-2}w-|u_n|^{p-2}u_n\Big)\,h_m(w)F_\eta\Big((w^{m+1}-u_n)^k\Big)\,\mathrm{d}x$$

which is less than

$$\Big(1-\frac{\ln k}{\ln \eta}\Big)\int_{m\le|w|\le m+1}\mathfrak{C}|\mathrm{D}w|\,\mathrm{d}x + \\ + \Big(1-\frac{\ln k}{\ln \eta}\Big)\int_\Gamma |\nu-\nu_n|\,\mathrm{d}H_{N-1} + \Big(1-\frac{\ln k}{\ln \eta}\Big)\int_\Omega |\mu-\mu_n|\,\mathrm{d}x + \\ + \int_{\eta<|w^{m+1}-u_n|\le k} C(h)|w-u_n|\,\Big[b_0(x)+|\mathrm{D}u_n|^{p-1}\Big]\times \\ \times \frac{1}{|w^{m+1}-u_n|\ln(1/\eta)}\,\big|\mathrm{D}(w^{m+1}-u_n)\big|\,\mathrm{d}x$$

where $\mathfrak{C} = \Big[C|\mathrm{D}u_n|^{p-1}+C|\mathrm{D}w|^{p-1}+C|u_n|^\sigma+C|w|^\sigma+2Ca_0(x)\Big]$; hypothesis (A4) is uded with $h=k+\|u_n\|_\infty$.

Now, let m go to infinity (η, n, k fixed); we get the following convergences:

$$\int_{m\le|w|\le m+1}\mathfrak{C}|\mathrm{D}w|\,\mathrm{d}x \longrightarrow 0\,,$$

$$\int_\Omega \Big(|w|^{p-2}w-|u_n|^{p-2}u_n\Big)\,h_m(w)F_\eta\Big((w^{m+1}-u_n)^k\Big)\,\mathrm{d}x \\ \longrightarrow \int_\Omega \Big(|w|^{p-2}w-|u_n|^{p-2}u_n\Big)\,F_\eta\Big((w-u_n)^k\Big)\,\mathrm{d}x\,,$$

$$\int_{|w^{m+1}-u_n|\le k} (\widehat{a}(x,w,\mathrm{D}w)-\widehat{a}(x,w,\mathrm{D}u_n))h_m(w)\times \\ \times \mathrm{D}(w^{m+1}-u_n)F'_\eta\Big((w^{m+1}-u_n)^k\Big)\,\mathrm{d}x$$

has a limit which is non-negative (A3),

$$\int_{\eta<|w^{m+1}-u_n|\le k} C(h)|w-u_n|\,\Big[b_0(x)+|\mathrm{D}u_n|^{p-1}\Big]\times \\ \times \frac{1}{|w^{m+1}-u_n|\ln(1/\eta)}\,\big|\mathrm{D}(w^{m+1}-u_n)\big|\,\mathrm{d}x \\ \longrightarrow \int_{\eta<|w-u_n|\le k} C(h)\,\Big[b_0(x)+|\mathrm{D}u_n|^{p-1}\Big]\,\frac{1}{\ln(1/\eta)}\,\big|\mathrm{D}(w-u_n)\big|\,\mathrm{d}x\,.$$

In a second step, $\eta \to \infty$:

$$\int_{\eta<|w-u_n|\leq k} C(h)\left[b_0(x)+\left|\mathrm{D}u_n\right|^{p-1}\right]\frac{1}{\ln(1/\eta)}\left|\mathrm{D}(w-u_n)\right|\mathrm{d}x \longrightarrow 0\,,$$

$$\int_\Omega\left(\left|w\right|^{p-2}w-\left|u_n\right|^{p-2}u_n\right)F_\eta\Big(\left(w-u_n\right)^k\Big)\,\mathrm{d}x$$
$$\longrightarrow \int_\Omega\left(\left|w\right|^{p-2}w-\left|u_n\right|^{p-2}u_n\right)\Big(\left(w-u_n\right)^k\Big)^+\,\mathrm{d}x\,.$$

In a third step $n \to \infty$ and finally $k \to \infty$,

$$\int_\Omega\left(\left|w\right|^{p-2}w-\left|u_n\right|^{p-2}u_n\right)\Big(\left(w-u_n\right)^k\Big)^+\,\mathrm{d}x$$
$$\longrightarrow \int_\Omega\left(\left|w\right|^{p-2}w-\left|u\right|^{p-2}u\right)\left(w-u\right)^+\mathrm{d}x\,.$$

We can conclude that $(w-u)^+ = 0$.

By an analogous manner we prove that $(u-w)^+ = 0$, and the Theorem 3 is proved. □

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