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# On the Novikov complex for Rational Morse Forms(*)

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RÉSUMÉ. — Soit $\omega$ une 1-forme fermée sur une variété compacte $M$. Supposons que les zéros de $\omega$ soient non dégénérés et que sa classe de cohomologie $[\omega] \in H^1(M, \mathbb{R})$ soit intégrale à constante multiplicative près. Soit $\widehat{p} : \widehat{M} \to M$ un revêtement avec groupe structural $G$, tel que $\widehat{p}^*\big([\omega]\big) = 0$.

À partir de ces données nous construisons un complexe de chaînes sur la complétion de Novikov de l'anneau $\mathbb{Z}G$. Ce complexe est libre et muni d'une base dont le nombre de générateurs en degré $k$ est égal au nombre de zéros de $\omega$ d'indice $k$ pour tout $k$. Le type simple d'homotopie de ce complexe est égal à celui de la complétion de Novikov du complexe de chaînes simpliciales de $\widehat{M}$.

ABSTRACT. — Let $\omega$ a closed 1-form with non-degenerate zeros on a compact manifold $M$. Assume that the cohomology class $[\omega] \in H^1(M, \mathbb{R})$ is integral up to a multiplicative constant. Let $\widehat{p} : \widehat{M} \to M$ be a covering with structure group $G$, such that $\widehat{p}^*\big([\omega]\big) = 0$.

To this data we associate a free based complex $C_*$ over the Novikov completion of $\mathbb{Z}G$. For every $k$ the number of free generatores of $C_k$ equals the number of zeros of $\omega$ of index $k$. The simple homotopy type of $C_*$ equals the simple homotopy type of the Novikov-completed simplicial chain complex of $\widehat{M}$.

## 0. Introduction

It is well known that, given a Morse function $f$ on a smooth manifold $M$, one can construct a chain complex $C_*(f)$ (the Morse complex), which is a complex of free abelian groups and the number of free generators in dimension $p$ is exactly the Morse number $m_p(f)$ (i.e. the number of critical points of index $p$). This complex computes the homology of $M$ itself: $H_*(C_*(f)) \approx H_*(M, \mathbb{Z})$.

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One can strengthen this result so as to associate to any regular covering $\widehat{M} \to M$ with structure group $G$ a free $\mathbb{Z}G$-complex $C_*(f,\widehat{M})$, such that the number $m_{\mathbb{Z}G}\big(C_p(f,\widehat{M})\big)$ of free $\mathbb{Z}G$-generators of the module $C_p(f,\widehat{M})$ equals $m_p(f)$. The simple homotopy type (over $\mathbb{Z}G$) of this complex appears to be the same as of the complex of simplicial chains of $\widehat{M}$ for the triangulation of $\widehat{M}$, coming from any smooth triangulation $\Delta$ of $M$. That is a useful tool for some kinds of nonsimply connected surgery ([2], [12]).

Approximately 10 years ago S. P. Novikov [7] proposed that Morse theory should be expanded to the case of Morse forms, that is closed 1-forms which are locally the derivatives of Morse functions. He suggested a construction of an appropriate analogue of the Morse complex. For the case when homology class of the 1-form $\omega$ is integral, this complex is a free complex over a ring $\mathbb{Z}[[t]][t^{-1}]$ of Laurent power series. This ring is a kind of completion of the group ring $\mathbb{Z}[\mathbb{Z}]$ and the Novikov complex computes the correspondingly completed homology of the infinite cyclic covering $\overline{M} \xrightarrow{\mathbb{Z}} M$, which corresponds to the homology class $[\omega] \in H^1(M,\mathbb{Z})$.

The existence of such a complex enables one to obtain the lower bounds for the Morse number $m_p(\omega)$ in homotopy invariant terms (here $m_p(\omega)$ stands for the number of zeros of $\omega$ of index $p$), and in some cases to prove their sharpness, thus pursuing the analogy with usual Morse theory further.

The sharpness of the arising inequalities was proved by Farber [1] for $\pi_1 M = \mathbb{Z}$, $\dim M \geq 6$ and by the author [8] for $\pi_1 M = \mathbb{Z}^s$, $\dim M \geq 6$ (under some restrictions on the homotopy type of $M$ and the cohomology class of the form in consideration).

I must note that up to the present there existed no detailed proof of the mentioned properties of Novikov complex. The papers [1] and [8] appealed only to the inequalities, which were proved in these papers without using the Novikov complex. But certainly what was underlying these proofs was the Novikov complex.

So in the present paper I suggest the full treatment of Novikov complex for the case $[\omega]$ is integral (up to a nonzero constant).

To formulate it here we need some notations. Let $\omega$ be a Morse 1-form on a closed manifold $M$, such that de Rham cohomology class $[\omega]$ is nonzero and is up to the multiplication by a constant, an element of $H^1(M,\mathbb{Z})$. Let $\widehat{p}: \widehat{M} \to M$ be a regular connected covering (with the structure group $G$), such that $\omega$ resolves on $\widehat{M}$, i.e.:

$$\widehat{p}^*\omega = d\widehat{f}, \quad \widehat{f}: \widehat{M} \to \mathbb{R}.$$

Then cohomology class $[\omega] : \pi_1 M \to \mathbb{R}$ is factored uniquely through some homomorphism $\xi : G \to \mathbb{R}$. The Novikov construction (generalized to the case of nonabelian groups by J.-Cl. Sikorav [13]) associates to any $\xi : G \to \mathbb{R}$ a completion $\Lambda_\xi^-$ of a group ring $\Lambda = \mathbb{Z}[G]$ (sect. 1). There is an appropriate notion of a Whitehead group over $\Lambda_\xi^-$, that is a factor of $K_1(\Lambda_\xi^-)$ by a subgroup of trivial units (sect. 1). By the appropriate version of the Kupka – Smale theorem (Proposition 2.0) we can choose the gradient-like vector field $v$ for $\omega$ in such a way that all the stable and unstable manifolds of $v$ are transversal. Let $\Delta$ be some smooth triangulation of $M$ (see [4]). It determines a smooth $G$-invariant triangulation of $\widehat{M}$ and the corresponding chain complex will be denoted $C_*^\Delta(\widehat{M})$. It is a free finitely generated complex over $\mathbb{Z}[G]$.

THE MAIN THEOREM (Theorem 2.2) .— *To this data there is associated the Novikov complex $C_*(v, \widehat{M})$ which is a free complex of $\Lambda_\xi^-$-modules, such that the number of free generators $m_{\Lambda_\xi^-}\big(C_p(v, \widehat{M})\big)$ is equal to $m_p(\omega)$ and $C_*(v, \widehat{M})$ is simply homotopy equivalent to $C_*^\Delta(\widehat{M}) \bigotimes_{\mathbb{Z}[G]} \Lambda_\xi^-$.*

Now we explain the main idea of the proof and present the contents of the paper.

The main instrument for proving the similar result for Morse functions [6] is an inductive argument. One uses the construction of a manifold cell by cell using a Morse function and proves by induction that each step of the construction is simply homotopy equivalent to the corresponding part of the Morse complex. The main difficulty in our situation is that although we can work with Morse functions, say $\widehat{f}$ on $\widehat{M}$, there is nothing to begin with, since $\widehat{f}$, for example, is not bounded from below and the descending discs of the critical points go infinitely downwards.

So we act in the opposite direction. Since we suppose that $[\omega]$ is a multiple of an integer class, there is an infinite cyclic covering $\overline{p} : \overline{M} \xrightarrow{\mathbb{Z}} M$, such that $\overline{p}^*\omega$ resolves:

$$\overline{p}^*\omega = \mathrm{d}\overline{f}\,,$$

where $\overline{f}$ is a Morse function. The covering $\widehat{p} : \widehat{M} \to M$ factors through $\overline{p}$ like

$$\widehat{M} \xrightarrow{Q} \overline{M} \xrightarrow{\overline{p}} M\,.$$

Now we invite the reader to look at the picture 4.1. The notations are clear from the picture itself (and also explained in the very beginning of Section 4). We only indicate that $t$ is a generator of a structure group $\mathbb{Z}$ of a covering, and we assume $\overline{f}(tx) < \overline{f}(x)$.

Let $G^-$ denote the submonoid of $G$, consisting of all the $g \in G$, such that $\xi(g) \leq 0$ and let $\mathbb{Z}G^-$ denote its group ring. The Novikov ring $\Lambda_\xi^-$ is formed by the power series in the elements of $G$ which are infinite to the direction where $\xi$ descends (a precise definition is given in Section 1). We denote by $\Lambda_\xi^=$ the ring of power series in the elements of $G^-$, which are infinite to the direction where $\xi$ descends (sect. 1).

Now consider the function $\overline{f}$, restricted to

$$W(n) = \{x \in \overline{M} \mid 0 \geq \overline{f}(x) \geq -(n+1)a\}.$$

It is a Morse function and we can define a Morse complex with respect to the covering $Q$, restricted to $W(n) \subset \overline{M}$ and a gradient-like vector field $v$, coming from the base. Since $v$ is $t$-invariant, one easily shows that it is actually a $\mathbb{Z}G^-$-complex, and since the elements $g \in G^-$ with $\xi(g) \leq -(n+1)a$ act trivially, it is also defined over $\mathbb{Z}G_n^-$ where

$$\mathbb{Z}G_n^- = \mathbb{Z}G^- / \{g \in G^- \mid \xi(g) \leq -(n+1)a\}.$$

We denote this complex $C_*^-(v,n)$. It is a free $\mathbb{Z}G_n^-$-complex. There is an obvious map $C_*^-(v,n) \to C_*^-(v,n-1)$, and the inverse limit will be denoted $C_*^-(v)$. It is a module over $\Lambda_\xi^=$ and one identifies easily the Novikov complex with $C_*^-(v) \otimes_{\Lambda_\xi^=} \Lambda_\xi^-$.

On the other hand, if we choose the triangulation of $M$ in such a way that $\overline{p}(V)$ is a subcomplex, then we obtain a triangulation $\Delta$ of $\overline{M}$, invariant under the action of $\mathbb{Z}$ and such that all the $W(n)$ are subcomplexes. The complexes $C_*(\widehat{V^-}, \widehat{t^{n+1}V^-})$, where $\widehat{\ }$ denotes passing to the preimage in $\widehat{M}$, are thus the free $\mathbb{Z}G_n$-complexes. There is an obvious projection map

$$C_*(\widehat{V^-}, \widehat{t^{n+1}V^-}) \to C_*(\widehat{V^-}, \widehat{t^nV^-})$$

and $C_*(\widehat{V^-}) \otimes_{\mathbb{Z}G^-} \Lambda_\xi^=$ is exactly the inverse limit of this system.

It suffices to prove the simple homotopy equivalence over $\Lambda_\xi^=$ of the two complexes $C_*^-(v)$ and $C_*(\widehat{V^-}) \otimes_{\mathbb{Z}G^-} \Lambda_\xi^=$.

That is done by comparison of two inverse systems. Denote $\mathrm{Ker}(\xi : G \to \mathbb{R})$ by $H$. The ordinary Morse theory gives a homotopy equivalence

$$h_n : C_*^-(v,n) \to C_*(\widehat{V^-}, t^{n+1}\widehat{V^-})$$

over $\mathbb{Z}H$.

We show that one can choose these $h_n$ in such a way, that in the diagram below all the squares are chain homotopy commutative and that $h_0$ is a simple homotopy equivalence.

$$\begin{array}{ccccccc} \dots & \longrightarrow & C_*^-(v,n) & \longrightarrow & C_*^-(v,n-1) & \longrightarrow & \dots \\ & & \Big\downarrow h_n & & \Big\downarrow h_{n-1} & & \\ \dots & \longrightarrow & C_*(\widehat{V^-}, t^{n+1}\widehat{V^-}) & \longrightarrow & C_*(\widehat{V^-}, t^{n}\widehat{V^-}) & \longrightarrow & \dots \end{array} \qquad (I.1)$$

From this we deduce in Section 3 that the inverse limits of the systems $\{C_*^-(v,n)\}$ and $\left\{C_*(\widehat{V^-}, t^{n+1}\widehat{V^-})\right\}$ are chain homotopy equivalent and the condition on $h_0$ guarantees that the resulting chain homotopy will be simple over $\Lambda_\xi^=$.

Now we present the plan of the paper.

The definition of the Novikov ring is given in Section 1. In Section 2 we state the main theorem. Section 3 is purely algebraic. Here we first develop some simple formalism for working with chain complexes, endowed with filtrations, similar to those arising from $t$-ordered Morse functions. This formalism enables us to prove homotopical uniqueness (in a certain sense) of the homotopy equivalence of the Morse complex of a Morse function and of the chain complex of a triangulation. In Section 4 we start proving the main theorem. There are mainly the explanations and the reduction to the existence of the diagram (I.1). This existence is proved in Section 5.

The Appendix contains all the information about Morse functions and Morse complexes for Morse functions. In particular there are the full proofs of the theorems cited in the very beginning of this introduction.

It is well known, that the Morse (Thom – Smale) complex of a Morse function is simply homotopy equivalent to the complex of simplicial chains of the universal covering of the manifold. The point, which is missing, is that this chain equivalence does not (up to homotopy) depend on the choices involved. (It is essential for our purposes, since we want to get the homotopy commutativity of the square (I.1). We need this result in the framework of $t$-equivariant Morse functions on the cyclic covering, and we prove it in the Section 5. Still we believe, that this sort of result is interesting in itself even in the non-equivariant setting, and we have included Propositions A.9 and A.11 in the Appendix, although they are not used in the principal text.

The main result of the paper was predicted (in a slightly weaker form) by S. P. Novikov long ago. So the main aim of the present paper is to give a complete and detailed proof of this result, including the theorems, which are more or less folklore, but which are necessarily used in the proof. This concerns first of all the Appendix, which seems to be a first text, working out in details the simple homotopy equivalence (well defined up to a homotopy) of the Morse complex of a function and the chain complex of the universal covering.

This work was done during my stay in Odense and Århus Universities in February 1991, and the present paper is a revised version of my preprint [9].

The applications of the result of the present will appear in the forthcoming paper [10], which contains the surgery theory for maps $M \to S^1$ in terms of the Novikov complex.

It seems that F. Latour has another approach to the main theorem of this paper [14, p. 14]. One first checks up that the simple homotopy type of the complex $C_*(f, v)$ does not change under the deformations of the function $f : M \to S^1$. That can be done following the ideas of Cerf's paper [3] on pseudoisotopies, or (the suggestion of J.-Cl. Sikorav) using the Floer's argument. Afterwards J.-Cl. Sikorav shows, that in the homotopy class $[f]$ there always exists a Morse map, having the same Novikov complex as a Morse function $g : M \to \mathbb{R}$. (One takes the rational Morse form $C\,\mathrm{d}g + \mathrm{d}f$. It is cohomologous to $\mathrm{d}f$ and for $C \to \infty$ it has the same Novikov complex as $g$).

This idea has the advantage to cope also with the irrational Morse forms although it could present some technical difficulties to overcome.

I take here the opportunity to note that the group $Wh(G, \xi)$ (sect. 1) was first introduced by F. Latour in his Orsay talk in 1990. The above ideas seem to be more recent.

## 1. The Novikov ring

In this section we recall the definition and the properties of the Novikov ring.

Suppose that:

- $G$ is a (discrete) group,
- $\xi : G \to \mathbb{R}$ is a homomorphism of the group $G$ to the additive group of real numbers,
- $A$ is a commutative ring with a unit.

The object, which we are going to construct, is a special completion of the group ring $A[G]$ (denoted $\Lambda$ for short), with respect to $\xi$.

Namely, denote by $\widehat{\widehat{\Lambda}}$ the abelian group of all the linear combinations of the type $\lambda = \sum n_g g$, where $g \in G$, $n_g \in A$ and the sum may be infinite.

For any $\lambda \in \widehat{\widehat{\Lambda}}$ we denote by $\operatorname{supp}\lambda$ the subset of $G$, consisting of all the elements $g$, for which $n_g \neq 0$. For a real number $c \in \mathbb{R}$ denote by $G_c$ the subset of $G$, consisting of all the elements $g$, for which $\xi(g) \geq c$.

Now we denote by $\Lambda_\xi^-$ the subset of $\widehat{\widehat{\Lambda}}$, consisting of all the elements $\lambda \in \widehat{\widehat{\Lambda}}$ such that for any $c \in \mathbb{R}$ the set $\operatorname{supp}\lambda \cap G_c$ is finite. It is obvious that $\Lambda_\xi^-$ is a subgroup of $\widehat{\widehat{\Lambda}}$, containing $\Lambda$.

Moreover, $\Lambda_\xi^-$ possesses the natural ring structure. Namely, let $\lambda = \sum n_g g$, $\mu = \sum m_h h$ belong to $\Lambda_\xi^-$. For any $f \in G$ we set $\ell_f \in A$ to be $\sum_{gh=f} n_g m_h$. One checks up easily, that this sum is finite, the element $\nu = \sum \ell_f f$ belongs to $\Lambda_\xi^-$ and the operation $\nu = \lambda \cdot \mu$ endows $\Lambda_\xi^-$ with a ring structure.

Note that $\operatorname{supp}(\lambda \cdot \mu) \subset \operatorname{supp}\lambda \cdot \operatorname{supp}\mu$.

The ring $\Lambda_\xi^-$ is called Novikov ring. It was introduced by Novikov [7] for the case $G = \mathbb{Z}^m$ and by J.-Cl. Sikorav [13] in the general case.

The basic example one should have in mind is the following: $G = \mathbb{Z}$, the homomorphism $\xi$ is the identity. Then $\Lambda_\xi^- = \mathbb{Z}[[t]][t^{-1}]$ (the integer Laurent power series ring).

If $G$ is finitely generated, the rank of $\operatorname{Im}\xi \in \mathbb{R}$ is finite and is denoted by $\operatorname{rk}\xi$. We shall be interested in this paper only in the case $\operatorname{rk}\xi = 1$. Now we introduce more definitions for this case.

The homomorphism $\xi : G \to \mathbb{R}$ factors uniquely as $\bar{\xi} \cdot q$, where $q : G \to \mathbb{Z}$ is an epimorphism, $\bar{\xi} : \mathbb{Z} \to \mathbb{R}$ is a monomorphism. Let $t$ be a generator of $\mathbb{Z}$, such that $\bar{\xi}(t)$ is negative. Choose and fix an element $\theta \in G$, such that $q(\theta) = t$. Denote $\xi(\theta)$ by $(-a)$, $a$ is a positive real number. Denote $\operatorname{Ker} \xi$ by $H$.

The monoid $\{g \in G \mid \xi(g) \leq 0\}$ will be denoted by $G^-$.

Let $\mathbb{Z}G^-$ and $\Lambda_\xi^=$ denote the subrings of $\mathbb{Z}G$ (and correspondingly of $\Lambda_\xi^-$), consisting of finite linear combinations (correspondingly, power series) with the supports, contained in $\{\xi(q) \leq 0\}$. The elements $x$ of $\mathbb{Z}G^-$ (resp. $\Lambda_\xi^=$), for which $\operatorname{supp} x \in \{\xi(q) \leq -(n+1)a\}$, form the double-sided ideal $I_n$ of $\mathbb{Z}G^-$ (resp. $I_n^-$ of $\Lambda_\xi^=$), which equals $\theta^{n+1} \cdot \mathbb{Z}G^- = \mathbb{Z}G^- \cdot \theta^{n+1}$ (resp. $\theta^{n+1} \cdot \Lambda_\xi^= = \Lambda_\xi^= \cdot \theta^{n+1}$). The natural embedding $\mathbb{Z}G^- \hookrightarrow \Lambda_\xi^=$ induces the isomorphism $\mathbb{Z}G^-/I_n \to \Lambda_\xi^=/I_n^-$, which preserves the ring (as well as $\mathbb{Z}G^-$-bimodule) structure. For $n = 0$ these quotients are isomorphic to $\mathbb{Z}H$ (an isomorphism preserves the ring and $\mathbb{Z}G^-$-bimodule structures). These quotients are denoted by $\mathbb{Z}G_n^-$.

Every right $\mathbb{Z}G_n^-$-module is therefore a right $\Lambda_\xi^=$-module. For any free f.g. right $\mathbb{Z}G^-$-module $F$ the module $F \bigotimes_{\mathbb{Z}G^-} \Lambda_\xi^=$ is the inverse limit of the sequence $F/I_1F \leftarrow F/I_2F \leftarrow \ldots$ of the right $\Lambda_\xi^=$-modules.

Vice versa, every free f.g. right $\Lambda_\xi^=$-module $\mathcal{F}$ is the inverse limit of the modules $\mathcal{F}/I_1\mathcal{F} \leftarrow \mathcal{F}/I_2\mathcal{F} \leftarrow \ldots$

Next we need the notions of simple homotopy type. For that we need an analogue of the Whitehead group. Denote by $U(G,\xi)$ the multiplicative group of units of the ring $\Lambda_\xi^-$ of the form $\pm g + \lambda$, where $g \in G$ and $\operatorname{supp} \lambda \subset \{\xi(g) < 0\}$. Now we set:

$$Wh(G,\xi) = K_1\Lambda_\xi^- / U(G,\xi) .$$

As usual two f.g. free $\Lambda_\xi^-$-complexes $C_1$, $C_2$ with fixed bases will be called simply homotopy equivalent if there exists a homotopy equivalence $f : C_1 \to C_2$ such that the torsion of $f$ vanishes in $Wh(G,\xi)$.

Let $U(G,\xi)^-$ denote the multiplicative group of power series $x \in \Lambda_\xi^=$ of the form $x = \pm h + x'$, where

$$h \in H \,, \quad \operatorname{supp} x' \subset \{\xi(g) < 0\} .$$

The group $K_1(\Lambda_\xi^=)/U(G,\xi)^-$ is denoted $Wh^-(G,\xi)$. The projection $\pi : \Lambda_\xi^= \to \mathbb{Z}H$ sends $U(G,\xi)^-$ to $\{\pm h \mid h \in H\}$, and therefore induces the homomorphism $\pi_* : Wh^-(G,\xi) \to Wh(H)$.

LEMMA 1.1.— *$\pi_*: Wh^-(G,\xi) \to Wh(H)$ is an isomorphism.*

*Proof.*— Surjectivity is obvious. Suppose next that for some matrix $A$ over $\Lambda_\xi^=$ the matrix $A \bigotimes_{\Lambda_\xi^=} \mathbb{Z}H$ is equivalent to $(\pm h) \oplus E$ (where $h \in H$, $(\pm h)$ represents a matrix in GL(1), $E$ stands for the unit matrix) via several elementary transformations. Performing the same elementary transformations over $\Lambda_\xi^=$ (note that $\mathbb{Z}H \subset \Lambda_\xi^=$) we get that $A$ is equivalent to the matrix $B = (b_{ij})$, where $b_{00} = \pm h + \lambda_{00}$, $b_{ii} = 1 + \lambda_{ii}$ for $i \neq 0$ and all the $\lambda_{ii}$ have the support, contained in $\{\xi(g) < 0\}$. Since the elements $b_{ii}$ are invertible in $\Lambda_\xi^=$, the matrix $B$ is equivalent to a diagonal matrix $B' = (b'_{ij})$, where $b'_{ii} = b_{ii}$. This matrix is clearly 0 in $Wh(\Lambda_\xi^=)$. □

COROLLARY 1.2.— *A chain homotopy equivalence $h: C_* \to D_*$ of the free f.g. $\Lambda_\xi^=$-complexes is simple if and only if $h \bigotimes_{\Lambda_\xi^=} \mathbb{Z}H$ is simple.* □

## 2. The statement of the main theorem

The manifold $M$ is supposed to be compact connected and without boundary. We denote dim $M$ by $m$.

Recall that a closed 1-form $\omega$ on a manifold $M^m$ is called *Morse form*, if locally $\omega$ is an exterior derivative of a Morse function (defined locally). The zeros of a Morse form $\omega$ are isolated and the index of each zero is defined. For each zero $c$ of index $p$ we fix a coordinate system $\{x_i\}$ in a neighborhood $U(c)$ of $c$, such that $c$ corresponds to $\{x_i = 0\}$ and the differential of the standard quadratic form

$$-x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_m^2$$

equals $\omega$ in $U(c)$. For $\epsilon > 0$ sufficiently small we denote by $B(c,\epsilon)$ the open $\epsilon$-disc around $c$ in the standard coordinate system.

A vector field $v$ on $M^m$ is called *gradient-like vector field for $\omega$*, if:

(1) $\omega(v) > 0$ apart from zeros of $\omega$;

(2) for every zero $c$ of $\omega$ the field $v$ has the coordinates

$$(-2x_1, \ldots, -2x_p, 2x_{p+1}, \ldots, 2x_m)$$

in the coordinate system above.

The gradient-like fields exist, see Milnor [5].

For each zero $c$ we denote by $B^-(c)$ (resp. $B^+(c)$) the set of all points $x \in M$, such that the $v$-trajectory (resp. $(-v)$-trajectory), starting at $x$, converges to $c$ for $t \to \infty$. These are injectively immersed discs of dimensions $\operatorname{ind} c$, resp. $(m - \operatorname{ind} c)$. We say that a vector field $v$ satisfies the transversality assumption, if for each two zeros $c$, $d$ the manifolds $B^+(c)$ and $B^-(c)$ are transversal. The following result is a version of Kupka – Smale theorem, and the usual argument, given for example, in [11], provides the proof for this result as well.

PROPOSITION 2.0. — *Let $v$ be a gradient-like vector field for $f$. Fix $\lambda$, $\epsilon > 0$ sufficiently small, and such that $\lambda < \epsilon$. Then in every neighborhood $U$ of $v$ in the set of all the vector fields there is gradient-like vector field $w$ for $f$, such that:*

*(1)* $\operatorname{supp}(v - w)$ *belongs to the union of the sets $B(c,\epsilon) \setminus B(c,\lambda)$ over all the critical points $c$ of $f$;*

*(2) $w$ satisfies the transversality assumption.*

Let $[\,a\,,\,b\,]$ be a segment of the real line, where $a$ may equal to $-\infty$, $b$ to $\infty$. We say that a trajectory $\gamma(t)$ of the gradient-like vector field $v$ starts at $x \in M$ (correspondingly, finishes at $y \in M$) if either $a$ is finite and $\gamma(a) = x$ (correspondingly $b$ is finite and $\gamma(b) = y$) or $a = -\infty$, $v(x) = 0$ and $\lim_{t\to-\infty} \gamma(t) = x$ (correspondingly, $b = \infty$, $v(y) = 0$ and $\lim_{t\to\infty} \gamma(t) = y$).

A connected regular covering $p : \overline{M} \to M$ with a structure group $G$ will be called *$\omega$-resolving* if $p^*\omega = d\overline{f}$, where $\overline{f}$ is a $\mathbb{R}$-valued smooth function on $M$. Since $\omega$ is a Morse form, $\overline{f}$ is a Morse function. In the set of all $\omega$-resolving coverings there is one, which is minimal in the sense that any other one factors through it. That is the covering, corresponding to the subgroup $\operatorname{Ker}\big([\omega] : \pi_1 M \to \mathbb{R}\big)$ (here we consider $[\omega] \in H^1(M,\mathbb{R})$ as a homomorphism of $\pi_1 M$ to $\mathbb{R}$); its structure group is $\pi_1 M/\operatorname{Ker}[\omega]$, which is finitely generated subgroup of $\mathbb{R}$, hence a f.g. free abelian group. We denote this covering by $\overline{p} : \overline{M}_{[\omega]} \to M$. Denote $[\omega]$ by $\xi$.

In what follows we consider only the forms for which the rank of that group is 1, i.e. the cohomology class $\xi$ is up to a positive constant, an integer cohomology class. In this case the covering $\overline{p} : \overline{M}_\xi \to M$ is an infinite cyclic covering.

Suppose now that $\widehat{p} : \widehat{M} \to M$ is any $\omega$-resolving covering with the structure group $G$. The resolving function will be denoted $\widehat{f} : \widehat{M} \to \mathbb{R}$.

The homomorphism $[\omega] : \pi_1 M \to \mathbb{R}$ is factored uniquely through $G$ and the resulting homomorphism will be denoted by $\xi : G \to \mathbb{R}$.

We choose and fix:

(1) the gradient-like vector field $v$ for $\omega$, such that $v$ satisfies transversality assumption,

(2) for each zero $c_i$ of $\omega$ the orientation of the stable disc $B^-(c_i)$.

To this data we shall associate a free finitely generated chain complex over $\Lambda_\xi^-$, where $\Lambda = \mathbb{Z}G$, which is called Novikov complex. It has the properties, listed in Theorem 2.2 below.

We proceed to the definition. Let $C_i$ be a free $\mathbb{Z}$-module, generated by the critical points of $\widehat{f}$ of index $i$. Let $\widehat{C}_i$ be the abelian group, consisting of all formal $\mathbb{Z}$-linear combinations $\lambda$ of critical points, such that above any given level surface $\widehat{f}^{-1}(a)$ of $\widehat{f}$ the combination $\lambda$ contains only a finite number of points. One checks up easily, that $\widehat{C}_i$ is a free right module over $\Lambda_\xi^-$, and that any choice of liftings $\widehat{c}$ to $\widehat{M}$ of all zeros $c$ of $\omega$ gives the system of free $\Lambda_\xi^-$-generators of $\widehat{C}_*$. Let $c$, $d$ be two critical points of $\widehat{f}$, $\operatorname{ind} c = \operatorname{ind} d + 1$, and let $\gamma$ be any trajectory of the vector field $(-v)$, starting at $c$ and finishing at $d$. For any liftings $\widehat{c}$ to $\widehat{M}$ of a zero $c$ of $\omega$ we denote by $\widehat{B}^+(\widehat{c})$, $\widehat{B}^-(\widehat{c})$ the stable and unstable discs of $\widehat{c}$ which are the liftings to $\widehat{M}$ of the discs $B^+(c)$, $B^-(c)$. In any point $r$ of $\gamma$ the tangent spaces $T_c$ and $T_d$ to $\widehat{B}^+(c)$, resp. $\widehat{B}^-(d)$ are transversal,

$$\dim T_c + \dim T_d = \dim M + 1 ,$$

$T_c$ is oriented, $T_d$ is cooriented and the intersection $T_c \cap T_d$ is generated by $v$, hence also oriented. This defines the sign $\pm$, attributed to the trajectory $\gamma$ (one verifies easily, that it does not depend on the choice of $r$). We denote it by $\varepsilon(\gamma)$.

LEMMA 2.1.— *Let $c$, $d$ be the critical points of $\widehat{f}$ in $\widehat{M}$,* $\operatorname{ind} c = \operatorname{ind} d + 1$. *Then:*

*(1) there is at most finite number of $(-v)$-trajectories, joining $c$ with $d$.*(*) *The sum of the signs $\varepsilon(\gamma)$ over all these trajectories will be denoted $n(c,d)$.*

*(2) For each critical point $c$ of $\widehat{f}$ of index $k$ the formal linear combination $\partial c = \sum n(c,d)d$, where $d$ runs over all the critical points of $\widehat{f}$ of index $k-1$, belongs to $\widehat{C}_{k-1}$.*

(*) Here and else where we identify two trajectories, which differ by a parameter change

We shall prove this lemma in Section 4.

The vector field $v$ on $\widehat{M}$ is $G$-invariant, which implies that $\partial$ commutes with $G$-action. From this one deduces easily that for each element of $\widehat{C}_i$ of the form $\lambda = \sum n_i c_i$ the element

$$\partial\lambda = \sum n_i \partial c_i$$

is a correctly defined element of $\widehat{C}_{i-1}$. This defines a homomorphism $\partial_i : \widehat{C}_i \to \widehat{C}_{i-1}$ of the right $\Lambda_\xi^-$-modules. If we want to stress that $\partial_i$ depends on $v$, we write it as $\partial_i(v)$. Any particular choice of liftings of zeros of $\omega$ to $\widehat{M}$ determines a $\Lambda_\xi^-$-basis in $\widehat{C}_*$, and therefore gives it a structure of a based $\Lambda_\xi^-$-complex.

THEOREM 2.2

*(1)* $\partial_i \circ \partial_{i+1} = 0$.

*(2) The free based chain complex* $C_*(v) = \{\widehat{C}_i, \partial_i(v)\}$ *is simply homotopy equivalent to* $C_*^\Delta(\widehat{M}) \bigotimes_\Lambda \Lambda_\xi^-$.

The proof of this theorem occupies the rest of the present paper.

## 3. Preliminaries on chain complexes

This section is purely algebraic. The part **A** deals with some special filtrations in chain complexes, the part **B** — with chain cylinders and telescopes.

**A.** A self-indexing Morse function on a manifold $M$ determines the filtration in the singular chain complex of $M$, and also of any covering of $M$ (for a precise definition of the filtration see Appendix). This filtration $\{F_n\}$ possesses a special property that the homology of $F_n/F_{n-1}$ vanishes except in dimension $n$. The chain complexes with the filtrations like that have some natural and simple properties which we treat in this section.

In this section all the chain complexes are supposed to begin from zero dimension, i.e. to be of the form $\{0 \leftarrow C_0 \leftarrow C_1 \leftarrow \ldots\}$, and the filtrations to be indexed by natural numbers, i.e. be of the form

$$0 = F_*^{(-1)} \subset F_*^{(0)} \subset F_*^{(1)} \subset F_*^{(2)} \subset \cdots$$

and to be exhausting, i.e.

$$\bigcup_i F_*^{(i)} = C_* .$$

Let $A$ be a ring and

$$C_* = \{0 \leftarrow C_0 \leftarrow C_1 \leftarrow \ldots \leftarrow C_n \leftarrow \ldots\}$$

be a chain complex of right $A$-modules.

DEFINITION 3.1.— *A filtration* $0 = C_*^{(-1)} \subset C_*^{(0)} \subset C_*^{(1)} \subset \cdots \subset C_*^{(n)} \subset \cdots$ *of* $C_*$ *by subcomplexes* $C_*^{(i)}$, *where* $\bigcup_i C_*^{(i)} = C_*$ *is called good if* $H_i\big(C_*^{(n)}/C_*^{(n-1)}\big)$ *is zero for* $i \neq n$.

Note that for a good filtration $H_i\big(C_*^{(n)}\big)$ is zero for $n < i$.

To any filtration $\{C_*^{(i)}\}$ of some complex $C_*$ we associate a complex $C_*^{gr}$, setting $C_n^{gr} = H_n\big(C_*^{(n)}/C_*^{(n-1)}\big)$, and introducing differential $\partial_n : C_n^{gr} \to C_{n-1}^{gr}$ to be that of the exact sequence of the triple $\big(C_*^{(n)}, C_*^{(n-1)}, C_*^{(n-2)}\big)$.

An obvious example of a good filtration is the filtration of the complex $D_*$ by the subcomplexes

$$D_*^{(i)} = \{0 \leftarrow D_0 \leftarrow D_1 \leftarrow \ldots \leftarrow D_i \leftarrow 0 \leftarrow \ldots\} .$$

The associated complex $D_*^{gr}$ is $D_*$ itself. This filtration will be called *trivial*.

LEMMA 3.2.— *Suppose that* $C_*^{(n)}$ *is a good filtration of a complex* $C_*$. *Let* $D_* = \{0 \leftarrow D_0 \leftarrow D_1 \leftarrow \ldots\}$ *be a chain complex of free right A-modules and* $\varphi : D_* \to C_*^{gr}$ *be a chain map. Then there exists a chain map* $f : D_* \to C_*$, *preserving filtrations (we imply that* $D_*$ *is good-filtered trivially) and inducing the map* $\varphi$ *in the graded complexes. This chain map* $f$ *is unique up to chain homotopy, preserving filtrations.*

*Proof.*— The proof is a standard diagram chasing. We will give only the construction of $f$.

Suppose by induction that we have constructed the maps $f_i : D_i \to C_i$ where $i \leq n-1$, commuting with the differentials $\partial_*$ in $C_*$ and $\mathrm{d}_*$ in $D_*$, preserving filtrations (i.e. $\operatorname{Im} f_i \subset C_i^{(i)}$) and inducing $\varphi$ in the graded groups.

It suffices to define $f_n$ on the free generators of $D_n$. Let $e$ be a generator of $D_n$. Let $x$ be any element in $C_n^{(n)}$, representing in the group $F_n^{gr} = H_n\big(C_n^{(n)}/C_n^{(n-1)}\big)$ the element $\varphi(e)$. Consider the element

$$z = \partial x - f_{n-1}(\mathrm{d}e) \in C_{n-1}^{(n-1)}.$$

We have $\partial z = 0$. The homology class of $z$ in $H_{n-1}\big(C_*^{(n-1)}/C_*^{(n-2)}\big)$ is zero, since the class of $f_{n-1}(\mathrm{d}e)$ is equal to the boundary of $\varphi(e)$ in $C_*^{gr}$ which is $\partial x$. Hence $z = \partial u + v$ where $u \in C_n^{(n-1)}$, $v \in C_{n-1}^{(n-2)}$. Note that $\partial v = 0$ (in $C_*$ itself), hence $v = \partial w$, $w \in C_n^{(n-2)}$, by the remark, following the definition 3.1. Now we put

$$f_n(e) = x - (u + w)$$

The class of $x - (u + w)$ in $C_n^{(n)}/C_n^{(n-1)}$ is equal to that of $x$ and $\partial f_n(e) = f_{n-1}(\mathrm{d}e)$; all the conditions are satisfied. □

DEFINITION 3.3.— *A good filtration $C_*^{(i)}$ of a complex $C_*$ is called nice if every module $H_n\big(C_*^{(n)}/C_*^{(n-1)}\big)$ is a free right $A$-module.*

COROLLARY 3.4.— *For a nice filtration $C_*^{(i)}$ of a complex $C_*$ there exists a homotopy equivalence $C_*^{gr} \to C_*$, functorial up to chain homotopy in the category of nicely filtered complexes.*

*Proof.*— The homotopy equivalence $C_*^{gr} \to C_*$ follows from Lemma 3.2 if we set $D_* = C_*^{gr}$ and let $f : D_* \to C_*^{gr}$ be the identity.

To prove the functoriality suppose that $C_*$ and $D_*$ are nicely filtered complexes with filtrations $C_*^{(i)}$, $D_*^{(i)}$ and $C_* \to D_*$ is a chain map, preserving filtrations. Denote by

$$\varphi : C_*^{gr} \to D_*^{gr}$$

the chain map, induced by $f$, and by $g : C_*^{gr} \to C_*$, $h : D_*^{gr} \to D_*$ the chain homotopy equivalences, preserving filtrations (recall that $C_*^{gr}$, $D_*^{gr}$ are trivially filtered). Then $f \circ g$ and $h \circ \varphi$ are chain maps from $C_*^{gr}$ to $D_*$, preserving filtrations and inducing the same map in graded homology, namely $\varphi$. Hence they are chain homotopic via the homotopy, preserving filtrations. □

Now we pass to the category of based complexes.

We need one more notation. For a given nice filtration $\{C_*^{(n)}\}$ of a complex $C_*$ we denote by $R_*^{(n)}$ the complex, which vanishes in all the dimensions except $* = n$ and is equal to $H_n(C_*^{(n)}/C_*^{(n-1)})$ for $* = n$. By Lemma 3.2 there exists a (uniquely defined up to homotopy) homotopy equivalence

$$\kappa_n : R_*^{(n)} \to C_*^{(n)}/C_*^{(n-1)},$$

including identity in homology.

The base ring now is $\mathbb{Z}G$. Let

$$C_* = \{0 \leftarrow C_0 \leftarrow C_1 \leftarrow \ldots \leftarrow C_n \leftarrow 0\}$$

be a free f.g. based complex of right $\mathbb{Z}G$-modules.

Let $C_*^{(i)}$ be a filtration of $C_*$.

DEFINITION 3.5. — *The filtration $C_*^{(i)}$ is called perfect if the conditions (1)-(4) below hold.*

*(1) The complexes $C_*^{(i)}$ and the quotient complexes $C_*^{(i)}/C_*^{(i-1)}$ are free f.g. complexes; the filtration $C_*^{(i)}$ is finite.*

*(2) All the complexes $C_*^{(i)}$ and the quotient complexes $C_*^{(i)}/C_*^{(i-1)}$ are endowed with the classes of preferred bases, compatible in the sense, that a preferred basis for $C_*^{(i-1)}$ and a preferred basis for $C_*^{(i)}/C_*^{(i-1)}$ form a preferred basis for $C_*^{(i)}$. The preferred basis for the final $C_*^{(i)}$ is a preferred basis for $C_*$.*

*(3) $C_*^{(i)}$ is nice and $R_n^{(n)} = H_n(C_*^{(n)}/C_*^{(n-1)})$ is endowed with a preferred class of bases.*

*(4) The map $\kappa_n : R_*^{(n)} \to C_*^{(n)}/C_*^{(n-1)}$, introduced above is a simple homotopy equivalence.*

Note that if $C_*$ is perfectly filtered, then every $C_*^{(n)}$ inherits from it a perfect filtration; the corresponding graded complex is $(C_*^{gr})^{(n)}$.

LEMMA 3.6. — *For a perfect filtration $\{C_*^{(i)}\}$ of a complex $C_*$ the homotopy equivalence $C_*^{gr} \to C_*$ (existing by Lemma 3.4) is a simple homotopy equivalence.*

*Proof*.— Induction in the length of the filtration. □

**B.** In this part we prove a technical result on chain complexes which will be of use in Section 4. Let $\{C_*^n\}$ and $\{D_*^n\}$ $(n \geq 0)$ be two inverse systems of chain complexes of right modules over some ring $R$, such that all the maps

$$f^i : C_*^i \to C_*^{i-1}$$

and

$$g_i : D_*^i \to D_*^{i-1}$$

are epimorphic. Let $h_n : C_*^n \to D_*^n$ be chain homotopy equivalences, such that all the following diagrams commute up to chain homotopy:

$$\begin{array}{ccccccc} C_*^0 & \xleftarrow{f^1} & C_*^1 & \xleftarrow{f^2} & C_*^2 & \leftarrow & \cdots \\ \downarrow h_0 & & \downarrow h_1 & & \downarrow h^2 & & \\ D_*^0 & \xleftarrow{g^1} & D_*^1 & \xleftarrow{g^2} & D_*^2 & \leftarrow & \cdots \end{array} \tag{3.1}$$

Let

$$\Gamma_* = \varprojlim C_*^n\,, \quad \Delta_* = \varprojlim D_*^n\,.$$

PROPOSITION 3.7

*(1) There exists a chain complex $Z_*$ and two chain maps $A : \Gamma_* \to Z_*$, $B : \Delta_* \to Z_*$ which induce isomorphisms in homology.*

*(2) If $\Gamma_*$ and $\Delta_*$ are free chain complexes, then there exists a homotopy equivalence $G : \Gamma_* \to \Delta_*$, such that the diagram*

$$\begin{array}{ccc} \Gamma_* & \xrightarrow{G} & \Delta_* \\ \downarrow & & \downarrow \\ C_*^0 & \xrightarrow{h_0} & D_*^0 \end{array} \tag{3.2}$$

*is homotopy commutative.*

For the proof we shall need some preliminaries on telescopic constructions for chain maps.

All the differentials will be denoted by a single letter $\partial$ since there is no possibility of confusion. Normally the element of, say, $C_*$ of degree $n$ will be denoted $c_n$. The sign $\sim$ denotes "chain homotopic" or "chain homotopy equivalent".

Recall that for a chain map $f : C_* \to D_*$ of chain complexes the cylinder $Z(f)_*$ is a chain complex defined by

$$Z(f)_n = D_n \oplus C_n \oplus C_{n-1} ,$$

$$\partial(d_n, c_n, c_{n-1}) = (\partial d_n - f(c_{n-1}), \partial c_n + c_{n-1}, -\partial c_{n-1}) .$$

Let:

$$i : C_* \to Z(f)_* , \quad \pi : Z(f)_* \to D_* , \quad j : D_* \to Z(f)_*$$

denote the maps, defined by

$$i(c_n) = (0, c_n, 0) , \quad \pi(d_n, c_n, c_{n-1}) = f(c_n) + d_n , \quad j(d_n) = (d_n, 0, 0) .$$

The following lemma is trivial.

LEMMA 3.8.— *$Z(f)_*$ is indeed a chain complex with that differential. The maps $i$, $\pi$, $j$ are the chain maps and have the following properties: $\pi j = \mathrm{id}$, $j\pi \sim \mathrm{id}$, $\pi i = f$.* □

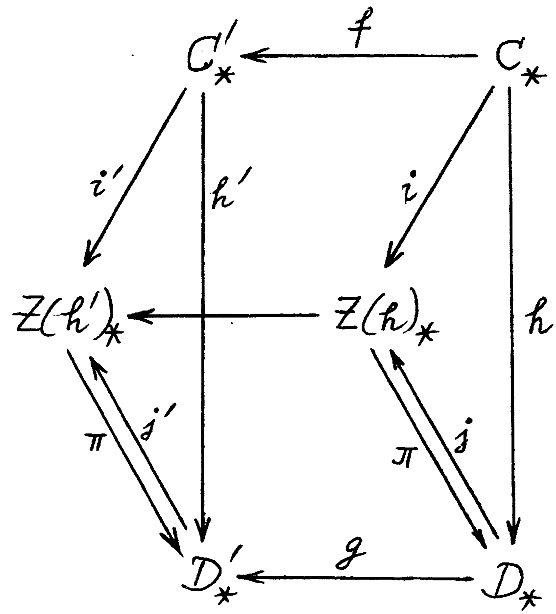
We shall need a homotopy functoriality property for the cylinder construction.

LEMMA 3.9.— *Let:*

$$\begin{array}{ccc} C'_* & \xleftarrow{\ f\ } & C_* \\ \big\downarrow{\scriptstyle h'} & & \big\downarrow{\scriptstyle h} \\ D'_* & \xleftarrow{\ g\ } & D_* \end{array}$$

*be a homotopy commutative square of chain complexes. Then there exists a chain map $F : Z(h)_* \to Z(h')_*$ such that in the diagram below all the parallelograms, squares and triangles are homotopy commutative and, moreover, two parallelograms are strictly commutative $(F \circ i = i' \circ f , F \circ j = j' \circ g)$.*

*If f and g are epimorphic then F can be chosen to be epimorphic*




*Proof.*— Let $H : C_* \to D'_{*+1}$ be a homotopy between $gh$ and $h'f$. We set:

$$F(d_n, c_n, c_{n-1}) = \big(g(d_n) + H(c_{n-1}),\ f(c_n),\ f(c_{n-1})\big)\,.$$

All the properties follow immediately. □

*Proof of proposition 3.7. Part (1).*— Set $Z^i_* = Z(h_i)_*$. Applying the preceding lemma to each square of (3.1) we get an inverse system

$$\cdots \leftarrow Z^{i-1}_* \xleftarrow{F^i} Z^i_* \leftarrow \cdots$$

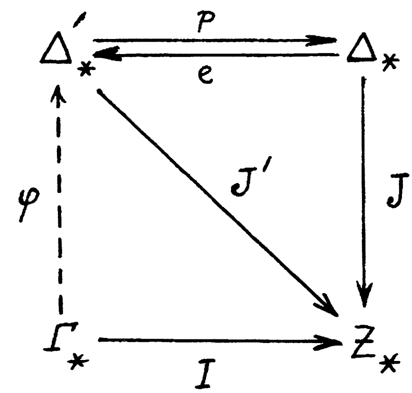
where each $F^i$ is an epimorphism and the maps $i^n : C^n_* \to Z^n_*$, $j^n : D^n_* \to Z^n_*$, producing the strictly commutative maps of inverse systems:

$$\{i^n\} : \{C^n_*\} \to \{Z^n_*\}$$
$$\{j^n\} : \{D^n_*\} \to \{Z^n_*\}$$

Thus we obtain a morphism of inverse limits $I : \Gamma_* \to Z_*$, $J : \Delta_* \to Z_*$ (where $Z_* = \varprojlim Z^n_*$).

Moreover, $j^n h^n \sim i^n$, therefore each $j^n$, $i^n$ induces an isomorphism in homology. Since all the three systems were epimorphic, this implies that $I_*$, $J_*$ also induce the isomorphisms in homology. □

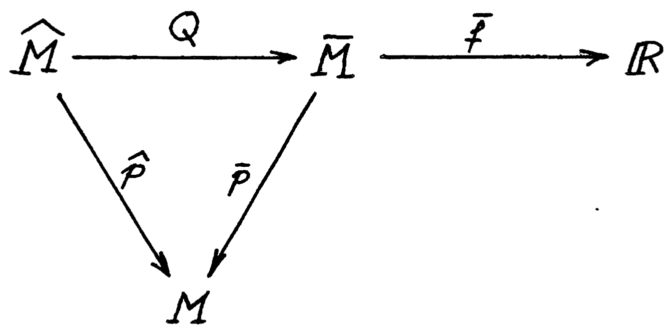
*Proof of proposition 3.7. Part (2).*— We shall construct a chain map $G : \Gamma_* \to \Delta_*$, such that $J \circ G \sim I$. We add to $\Delta_*$ a collapsible free complex in order to get a complex $\Delta'_*$ with the epimorphic chain map $J'$ and the diagram below, where $p$ is a projection, $e$ is an embedding, $J'e = J$, $Jp \sim J'$.




Note that $J'$ induce an isomorphism in homology, hence $\left(\operatorname{Ker} J'\right)_*$ is an acyclic complex and by an easy induction argument we construct a map $\varphi : \Gamma_* \to \Delta'_*$, such that $J'\varphi = I$. It is easy to check that $G = p\varphi$ satisfies our conclusion. Moreover, since $JG \sim I$, the map $G$ induces an isomorphism in homology, therefore it is a homotopy equivalence. The commutativity of the square (3.2) is obvious. □

## 4. The proof of the main theorem. Part 1

Recall that $\omega$ stands for a Morse form on a closed manifold $M$ and the rank of $\operatorname{Im}([\omega] : \pi_1 M \to \mathbb{R})$ equals 1. We fix any $\omega$-resolving covering $\widehat{p} : \widehat{M} \xrightarrow{G} M$ and denote by $\overline{p} : \overline{M} \xrightarrow{\mathbf{Z}} M$ the minimal $\omega$-resolving covering, which is infinite cyclic. There exists a commutative diagram




corresponding to the epimorphism of groups $q : G \to \mathbb{Z}$. $Q$ is a covering with the structure group $H = \operatorname{Ker} q$. Let $\overline{f} : \overline{M} \to \mathbb{R}$ be the corresponding Morse function;

$$\mathrm{d}\overline{f} = \overline{p}^* \omega ,$$

and denote $\overline{f} \circ Q$ by $\widehat{f}$.

Let $t$ be the generator of $\mathbb{Z}$, uniquely determined by the condition $\overline{f}(tx) < \overline{f}(x)$. Choose and fix an element $\theta \in G$, such that $q(\theta) = t$.

Recall that we denote by $\xi$ the homomorphism $G \to \mathbb{Z}$ induced by $[\omega] : \pi_1 M \to \mathbb{R}$. The class $[\omega]$ factors also through $\mathbb{Z}$ and we denote by $\overline{\xi} : \mathbb{Z} \to \mathbb{R}$ the corresponding homomorphism.

Choose now any regular value of $\overline{f}$. Since we can change $\overline{f}$ by adding a constant, we can assume that this value is zero. Let $a$ denote $\overline{f}(x) - \overline{f}(tx)$. The preimage $\overline{f}^{-1}([-a\,,\,0])$ is a compact manifold $W$ with boundary $\partial W = V \sqcup tV$, where $V$ stands for $\overline{f}^{-1}(0)$, and $\overline{f}$ is a Morse function on a cobordism $W$. Let $n$ be a natural number and denote by $V^-$ the space $\{\overline{f}(x) \leq 0\}$, by $V^+$ the space $\{\overline{f}(x) \geq 0\}$, and by $W_n$ the preimage:

$$\overline{f}^{-1}\left([-(n+1)a\,,\,-na\,]\right) ,$$

so that $W_0 = W$. Let $W(n)$ denote the preimage $\overline{f}^{-1}\left([-(n+1)a\,,\,0\,]\right)$, that is:

$$W(n) = \bigcup_{i=0}^{n} W_i \,.$$

Note that for any $n$ the restriction $Q \mid Q^{-1}(W(n)) : Q^{-1}(W(n)) \to W(n)$ is a regular covering with the structure group $H$.

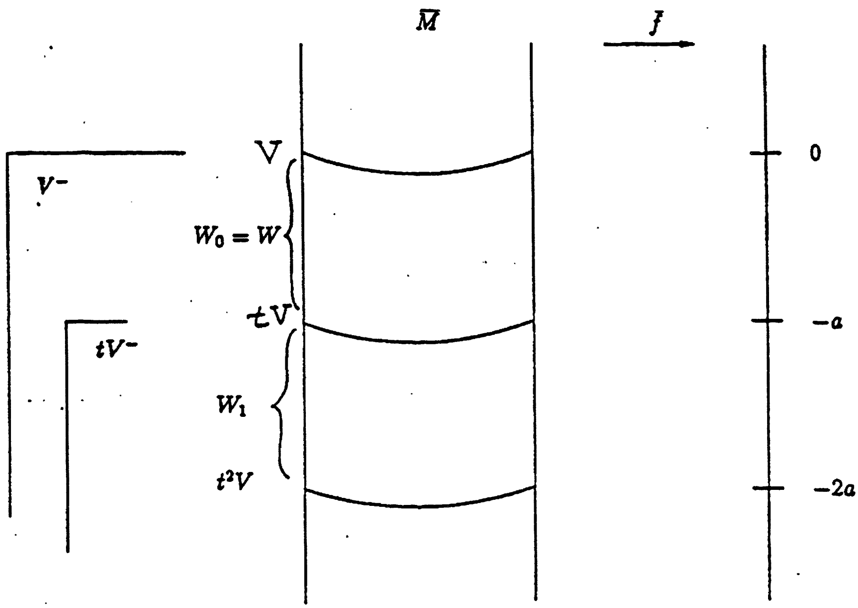



**Fig. 4.1**

The liftings of a vector field $v$ to $\overline{M}$ and to $\widehat{M}$ will be denoted by the same letter $v$.

*Proof of Lemma 2.1.*— To prove both (1) and (2) it is enough to show that for each critical point $c$ of $\widehat{f}$ and for each $a \in \mathbb{R}$ there is at most finite number of $(-v)$-trajectories, joining $c$ with the critical points of $\widehat{f}$, lying above the level $a$. The function $\widehat{f}$ and the vector field $v$ are induced from $\overline{M}$, so it is enough to check the corresponding assertion on $\overline{M}$. But it follows directly from Lemma A.3 of the Appendix. □

The proof of Theorem 2.2 will occupy the rest of this section and the following one.

First we shall reduce the proof to the study of the Morse function $\overline{f}$, restricted to $V^-$.

We denote the preimages in $\widehat{M}$ of the sets in $\overline{M}$ by adding the sign $\widehat{\ }$. Let $C_i^-$ be a free $\mathbb{Z}$-module, generated by the critical points of $\widehat{f}$, belonging to $\widehat{V^-}$. Let $\widehat{C}_i^-$ be the abelian group, consisting of all the formal $\mathbb{Z}$-linear combinations $\lambda$ of critical points in $\widehat{V^-}$, such that above any given level surface $\widehat{f}^{-1}(a)$ of $\widehat{f}$ there is only a finite number of points in $\lambda$. One checks that $\widehat{C}_i^-$ is a free right $\Lambda_\xi^=$-module and that one can choose as a free basis for $\widehat{C}_i^-$ the liftings to $\widehat{M}$ of the critical points of $\overline{f}$ in $W_0$. In the same way as in Section 2 one defines the homomorphism

$$\partial_p^- : \widehat{C}_p^- \to \widehat{C}_{p-1}^-$$

(using Lemma 2.1). Similarly to the section 2, if we want to stress the dependance of $\partial_p^-$ on $v$ we write $\partial_p^-(v)$. One sees easily that the $\Lambda_\xi^-$-module $\widehat{C}_p$ is the tensor product $\widehat{C}_p^- \bigotimes_{\Lambda_\xi^=} \Lambda_\xi^-$, and the homomorphism $\partial_p$ equals $\partial_p^- \otimes id$.

If we choose a smooth triangulation of $M$ in such a way, that $V$ is a subcomplex, then it induces a triangulation of $\overline{M}$, such that all the $W_i$, $W(i)$, $V^-$, $t^iV^-$ are the subcomplexes. The chain complex of $\widehat{V^-}$ is a free based f.g. chain complex over $\mathbb{Z}G^-$ and the chain complex:

$$C_*^\Delta(\widehat{V^-})/\theta C_*(\widehat{V^-}) = C_*^\Delta(\widehat{V^-}) \bigotimes_{\mathbb{Z}G^-} \mathbb{Z}H$$

is just $C_*^\Delta(\widehat{W_0}, \widehat{tV})$.

We claim that in order to prove Theorem 2.2 it is enough to prove the following theorem.

THEOREM 4.1

*(1)* $\partial_p^- \circ \partial_{p+1}^- = 0$.

*(2) The free based complex* $C_*^-(v) = \{\widehat{C}_i^-, \partial_i^-(v)\}$ *is homotopy equivalent to* $C_*(\widehat{V^-}) \bigotimes_{\mathbb{Z}G^-} \Lambda_\xi^=$.

*(3) The homotopy equivalence can be chosen in such a way, that it gives a simple homotopy equivalence, when tensored with* $\mathbb{Z}H$.

*Proof of theorem 2.2 from theorem 4.1.* — Note that by the definition of the $Wh(G,\xi)$ the simple homotopy type of $C_*(v)$ does not depend on the particular choice of liftings $\widehat{c}$ of the $\omega$-zeros $c$ to the covering $\widehat{M}$. Thus we can choose the liftings in such a way, that their $Q$-projections belong to $W$. Then the based complexes $C_*^-(v) \bigotimes_{\Lambda_\xi^=} \Lambda_\xi^-$ and $C_*(v)$ are isomorphic by the isomorphism, preserving the bases. The same is true about

$$C_*(\widehat{M}) \bigotimes_{\mathbb{Z}G} \Lambda_\xi^- \quad \text{and} \quad \left(C_*(\widehat{V^-}) \bigotimes_{\mathbb{Z}G^-} \Lambda_\xi^=\right) \bigotimes_{\Lambda_\xi^=} \Lambda_\xi^- .$$

**Thus we have only to show that the condition (3) in the statement of our theorem implies that the homotopy equivalence:**

$$C_*(\widehat{V}) \bigotimes_{\mathbf{Z}G^-} \Lambda_\xi^= \to C_*^-(v)$$

**is simple. But it follows from Lemma 1.1.**

**To prove Theorem 4.1 we consider the Morse complex $C_*(n,v)$ of the Morse function $\overline{f}$, restricted to the cobordism $W(n)$, with respect to the covering $Q$. It is obvious that $C_*(n,v)$ is the quotient of $C_*^-(v)$ by the subcomplex, generated by all the critical points of $\widehat{f}$, lying below the level $-(n+1)a$. Therefore, $C_*(n,v)$ obtains a structure of $\Lambda_\xi^=$-module and it is obvious that the projection $p_n : C_*(n+1\,,\,v) \to C_*(n,v)$ preserves this structure and that:**

$$C_*^-(v) = \varprojlim C_*(n,v)\,.$$

**Note that the simplicial chain complexes $C_*^\Delta(\widehat{V^-}, \widehat{t^{n+1}V^-})$ also form the inverse system with the inverse limit $C_*^\Delta(\widehat{V^-}) \otimes_{\mathbf{Z}G^-} \Lambda_\xi^=$. □**

Lemma 4.2. — *There exists a diagram*

$$\begin{array}{ccccccc}
\cdots \to & C_*(n,v) & \longrightarrow & C_*(n-1\,,\,v) & \to \cdots \\
& \Big\downarrow h_n & & \Big\downarrow h_{n-1} & \\
\cdots \to & C_*^\Delta(\widehat{V^-}, \widehat{t^{n+1}V^-}) & \longrightarrow & C_*^\Delta(\widehat{V^-}, \widehat{t^nV^-}) & \to \cdots
\end{array} \qquad (4.1)$$

*such that:*

*(1) the maps $h_n$ are chain homotopy equivalences over $\Lambda_\xi^=$;*

*(2) the squares of (4.1) are homotopy commutative;*

*(3) the map $h_0$ is a simple homotopy equivalence over $\mathbf{Z}H$.*

*Proof of Theorem 4.1 from Lemma 4.2.* — **From Proposition 3.7, we deduce the existence of the homotopy commutative diagram**

$$\begin{array}{ccc}
\varprojlim C_*(n,v) & \xrightarrow{\;g\;} & \varprojlim C_*^\Delta(\widehat{V^-}, \widehat{t^nV^-}) \\
\Big\downarrow \mathrm{pr}_1 & & \Big\downarrow \mathrm{pr}_2 \\
C_*(W_0,v) & \xrightarrow{\;h_0\;} & C_*(\widehat{V^-}, \widehat{tV^-})
\end{array}$$

where $g$ is a $\Lambda_\xi^=$-homotopy equivalence. Since any chain $\Lambda_\xi^=$-map of a free $\Lambda_\xi^=$-module $F$ to a $\mathbb{Z}H$-complex factors uniquely though $F\bigotimes_{\Lambda_\xi^=}\mathbb{Z}H$ we get, that $h_0\sim g\bigotimes_{\Lambda_\xi^=}\mathbb{Z}H$, hence $g\bigotimes_{\Lambda_\xi^=}\mathbb{Z}H$ is a simple homotopy equivalence, and Theorem 4.1 is proved. □

The proof of Lemma 4.2 occupies the next section.

## 5. The proof of the main theorem. Part 2

In this section we give a proof of Lemma 4.2.

We note first, that if we demanded only that $h_n$ be the chain homotopy equivalences over $\mathbb{Z}H$, this lemma would follow from the results of Appendix. We need more, because we want $h_n$ to commute with the actions of $\Lambda_\xi^=$. Actually both $C_*(n,v)$ and

$$C_*^\Delta\left(\widehat{V^-},\widehat{t^{n+1}V^-}\right)=C_*^\Delta\left(\widehat{W(n)},\widehat{t^{n+1}V}\right)$$

are modules over $\mathbb{Z}G_n^-$, hence it suffices to construct the $h_n$ of the diagram (4.1) as $\mathbb{Z}G_n^-$-chain homotopy equivalences and to check the chain homotopy commutativity of the squares over $\mathbb{Z}G_n^-$ (note that $\mathbb{Z}G_{n-1}^-$-modules and maps are obviously $\mathbb{Z}G_n^-$-modules and maps via the obvious projection $\mathbb{Z}G_n^-\to\mathbb{Z}G_{n-1}^-$).

First we need an ordered Morse function on $W(n)$ which respects the action of $t$.

LEMMA 5.1.— *There exists an ordered Morse function $F:W(n)\to\mathbb{R}$, such that $v$ is a gradient-like vector field for $F$, $F$ is constant on $V$ and $t^{n+1}V$ and if $x\in W(n)$ and $tx\in W(n)$, then $F(tx)<F(x)$.*

*Proof.*— We need some more notations. We denote by $f:M\to S_a^1=\mathbb{R}/a\mathbb{Z}$ the quotient of $\overline{f}$ by the action of $\mathbb{Z}$. A Morse function $g:M\to S_a^1=\mathbb{R}/a\mathbb{Z}$ is called admissible, if:

(1) $g$ is homotopic to $f$ and $v$ is a gradient-like vector field for $g$,

(2) the set of critical values of $g$ consists of $N\le n$ points, equidistantly placed on $S_a^1$,

(3) for two critical points $x,y$ of $g$ the values $g(x),g(y)$ are the same if and only if $\operatorname{ind}x=\operatorname{ind}y$.

A Morse function $\overline{g} : \overline{M} \to \mathbb{R}$ is called admissible, if $\overline{g}(tx) = \overline{g}(x) - a$ and the quotient $g : M \to S^1_a$ is admissible.

The standard procedure of ordering of the Morse function $\overline{f} \mid W(0)$ [5, § 4] implies, that admissible functions exist. Now we shall describe a procedure of modification of an admissible function. We shall assume that the number of different indices of the zeros of $v$ is not less than 2 (otherwise the initial Morse function satisfies the conclusions of 5.1).

Let $g : M \to S^1_a$ be an admissible function, $\Gamma \subset S^1_a$ be its set of critical values. Assume that $\gamma_1, \gamma_2 \in \Gamma$, that $\gamma_2 = \gamma_1 + a/N$ (that means that $\gamma_2$ is the next critical value after $\gamma_1$), and that $\operatorname{ind}\gamma_2 < \operatorname{ind}\gamma_1$ (the condition (3) above) implies that for a critical value $\gamma \in \Gamma$ the notion of index is well defined).

For an $\varepsilon$ small enough let $N$ be

$$g^{-1}([\gamma_1 - \varepsilon , \gamma_2 + \varepsilon]) ,$$

$$\partial N = g^{-1}(\gamma_2 + \varepsilon) \sqcup g^{-1}(\gamma_1 - \varepsilon) .$$

The map $g$, restricted to $N$, provides a Morse function $N \to [\gamma_1 - \varepsilon , \gamma_2 + \varepsilon]$ with two critical values $\gamma_1 < \gamma_2$, $\operatorname{ind}\gamma_2 < \operatorname{ind}\gamma_1$. The standard ordering procedure [5, § 4] provides us with a Morse function $r : N \to [\gamma_1 - \varepsilon , \gamma_2 + \varepsilon]$ which:

(1) coincides with $g$ in the neighborhood of $g^{-1}(\gamma_2 + \varepsilon)$ and $g^{-1}(\gamma_1 - \varepsilon)$,

(2) has $v$ as a gradient-like vector field,

(3) $r \mid g^{-1}(\gamma_2) = \gamma_1$, $r \mid g^{-1}(\gamma_1) = \gamma_2$.

We set $g_1 \mid (M \setminus N) = g$, $g_1 \mid N = r$. It is again an admissible Morse function $M \to S^1_a$.

We call this procedure elementary modification of the admissible function $g$ and the function $g_1$ is called the result of the elementary modification. The number $\operatorname{ind}\gamma_2$ is called the lower index of the modification, the $\operatorname{ind}\gamma_1$ the upper index.

If $\overline{g} : \overline{M} \to \mathbb{R}$ is a lifting of $g$ to $\overline{M}$, then we fix a lifting $\overline{g}_1 : \overline{M} \to \mathbb{R}$, setting $\overline{g}_1 = \overline{g}$ in $\overline{p}^{-1}(M \setminus N)$.

Note that for this particular lifting $\overline{g}_1$ the values of critical points are as follows: $\overline{g}_1(w) < \overline{g}(w)$, if $\operatorname{ind} w = \operatorname{ind}\gamma_2$, $\overline{g}_1(w) > \overline{g}(w)$, if $\operatorname{ind} w = \operatorname{ind}\gamma_1$, $\overline{g}_1(w) = \overline{g}(w)$ in the other case.

We shall always choose this lifting, so the admissible modification will be applied to a pair $(g, \overline{g})$ to get a new pair $(g_1, \overline{g_1})$.

*Remark.*— One proves easily, that if $y$ and $z$ are critical points of $\overline{g}$, such that $\operatorname{ind} y < \operatorname{ind} z$ and $\overline{g}(y) \leq \overline{g}(z)$, then $y$ and $z$ are again the critical points of $g_1$ and the inequalities $\operatorname{ind} y < \operatorname{ind} z$ and $\overline{g}_1(y) < \overline{g}_1(z)$ still hold.

LEMMA 5.2.— *Let $f : M \to S^1_a$ be an admissible function, $A$ be a finite set of critical points of $\overline{f} : \overline{M} \to \mathbb{R}$.*

*Then there exists a finite sequence of elementary modifications of $f$, resulting with an admissible function $g : M \to S^1_a$, such that if $x$, $y \in A$ and $\operatorname{ind} x < \operatorname{ind} y$, then $\overline{g}(x) < \overline{g}(y)$.*

*Proof.*— Denote by $A_g \subset A \times A \setminus \Delta$ the set of all the pairs $(x, y) \in A \times A$, such that $\operatorname{ind} x < \operatorname{ind} y$ and $\overline{g}(x) > \overline{g}(y)$. By the above remarks the set $A_{g_1}$ is contained in $A_g$. So to prove the lemma it suffices to construct a sequence of elementary modifications with the resulting $A_{g_n} \subset A_g$ and $A_{g_n} \neq A_g$. For that we need one more lemma.

LEMMA 5.3.— *Let $g : M \to S^1_a$ be admissible and let $x$, $y$ be critical points of $\overline{g} : \overline{M} \to \mathbb{R}$, $\operatorname{ind} x < \operatorname{ind} y$, $\overline{g}(x) > \overline{g}(y)$.*

*Then there is a sequence of elementary modifications with lower indices $\leq \operatorname{ind} x$, such that for the resulting admissible function $h : M \to S^1$ we have $\overline{g}(y) \leq \overline{h}(y)$, $\overline{h}(x) < \overline{g}(x)$.*

*Proof.*— Denote by $N$ the number of critical values of $g$. Note that this number is not changed under an elementary modification. We prove the lemma by induction in $\big(\overline{g}(x) - \overline{g}(y)\big)(N/a)$, which is an integer, because $g$ is admissible.

If $\overline{g}(x) - \overline{g}(y) = (a/N)$, then the elementary modification, changing $\overline{g}(x)$ and $\overline{g}(y)$ finishes the proof.

If not, then we distinguish two cases.

$\alpha$) For each critical point $w$ of $\overline{g}$ between $\overline{g}(y)$ and $\overline{g}(x)$ we have $\operatorname{ind} w \leq \operatorname{ind} x$. Then we apply the elementary modification to the pair of critical levels $\overline{g}(y)$, $\overline{g}(y) + (a/N)$. After the modification the level of $x$ did not grow up, the level of $y$ did grow up, and we are over.

$\beta$) There are critical points $w$ between $\overline{g}(y)$ and $\overline{g}(x)$, such that $\operatorname{ind} w > \operatorname{ind} x$. Then we apply the induction assumption to the pair of critical points $w$, $x$. We get an admissible function $h$, such that $\overline{g}(w) \leq \overline{h}(w)$,

$\overline{h}(x) < \overline{g}(x)$. Note that the lower indices of the modifications were $\leq \operatorname{ind} x$, therefore the level of $y$ did not lower.

Lemma 5.3 is proved, as well as Lemma 5.2. □

*End of the proof of Lemma 5.1.*— Let $f : M \to S^1_a$ be an admissible Morse function and $W(n)$, as usual, be $\overline{f}^{-1}([-(n+1)a\,,\,0])$. We apply Lemma 5.2 to the function $f$ and to the set $A$ of all the critical points of $\overline{f}$ in $W(n)$. We get as a result an admissible function $g : M \to S^1_a$, such that the lifting $\overline{g} : \overline{M} \to \mathbb{R}$ satisfies $\overline{g}(x) < \overline{g}(y)$ if $\operatorname{ind} x < \operatorname{ind} y$ and $x, y \in W(n)$.

We restrict the function $\overline{g}$ to $W(n)$ and apply to $\overline{g}$ the damping construction (see the Appendix Prop. A.10). It is easy to see that the resulting function $F : W(n) \to \mathbb{R}$ satisfies the conclusions of the lemma. Indeed, the only thing to check is the condition $F(tx) < F(x)$ if $x, tx \in W(n)$. If both $x$ and $tx$ do not belong neither to $U_0$ nor to $U_1$, then:

$$F(tx) = \overline{g}(tx) = \overline{g}(x) - a < \overline{g}(x) = F(x)\,.$$

(For the definition of $U_0$ and $U_1$ see the damping construction of the Appendix). If $tx$ belongs to $U_0$, then $x$ does not, therefore

$$F(tx) \leq \overline{g}(tx) = \overline{g}(x) - a < F(x)\,.$$

The case $x \in U_1$ is similar. □

Now we can proceed directly to the proof of Lemma 4.2. The functions $F : W(n) \to \mathbb{R}$, satisfying the assumptions of Lemma 5.1 will be called $t$-ordered. For any $t$-ordered function $F : W(n) \to [\alpha\,,\,\beta]$ with an ordering sequence $\alpha < \alpha_1 < \cdots < \alpha_m < \beta$ we consider the filtration of the pair $(V^-, t^{n+1}V^-)$ by the subpairs $(W_p, t^{n+1}V^-)$, where

$$W_p = F^{-1}([\,\alpha\,,\,\alpha_p\,]) \cup t^{n+1}V^-\,.$$

Note that this filtration is $t$-invariant, therefore the filtration $C_*^{(p)}$ of the singular chain complex

$$C_* = C_*^s(\widehat{V^-}\,,\,\widehat{t^{n+1}V^-})$$

defined by

$$C_*^{(p)} = C_*^s(\widehat{W_{p+1}}\,,\,\widehat{t^{n+1}V^-})$$

is the filtration by $\mathbb{Z}G_n^-$-modules.

Note that it is a nice filtration. Indeed, consider the free $\mathbb{Z}$-module $C_p(n,v)$ generated by the zeros of $v$ of index $p$, belonging to $\widehat{W(n)}$, and the homomorphism $J : C_p(n,v) \to H_p(\widehat{W_{p+1}}, \widehat{W_p})$, which sends each zero $c$ to the homology class of the intersection of the descending disc $D(c)$ with $\widehat{F}^{-1}([\alpha_p, \beta])$ modulo $D(c) \cap \widehat{F}^{-1}(\alpha_p)$. (Here $\widehat{F}$ stands for $F \circ Q$). By the standard Morse theory $J$ is an isomorphism of $\mathbb{Z}H$-modules. It is easy to see, that, since $F$ is $t$-ordered, the isomorphism $J$ commutes with the action of $\theta$ and, therefore, of $\mathbb{Z}G_n^-$.

By Lemma A.7 of the Appendix, $J$ is an isomorphism of chain complexes. From Corollary 3.4 it follows that the complex $C_*^{gr}$ is $\mathbb{Z}G_n^-$-homotopy equivalent to $C_*^s(\widehat{V^-}, \widehat{t^{n+1}V^-})$. Thus we get a $\mathbb{Z}G_n^-$-homotopy equivalence:

$$J(F) : C_*(n,v) \to C_*^s(\widehat{V^-}, \widehat{t^{n+1}V^-}) .$$

Furthermore, it does not depend on the particular choice of $t$-ordered function $F$. Indeed, let $G : W(n) \to [a, b]$ be another $t$-ordered function with an ordering sequence $a < a_1 < \cdots < a_m < b$, generating the filtration:

$$U_p = G^{-1}([a, a_p]) \cup t^{n+1}V^- \quad \text{of} \quad V^- .$$

Since $v$ is a gradient-like vector field for both $G$ and $F$, there are no zeros of $v$ in the sets $G^{-1}([a, a_p]) \cap F^{-1}([\alpha_p, \beta])$ and $G^{-1}([a_p, b]) \cap F^{-1}([\alpha, \alpha_p])$, which implies that $U_p$ and $W_p$ are both deformation retracts (along the $(-v)$-trajectories) of $U_p \cup W_p$. Therefore $\{U_p \cup W_p\}$ is a $t$-invariant filtration of $V^-$, and it determines a nice $\mathbb{Z}G_n^-$-filtration

$$\left\{ C_*^s\left(\widehat{U_p \cup W_p}, \widehat{t^{n+1}V^-}\right)\right\} \quad \text{of} \quad C_* .$$

Both $J(F)$ and $J(G)$ preserve this filtration. Therefore in order to show that $J(F)$ is homotopic to $J(G)$ it is sufficient (by Lemma 3.2) to show that they induce the same map in graded homology. To see this notice, that the $J(F)$-image (respectively, $J(G)$-image) of a zero $c$ of $v$ of index $p$ in

$$H_p\left(\widehat{W_{p+1} \cup U_{p+1}}, \widehat{W_p \cup U_p}\right)$$

equals to the homology class of

$$\left(D(c) \cap F^{-1}([\alpha_p, \infty]), D(c) \cap F^{-1}(\alpha_p)\right) ,$$

respectively, of

$$\left(D(c) \cap G^{-1}([\,a_p\,,\,\infty\,])\,,\; D(c) \cap G^{-1}(a_p)\right)\,,$$

in

$$\left(\widehat{W_{p+1} \cup U_{p+1}}\,,\; \widehat{W_p \cup U_p}\right).$$

But since the sets:

$$F^{-1}\big([\alpha_p\,,\,\infty) \cap G^{-1}(-\infty\,,\,a_p\,]\big)\,,$$

respectively

$$F^{-1}\big((-\infty\,,\,\alpha_p\,]\big) \cap G^{-1}\big([\,a_p\,,\,\infty\,)\big)\,,$$

contain no zero of $v$, it is obvious that these classes are even homotopic via a homotopy along the $v$-trajectories.

Therefore $J(F)$ is $\mathbb{Z}G_n^-$-chain homotopic to $J(G)$.

Now we proceed to check the homotopy commutativity of the following square

$$\begin{array}{ccc} C_*(n,v) & \longrightarrow & C_*(n-1,v) \\ \Big\downarrow {\scriptstyle J(F_n)} & & \Big\downarrow {\scriptstyle J(F_{n-1})} \\ C_*^s\big(\widehat{V^-}\,,\,\widehat{t^{n+1}V^-}\big) & \longrightarrow & C_*^s\big(\widehat{V^-}\,,\,\widehat{t^{n}V^-}\big) \end{array}$$

where $F_n$, $F_{n-1}$ are the arbitrary $t$-ordered functions on $W(n)$, resp. $W(n-1)$.

Note that by the above we can choose any function $F_{n-1}$ we like. We shall take as $F_{n-1}$ the lower damping of $F_n \mid W(n-1)$. (One easily shows that it is really a $t$-ordered Morse function on $W(n-1)$.) In this case the lower horizontal arrow preserves the filtrations and the homotopy commutativity follows again from Lemma 3.2.

To get finally the maps $h_n : C_*(n,v) \to C_*^{\Delta}\big(\widehat{V^-}\,,\,\widehat{t^{n+1}V^-}\big)$ as demanded by Lemma 4.2 we compose $J(F_n)$ with an arbitrary chain homotopy inverse of the natural $\mathbb{Z}G_n^-$-homotopy equivalence:

$$\mu : C_*^{\Delta}\big(\widehat{V^-}\,,\,\widehat{t^{n+1}V^-}\big) \to C_*^s\big(\widehat{V^-}\,,\,\widehat{t^{n+1}V^-}\big)\,,$$

and thus the points (1) and (2) are achieved.

To get (3) we consider the simple $\mathbb{Z}H$-homotopy equivalence $R$ : $C_*(0,v) \to C_*^{\Delta}\big(\widehat{V^-}\,,\,\widehat{tV^-}\big)$, constructed in the proof of Theorem A.5 with respect to an ordered function $F_0$. Then the composition $\mu \circ R$ is homotopic to $J(F_0)$, hence $R \sim h_0$. Lemma 4.2 is proved. □

# Appendix

## Morse complex of a Morse function

Let $(W\,;\,V_0, V_1)$ be an $m$-dimensional compact manifold with boundary $\partial W = V_0 \sqcup V_1$. Let $f : W \to \mathbb{R}$ be a Morse function such that $\operatorname{Im} f = [\,a\,,\,b\,]$, $f^{-1}(a) = V_0$, $f^{-1}(b) = V_1$ and all the zeros of $\mathrm{d}f$ belong to $\overset{\circ}{W} = W \setminus \partial W$.

Pick any gradient-like vector field $v$ for $f$. For a critical point $c$ of $f$ we denote by $B^-(c)$, resp. $B^+(c)$, the set of all points $x \in \overset{\circ}{W}$ such that the $v$-trajectory, starting at $x$, converges to $c$, when the parameter tends to $+\infty$ (resp. to $-\infty$). We shall denote $B^-(c) \cap f^{-1}(\lambda)$ by $B^-_\lambda(c)$, resp. $B^+(c) \cap f^{-1}(\lambda)$ by $B^+_\lambda(c)$. If we want to stress the role of $v$, we write $B^-_\lambda(c, v)$, etc.

The following lemma is well known.

LEMMA A.0

*(1) The set $B^-(c)$ (respectively, $B^+(c)$) is a submanifold of $\overset{\circ}{W} = W \setminus (V_0 \cup V_1)$ of the dimension $p$ (respectively, $m - p$), where $p$ is the index of $c$.*

*(2) For each regular value $\lambda$ of $f$ the set $B^-(c) \cap f^{-1}(\lambda)$, resp. $B^+(c) \cap f^{-1}(\lambda)$, is the submanifold of $f^{-1}(\lambda)$ of dimension $p - 1$, resp. $m - p - 1$.* □

We say that a gradient-like vector field $v$ satisfies the transversality assumption if for any two zeros $c, d$ the manifolds $B^-(c), B^+(d)$ are transversal.

Note, that the transversality of $B^-(c)$ to $B^+(d)$ is equivalent to the transversality of $B^-_\lambda(c)$ to $B^+_\lambda(d)$ in the manifold $f^{-1}(\lambda)$, where $\lambda$ is any regular value of $f$ between $c$ and $d$.

LEMMA A.1.— *There exists a gradient-like vector field $v$ for $f$, satisfying the transversality assumption.*

*Proof.*— It is almost the same as that of theorem 5.2 in [5].

We explain first a procedure of "elementary change" of a gradient-like vector field. Let $a$ be a critical value of $f$. Choose the regular value $b$, such that $b < a$ and there are no critical values of $f$ in $(b, a)$. Consider the union of all $B^-(d, v)$, where $d$ is a critical point on the level $a$ and denote by $N(v)$ the intersection of this set with $f^{-1}(b)$. It is a union of compact submanifolds of $f^{-1}(b)$, each having trivial normal fibration. If $c$ is a critical point of $f$ below $a$, the intersection of $B^+(c)$ with $f^{-1}(b)$ is a smooth submanifold (not necessarily compact) of $f^{-1}(b)$. Exactly in the same way as in [5, lemma 5.3], we find an isotopy $\varphi$ of $f^{-1}(b)$, close to identity, such that $\varphi(N(v))$ is transversal to all the $B^+(c)$. By [5, lemma 4.6], we obtain the vector field $w$ which coincides with $v$ everywhere, except $f^{-1}([b, b+\varepsilon])$, such that $N(w)$ equals $\varphi(N(v))$. Thus the intersection $B^-(c) \cap B^+(c')$ for $c$, $c'$ below $b$ did not change and the intersections $B^+(c) \cap B^-(c')$ are transversal if $f(c') = a$.

Now the easy induction argument proves the lemma. □

Note that for the principal part of our paper we do not need to apply this lemma since the gradient-like vector field satisfying transversality assumption is provided by the gradient-like vector field for the form $\omega$, satisfying the assumption. We give here the proof for the sake of completness.

From now on the vector field $v$, satisfying the transversality assumption, will be fixed.

LEMMA A.2.— *There exists a Morse function $g : W \to [c, d]$, such that $v$ is a gradient-like vector field for $g$ and that all the critical points of the same index have the same value, increasing with the index.*

*Proof*.— Apply several times [5, Theorem 4.1]. □

Such functions will be called regular with respect to $v$.

LEMMA A.3.— *Let $c_1$ and $c_2$ be the zeros of $v$ of the index $p$ and, respectively, $p-1$. Then there is at most a finite set of trajectories of $(-v)$, starting at $c_1$ and finishing at $c_2$.*(*)

*Proof*.— Choose the Morse function $g$ so as to satisfy the conditions of Lemma A.2. Let $d$ be any value in the interval $(g(c_1), g(c_2))$. The set of $(-v)$-trajectories going from $c_1$ to $c_2$ is in bijective correspondence with the set of points in the intersection of $(B^-(c_1) \cap g^{-1}(d))$ and $(B^+(c_2) \cap g^{-1}(d))$. These manifolds both are compact submanifolds of $g^{-1}(d)$, transversal to each other and the sum of dimensions is $m-1$. □

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(*) We identify the trajectories, which differ by a parameter change.

Now we shall describe two versions of Morse complex and prove that they are isomorphic. To define the first version we need to fix the following data:

(1) A g.-l. vector field $v$ for the function $f$, satisfying the transversality assumption.

(2) For each critical point $c$ of $f$ an orientation of the manifold $B^-(c)$.

(3) A regular covering $\widehat{p} : \widehat{W} \to W$ with the structure group $G$ (we do not assume, that $\widehat{W}$ is connected).

With these data fixed we proceed as follows. The lift of the vector field $v$ to $\widehat{W}$ will be denoted by the same letter $v$. It is a gradient-like vector field for the Morse function $\widehat{f} = f \circ \widehat{p}$. If $c$, $d$ are two critical points of $\widehat{f}$ such that $\operatorname{ind} c = \operatorname{ind} d + 1$, $\gamma$ is a $v$-trajectory, joining $c$ and $d$, and $z$ is a point, belonging to $\gamma$, then the tangent space $T_z B^-(c)$ is oriented, the tangent space $T_z B^+(d)$ is cooriented, their intersection is of dimension 1, generated by the vector $v$, hence oriented. Thus we obtain a sign $\pm$, which is easily seen not to depend on $z$, and is denoted $\varepsilon(\lambda)$. Lemma A.3 implies easily, that there is only a finite number of $(v)$-trajectories, joining $c$ and $d$. The sum of the signs $\varepsilon(\lambda)$ corresponding to these $\gamma$, is denoted $n(c, d)$.

We denote by $C_p$ the free abelian group, generated by the critical points of $\widehat{f}$ of index $p$. For each generator $c$ of $C_p$ we denote by $\partial_p c$ the element $\sum_d n(c,d)d$, where $d$ runs over all the critical points of $\widehat{f}$ of index $p-1$. (Again Lemma A.3 implies that the sum is well defined.) Note that $C_*$ is a right $\mathbb{Z}G$-module and that the operator $\partial$ commutes with the $\mathbb{Z}G$-action. Thus we get a map $\partial_p : C_p \to C_{p-1}$ of $\mathbb{Z}G$-modules. Now we fix:

(4) For each critical point $c$ of $f$ a lifting $\widehat{c}$ of $c$ to $\widehat{W}$.

It is obvious, that these $\widehat{c}$ form a free basis of $C_*$, and thus, with (4) we obtain the homomorphism $\partial_p : C_p \to C_{p-1}$ of the free based right $\mathbb{Z}G$-modules. If we want to stress the role of the choices (1)-(4) we write $C_*(v)$ or even $C_*(v, \widehat{W}, \mathcal{O}, \ell)$, where $\mathcal{O}$ is the choice of orientations, $\ell$ — the choice of liftings.

*Remark* A.4. — If $g$ is another Morse function with the same g.-l. vector field $v$, the corresponding modules and homomorphisms are the same.

THEOREM A.5. — *$\partial_p \circ \partial_{p+1} = 0$ and the resulting free based $\mathbb{Z}G$-complex $C_*(v)$ is simply homotopy equivalent to the chain complex of $(\widehat{W}, \widehat{V_0})$, induced by any smooth triangulation of the pair $(W, V_0)$.*(*)

(*) For a subset $X$ of $W$ we denote by $\widehat{X}$ the preimage of $X$ in $\widehat{W}$.

Before beginning the proof we shall present another version of Morse complex, which will be of use.

If for a Morse function $\varphi : W \to [\alpha, \beta]$ there is a sequence of regular values $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_m < \alpha_{m+1} = \beta$, such that all the critical points of $\varphi$ of index $p$ lie in $\varphi^{-1}\big((\alpha_p, \alpha_{p+1})\big)$ then we say that $\varphi$ is ordered [3, p. 95] and that $\{\alpha_i\}$ is a ordering sequence for $\varphi$. (We do not suppose, that the critical points of the same index belong to the same level.)

The following lemma is standard.

LEMMA A.6. — *Let $\varphi : W \to [\alpha, \beta]$ be an ordered Morse function with an ordering sequence $\{\alpha_i\}$ and a gradient-like vector field $v$, satisfying the transversality assumption. Then:*

*(1) for every critical point $c$ of $\varphi$ of index $p$ the pair*

$$\Big(B^-(c) \cap \varphi^{-1}([\alpha_p, \beta]), \ B^-(c) \cap \varphi^{-1}(\alpha_p)\Big)$$

*is diffeomorphic to $(D^p, S^{p-1})$; it will be denoted by $\big(D^p(c), S^{p-1}(c)\big)$;*

*(2) The pair*

$$\varphi^{-1}([\alpha, \alpha_p]) \cup \left(\bigcup_c D^p(c)\right),$$

*where $c$ runs through the critical points of index $p$ is a deformation retract of $\varphi^{-1}\big([\alpha, \alpha_{p+1}]\big)$.* □

Now for an arbitrary Morse function $f : W \to [a, b]$ with a gradient-like vector field $v$, satisfying the transversality assumption, we choose an ordered Morse function $\varphi : W \to [\alpha, \beta]$ with the same gradient-like vector field and with a ordering sequence $\{\alpha_p\}$. We denote $\varphi \circ \widehat{p}$ by $\widehat{\varphi}$ and the preimages $\widehat{\varphi}^{-1}\big([\alpha, \alpha_p]\big)$ by $\widehat{W_p}$. The above lemma implies, that the homology $H_*\big(\widehat{W_{p+1}}, \widehat{W_p}\big)$ vanishes for $* \neq p$. Denote $H_p\big(\widehat{W_{p+1}}, \widehat{W_p}\big)$ by $C_p(\varphi)$. It is obviously a $\mathbb{Z}G$-module. The differential $\mathrm{d} : C_p(\varphi) \to C_{p-1}(\varphi)$ of the exact sequence of the triple $\big(\widehat{W_{p+1}}, \widehat{W_p}, \widehat{W_{p-1}}\big)$ commutes obviously with the $\mathbb{Z}G$-action and determines the $\mathbb{Z}G$-complex $C_*(\varphi)$. If for each critical point $c$ of $f$ of index $p$ we choose an orientation of $B^-(c)$ and a lifting $\widehat{c}$ of $c$ to $\widehat{W}$, the corresponding liftings of the pairs $\big(D^p(c), S^{p-1}(c)\big)$ will give a free $\mathbb{Z}G$-basis of $C_p(\varphi)$.

Thus, having fixed an ordered Morse function $\varphi$ with the same gradient-like vector field $v$ as for $f$, as well as the choices (2)-(4) above we construct a free based $\mathbb{Z}G$-complex $C_*(\varphi)$.

The following lemma is an extension to the non-simply connected case of the assertion of [5, corollary 7.3]. The proof is similar to that of [5] and will be omitted.

LEMMA A.7. — *There is an isomorphism $J : C_*(v) \to C_*(\varphi)$ of graded $\mathbb{Z}G$-modules, such that it preserves the chosen bases and that $J \circ \partial = \mathrm{d} \circ J$.* □

*Proof of Theorem A.5.* — The property $\partial_p \circ \partial_{p+1} = 0$ follows from Lemma A.7, which implies also that it suffices to prove the simple homotopy equivalence $C_*(\varphi) \sim C_*^{\Delta}(\widehat{W}, \widehat{V_0})$, where $\Delta$ stands for some smooth triangulation of $W$, such that $V_0$ is a subcomplex.

First we note that by the standard argument the simple homotopy type of $C_*^{\Delta}(\widehat{W}, \widehat{V_0})$ indeed does not depend on the chosen triangulation $\Delta$ (see [4, th. 10.4] and [6, th.7.1]). Thus we can choose any smooth triangulation of $(W, V_0)$.

LEMMA A.8. — *There exists a smooth triangulation of $(W, V_0)$ such that:*

*(1) $V_0$, $V_1$ and all the $\varphi^{-1}(\alpha_i)$ are simplicial subcomplexes;*

*(2) for each critical point $c$ of index $p$ the disc $D^p(c)$ is a simplicial subcomplex;*

*(3) the embedding*

$$\varphi^{-1}([\alpha, \alpha_p]) \cup \left(\bigcup_c D^p(c)\right) \subset \varphi^{-1}([\alpha, \alpha_{p+1}]),$$

*where $c$ runs through all the critical points of index $p$, is a simple homotopy equivalence.*

*Proof.* — Of course this lemma is essentially well known, and in a sense it is implicit in [6, sect. 9]. We give a proof for the sake of completeness.

Fix a natural number $p$: $0 \le p \le m$. We assume for simplicity that there is only critical point $c$ of index $p$ in $\varphi^{-1}([\alpha_p, \alpha_{p+1}])$. We assume $\varphi(c) = 0$. We start with triangulating the manifold $U = \varphi^{-1}([-\varepsilon, \varepsilon])$ in such a way that $D^p(c) \cap U$ is a subcomplex. To do that we suppose that $\varepsilon$ is small enough and consider the standard coordinate neighborhood $U(c) \subset \mathbb{R}^{m-p} \times \mathbb{R}^p$ around $c$ (fig. A1). Here

$$\varphi(\vec{x}, \vec{y}) = |\vec{y}|^2 - |\vec{x}|^2 .$$

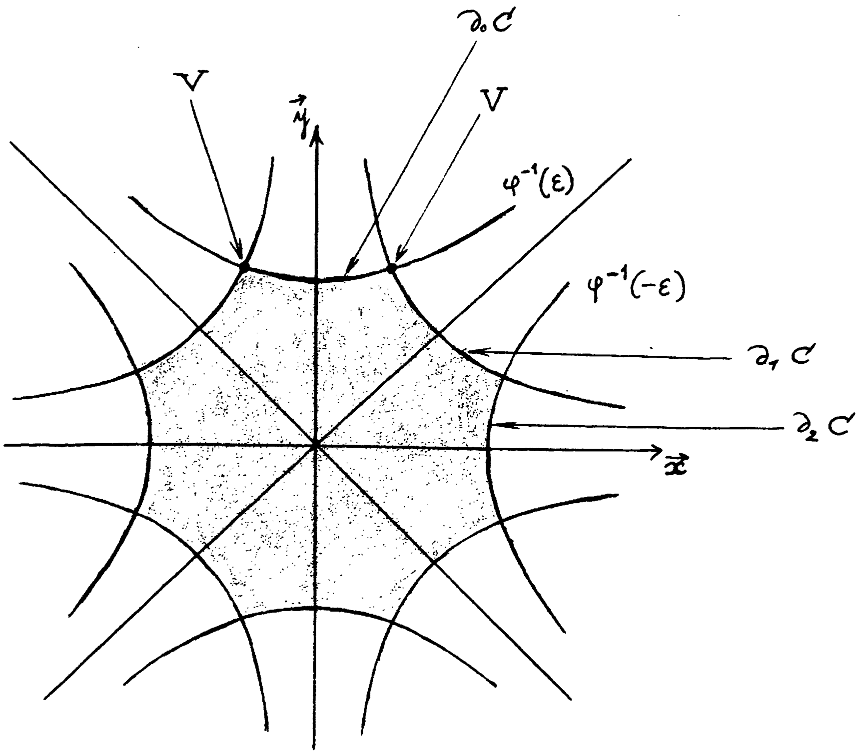



**Fig. A1**

Denote $|\vec{x}|\,|\vec{y}|$ by $\lambda(\vec{x}, \vec{y})$ and consider the set $C = \varphi^{-1}([-\varepsilon, \varepsilon]) \cap \lambda^{-1}([0, \varepsilon])$ (shaded on the fig. A1). It is the conical subset of $\mathbf{R}^m$ with the boundary, split into three parts:

$$\partial_0 C = \varphi^{-1}(\varepsilon) \cap \lambda^{-1}([0, \varepsilon]),$$
$$\partial_1 C = U \cap \lambda^{-1}(\varepsilon)$$

and

$$\partial_2 C = \varphi^{-1}(-\varepsilon) \cap \lambda^{-1}([0, \varepsilon]).$$

We denote by $V$ the intersection:

$$\partial_0 C \cap \partial_1 C = \partial(\partial_0 C) = \varphi^{-1}(\varepsilon) \cap \lambda^{-1}(\varepsilon).$$

The vector field $v = \operatorname{grad} \varphi = (-2\vec{x}, +2\vec{y})$ is tangent to $\partial_1 C$ and determines a diffeomorphism $V \times [0, 1] \approx \partial_1 C$, which is restriction of

$$U \setminus \operatorname{Int} C \approx (\varphi^{-1}(\varepsilon) - \operatorname{Int} \partial_0 C) \times I.$$

Note also that $\partial_2 C$ is diffeomorphic to $S^{p-1} \times D^{m-p}$ by means of a diffeomorphism, which sends $\partial_2 C \cap \mathbb{R}^p$ to $S^{p-1} \times \{0\}$ and $\partial(\partial_2 C)$ to $S^{p-1} \times \partial D^{m-p} = S^{p-1} \times S^{m-p-1}$.

Now we choose (by [4, th. 10.6 ]) any smooth triangulation of $\varphi^{-1}(\varepsilon)$, such that $V$ (and, henceforth $\partial_0 C$ and $\varphi^{-1}(\varepsilon)\setminus \operatorname{Int} \partial_0 C$) is a subcomplex. We triangulate $U \setminus \operatorname{Int} C$ as a product. Now the intersection $\varphi^{-1}(-\varepsilon) \cap \partial_1 C = \partial(\partial_2 C)$ is triangulated. We expand this triangulation to a triangulation of $\partial_2 C$ in such a way, that $\mathbb{R}^p \cap \partial_2 C$ is a subcomplex.

Now we have triangulated $\partial C$ completely and we take a cone on this triangulation to get the triangulation of $C$. Note that the $D^p(c) \cap U$, being a cone over $S^{p-1}$, is a subcomplex, and that our triangulation fits with the product triangulation of $U \setminus \operatorname{Int} C$, so that we can expand it to triangulate all the $U$.

Now using the shift along the $(-v)$-trajectories one easily constructs a diffeomorphism $\varphi^{-1}([-\varepsilon\,,\,\varepsilon])$ to $\varphi^{-1}([\alpha_p\,,\,\varepsilon])$, which leaves $\varphi^{-1}(\varepsilon)$ fixed and sends $D^p(c) \cap \varphi^{-1}([-\varepsilon\,,\,\varepsilon])$ to $D^p(c) \cap \varphi^{-1}([\alpha_p\,,\,\varepsilon])$.

Applying the same procedure to all the critical levels $\gamma_s \in [\alpha_s\,,\,\alpha_{s+1}]$, $0 \le s \le m$ (we assume, that all the critical points of the same index belong to the same level) we get the triangulations of $\varphi^{-1}([\alpha_s\,,\,\gamma_s + \varepsilon])$. Now we triangulate the manifolds $\varphi^{-1}([\gamma_s + \varepsilon\,,\,\alpha_{s+1}])$ arbitrarily by [4, th. 10.6], and the proof of (1) and (2) is finished.

The proof of (3) is reduced, by the homeomorphism invariance of Whitehead torsion to the proof of simple equivalence $\varphi^{-1}(-\varepsilon) \cup D(c) \hookrightarrow \varphi^{-1}([-\varepsilon\,,\,\varepsilon])$ in the situation above. This is done in [6, sect. 3]. □

We can now finish the proof of Theorem A.5. It is obviously enough to prove it for $W$ connected and in this case — for the covering $\widehat{W} \to W$ being universal.

The filtration of $(W, V_0)$ by the pairs $(W_p, V_0)$, where $W_p = \varphi^{-1}([\alpha\,,\,\alpha_p])$ is now a filtration by subcomplexes, hence it induces a filtration in $C_*^\Delta(\widehat{W}, \widehat{V_0})$ which is nice. By the preceding lemma it is even perfect. Indeed, denote by $C_*$ the chain complex $C_*^\Delta(\widehat{W}, \widehat{V_0})$, by $C_*^{(p)}$ the subcomplex $C_*^\Delta(\widehat{W_{p+1}}, \widehat{V_0})$, by $Y_p$ the simplicial subcomplex

$$W_p \cup \left( \bigcup_c D^p(c) \right) \quad \text{of} \quad W_{p+1}\,.$$

We give the $\mathbb{Z}G$-module $C_p^{gr} = H_p(\widehat{W_{p+1}}, \widehat{W_p})$ the basis, formed by the liftings $\Delta(c)$ of the discs $(D^p(c), S^{p-1}(c))$ to $(\widehat{W_{p+1}}, \widehat{W_p})$, where $c$ runs through the critical points of $f$ of index $p$. Note, that the map

$$K_p : C_p^{gr} \to C_*^{(p)}/C_*^{(p-1)} = C_*^{\Delta}(\widehat{W_{p+1}}, \widehat{W_p}),$$

which sends the homology class of each lifting $\Delta(c)$ to $\Delta(c)$ itself, induces the identity in homology, therefore it is homotopic to the homotopy equivalence $\kappa_p : R_*^{(p)} \to C_*^{(p)}/C_*^{(p-1)}$ (in the notations of section 3). To check, that $\kappa_p$ is a simple homotopy equivalence, consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & C_*^{\Delta}(\widehat{W_p}) & \longrightarrow & C_*^{\Delta}(\widehat{Y_p}) & \longrightarrow & C_*^{\Delta}(\widehat{Y_p}, \widehat{W_p}) & \to 0 \\ & \downarrow id & & \downarrow & & \downarrow K_p & \\ 0 \to & C_*^{\Delta}(\widehat{W_p}) & \longrightarrow & C_*^{\Delta}(\widehat{W_{p+1}}) & \longrightarrow & C_*^{\Delta}(\widehat{W_{p+1}}, \widehat{W_p}) & \to 0 \end{array}$$

The middle vertical arrow is a simple homotopy equivalence by A.8, therefore $K_p$ is a simple homotopy equivalence, and the filtration $\{C_*^{(p)}\}$ is perfect.

Now Lemma 3.6 implies our theorem. □

Note that the resulting homotopy equivalence $C_*(v) \to C_*^{\Delta}(\widehat{W}, \widehat{V_0})$ depends on the choice of $\varphi$. To be able to compare the equivalences for different choices of $\varphi$ (and $\Delta$) we denote by $J(\varphi)$ the composition

$$C_*(v) \to C_*^{\Delta}(\widehat{W}, \widehat{V_0}) \to C_*^{s}(\widehat{W}, \widehat{V_0})$$

(where the latter module is the singular chain complex) and formulate one more version of Theorem A.5.

THEOREM A.5′.— *Let $\varphi : W \to [\alpha, \beta]$ be an ordered Morse function on the cobordism $W$, $v$ be a gradient-like vector field $\varphi$, $\varphi^{-1}(\alpha) = V_0$, $\varphi^{-1}(\beta) = V_1$ and let $\alpha < \alpha_1 < \cdots < \alpha_m < \beta$ be the ordering sequence. Let $\widehat{p} : \widehat{W} \to W$ be a regular covering with the structure group $G$ and denote by $W_p$ the space $\varphi^{-1}([\alpha, \alpha_p])$.*

*Then there exists a homotopy equivalence* $J(\varphi) : C_*(v) \to C_*^s(\widehat{W}, \widehat{V_0})$, *such that:*

(1) $J(\varphi)(C_p(v)) \subset C_*^s(\widehat{W_{p+1}}, \widehat{V_0})$;

(2) *for each critical point* $c$ *of* $\widehat{\varphi}$ *of index* $p$ *the image of* $J(\varphi)(c)$ *in the group* $H_p(\widehat{W_{p+1}}, \widehat{W_p})$ *equals the class of descending disc* $(D^p(c), S^{p-1}(c))$ *of the point* $c$.

*The homotopy class of* $J(\varphi)$ *is uniquely determined by (1) and (2).*

**All the statements of this theorem are already proved, except the uniqueness of $J(\varphi)$, which follows from Lemma 3.2.** □

PROPOSITION A.9.— *Let* $\varphi$ *and* $\varphi'$ *be two ordered Morse functions on the cobordism* $W$ *with the same gradient-like vector field* $v$, *and* $\widehat{p} : \widehat{W} \to W$ *be a regular covering with the structure group* $G$.

*Then the chain homotopy equivalences* $J(\varphi)$, $J(\varphi') : C_*(v) \to C_*^s(\widehat{W}, \widehat{V_0})$ *are chain homotopic.*

*Proof.*— **Let $\alpha < \alpha_1 < \cdots < \alpha_m < \beta$ and $\alpha' < \alpha_1' < \cdots < \alpha_m' < \beta'$ be the ordering sequences for $\varphi$ and $\varphi'$. Let:**

$$W_p = \varphi^{-1}([\alpha, \alpha_p]),$$
$$W_p' = \varphi'^{-1}([\alpha', \alpha_p'])$$

**and denote by $U_p$ the union $W_p \cup W_p'$. The pairs $(U_p, V_0)$ form a filtration of $(W, V_0)$. Note that the corresponding filtration $C_*^s(\widehat{U_p}, \widehat{V_0})$ in the singular chain complex is nice. Indeed, by the definition of ordering sequence, there are no critical points in the domain**

$$\varphi^{-1}([\alpha_p, \beta]) \cap \varphi'^{-1}([\alpha', \alpha_p']),$$

**which implies that the shift along the $(-v)$-trajectories defines the deformation retraction of $U_p$ onto $W_p$. (Similarly, there is a deformation retraction of $U_p$ onto $W_p'$.) Now the five-lemma implies that the inclusion $(W_p, W_{p-1}) \hookrightarrow (U_p, U_{p-1})$ induces an isomorphism in the homology of the coverings.**

**Now the maps $J(\varphi)$, $J(\varphi')$ preserve the filtrations (where $C_*(v)$ is filtered trivially), thus we have only to show, that for each critical point $c$ of $\widehat{\varphi}$ (or $\widehat{\varphi}'$) of index $p$ the images $J(\varphi)(c)$ and $J(\varphi')(c)$ in the homology of $C_*^s(\widehat{U_{p+1}}, \widehat{U_p})$ are the same. An easy argument, using the shift along**

the $v$-trajectories shows that the intersection of the descending disc of $c$ with $\varphi^{-1}([\alpha_p, \beta])$ is homotopic to its intersection with $\varphi^{-1}([\alpha_p, \beta]) \cap \varphi'^{-1}([\alpha'_p, \beta'])$. In the course of this homotopy the boundary of the disc rests always in $\varphi^{-1}([\alpha, \alpha_p]) \cup \varphi'^{-1}([\alpha', \alpha'_p])$. The same for the disc $D(c) \cap \varphi'^{-1}([\alpha'_p, \beta'])$. The lemma is proved. □

Suppose now, that $f : W \to \mathbb{R}$ is a Morse function, and $\alpha \in (a, b)$ is a regular value. By Theorem A.5′ there are chain homotopy equivalences

$$C_*\left(v \mid f^{-1}([\alpha, b])\right) \to C_*^{\Delta}\left(f^{-1}([\alpha, b]), f^{-1}(\alpha)\right)$$

and

$$C_*(v) \to C_*^{\Delta}(\widehat{W}, \widehat{V_0}).$$

We want to establish relations between these maps. For that we need one more notion.

Suppose that $W$ is a compact cobordism, $\partial W = V_0 \sqcup V_1$, $f : W \to \mathbb{R}$ is a Morse function, $v$ is a gradient-like vector field for $f$, such that $v$ is transversal to $\partial W$ and points inwards $W$ on $V_0$ and outwards $W$ on $V_1$. We *do not* suppose that $f$ is constant on $V_0$ or on $V_1$. For a small $\varepsilon$ consider the diffeomorphism $F_0$ to $V_0 \times [0, \varepsilon]$ onto a neighborhood of $V_0$ in $W$, defined by $F_0(x,t) = \gamma_x(t)$, where $\gamma_x$ is the $v$-trajectory, starting in $x$ at the moment $t = 0$. Similarly consider the diffeomorphism $F_1$ of $V_1 \times [0, \varepsilon]$ onto a neighborhood of $V_1$ in $W$, given by

$$F_1(x,t) = \widetilde{\gamma}_x(t),$$

where $\widetilde{\gamma}$ is the $(-v)$-trajectory, starting in $x$ at the moment $t = 0$. Denote $\operatorname{Im} F_0$ by $U_0$, $\operatorname{Im} F_1$ by $U_1$.

PROPOSITION A.10.— *There exist two functions $\overline{f}$ and $\underline{f}$ on the cobordism $W$, satisfying the following properties:*

*(1) $\overline{f}(x) \geq f(x)$, and if $x \notin U_1$, then $\overline{f}(x) = f(x)$,*
*$\underline{f}(x) \leq f(x)$, and if $x \notin U_0$, then $\underline{f}(x) = f(x)$;*

*(2) $\overline{f}$ and $\underline{f}$ are Morse functions and $v$ is a gradient-like vector field for both;*

*(3) $\overline{f}$ is constant on $V_1$ with the value $\max_{x \in V_1} f(x)$. $\underline{f}$ is constant on $V_0$ with the values $\min_{x \in V_0} f(x)$.*

The proof is easy and will be omitted. □

The functions $\overline{f}$ and $\underline{f}$ will be called upper damping and lower damping. The result of two consequent operations, applied to $f$ will be called damping of $f$ and denoted by $\overline{\underline{f}}$. These dampings depend on some choices (as $U_0$, $U_1$) but we never consider more then one, so no possibility of confusion occurs.

Consider now a cobordism $W$, $\partial W = V_0 \sqcup V_1$, $f : W \to [a, b]$ a Morse function, $f^{-1}(a) = V_0$, $f^{-1}(b) = V_1$. Let $v$ be an arbitrary gradient-like vector field for $f$. Let $\alpha \in [a, b]$ be any regular value for $f$. Denote by $V_\alpha$ the preimage $f^{-1}(\alpha)$, by $W_0$ the cobordism $f^{-1}([a, \alpha])$, by $W_1$ the cobordism $f^{-1}([\alpha, b])$. Any ordered Morse function $\varphi$ on $W$ (with the same gradient-like vector field $v$) defines a chain homotopy equivalence:

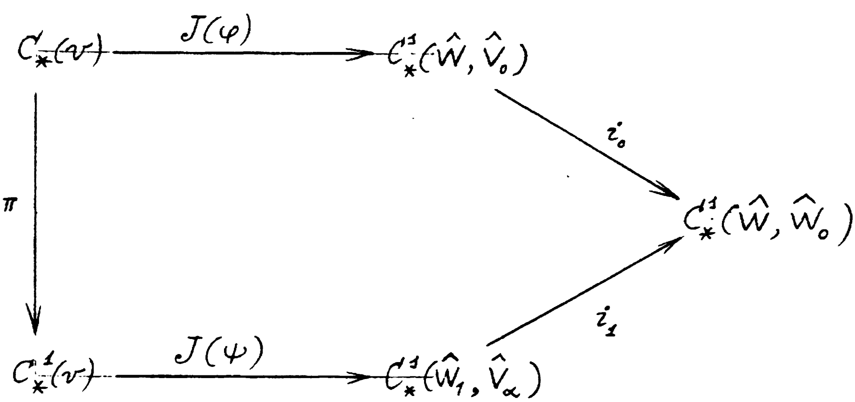
$$J(\varphi) : C_*(v) \to C_*^s(\widehat{W}, \widehat{V_0}).$$

Any ordered Morse function $\psi$ on $W_1$ defines a chain homotopy equivalence

$$J(\psi) : C_*^1(v) \to C_*^s(\widehat{W_1}, \widehat{V_\alpha}),$$

where $C_*^1(v)$ is the Morse complex of $f \mid W_1$, which is a natural quotient of $C_*(v)$, the projection denoted by $\pi$.

PROPOSITION A.11. — *The following diagram is commutative up to a chain homotopy :*




*($i_0$ and $i_1$ are the natural inclusions).*

*Proof*.— By Proposition A.9, the chain map $J(\psi)$ does not depend on particular choice of $\psi$. Thus we can take as $\psi$ the lower damping of $\varphi \mid W_1$. It is easy to verify, that if $a < a_1 < \cdots < a_m < b$ is an ordering sequence for $\varphi$, then the sequence $\alpha < a_r < \cdots < a_m < b$, where $a_{r-1} < \alpha$ is an ordering sequence for $\psi$. All the three complexes $C_*^s(\widehat{W}, \widehat{V_0})$, $C_*^s(\widehat{W}, \widehat{W_0})$, $C_*^s(\widehat{W_1}, \widehat{V_\alpha})$ are nicely filtered: the first — due to the ordering sequence of $\varphi$, the third — due to the ordering sequence of $\psi$, the second obtains the filtration from the third. The homomorphisms $i_0$, $i_1$ obviously preserve the filtrations, $i_0$ — because $\varphi \leq \psi$ on $W_1$, $i_1$ — by the obvious reasons. Therefore $I_0 = i_0 \circ J(\varphi)$ and $I_1 = i_1 \circ J(\psi) \circ \pi$ preserve the filtrations and to prove that they are homotopic, we are left to prove that the $I_0$ and $I_1$-images of each critical point $c$ of index $p$ of $\widehat{\varphi}$ in the homology group

$$H_p\left(\widehat{\psi}^{-1}([a, a_{p+1}]) \cup \widehat{W_0}, \widehat{\psi}^{-1}([a, a_p]) \cup \widehat{W_0}\right)$$

are the same. But these images are: the intersections of $D(c)$ with $\widehat{\psi}^{-1}([a_p, b])$ modulo the boundary $D(c) \cap \widehat{\psi}^{-1}(a_p)$ and the intersections of $D(c)$ with $\widehat{\varphi}^{-1}([a_p, b])$ modulo the boundary $D(c) \cap \widehat{\varphi}^{-1}([a_p, b])$. They are obviously homotopic by means of the shift along the $(-v)$-trajectories. □

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## References

[1] FARBER (M.) .— *The exactness of Novikov inequalities*, Funktional Anal. i Pril. **19**, No 1 (1985), pp. 49-59. English Transl. in Functional Anal. Appl. **19** (1985).

[2] KERVAIRE (M.) .— *Le théorème de Barden-Mazur-Stallings*, Comment. Math. Helv. **40**, No 1 (1965), pp. 31-42.

[3] CERF (J.) .— *La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie*, Publ. Maht. I.H.E.S. **39** (1970), pp. 7-172.

[4] MUNKRES (I.) .— *Elementary differential topology*, Ann. Math. Studies **54**, Princeton, New Jersey, Princeton University Press (1966).

[5] MILNOR (J.) .— *Lectures on the h-cobordism theorem*, Princeton University Press (1965).

[6] MILNOR (J.) .— *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1966), pp. 358-426.

[7] NOVIKOV (S. P.) .— *The Hamilton formalism and a multivalued analog of Morse theory*, Uspekhi Mat. Nauk **37**, No 5 (227) (1982), pp. 3-49. English translation in Russian Math. Surveys **37** (1982).

[8] PAZHITNOV (A. V.) .— *On the sharpness of Novikov type inequalities of manifold with free abelian fundamental groups*, Mat. Sbornik. **180**, No 11 (1989), pp. 1486-1532. English translation Math. USSR Sbornik **68**, No 2 (1991).

[9] PAZHITNOV (A. V.) .— *On the Novikov complex for rational Morse forms*, Preprint of Odense University **12** (October 1991).

[10] PAZHITNOV (A. V.) .— *Surgery on the Novikov complex*, Preprint of the Nantes University, 1993.

[11] PEIXOTO (M. M.) .— *On an approximation theorem of Kupka and Smale*, Journal of differential equations **3** (1966), pp. 214-227.

[12] SHARKO (V. V.) .— *Stable algebra of Morse theory*, Izv. AN SSSR **3** (1990), pp. 607-632.

[13] SIKORAV (J.-CL.) .— *Homologie de Novikov et formes de Morse*, Preprint Orsay, 1987.

[14] SIKORAV (J.-CL.) .— *Homologie, associée à une fonctionnelle (d'après A. Floer)*, Séminaire Bourbaki **733** (1990-91).