ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 6e série, tome 3, nº 3 (1994), p. 411-448

<http://www.numdam.org/item?id=AFST_1994_6_3_3_411_0>

Annales de la Faculté des Sciences de Toulouse Vol. III, n° 3, 1994

Global stability of saddle-node bifurcation of a periodic orbit for vector fields(*)

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RÉSUMÉ. — On étudie la stabilité d'une famille générique de champs de vecteurs ayant une orbite périodique de type col-nœud.

ABSTRACT. — In a bifurcation value, the global stability of families of vector fields which have a generically unfolding saddle-node periodic orbit is studied.

Introduction

In this paper we study the global stability of families of vector fields which have a saddle-node periodic orbit which unfolds generically. We recall that a generic characterization of stable families for the stability of one-parameter families of gradient vector fields was obtained by J. Palis and F. Takens [9]. We also recall that for one-parameter families of vector fields with simple recurrences and no-cycles, the global stability for those which have bifurcations due to quasi-transversal orbits was studied by R. Labarca [4] for the cases in which the bifurcation is due to a saddle-node (or Hopf) singularity or a flip periodic orbit, under generic conditions, the global stability follows from results of S. Newhouse, J. Palis and F. Takens [8]. When the bifurcation is due to a saddle-node periodic orbit, the global stability has only been studied in the case of two-dimensional manifolds by I. Malta and J. Palis [6]. Before stating our results, we recall some concepts and results on one-parameter families of vector fields.

(*) Reçu le 14 juin 1992

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Part of this work was supported by IMPA-CNPq (Brazil), by Fondecyt Grants N° 1032-88 and N° 0901-90, and by Dicyt-USACH

Let M be a C^∞ boundaryless compact manifold. We let $\mathcal{X}^\infty(M)$ denote the space of C^∞ vector fields on M and $\mathcal{X}_1^\infty(M)$ denote the space of C^∞ arcs, $\xi : I = [-1, 1] \mapsto \mathcal{X}^\infty(M)$ both endowed with the C^∞ Whitney topology. We use the notation $\xi = \{X_\mu\}$, where for each $\mu \in I$, $\xi(\mu) = X_\mu$. Now, let $\{X_\mu\}$, $\{\widetilde{X}_\mu\} \in \mathcal{X}_1^\infty(M)$ and $\overline{\mu}$, $\widetilde{\mu} \in I$; we say that $\{X_\mu\}$ at $\overline{\mu}$ is equivalent to $\{\widetilde{X}_\mu\}$ at $\widetilde{\mu}$ if there are homeomorphisms $\rho : (I, \overline{\mu}) \mapsto (I, \widetilde{\mu})$, increasing, and $H_\mu : M \mapsto M$, which depends continuously on μ, such that, for each μ near $\overline{\mu}$, H_μ is a topological equivalence between X_μ and $\widetilde{X}_{\rho(\mu)}$, *i.e.* H_μ sends orbits of X_μ into orbits of $\widetilde{X}_{\rho(\mu)}$ preserving the sense of the trajectories. We say that $\{X_\mu\}$ is stable at $\overline{\mu}$ if there exists a neighborhood $\mathcal{U} \subseteq \mathcal{X}_1^\infty(M)$ of $\{X_\mu\}$ such that, for each $\{\widetilde{X}_\mu\} \in \mathcal{U}$, there exists $\widetilde{\mu} \in I$ near $\overline{\mu}$ such that $\{X_\mu\}$ at $\overline{\mu}$ is equivalent to $\{\widetilde{X}_\mu\}$ at $\widetilde{\mu}$.

Let $\Gamma \subseteq \mathcal{X}_1^\infty(M)$ be characterized by :

- for each $\mu \in I$, the non-wandering set of X_μ is constituted by a finite number of critical elements (*i.e.* singularities and periodic obtits) of X_μ;
- there are no cycles among the critical elements of X_μ, (*i.e.* there is no sequence $\alpha_1, \ldots, \alpha_k$, $k \geq 1$, of critical elements of X_μ such that $\alpha_k = \alpha_1$ and $\mathcal{W}^u(\alpha_i) \cap \mathcal{W}^s(\alpha_{i+1}) \neq \emptyset$, $i = 1, \ldots, k-1$).

Let $\{X_\mu\} \in \Gamma$ and $\overline{\mu} \in B(\{X_\mu\}) = \{\mu \in I \mid X_\mu \text{ is not stable}\}$ such that $X_{\overline{\mu}}$ has a saddle-node periodic orbit θ. Let $\Sigma_q \subseteq M$ be a transversal section for $X_{\overline{\mu}}$ at $q \in \theta$, and $X(x, \mu) = (X_\mu(x), 0)$. There exists an interval $I_1 \subseteq I$, $\overline{\mu} \in I_1$, such that $\Sigma = \Sigma_q \times I_1$ is a transversal section to X at $(q, \overline{\mu})$. We let $\mathcal{P} : \Sigma \mapsto \Sigma$ denote the Poincaré map of X in the non-hyperbolic periodic orbit $(\theta, \overline{\mu})$, and $\mathcal{P}_\mu = \mathcal{P}/\Sigma_q \times \{\mu\}$ denote the Poincaré map of X_μ. Then $\mathcal{P} = \{\mathcal{P}_\mu\}$ is an arc of saddle-node diffeomorphisms (see [8]). From the theory of invariant manifolds (see [3]) we have that there exists C^r, $1 \leq r < \infty$, $\mathcal{P}$-invariant manifolds $\mathcal{W}^{cs}(q)$, $\mathcal{W}^{cu}(q)$ and $\mathcal{W}^c(q)$. Set

$$\mathcal{W}^{cs}_\mu(q) = \mathcal{W}^{cs}(q) \cap \Sigma_q \times \{\mu\}\,;$$

analogously for $\mathcal{W}^{cu}_\mu(q)$ and $\mathcal{W}^c_\mu(q)$. Let

$$\mathcal{W}^s(q) = \{(x, \overline{\mu}) \in \Sigma_q \times \{\overline{\mu}\} \mid \mathcal{P}^n(x, \overline{\mu}) \mapsto (q, \overline{\mu})\,,\ n \mapsto +\infty\}$$

and

$$\mathcal{W}^u(q) = \{(x, \overline{\mu}) \in \Sigma_q \times \{\overline{\mu}\} \mid \mathcal{P}^{-n}(x, \overline{\mu}) \mapsto (q, \overline{\mu})\,,\ n \mapsto +\infty\}$$

be the stable and unstable manifolds for q, respectively. We have that

$$\mathcal{W}^s(q) \subseteq \mathcal{W}^{cs}_{\overline{\mu}}(q) \quad \text{and} \quad \mathcal{W}^u(q) \subseteq \mathcal{W}^{cu}_{\overline{\mu}}(q)$$

are C^r, $r \geq 1$, injectively immersed manifolds of M with boundary

$$\partial\mathcal{W}^s(q) = \mathcal{W}^{ss}(q)\,, \quad \partial\mathcal{W}^u(q) = \mathcal{W}^{uu}(q)\,,$$

are called strong stable and strong unstable manifolds, respectively. In addition, in $\mathcal{W}^s(q)$ there exists a (unique) $\mathcal{P}$-invariant C^1 codimension one foliation $\mathcal{F}^{ss}(q)$ with space of leaves $\mathcal{W}^{c}_{\overline{\mu}}(q)$; analogously, there exists a foliation $\mathcal{F}^{uu}(q)$ in $\mathcal{W}^u(q)$. Let α be a hyperbolic critical element of $X_{\overline{\mu}}$ such that $\mathcal{W}^u(\alpha)$ intersects $\mathcal{W}^s(q)$, we say that α is s-critical, or that $\mathcal{W}^u(\alpha)$ has s-criticalities if $\mathcal{W}^u(\alpha) \cap \Sigma_q \times \{\overline{\mu}\}$ has a non-transversal intersection with some leaf of $\mathcal{F}^{ss}(q)$; analogously, we define u-criticalities. We say that the s-criticality between $\mathcal{W}^u(\alpha)$ and $\mathcal{F}^{ss}(q)$ is generic if there exists a unique tangency orbit between $\mathcal{W}^u(\alpha)$ and $\mathcal{F}^{ss}(q)$ along which they have a quasi-transversal contact.

Excepting certain additional conditions which will be specified later, our results on the stability of families $\{X_\mu\} \in \Gamma$ which have saddle-node periodic orbits are the following.

PROPOSITION 1

(a) If there exists a hyperbolic critical element α of $X_{\overline{\mu}}$ such that $\mathcal{W}^u(\alpha)$ has non-generic s-criticalities with $\mathcal{F}^{ss}(q)$, then $\{X_\mu\}$ is non-stable at $\overline{\mu}$.

(b) If there exists two hyperbolic critical elements α_1, α_2 of $X_{\overline{\mu}}$ such that $\mathcal{W}^u(\alpha_1)$ and $\mathcal{W}^u(\alpha_2)$ have generic s-criticalities with $\mathcal{F}^{ss}(q)$, then $\{X_\mu\}$ is non-stable at $\overline{\mu}$.

THEOREM 1

(a) If α is a hyperbolic periodic orbit of $X_{\overline{\mu}}$ such that there exists the weakest contraction in α and $\mathcal{W}^u(\alpha)$ has a generic s-criticality, then $\{X_\mu\}$ is non-stable at $\overline{\mu}$. Moreover, in this case the weakest contraction in α is a modulus of stability for $X_{\overline{\mu}}$.

(b) If α is a hyperbolic singularity with complex weakest contraction of $X_{\overline{\mu}}$ such that $\mathcal{W}^u(\alpha)$ has a generic s-criticality, then $\{X_\mu\}$ is non-stable at $\overline{\mu}$. Moreover, if $a \pm ib$, $a < 0$, $b \neq 0$, denotes the weakest contraction in α, then $\rho = a/b$ is modulus of stability for $X_{\overline{\mu}}$.

THEOREM 2

(a) If the hyperbolic critical elements of $X_{\overline{\mu}}$ have no criticalities, then $\{X_\mu\}$ is stable at $\overline{\mu}$.

(b) If there exists a unique hyperbolic singularity α which has real weakest contraction and $\mathcal{W}^u(\alpha)$ has a generic s-criticality, and the remaining hyperbolic critical elements of $X_{\overline{\mu}}$ have no criticalities, then $\{X_\mu\}$ is stable at $\overline{\mu}$.

(c) If there exist hyperbolic singularities α, β ($\alpha \neq \beta$) of $X_{\overline{\mu}}$ such that the weakest contraction in α and the weakes expansion in β are real, and $\mathcal{W}^u(\alpha)$, respectively $\mathcal{W}^s(\beta)$, has a generic s-criticality with $\mathcal{F}^{ss}(q)$, respectively has a generic u-criticality with $\mathcal{F}^{uu}(q)$) and the remaining hyperbolic critical elements of $X_{\overline{\mu}}$ have no criticalities, then $\{X_\mu\}$ is stable at $\overline{\mu}$.

From the methods developed in the proof of Theorem 2 (Section 3) and in [6], the following result follows.

THEOREM 3.— *If there exist hyperbolic singularities*

$$\alpha_1, \ldots, \alpha_k, \quad \beta_1, \ldots, \beta_\ell$$

such that the weakest contractions in the α_i's and the weakest expansions in the β_j's are real, and $\mathcal{W}^u(\alpha_s)$ for some $1 \leq s \leq k$ (resp. $\mathcal{W}^s(\beta_r)$ for some $1 \leq r \leq \ell$) has a generic s-criticality , respectively has a generic u-criticality, and the remaining hyperbolic critical elements of $X_{\overline{\mu}}$ have no criticalities, then $\{X_\mu\}$ has finite modulus of stability.

1. Basic concepts

In this section we will recall some basic concepts and state the above results in greater detail. Let M be a C^∞ boundaryless compact manifold. We will denote by $\mathcal{X}^\infty(M)$ the space of C^∞ vector fields on M endowed with the C^∞ Whitney topology. Let $x \in M$ be a singularity of $X \in \mathcal{X}^\infty(M)$, *i.e.* $X(x) = 0$, we say x is hyperbolic if $\mathrm{D}X(x)$ has no eigenvalues with null real part. Let σ be a periodic orbit of X and Σ_q be a transversal section to X at $q \in \sigma$ and $\mathcal{P} : (\Sigma_q, q) \mapsto (\Sigma_q, q)$ be the Poincaré map for X at σ, we say σ is hyperbolic if $\mathrm{D}\mathcal{P}(q)$ has no eigenvalues with norm equaling one. The singularities and periodic orbits of X will be called critical elements of X.

Let θ be a hyperbolic critical element of X, then

$$\mathcal{W}^s(\theta) = \{x \in M \mid X_t(x) \mapsto \theta\,,\ t \mapsto +\infty\}$$

and

$$\mathcal{W}^u(\theta) = \{x \in M \mid X_t(x) \mapsto \theta\,,\ t \mapsto -\infty\}$$

(X_t denotes the flow of X) are C^∞ submanifolds injectively immersed in M called stable and unstable manifolds of θ, respectively.

Let $x \in M$, we say x is a non-wandering point of X if, for each neighborhood U of X and each $t_0 > 0$, there exists $t_1 > t_0$ such that

$$X_{t_1}(U) \cap U \neq \emptyset\,.$$

We let $\Omega(X)$ denote the set of non-wandering points of X. If $\Omega(X)$ is constituted by a finite number of critical elements of X, we say X has simple recurrences. The interior of the set of the vector fields with simple recurrences will be denoted by $WR^\infty(M)$.

Let α, β be critical elements of X, we say $\mathcal{W}^u(\alpha)$ and $\mathcal{W}^s(\beta)$ are transversal if

$$T_x\mathcal{W}^u(\alpha) + T_x\mathcal{W}^s(\beta) = T_xM\,,$$

for each $x \in \mathcal{W}^u(\alpha) \cap \mathcal{W}^s(\beta)$. A cycle for X is a sequence $\alpha_1, \ldots, \alpha_k$, $k \geq 1$, $\alpha_k = \alpha_1$, of critical elements of X such that $\mathcal{W}^u(\alpha_i) \cap \mathcal{W}^s(\alpha_{i+1}) \neq \emptyset$, $i = 1, \ldots, k-1$. We let $WC^\infty(M)$ denote the set of vector fields which do not have cycles among their critical elements.

Let $\sigma \in M$ be a hyperbolic singularity (resp. periodic orbit) of $X \in \mathcal{X}^\infty(M)$, we say the weakest contraction at σ is defined if among the contractive eigenvalues of $\mathrm{D}X(\sigma)$ (resp. $\mathrm{D}\mathcal{P}(q)$, $q \in \sigma$, $\mathcal{P}$ the Poincaré map) the one with biggest real part (resp. norm) is simple.

We let $\mathcal{X}_1^\infty(M)$ denote the space of C^∞ arcs,

$$\xi : I = [-1\,,\,1] \mapsto \mathcal{X}^\infty(M)$$

endowed with the C^∞ Whitney topology, we use the notation $\xi = \{X_\mu\}$, $\xi(\mu) = X_\mu$.

We say $\overline{\mu} \in I$ is a bifurcation value for $\{X_\mu\}$ if $X_{\overline{\mu}}$ is non-stable. We set

$$B\left(\{X_\mu\}\right) = \{\overline{\mu} \in I \mid \overline{\mu} \text{ is a bifurcation value for } \{X_\mu\}\}\,.$$

DEFINITION 1.1.— *Let* $\{X_\mu\} \in \mathcal{X}_1^\infty(M)$ *and* $\overline{\mu} \in B(\{X_\mu\})$ *such that* $X_{\overline{\mu}}$ *has a non-hyperbolic periodic orbit* θ, *we say* θ *is a saddle-node periodic orbit which unfolds generically at* $\mu = \overline{\mu}$ *if there exists a* C^r $(1 \leq r < \infty)$, μ*-depending center manifold* $\mathcal{W}^c(q)$ *through* $q \in \theta$ $(\dim \mathcal{W}^c(q) = 2)$ *such that the Poincaré map* $\mathcal{P}(x,\mu) = (\mathcal{P}_\mu(x)\,,\,\mu)$ *associated to the vector field* $X(x,\mu) = (X_\mu(x)\,,\,0)$ *in* $(\theta, \overline{\mu})$ *restricted to* $\mathcal{W}^c(q)$ *has the form*

$$\mathcal{P}_\mu(x) = b(\mu - \overline{\mu}) + v_1(\mu) + v_2(\mu)x^2 + O(x^2)\,,$$

$$v_1(\overline{\mu}) = 1\,, \quad \frac{\mathrm{d}v_1(\mu)}{\mathrm{d}\mu}\Big|_{\mu=\overline{\mu}} \neq 0\,, \quad b \neq 0\,, \quad v_2(\mu) \in \mathbb{R} \setminus \{0\}\,.$$

DEFINITION 1.2.— *Let* $\{X_\mu\} \in \mathcal{X}_1^\infty(M)$ *and* $\overline{\mu} \in B(\{X_\mu\})$ *such that* $X_{\overline{\mu}}$ *has a non-transversal intersection orbit,* $\gamma \subseteq \mathcal{W}^u(\alpha) \cap \mathcal{W}^s(\beta)$, α, β *hyperbolic critical elements of* $X_{\overline{\mu}}$. *We say* γ *is a quasi-transversal intersection orbit which unfolds generically at* $\mu = \overline{\mu}$ *for* $\{X_\mu\}$ *if for each* $x \in \gamma$ *there exist* C^∞, μ*-depending coordinates,* $\psi_\mu : (V, x) \mapsto (\mathbb{R}^n, 0)$, V *a neighborhood of* x $(n = \dim M)$ *such that:*

(a) $(\psi_\mu)_\star(X_\mu) = \dfrac{\partial}{\partial y_n}$;

(b) $\psi_\mu(\mathcal{W}^u(\alpha_\mu) \cap V) = \{y_1 = \cdots = y_{n-u} = 0\}$;

(c) if $n - u = s$,

$$\psi_\mu(\mathcal{W}^s(\beta_\mu) \cap V) = \{y_1 = \varepsilon(\mu),\, y_{s+1} = \cdots = y_{n-1} = 0\}\,;$$

if $n - u \leq s - 1$,

$$\begin{aligned}\psi_\mu(\mathcal{W}^s(\beta_\mu) \cap V) &= \\ &= \{y_1 = Q(y_{n-u}, \ldots, y_s) + \varepsilon(\mu),\, y_{s+1} = \cdots = y_{n-1} = 0\}\,,\end{aligned}$$

where $u = \dim \mathcal{W}^u(\alpha)$, $s = \dim \mathcal{W}^s(\beta)$ *and for each* μ *near* $\overline{\mu}$, α_μ *and* β_μ *denote the hyperbolic critical elements of* X_μ *near* α *and* β, *respectively,* Q *is a Morse function and* ε *is a* C^∞ *function such that*

$$\varepsilon(\overline{\mu}) = 0 \quad \textit{and} \quad \frac{\mathrm{d}\varepsilon(\mu)}{\mathrm{d}\mu}\Big|_{\mu=\overline{\mu}} \neq 0\,.$$

Now let $\Gamma_{sn} = \Gamma_{sn}(M) \subseteq \mathcal{X}_1^\infty(M)$ be the set of arcs $\{X_\mu\} \in \mathcal{X}_1^\infty(M)$ such that:

- $B(\{X_\mu\}) \subseteq]-1, 1[$ is at most countable;
- for each $\overline{\mu} \in B(\{X_\mu\})$, $X_{\overline{\mu}}$ has a unique orbit $\theta_{\overline{\mu}}$ along which it is non-stable in one of the following senses:

  (a) if $\theta_{\overline{\mu}}$ is a non-hyperbolic periodic orbit of $X_{\overline{\mu}}$, then $\theta_{\overline{\mu}}$ is a saddle-node periodic orbit unfolding generically at $\mu = \overline{\mu}$,

  (b) if $\theta_{\overline{\mu}}$ is a non-transversal intersection orbit of an unstable and a stable manifolds of hyperbolic critical elements of $X_{\overline{\mu}}$, then $\theta_{\overline{\mu}}$ is a quasi-transversal intersection orbit unfolding generically at $\mu = \overline{\mu}$;
- for each $\mu \in I$, $X_\mu \in WR^\infty(M) \cap WC^\infty(M)$.

Let $\{X_\mu\} \in \Gamma_{sn}$ and $\overline{\mu} \in B(\{X_\mu\})$ such that $X_{\overline{\mu}}$ has a saddle-node periodic orbit $\theta_{\overline{\mu}}$, we have the following theorem.

THEOREM (I. Malta, J. Palis [6]).— *Assume* $\dim M = 2$. *Let* $\{X_\mu\} \in \Gamma_{sn}$ *and* $\overline{\mu} \in B(\{X_\mu\})$ *as above. Then:*

(a) if there exists more than one saddle separatrix (stable or unstable manifold of a saddle) accumulating on the same side of $\theta_{\overline{\mu}}$, *then* $\{X_\mu\}$ *is non-stable at* $\overline{\mu}$; *moreover, under generic conditions,* $X_{\overline{\mu}}$ *has finite stability modulus;*

(b) if there exists at most one saddle separatrix accumulating at $\theta_{\overline{\mu}}$, *then* $\{X_\mu\}$ *is stable at* $\overline{\mu}$;

(c) if there exists at most one unstable saddle separatrix and one stable saddle separatrix of $X_{\overline{\mu}}$ *accumulating at* $\theta_{\overline{\mu}}$ *then* $\{X_\mu\}$ *is stable at* $\overline{\mu}$.

In this work we generalize the above theorem for $\dim M > 2$.

Let $\{X_\mu\} \in \Gamma_{sn}$ and $\overline{\mu} \in B(\{X_\mu\})$ such that $X_{\overline{\mu}}$ has saddle-node periodic orbit θ. Let $\Sigma_q \subseteq M$ be a transversal section to $X_{\overline{\mu}}$ at $q \in \theta$ and $I_1 \subseteq I$ be a neighborhood of $\overline{\mu}$. We let $\mathcal{P} : \Sigma_q \times I_1 \mapsto \Sigma_q \times I_1$ be the Poincaré map of the vector field $X(x,\mu) = (X_\mu(x), 0)$ at $(\theta, \overline{\mu})$. Set $\mathcal{P}_\mu = \mathcal{P}/\Sigma_q \times \{\mu\}$; then $\mathcal{P} = \{\mathcal{P}_\mu\}$ is an arc of saddle-node diffeomorphisms (see [8]).

In what follows, we will assume that for $\mu < \overline{\mu}$ there exist two hyperbolic periodic orbits, $\theta_{1,\mu}$, $\theta_{2,\mu}$, of X_μ which collapse at $\mu = \overline{\mu}$ originating the saddle-node periodic orbit $\theta = \theta_{1,\overline{\mu}} = \theta_{2,\overline{\mu}}$ of $X_{\overline{\mu}}$, and disappearing for $\mu > \overline{\mu}$. Thus $\mathcal{P} = \{\mathcal{P}_\mu\}$ has, for $\mu < \overline{\mu}$, two hyperbolic fixed points $q_{1,\mu}$, $q_{2,\mu}$, which collapse at $\mu = \overline{\mu}$ originating the saddle-node fixed point $q = q_{1,\overline{\mu}} = q_{2,\overline{\mu}}$ of $\mathcal{P}_{\overline{\mu}}$ and disappearing for $\mu > \overline{\mu}$.

From [8] it follows that for $\mathcal{P} = \{\mathcal{P}_\mu\}$ we have, in $\Sigma_q \times I_1$, there are C^r $(1 \leq r < \infty)$ $\mathcal{P}$-invariant submanifolds $\mathcal{W}^c(q)$, $\mathcal{W}^{cs}(q)$ and $\mathcal{W}^{cu}(q)$,

$$\mathcal{W}^c(q) \subseteq \mathcal{W}^{cs}(q) \cap \mathcal{W}^{cu}(q),$$

called center, center-stable and center-unstable manifolds respectively. In addition, in $\mathcal{W}^{cs}(q)$, there exists a C^r $(1 \leq r \leq \infty)$ $\mathcal{P}$-invariant, strong stable foliation $\mathcal{F}^{ss}(q)$. Analogously in $\mathcal{W}^{cu}(q)$, there exists a strong unstable foliation $\mathcal{F}^{uu}(q)$. Furthermore at $\mu = \overline{\mu}$, $\mathcal{F}^{ss}(q)/\mathcal{W}^s(q)$ is unique and should be preserved by conjugations of arcs of saddle-node diffeomorphisms; similarly for $\mathcal{F}^{uu}(q)/\mathcal{W}^u(q)$. This is a necessary condition (rigidity) for the construction of conjugations of arcs of saddle-node diffeomorphisms and, therefore, a necessary condition (rigidity) for the construction of equivalences for arcs in Γ_{sn} for the parameter values in which there are saddle-node periodic orbits.

We next consider the arc of diffeomorphisms $\mathcal{P}^c = \{\mathcal{P}^c_\mu\}$ where

$$\mathcal{P}^c_\mu = \mathcal{P}/\mathcal{W}^c_\mu(q) : \mathcal{W}^c_\mu(q) \mapsto \mathcal{W}^c_\mu(q).$$

Thus $\mathcal{P}^c = \{\mathcal{P}^c_\mu\}$ is an arc of saddle-node diffeomorphisms in $\mathbb{R}$. Therefore, from [12], there exists a unique C^∞ vector field Z defined in a neighborhood of $(q, \overline{\mu})$ in $\mathcal{W}^c_{\overline{\mu}}(q)$ such that

$$\mathcal{P}^c_{\overline{\mu}} = Z_{t=1}. \tag{1}$$

And, in addition, if $\{\widetilde{X}_\mu\} \in \Gamma_{sn}$ and $\widetilde{\mu} \in I$ have the same characteristics as $\{X_\mu\}$ and $\overline{\mu}$. Respectively, we denote by $\widetilde{\mathcal{P}}^c = \{\widetilde{\mathcal{P}}^c_\mu\}$ the corresponding arc of saddle-node diffeomorphisms in $\mathbb{R}$, and by $\widetilde{Z}$ the unique C^∞ vector field defined in a neighborhood of $(\widetilde{q}, \widetilde{\mu})$ in $\mathcal{W}^c_{\widetilde{\mu}}(\widetilde{q})$. If $h : \mathcal{W}^c(q) \mapsto \mathcal{W}^c(\widetilde{q})$ is a conjugation between $\mathcal{P}^c = \{\mathcal{P}^c_\mu\}$ and $\widetilde{\mathcal{P}}^c = \{\widetilde{\mathcal{P}}^c_\mu\}$, $h = (h_\mu, \rho)$, $\rho : (I, \overline{\mu}) \mapsto (I, \widetilde{\mu})$ reparametrization, h_μ conjugation between $\mathcal{P}^c_\mu$ and $\widetilde{\mathcal{P}}^c_{\rho(\mu)}$, then

$$h_{\overline{\mu}} : \mathcal{W}^c_{\overline{\mu}}(q) \mapsto \mathcal{W}^c_{\widetilde{\mu}}(\widetilde{q})$$

is a conjugation between the flows Z_t and $\widetilde{Z}_t$; this is another rigidity condition for the construction of equivalences for arcs in Γ_{sn} in the parameter values in which there are saddle-node periodic orbits.

DEFINITION 1.3.— *Let* $\{X_\mu\} \in \Gamma_{sn}$ *and* $\overline{\mu} \in B(\{X_\mu\})$ *such that* $X_{\overline{\mu}}$ *has saddle-node periodic orbit* θ. *We say* θ *is s-critical if there exists a hyperbolic critical element* α *of* $X_{\overline{\mu}}$ *such that* $\mathcal{W}^u(\alpha)$ *has a non-transversal intersection with some leaf of* $\mathcal{F}^{ss}(q)$, *and we say this s-criticality is generic if there exists a unique tangency orbit between* $\mathcal{W}^u(\alpha)$ *and* $\mathcal{F}^{ss}(q)$ *along which they have a quasi-transversal contact.*

PROPOSITION 1.— *Let* $\{X_\mu\} \in \Gamma_{sn}$ *and* $\overline{\mu} \in B(\{X_\mu\})$ *such that* $X_{\overline{\mu}}$ *has saddle-node periodic orbit* θ. *Then:*

(a) *if* α *is a hyperbolic critical element of* $X_{\overline{\mu}}$ *such that* $\mathcal{W}^u(\alpha)$ *has non-generic s-criticalities, then* $\{X_\mu\}$ *is non-stable at* $\overline{\mu}$;

(b) *if* α_1, α_2 $(\alpha_1 \neq \alpha_2)$ *are hyperbolic critical elements of* $X_{\overline{\mu}}$ *such that* $\mathcal{W}^u(\alpha_1)$ *and* $\mathcal{W}^u(\alpha_2)$ *have generic s-criticalities, then* $\{X_\mu\}$ *is non-stable at* $\overline{\mu}$.

THEOREM 1.— *Let* $\{X_\mu\} \in \Gamma_{sn}$ *and* $\overline{\mu} \in B(\{X_\mu\})$ *be as in Proposition 1.*

(a) *If* α *is a hyperbolic periodic orbit of* $X_{\overline{\mu}}$ *such that there exists the weakest contraction in* α *and* $\mathcal{W}^u(\alpha)$ *has a generic s-criticality, then* $\{X_\mu\}$ *is non-stable at* $\overline{\mu}$. *Moreover, if* A *denotes the weakest contraction of* $X_{\overline{\mu}}$ *in* α, *then* A *is stability modulus for* $X_{\overline{\mu}}$.

(b) *If* α *is a hyperbolic singularity with complex weakest contraction of* $X_{\overline{\mu}}$ *such that* $\mathcal{W}^u(\alpha)$ *has a generic s-criticality, then* $\{X_\mu\}$ *is non-stable at* $\overline{\mu}$. *Moreover, if* $a \pm ib$, $a < 0$, $b \neq 0$, *denotes the weakest contraction in* α, *then* $\rho = a/b$ *is modulus of stability for* $X_{\overline{\mu}}$.

We next impose some conditions on the families $\{X_\mu\} \in \Gamma_{sn}$ we are considering:

SN1 If α is a saddle-type hyperbolic critical element of $X_{\overline{\mu}}$ $(\alpha \neq \theta)$ and $\mathcal{W}^u(\alpha) \cap \mathcal{W}^s(q) \neq \emptyset$ then $\mathcal{W}^u(\alpha)$ is transversal to $\mathcal{W}^{ss}(q)$ and $\mathcal{W}^u(\alpha) \cap \mathcal{W}^c_{\overline{\mu}}(q) = \emptyset$; analogously for $\mathcal{W}^s(\alpha)$.

SN2 Let α be a hyperbolic critical element of $X_{\overline{\mu}}$ such that $\mathcal{W}^u(\alpha)$. has a generic s-criticality with $\mathcal{F}^{ss}(q)$; then:

- $\mathcal{W}^u(\alpha)$ is transversal to $\mathcal{W}^s(q)$ in $\Sigma_q \times \{\overline{\mu}\}$; moreover, there exists C^r-linearisations $(r \geq 2)$ coordinates in a neighborhood of α;
- the weakest contraction is defined in α which is real;

- there exists a C^r $(r \geq 1)$ center-unstable manifold $\mathcal{W}^{cu}(\alpha)$ which is transversal to $\mathcal{F}^{ss}(q)$ in a neighborhood of the s-criticality; analogously for $\mathcal{W}^s(\alpha)$, $\mathcal{W}^{cs}(\alpha)$ and $\mathcal{F}^{uu}(q)$.

SN3 Let α be a hyperbolic critical element of $X_{\overline{\mu}}$ such that $\mathcal{W}^u(\alpha)$ has an s-criticality, and $\mathcal{D}^{cs}_{\overline{\mu}}(q)$ be a fundamental domain for $\mathcal{P}/\mathcal{W}^{cs}_{\overline{\mu}}(q)$. Then the s-criticality between $\mathcal{W}^u(\alpha)$ and $\mathcal{F}^{ss}(q)$ is generic and the tangency is a unique point.

We let $\Gamma^1_{sn} \subseteq \Gamma_{sn}$ denote the set characterized by $\{X_\mu\} \in \Gamma^1_{sn}$ if and only if, for each $\overline{\mu} \in B(\{X_\mu\})$, $X_{\overline{\mu}}$ has a unique saddle-node periodic orbit, and the conditions SN1, SN2 and SN3 are satisfied.

THEOREM 2.— *Let $\{X_\mu\} \in \Gamma^1_{sn}$ and $\overline{\mu} \in B(\{X_\mu\})$ such that $X_{\overline{\mu}}$ has a saddle-node periodic orbit θ, we have:*

(a) if θ is not s-critical nor u-critical, then $\{X_\mu\}$ is stable at $\overline{\mu}$ (in this case $\overline{\mu}$ is isolated);

(b) if there exists a unique hyperbolic singularity σ of $X_{\overline{\mu}}$ such that $\mathcal{W}^u(\sigma)$ is s-critical, and the remaining hyperbolic critical elements of $X_{\overline{\mu}}$ do not have criticalities, then $\{X_\mu\}$ is stable at $\overline{\mu}$ (in this case $\overline{\mu}$ is isolated);

(c) if there exist hyperbolic singularities σ_1, σ_2 $(\sigma_1 \neq \sigma_2)$ of $X_{\overline{\mu}}$ such that $\mathcal{W}^u(\sigma_1)$ is s-critical and $\mathcal{W}^s(\sigma_2)$ is u-critical, and if the remaining hyperbolic critical elements of $X_{\overline{\mu}}$ do not have criticalities, then $\{X_\mu\}$ is stable at $\overline{\mu}$; in this case there exists a strictly monotone sequence $(\overline{\mu}_n)$ of parameter values which converges to $\overline{\mu}$ such that, for each $n \in \mathbb{N}$, the vector field $X_{\overline{\mu}_n}$ has a unique orbit of quasi-transversal intersection between $\mathcal{W}^s(\sigma_1, \overline{\mu}_n)$ and $\mathcal{W}^s(\sigma_2, \overline{\mu}_n)$ which unfolds generically at $\mu = \overline{\mu}$.

Now let $\{X_\mu\} \in \Gamma^1_{sn}$ and $\overline{\mu} \in B(\{X_\mu\})$ such that $X_{\overline{\mu}}$ has a saddle-node periodic orbit θ. Suppose that there exist hyperbolic critical elements $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_k$ of $X_{\overline{\mu}}$ such that, for each $i = 1, \ldots, m$ (resp. $j = 1, \ldots, k$), $\mathcal{W}^u(\alpha_i)$ (resp. $\mathcal{W}^s(\beta_j)$) has a generic s-criticality (resp. u-criticality). We let π^{ss} (resp. π^{uu}) denote the projection on $\mathcal{W}^c_{\overline{\mu}}(q)$ via leaves of $\mathcal{F}^{ss}(q)$ (resp. $\mathcal{F}^{uu}(q)$) and $x_1, \ldots, x_m$ (resp. $y_1, \ldots, y_k$) denote the points of s-criticality (resp. u-criticality) between $\mathcal{W}^u(\alpha_i)$ and $\mathcal{F}^{ss}(q)$ (resp. $\mathcal{W}^s(\beta_j)$ and $\mathcal{F}^{uu}(q)$), $i = 1, \ldots, m$ (resp. $j = 1, \ldots, k$), in a fundamental domain for $\mathcal{P}/\mathcal{W}^{cs}_{\overline{\mu}}(q)$ (resp. $\mathcal{P}/\mathcal{W}^{cu}_{\overline{\mu}}(q)$). Let $\pi^{ss}(x_i) = x^c_i$ (resp. $\pi^{uu}(y_j) = y^c_j$) $i = 1, \ldots, m$ (resp. $j = 1, \ldots, k$). We impose the following conditions.

- $x_i^c \neq x_\ell^c$ and $y_j^c \neq y_s^c$ $i, \ell = 1, \ldots, m$, $j, s = 1, \ldots, k$, $i \neq \ell$, $j \neq s$.
- x_i^c, y_j^c belong to the interior of $D_{\overline{\mu}}^{cs}(q)$, $D_{\overline{\mu}}^{cu}(q)$ respectively, $i = 1, \ldots, m$, $j = 1, \ldots, k$.

We now define the numbers t_i, s_j $(i = 1, \ldots, m,\ j = 1, \ldots, k)$ by the equations

$$Z_{t_i}(x_1^c) = x_i^c \quad \text{and} \quad Z_{s_j}(y_1^c) = y_j^c .$$

We furthermore suppose that the ramaining hyperbolic critical elements of $X_{\overline{\mu}}$ do not have criticalities.

THEOREM 3.— *Let $\{X_\mu\} \in \Gamma_{sn}^1$ and $\overline{\mu} \in B(\{X_\mu\})$ be as above. Suppose that*

$$|t_i - t_\ell| \neq |s_s - s_r| \quad \textit{and that} \quad 1 - |t_i - t_j| \neq |s_s - s_r| ,$$

for each $i, \ell = 1, \ldots, m$, $i \neq \ell$ and each $j, r = 1, \ldots, k$, $j \neq r$; then $\{X_\mu\}$ has finite stability modulus.

Remark.— In the two-dimensional case Theorem 3 was proved by I. Malta and J. Palis [6]. For $\dim M \geq 3$, the proof of Theorem 3 follows from the proof of Theorem 2 and the two-dimensional case.

2. Proof of Proposition 1 and Theorem 1

Proof of Proposition 1

Let $\Sigma \subseteq M$ be a cross section to $X_{\overline{\mu}}$ at $q \in \theta$. We let $\pi^{ss} : \mathcal{W}^{cs}(q) \mapsto \mathcal{W}^c(q)$ denote the projection through leaves of $\mathcal{F}^{ss}(q)$.

Case (a) Let x a point of non-generic s-criticality. We now make a small perturbation of $\mathcal{W}^u(\alpha)$ in a small neighborhood of x in $\mathcal{W}^s(q)$ in such a way as to produce generic s-criticalities among $\mathcal{W}^u(\alpha)$ and two or more leaves of $\mathcal{F}^{ss}(q)$.

Case (b) Let x_1, x_2 be the points of generic s-criticalities of $\mathcal{W}^u(\alpha_1)$ and $\mathcal{W}^u(\alpha_2)$ with $\mathcal{F}^{ss}(q)$, respectively. Again, we make a small perturbation of $\mathcal{W}^u(\alpha_2)$ in a small neighborhood of x_2 in $\mathcal{W}^s(q)$ such that the new s-criticality x_2' is in leaf $\mathcal{F}^{ss}(x_2')$, with

$$\mathcal{F}^{ss}(x_2') \neq \mathcal{F}^{ss}(x_2)$$

(we note that we may suppose $\pi^{ss}(x_1) \neq \pi^{ss}(x_2)$).

In cases *(a)* and *(b)* these perturbations may be obtained as perturbations of $\{X_\mu\}$. Therefore, we obtain a family $\{\widetilde{X}_\mu\} \in \Gamma_{sn}$ near $\{X_\mu\}$; it is clear that they are not equivalent by the rigidity condition on $\mathcal{W}^c(q)$.

Proof of Theorem 1

In part *(a)* as well as in part *(b)* of the theorem, the unstability of $\{X_\mu\}$ at $\overline{\mu}$ follows from the existence of modulus of stability for $X_{\overline{\mu}}$ (which we will prove). In what follows we will write Y for $X_{\overline{\mu}}$.

Part (a) Let $\widetilde{Y}$ be a vector field near Y with the same characteristics of Y. We let $\widetilde{A}$ denote the weakest contraction of the hyperbolic periodic orbit $\widetilde{\alpha}$ of $\widetilde{Y}$. Each time we make a construction for Y, we will suppose it made for $\widetilde{Y}$.

Let Σ_{q_1} be a cross section for Y at $q_1 \in \alpha$ and $\mathcal{P}_{q_1} : (\Sigma_{q_1}, q_1) \mapsto (\Sigma_{q_1}, q_1)$ be the Poincaré map of Y at α. Similarly, let Σ_q be a cross section of Y at $q \in \theta$ and $\mathcal{P}_q : (\Sigma_q, q) \mapsto (\Sigma_q, q)$ be the Poincaré map of Y at θ. We let $D^{cs}(q)$ denote a fundamental domain for $\mathcal{P}_q$ in $\mathcal{W}^{cs}(q)$. Let x be the point of generic s-criticality between $\mathcal{W}^u(\alpha)$ and $\mathcal{F}^{ss}(q)$ in $D^{cs}(q)$. Suppose

$$x^c = \pi^{ss}(x) \in \operatorname{Int}\big(D^{cs}(q) \cap \mathcal{W}^c(q)\big).$$

If H is a topological equivalence between Y and $\widetilde{Y}$ in a neighborhood V of the closure of $O_Y(x)$ in M, then without loss of generality, we may suppose that $H(\Sigma_{q_1}) = \Sigma_{\widetilde{q}_1}$ and that $H(\Sigma_q) = \Sigma_{\widetilde{q}}$. Then

$$\big(H/\Sigma_{q_1}\big) \circ \mathcal{P}_{q_1} = \mathcal{P}_{\widetilde{q}_1} \circ \big(H/\Sigma_{q_1}\big) \quad \text{and} \quad \big(H/\Sigma_q\big) \circ \mathcal{P}_q = \mathcal{P}_{\widetilde{q}} \circ \big(H/\Sigma_q\big) .$$

We will write $h_1 = H/\Sigma_{q_1}$ and $h = H/\Sigma_q$. Now, since H preserves $\mathcal{F}^{ss}$ and $H\big(\mathcal{W}^u(\alpha)\big) = \mathcal{W}^u(\widetilde{\alpha})$, we have that $H(x) = \widetilde{x}$ (see [1]). As a first case we suppose A is real and positive; similarly for $\widetilde{A}$. Let $\varphi = (y_1, \ldots, y_{n-1})$ be C^2 linearizing coordinates in a neighborhood U of q_1 in Σ_{q_1} such that y_1-axis is the eigenspace corresponding to the weakest contraction. Let $\mathcal{W}^{cu}(q_1)$ be a C^1 center-unstable manifold of q_1; it induces a C^1 center-unstable manifold $\mathcal{W}^{cu}(\alpha)$ of α. Since $\mathcal{W}^u(\alpha)$ is a codimension one submanifold of $\mathcal{W}^{cu}(\alpha)$, there exists a distance $\widetilde{d}$ in $\Sigma_{\widetilde{q}}$ and two real numbers, $c > 0$ (large enough) and $\tau > 1$ (close to 1) such that

$$h\big(\mathcal{W}^{cu}(\alpha) \cap \Sigma_q\big) \cap \widetilde{V} \subseteq K^{cu}(\widetilde{q}),$$

where $\widetilde{V}$ is a neighborhood of $\widetilde{x}$ in $\Sigma_{\widetilde{q}}$ and $K^{cu}(\widetilde{q})$ is the cone

$$K^{cu}(\widetilde{q}) = \Big\{ y \in \Sigma_{\widetilde{q}} \mid \widetilde{d}\big(y, \mathcal{W}^{cu}(\widetilde{\alpha}) \cap \Sigma_{\widetilde{q}}\big) \leq c \left(\widetilde{d}\big(y, \mathcal{W}^{u}(\widetilde{\alpha}) \cap \Sigma_{\widetilde{q}}\big) \right)^{\tau} \Big\}.$$

We now reparametrize the flows Y_t and $\widetilde{Y}_t$ in a such a way that

$$Y_T(V) \subseteq U\,, \quad \widetilde{Y}_T(\widetilde{V}) \subseteq \widetilde{U}\,,\; T < 0\,.$$

We now consider a sequence
$x_n \in K^{cu}(q_1) = \{y \in \Sigma_q \mid d(y\,, \mathcal{W}^{cu}(\alpha) \cap \Sigma_q) \leq c\,(d(y\,, \mathcal{W}^u(\alpha) \cap \Sigma_q))^{\tau}\}$, $x_n \mapsto x$, such that the sequence $y_n = Y_T(x_n)$ satisfies $\mathcal{P}_{q_1}^{-k_n}(y_n) \mapsto s$, $s \in \mathcal{W}^s(q_1) \setminus \{q_1\}$; in fact, $s \in \mathcal{W}^s(q_1) \setminus \mathcal{W}^{ss}(q_1)$. Note that $k_n \mapsto \infty$ as $n \mapsto \infty$ and that

$$d(x_n\,, \mathcal{W}^u(\alpha)) \cong \sigma_{q_1} A^{k_n} |\pi_1(s)|\,, \tag{A*}$$

where $\pi_1(s)$ is the first coordinate of s and $\sigma_{q_1} > 0$ is a constant (transition constant) which does not depend on x nor d.

Notation.— Let (a_i) and (b_i) be real number sequences. Then $a_i \cong b_i$ means

$$\lim_{i\to\infty} \frac{a_i}{b_i} = 1\,.$$

Since the contact between $\mathcal{W}^u(\alpha) \cap \Sigma_q$ and $F^{ss}(x)$ is Morse type, in a neighborhood of x in $\mathcal{W}^{cu}(\alpha) \cap \Sigma_q$ we may write $\mathcal{W}^u(\alpha) \cap \Sigma_q$ as the graph of a Morse function Q with respect to the leaf $F^{ss}(x)$ of $\mathcal{F}^{ss}(q)$ which contains x. Under the above conditions we have that there are distances d, $\widetilde{d}$, in Σ_q and $\Sigma_{\widetilde{q}}$ respectively, such that:

(1) if, at x, Q has at x a saddle type critical point, then for each small $\delta > 0$ there exists a point $x_\delta \in \mathcal{W}^{cu}(\alpha) \cap \mathcal{W}^{cs}(q)$ such that

$$\delta = d(x_\delta\,, \mathcal{W}^u(\alpha) \cap \Sigma_q) \cong d(x_\delta\,, F^{ss}(x) \cap \mathcal{W}^{cu}(\alpha))$$

and that

$$\widetilde{d}(h(x_\delta)\,, \mathcal{W}^u(\widetilde{\alpha}) \cap \Sigma_{\widetilde{q}}) \cong \widetilde{d}(h(x_\delta)\,, F^{ss}(\widetilde{x}) \cap \mathcal{W}^{cu}(\widetilde{\alpha}))\,;$$

(2) if, at x, Q has at x, a maximal (minimal) type critical point, then for each small $\delta > 0$, there are points $x_\delta^i \in \mathcal{W}^{cu}(\alpha) \cap \mathcal{W}^{cs}(q)$, $i = 1, 2$, such that

$$\delta = d(x_\delta^i\,, \mathcal{W}^u(\alpha) \cap \Sigma_q) \cong d(x_\delta^i\,, F^{ss}(x) \cap \mathcal{W}^{cu}(\alpha))\,, \quad i = 1,\, 2\,,$$

and that

$$\tilde{d}\big(h(x_\delta^1),\, \mathcal{W}^u(\tilde{\alpha}) \cap \Sigma_{\tilde{q}}\big) < \tilde{d}\big(h(x_\delta^1),\, F^{ss}(\tilde{x}) \cap \mathcal{W}^{cu}(\tilde{\alpha})\big)\,,$$

and that

$$\tilde{d}\big(h(x_\delta^2),\, \mathcal{W}^u(\tilde{\alpha}) \cap \Sigma_{\tilde{q}}\big) > \tilde{d}\big(h(x_\delta^2),\, F^{ss}(\tilde{x}) \cap \mathcal{W}^{cu}(\tilde{\alpha})\big)\,.$$

Notation.— Let (a_i) and (b_i) be real number sequences. Then $a_i \lessapprox b_i$ means

$$0 < \sup_{i\in\mathbb{N}} \left\{ \frac{a_i}{b_i} \right\} \le 1\,.$$

Under the conditions (1), setting $\delta = 1/n$, $n \in \mathbb{N}$ large enough, we obtain a sequence $x_n \in \mathcal{W}^{cu}(\alpha) \cap \mathcal{D}^{cs}(q)$ such that

$$\frac{1}{n} = d\big(x_n\,,\, \mathcal{W}^u(\alpha) \cap \Sigma_q\big) \cong d\big(x_n\,,\, F^{ss}(x) \cap \mathcal{W}^{cu}(\alpha)\big)$$

and that

$$\tilde{d}\big(h(x_n)\,,\, \mathcal{W}^u(\tilde{\alpha}) \cap \Sigma_{\tilde{q}}\big) \cong \tilde{d}\big(h(x_n)\,,\, F^{ss}(\tilde{x}) \cap \mathcal{W}^{cu}(\tilde{\alpha})\big)\,.$$

Since h is C^1 and preserves $\mathcal{F}^{ss}$, we have

$$\tilde{d}\big(h(x_n)\,,\, F^{ss}(\tilde{x}) \cap \mathcal{W}^{cu}(\tilde{\alpha})\big) \cong \gamma d\big(x_n\,,\, F^{ss}(x) \cap \mathcal{W}^{cu}(\alpha)\big)\,. \tag{A1}$$

Now, by taking the sequences $y_n = Y_T(x_n)$, $\tilde{y}_n = \tilde{Y}_T\big(h(x_n)\big)$, we have that

$$\mathcal{P}_{q_1}^{-k_n}(y_n) \to s\,, \quad s \in \mathcal{W}^s(q_1) \setminus \mathcal{W}^{ss}(q_1)$$

$$\tilde{\mathcal{P}}_{\tilde{q}_1}^{-k_n}(\tilde{y}_n) \mapsto h_1(s)\,, \quad h_1(s) \in \mathcal{W}^s(\tilde{q}_1) \setminus \mathcal{W}^{ss}(\tilde{q}_1)$$

and that

$$d\big(x_n\,,\, \mathcal{W}^u(\alpha) \cap \Sigma_q\big) \cong \sigma_{q_1} A^{k_n} \left|\pi_1(s)\right| \tag{A*}$$

$$\tilde{d}\big(h(x_n)\,,\, \mathcal{W}^u(\tilde{\alpha}) \cap \Sigma_{\tilde{q}}\big) \cong \tilde{\sigma}_{\tilde{q}_1} \tilde{A}^{k_n} \left|\tilde{\pi}_1\big(h_1(s)\big)\right|\,. \tag{A**}$$

Combining (A*), (A**) and (A1) we obtain

$$\gamma \sigma_{q_1} A^{k_n} \left|\pi_1(s)\right| \cong \tilde{\sigma}_{\tilde{q}_1} \tilde{A}^{k_n} \left|\tilde{\pi}_1\big(h_1(s)\big)\right|\,.$$

Therefore, for $n \in \mathbb{N}$ large enough, we have

$$\left(\frac{A}{\tilde{A}}\right)^{k_n} \cong K\,, \quad K \text{ non zero constant, thus } A = \tilde{A}\,.$$

Under the conditions (2), we analogously obtain asymptotic inequalities from which $A = \tilde{A}$.

In the case A is negative or complex, the result is obtained in a similar way [1].

Part (b) Let $\widetilde{Y}$ be a vector field near Y with the same characteristics of Y. We let $\widetilde{a} \pm i\widetilde{b}$ ($\widetilde{a} < 0$, $\widetilde{b} \neq 0$) denote the weakest contraction of the hyperbolic singularity $\widetilde{\alpha}$ of $\widetilde{Y}$. Again, each time we make a construction for Y, we will suppose it made for $\widetilde{Y}$.

We first suppose $\dim M = 3$.

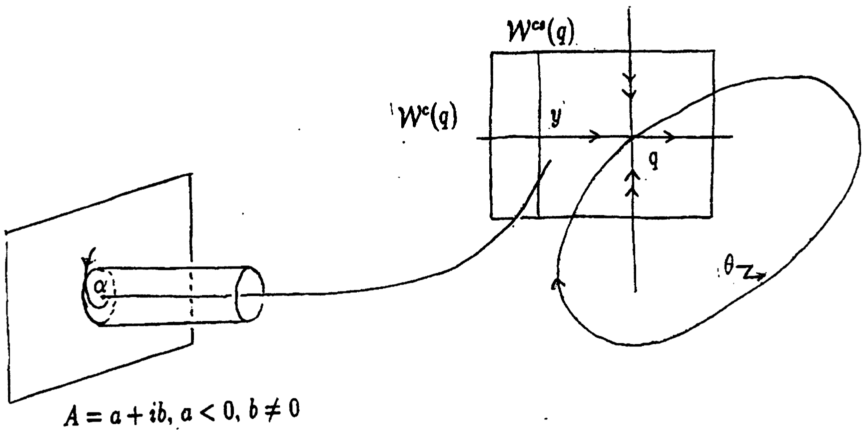

Fig. 1

Let H be an equivalence between Y and $\widetilde{Y}$ in a neighborhood U of the closure of $O_Y(x)$ in M. We let $\mathcal{P} : \Sigma_q \to \Sigma_q$ and $\widetilde{\mathcal{P}} : \Sigma_{\widetilde{q}} \to \Sigma_{\widetilde{q}}$ denote the Poincaré maps of the periodic orbits θ and $\widetilde{\theta}$ in the transversal sections Σ_q ($q \in \theta$), $\Sigma_{\widetilde{q}}$ ($\widetilde{q} \in \widetilde{\theta}$), respectively. We may suppose $H(\Sigma_q) = \Sigma_{\widetilde{q}}$; thus $(H/\Sigma_q) \circ \mathcal{P} = \widetilde{\mathcal{P}} \circ (H/\Sigma_q)$. Moreover, since H/Σ_q sends leaves of $\mathcal{F}^{ss}(q)/\mathcal{W}^s(q)$ into leaves of $\mathcal{F}^{ss}(\widetilde{q})/\mathcal{W}^s(\widetilde{q})$ and

$$H\big(\mathcal{W}^u(\alpha) \cap \mathcal{W}^s(q)\big) = \mathcal{W}^u(\widetilde{\alpha}) \cap \mathcal{W}^s(\widetilde{q}),$$

we have that H is completely determined in a part of $\mathcal{W}^c_{\overline{\mu}}(q)$.

Let V be a neighborhood of α and $\widetilde{V} = H(V)$ and let $f : V \to \mathbb{R}$, $\widetilde{f} : \widetilde{V} \to \mathbb{R}$ be C^∞ Lyapunov functions for Y and $\widetilde{Y}$ respectively. Given $\varepsilon > 0$ small, let $C_\varepsilon \subseteq f^{-1}(-\varepsilon)$ and $\widetilde{C}_\varepsilon \subseteq \widetilde{f}^{-1}(-\varepsilon)$ be transversal cylinders to $\mathcal{W}^s(\alpha)$ and $\mathcal{W}^s(\widetilde{\alpha})$ respectively. Since H is a equivalence between Y/V and $\widetilde{Y}/\widetilde{V}$, we have that $H(C_\varepsilon)$ is a cylinder topologically transversal to $\mathcal{W}^s(\widetilde{\alpha})$. We may suppose $H(C_\varepsilon) \subseteq \widetilde{C}_\varepsilon$.

Now in C_ε, we take a C^1 one-dimensional foliation $\mathcal{F}^c$ with leaves transversal to $\mathcal{W}^s(\alpha)\cap C_\varepsilon$. The foliation $\mathcal{F}^c$ induces a C^1 center-unstable foliation $\mathcal{F}^{cu}$ as follows

$$F^{cu}(z)=\bigcup_{t\in\mathbb{R}} Y_t\big(F^c(z)\big)\,,\quad z\in\mathcal{W}^s(\alpha)\cap C_\varepsilon\,.$$

If $z\in\mathcal{W}^s(\alpha)\cap C_\varepsilon$, we have that $H\big(F^c(z)\big)\subseteq\widetilde{C}_\varepsilon$ is a curve topologically transversal to $\mathcal{W}^s(\widetilde{\alpha})\cap\widetilde{C}_\varepsilon$. A curve of the form $H\big(F^c(z)\big)$ is denoted by $\widehat{F}^c\big(H(z)\big)$ and we define

$$\widehat{F}^{cu}\big(H(z)\big)=\bigcup_{t\in\mathbb{R}}\widetilde{Y}_t\left(\widehat{\mathcal{F}}^{cu}\big(H(z)\big)\right)\,.$$

Under the above conditions (see [13]), there exist $\delta=\delta\big(H(z)\big)>0$ and a cone $\widetilde{K}_{H(z)}\subseteq\widetilde{C}_\varepsilon$ with a vertex in $H(z)$ such that if $\widetilde{C}_\delta\subseteq\widetilde{C}_\varepsilon$ denotes a δ-size cylinder, then:

- $\widetilde{\mathcal{F}}^c\big(H(z)\big)\subseteq\widetilde{K}_{H(z)}$,
- $\widehat{\mathcal{F}}^{cu}\big(H(z)\big)\cap\widetilde{C}_\delta\subseteq\widetilde{K}_{H(z)}$.

Thus,

$$\widetilde{\mathcal{F}}^{cu}\big(H(z)\big)\cap\widetilde{f}^{-1}(\varepsilon)\quad\text{and}\quad\widehat{\mathcal{F}}^{cu}\big(H(z)\big)\cap\widetilde{f}^{-1}(\varepsilon)$$

approach each other when we come near $\mathcal{W}^u(\widetilde{\alpha})\cap\widetilde{f}^{-1}(\epsilon)$. In this case we say they are asymptotically undistinguishable. Furthermore, the curves

$$F^{cu}(z)\cap f^{-1}(\varepsilon)\quad\text{and}\quad\widetilde{F}^{cu}(\widetilde{z})\cap\widetilde{f}^{-1}(\varepsilon)\,,$$

$z\in\mathcal{W}^s(\alpha)\cap C_\varepsilon$ and $\widetilde{z}\in\mathcal{W}^s(\widetilde{\alpha})\cap\widetilde{C}_\varepsilon$, are spirals which we may suppose linear with contraction coefficients $\rho=a/b$ and $\widetilde{\rho}=\widetilde{a}/\widetilde{b}$ respectively, in linearization neighborhoods of α and $\widetilde{\alpha}$ respectively.

Now, we have that the intersections

$$\mathcal{F}^{cu}\cap\Sigma_q\,,\quad\widetilde{\mathcal{F}}^{cu}\cap\Sigma_{\widetilde{q}}\quad\text{and}\quad\widehat{\mathcal{F}}^{cu}\cap\Sigma_{\widetilde{q}}$$

are spirals (see [7] or [11]). We let γ denote the C^1 curve given by the tangencies between the leaves of $\mathcal{F}^{cu}\cap\Sigma_q$ and the leaves of $\mathcal{F}^{ss}(q)$ in a neighborhood of the s-criticality x in $\mathcal{W}^{cs}(q)$. We define $\widetilde{\gamma}$ analogously. Note that $H(\gamma)$ does not necessarily coincide with γ. But since

$$\pi^{ss}/\gamma:\gamma\mapsto\pi^{ss}(\gamma)\quad\text{and}\quad\widetilde{\pi}^{ss}/\widetilde{\gamma}:\widetilde{\gamma}\mapsto\widetilde{\pi}(\widetilde{\gamma})$$

are C^1 diffeomorphisms, we may suppose $H(\gamma)=\widetilde{\gamma}$.

Let $h = H/\gamma : \gamma \mapsto \widetilde{\gamma}$, h is C^1 and compatible with $H^c = H/\mathcal{W}^c(q)$. Using the spirals $\mathcal{F}^{cu} \cap \Sigma_q$, $\widetilde{\mathcal{F}}^{cu} \cap \Sigma_{\widetilde{q}}$ and $\widehat{\mathcal{F}}^{cu} \cap \Sigma_{\widetilde{q}}$, we define the maps $h^{cu} : \gamma \mapsto \gamma$, $\widehat{h}^{cu}$, $\widetilde{h}^{cu} : \widetilde{\gamma} \mapsto \widetilde{\gamma}$ as follows: given $y \in \gamma$, $h^{cu}(y)$ is the first point where $F^{cu}(y) \cap \Sigma_q$ intersects the component of $\gamma \setminus \{x\}$ which contains y. We define $\widetilde{h}^{cu}$ and $\widehat{h}^{cu}$ analogously. We have that

$$h^{cu}(x) = x\,, \quad \widetilde{h}^{cu}(\widetilde{x}) = \widehat{h}^{cu}(\widetilde{x}) = \widetilde{x}\,, \quad \widehat{h}^{cu}\big(h(y)\big) = h\big(h^{cu}(y)\big)$$

and that, in a neighborhood of x, respectively of $\widetilde{x}$,

$$h^{cu}(y) = e^{2\pi\rho}y\,, \quad \widetilde{h}^{cu}(z) = e^{2\pi\widetilde{\rho}}z\,,$$

where $\rho = a/b$, $\widetilde{\rho} = \widetilde{a}/\widetilde{b}$ (fig. 2).

We now fix $z_1, z_2 \in \mathcal{W}^s(\alpha) \cap C_\varepsilon$, z_1 near z_2. Let $u_1 = H(z_1)$, $u_2 = H(z_2)$. Associated to each u_i $(i = 1, 2)$, we have a cone $\widetilde{K}_{u_i} \subseteq \widetilde{C}_\varepsilon$ with vertex u_i, and a cylinder C_{δ_i} $(\delta_i = \delta(u_i))$. Let $0 < \delta < \min\{\delta_1, \delta_2\}$, C_δ denotes the corresponding cylinder. Let W be a neighborhood of u_1 and u_2 in $\mathcal{W}^u(\widetilde{\alpha}) \cap \widetilde{C}_\varepsilon$, in W we take the points v_1, v_2 as in figure 2. From this construction, we have that the spirals $\widehat{\mathcal{F}}^{cu}(u_1) \cap \Sigma_{\widetilde{q}}$ and $\widehat{\mathcal{F}}^{cu}(u_2) \cap \Sigma_{\widetilde{q}}$ are contained in a spiral neighborhood limited by the spirals $\widetilde{\mathcal{F}}^{cu}(v_1) \cap \Sigma_{\widetilde{q}}$ and $\widetilde{\mathcal{F}}^{cu}(v_2) \cap \Sigma_{\widetilde{q}}$ as in figure 3. We take a neighborhood of x (resp. $\widetilde{x}$) in $\mathcal{W}^{cs}(q)$ (resp. $\mathcal{W}^{cs}(\widetilde{q})$) small enough such that $\widehat{x}_0 = h(x_0)$ and $\widehat{y}_0 = h(y_0)$ are near $\widetilde{x}$. We now define the following sequences and intervals in γ (resp. $\widetilde{\gamma}$):

- $x_n = \big(h^{cu}\big)^n(x_0)$, $y_n = \big(h^{cu}\big)^n(y_0)$, $I_0 = [\,y_0\,,\,x_0\,]$, $I_n = [\,y_n\,,\,x_n\,]$;
- $u_n = \big(\widetilde{h}^{cu}\big)^n(\widetilde{u}_0)$, $\widetilde{v}_n = \big(\widetilde{h}^{cu}\big)^n(\widetilde{v}_0)$, $\widetilde{I}_0 = [\,\widetilde{v}_0\,,\,\widetilde{u}_0\,]$, $\widetilde{I}_n = [\,\widetilde{v}_n\,,\,\widetilde{u}_n\,]$;
- $\widehat{x}_n = \big(\widehat{h}^{cu}\big)^n(\widehat{x}_0)$, $\widehat{y}_n = \big(\widehat{h}^{cu}\big)^n(\widehat{y}_0)$, $\widehat{I}_0 = [\,\widehat{y}_0\,,\,\widehat{x}_0\,]$, $\widehat{I}_n = [\,\widehat{y}_n\,,\,\widehat{x}_n\,]$.

For each $n \in \mathbb{N}$, we have that

$$l(I_n) = e^{2n\pi\rho}l(I_0)\,, \quad l(\widetilde{I}_n) = e^{2n\pi\widetilde{\rho}}l(\widetilde{I}_0) \quad \text{and} \quad \widehat{I}_n \subseteq \widetilde{I}_n$$

(where $l(J)$ is the length of the interval J). Therefore $\widetilde{d}(\widehat{x}_n, \widehat{y}_n) = l(\widetilde{I}_n)$. On the other hand, we have:

$$\begin{aligned}
\widetilde{d}(\widehat{x}_n, \widehat{y}_n) &= \widetilde{d}\left(\big(\widehat{h}^{cu}\big)^n(\widehat{x}_0)\,,\ \big(\widehat{h}^{cu}\big)^n(\widehat{y}_0)\right)\\
&= \widetilde{d}\left(\big(\widehat{h}^{cu}\big)^n\big(h(x_0)\big)\,,\ \big(\widehat{h}^{cu}\big)^n\big(h(y_0)\big)\right)\\
&= \widetilde{d}\left(h\Big(\big(h^{cu}\big)^n(x_0)\Big)\ ,\ h\big(\big(h^{cu}\big)^n(y_0)\big)\right)\\
&= \widetilde{d}\big(h(x_n)\,,\ h(y_n)\big)\\
&= cd(x_n, y_n) \quad (\text{since } h \text{ is } C^1)\\
&= cl(I_n)\\
&= c\,e^{2n\pi\rho}l(I_0)\,,
\end{aligned}$$

where $0 < c\,e^{2n\pi\rho} l(I_0) \leq e^{2n\pi\widetilde{\rho}} l(\widetilde{I}_0)$; thus $0 < K \leq e^{2n\pi(\widetilde{\rho}-\rho)}$, $n \in \mathbb{N}$ large enough.

Through an analogous argument for H^{-1}, we have $0 < \widetilde{K} \leq e^{2n\pi(\rho-\widetilde{\rho})}$, $n \in \mathbb{N}$ large enough.

Therefore, if $n \in \mathbb{N}$ is large enough, we have $0 < C \leq e^{2n\pi(\widetilde{\rho}-\rho)} \leq \widetilde{C} < \infty$ from which $\rho = \widetilde{\rho}$.

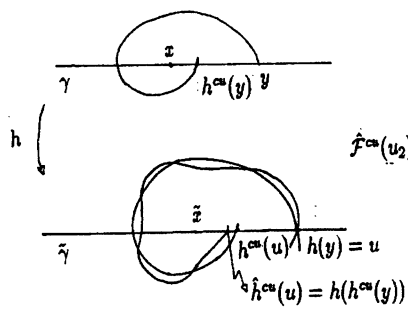

Fig. 2

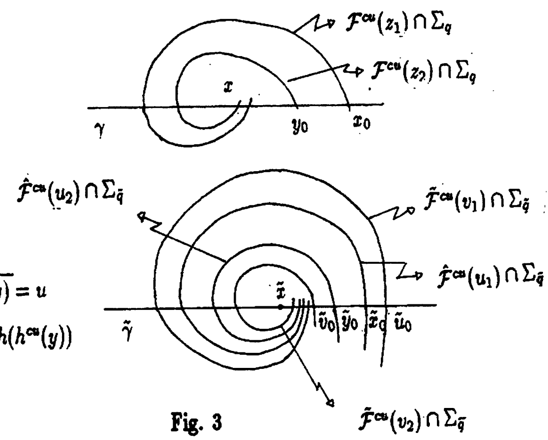

Fig. 3

The n-dimensional case is reduced to the 3-dimensional case.

3. Proof of Theorem 2

This section is devoted to the proof of Theorem 2. We only prove part *(c)* since the methods used can easily be adapted to parts *(a)* and *(b)*. We first recall some definitions of invariant foliation associated to vector fields.

Let $\{X_\mu\} \in \mathcal{X}_1^\infty(M)$, and let $\overline{\mu} \in I$.

DEFINITION 3.1. — *Let $p \in M$ be a hyperbolic singularity of $X_{\overline{\mu}}$ a (local) unstable foliation for $\{X_\mu\}$ (or for the vector field $X(x,\mu) = (X_\mu(x)\,,\,0)$) at $(p,\overline{\mu})$, $\mathcal{F}_p^u$ is a continuous foliation of $U_p \times I_1$, where $U_p \subseteq M$, $I_1 \subseteq I$, are neighborhoods of p and $\overline{\mu}$ respectively, such that:*

(a) the leaves of $\mathcal{F}_p^u$ are C^r discs, $r \geq 1$, varying continuously in the C^r topology with distinguished leaf

$$F^u(p_\mu,\mu) = \big(\mathcal{W}^u(p_\mu) \cap U_p\big) \times \{\mu\}\,;$$

(b) *each leaf* $F^u(x,\mu)$ *is contained in* $U_p \times \{\mu\}$;

(c) $\mathcal{F}^u$ *is invariant, i.e.,*

$$F^u\big(X_{\mu,t}(x)\,,\,\mu\big) \subset X_{\mu,t}\big(F^u(x,\mu)\big)\,, \quad t \geq 0\,.$$

A global unstable foliation $\mathcal{F}^u_p$ is just the positive saturation by the flow X_μ ($\mu \in I_1$) of the local unstable foliation. Similarly we define a stable foliation $\mathcal{F}^s_p$.

Now, let σ be a hyperbolic periodic orbit of $X_{\overline{\mu}}$, and let

$$\mathcal{P}_\mu : V_q \times I_1 \to \Sigma_q \times I_1$$

(V_q is a neighborhood of q in Σ_q) be the Poincaré map.

DEFINITION 3.2.— *A (local) unstable foliation for* $\{X_\mu\}$ *(or for* $X(x,\mu)$*) at* $(q,\overline{\mu})$, $\mathcal{F}^u_\sigma$ *is a continuous foliation of* $V_q \times I_1$ *such that:*

(a) *the leaves of* $\mathcal{F}^u_\sigma$ *are* C^r *discs,* $r \geq 1$, *varying continuously in the* C^r *topology with distinguished leaf*

$$F^u(q_\mu,\mu) = \big(\mathcal{W}^s(q_\mu) \cap V_q\big) \times \{\mu\}\,;$$

(b) *each leaf* $F^u(x,\mu)$ *is contained in* $V_q \times \{\mu\}$;

(c) $\mathcal{F}^u$ *is invariant, i.e.,*

$$F^u\big(\mathcal{P}_\mu(x)\,,\,\mu\big) \subseteq \mathcal{P}\big(F^u(x,\mu)\big)\,.$$

A global unstable foliation $\mathcal{F}^u_\sigma$ is just the positive saturation by the flow X_μ ($\mu \in I_1$) of the local unstable foliation. Similarly we define a stable foliation $\mathcal{F}^s_\sigma$.

If $\overline{\mu} \in B\big(\{X_\mu\}\big)$ such that $X_{\overline{\mu}}$ has a saddle-node periodic orbit θ, $\mathcal{P} = \{\mathcal{P}_\mu\}$ (the Poincaré map associated to the vector field X in the non-hyperbolic periodic orbit $(\theta,\overline{\mu})$) is an arc of saddle-node diffeomorphisms. Thus we have that, in $\mathcal{W}^{cs}(q)$ (resp. $\mathcal{W}^{cu}(q)$), there is a strong stable (resp. strong unstable) C^r foliation, $r \geq 1$, $\mathcal{F}^{ss}_q(q)$ (resp. $\mathcal{F}^{uu}_q(q)$).

DEFINITION 3.3. — *A strong unstable foliation for* $\mathcal{P} = \{\mathcal{P}_\mu\}$ *at the saddle-node* $(q, \overline{\mu}))$, $\mathcal{F}_q^{uu}$ *is a continuous foliation of* $\Sigma_q \times I_1$ *such that:*

(a) *the leaves of* $\mathcal{F}_q^{uu}$ *are* C^r *discs,* $r \geq 1$, *varying continuously in the* C^r *topology with distinguished leaf*

$$F^{uu}(q, \overline{\mu}) = \big(\mathcal{W}^{uu}(q) \cap \Sigma_q\big) \times \{\overline{\mu}\};$$

(b) *each leaf* $F^{uu}(x, \mu)$ *is contained in* $\Sigma_q \times \{\mu\}$;

(c) $\mathcal{F}_q^{uu}$ *is invariant, i.e.,*

$$F^{uu}\big(\mathcal{P}_\mu(x)\,,\,\mu\big) \subseteq \mathcal{P}\big(F^{uu}(x, \mu)\big)\,.$$

Similarly we define a strong stable foliation, $\mathcal{F}_q^{ss}$.

Let $\{X_\mu\} \in \Gamma_{sn}^1$ and $\overline{\mu} \in B\big(\{X_\mu\}\big)$ such that $X_{\overline{\mu}}$ has a saddle-node periodic orbit θ. Since for each $\mu \in I$, $X_\mu \in WR^\infty(M)$, we have that $\Omega(X_\mu)$ is the union of a finite number of critical elements of X_μ. For $\mu < \overline{\mu}$, let $\alpha_{1,\mu}, \ldots, \alpha_{k,\mu}, \theta_{1,\mu}, \theta_{2,\mu}, \beta_{1,\mu}, \ldots, \beta_{\ell,\mu}$ be the critical elements (hyperbolic) of X_μ, where $\theta_{1,\mu}$ and $\theta_{2,\mu}$ are the periodic orbits which originate the saddle-node periodic orbit

$$\theta = \theta_{1,\overline{\mu}} = \theta_{2,\overline{\mu}} \quad \text{of } X_{\overline{\mu}}\,.$$

On the other hand, for each $\mu \in I$, $X_\mu \in WC^\infty(M)$, then we may define a partial order $\leq$ among the critical elements of X_μ,

$$\sigma_1 \leq \sigma_2 \quad \text{if and only if} \quad \mathcal{W}^u(\sigma_1) \cap \mathcal{W}^s(\sigma_2) \neq \emptyset\,.$$

This partial order is extended to a total order and we assume it is the following

$$\alpha_{1,\mu} \leq \cdots \leq \alpha_{k,\mu} \leq \theta_{1,\mu} \leq \theta_{2,\mu} \leq \beta_{1,\mu} \leq \cdots \leq \beta_{\ell,\mu}\,.$$

Clearly, this order holds for $\mu = \overline{\mu}$ and even for $\mu > \overline{\mu}$ near $\overline{\mu}$. Therefore, for each $\mu \in I$, we order the critical elements of X_μ as follows

$$\begin{cases} \mu < \overline{\mu}\text{:} & \alpha_{1,\mu} \leq \cdots \leq \alpha_{k,\mu} \leq \theta_{1,\mu} \leq \theta_{2,\mu} \leq \beta_{1,\mu} \leq \cdots \leq \beta_{\ell,\mu} \\ \mu = \overline{\mu}\text{:} & \alpha_{1,\overline{\mu}} \leq \cdots \leq \alpha_{k,\overline{\mu}} \leq \theta \leq \beta_{1,\overline{\mu}} \leq \cdots \leq \beta_{\ell,\overline{\mu}} \\ \mu > \overline{\mu}\text{:} & \alpha_{1,\mu} \leq \cdots \leq \alpha_{k,\mu} \leq \beta_{1,\mu} \leq \cdots \leq \beta_{\ell,\mu}, \end{cases}$$

Definition 3.4. — *Let $\{X_\mu\} \in \Gamma^1_{sn}$ and $\overline{\mu} \in B(\{X_\mu\})$ be as above. We say the system of unstable and strong unstable foliations*

$$\mathcal{F}^u(\alpha_1), \ldots, \mathcal{F}^u(\alpha_k), \mathcal{F}^{uu}(q)$$

is compatible:

(a) *if a leaf F of $\mathcal{F}^u(\alpha_n)$ intersects a leaf E of $\mathcal{F}^u(\alpha_i)$ ($\alpha_n \leq \alpha_i \leq \alpha_k$) then $F \supseteq E$ and the restriction of the foliation $\mathcal{F}^u(\alpha_i)$ to each leaf of $\mathcal{F}^u(\alpha_n)$ is a C^2 foliation;*

(b) *if a leaf F of $\mathcal{F}^u(\alpha_i)$, $\alpha_i \leq \alpha_k$, intersects a leaf F^{uu} of $\mathcal{F}^{uu}(q)$, then $F \supseteq F^{uu}$ and the restriction of the foliation $\mathcal{F}^{uu}(q)$ to each leaf of $\mathcal{F}^u(\alpha_n)$ is a C^1 foliation.*

Similarly, we define a compatible system of stable and strong stable foliations $\mathcal{F}^{ss}(q)$, $\mathcal{F}^s(\beta_1), \ldots, \mathcal{F}^s(\beta_\ell)$.

Lemma 1. — (J. Palis and F. Takens [9]) *Let $\{X_\mu\} \in \Gamma^1_{sn}$ and $\overline{\mu} \in B(\{X_\mu\})$ such that $X_{\overline{\mu}}$ has a saddle-node periodic orbit θ. Then for $\{X_\mu\}$ there exist compatible systems of foliations:*

(a) $\mathcal{F}^u(\alpha_1), \ldots, \mathcal{F}^u(\alpha_k)$,

(b) $\mathcal{F}^s(\beta_1), \ldots, \mathcal{F}^s(\beta_\ell)$.

Now let $\{X_\mu\}$, $\{\widetilde{X}_\mu\} \in \Gamma^1_{sn}$ be close families, and let $\overline{\mu}$, $\widetilde{\mu} \in I$ their first bifurcation values, $\widetilde{\mu}$ close to $\overline{\mu}$. We have that $X_{\overline{\mu}}$ and $\widetilde{X}_{\widetilde{\mu}}$ have close saddle-node periodic orbits θ and $\widetilde{\theta}$, respectively. If $\rho : (I, \overline{\mu}) \to (I, \widetilde{\mu})$ is a reparametrization then from the existence of compatible systems of foliations

$$\mathcal{F}^u(\alpha_1), \ldots, \mathcal{F}^u(\alpha_k), \quad \mathcal{F}^s(\beta_1), \ldots, \mathcal{F}^s(\beta_\ell)$$

for $\{X_\mu\}$ and of the respective one for $\{\widetilde{X}_\mu\}$, we may suppose constructed the homeomorphisms:

$$H^s : \bigcup_{\mu \in I_1} \left(\bigcup_{i=1}^{k} (\mathcal{W}^s(\alpha_{i,\mu}) \times \{\mu\}) \right) \longrightarrow \bigcup_{\mu \in \widetilde{I}_1} \left(\bigcup_{i=1}^{k} (\mathcal{W}^s(\widetilde{\alpha}_{i,\mu}) \times \{\mu\}) \right) \quad (1)$$

such that, for each i ($1 \leq i \leq k$), the restrictions

$$H^s_{i,\mu} = H^s / \mathcal{W}^s(\alpha_{i,\mu}) \times \{\mu\} : \mathcal{W}^s(\alpha_{i,\mu}) \times \{\mu\} \longrightarrow \mathcal{W}^s(\widetilde{\alpha}_{i,\rho(\mu)}) \times \{\rho(\mu)\}$$

are conjugacies between the flows

$$X_{\mu,t}/\mathcal{W}^s(\alpha_{i,\mu}) \times \{\mu\} \quad \text{and} \quad \widetilde{X}_{\rho(\mu),t}/\mathcal{W}^s(\widetilde{\alpha}_{i,\rho(\mu)}) \times \{\rho(\mu)\}\,;$$

$$H^u : \bigcup_{\mu \in I_1} \left(\bigcup_{j=1}^{\ell} (\mathcal{W}^u(\beta_{j,\mu}) \times \{\mu\}) \right) \longrightarrow \bigcup_{\mu \in \widetilde{I}_1} \left(\bigcup_{j=1}^{\ell} (\mathcal{W}^u(\widetilde{\beta}_{j,\mu}) \times \{\mu\}) \right) \tag{2}$$

such that, for each j $(1 \leq i \leq k)$, the restrictions

$$H^u_{j,\mu} = H^u/\mathcal{W}^u(\beta_{j,\mu}) \times \{\mu\} : \mathcal{W}^u(\beta_{j,\mu}) \times \{\mu\} \longrightarrow \mathcal{W}^u(\widetilde{\beta}_{j,\rho(\mu)}) \times \{\rho(\mu)\}$$

are conjugacies between the flows

$$X_{\mu,t}/\mathcal{W}^u(\beta_{j,\mu}) \times \{\mu\} \quad \text{and} \quad \widetilde{X}_{\rho(\mu),t}/\mathcal{W}^u(\widetilde{\beta}_{j,\rho(\mu)}) \times \{\rho(\mu)\}\,.$$

Remark.— To construct a global equivalence between $\{X_\mu\}$ and $\{\widetilde{X}_\mu\}$ as above, we first extend the definition of H^s to $\mathcal{W}^{cs}(q)$ and that of H^u to $\mathcal{W}^{cu}(q)$ in a such way as to conjugate $\mathcal{P}/\mathcal{W}^{cs}(q)$ with $\widetilde{\mathcal{P}}/\mathcal{W}^{cs}(\widetilde{q})$, and $\mathcal{P}/\mathcal{W}^{cu}(q)$ with $\widetilde{\mathcal{P}}/\mathcal{W}^{cu}(\widetilde{q})$, and afterwards extend the definitions given in $\mathcal{W}^{cs}(q)$ and $\mathcal{W}^{cu}(q)$ to a neighborhood of $(q, \overline{\mu})$ in $\Sigma_q \times I_1$ in order to obtain a homeomorphism $H_q : \Sigma_q \times I_1 \to \Sigma_{\widetilde{q}} \times \widetilde{I}_1$ such that:

- it conjugates the Poincaré maps $\mathcal{P}$ and $\widetilde{\mathcal{P}}$;
- it is compatible with the constructions done above;
- it is possible to extend the definition of H^s (or H^u) to

$$\bigcup_{\mu \in I_1} \left(\bigcup_{j=1}^{\ell} (\mathcal{W}^s(\beta_{j,\mu}) \times \{\mu\}) \right) \quad \text{or} \quad \bigcup_{\mu \in I_1} \left(\bigcup_{i=1}^{k} (\mathcal{W}^u(\alpha_{i,\mu}) \times \{\mu\}) \right)\,.$$

DEFINITION 3.5.— *Let $\{X_\mu\} \in \Gamma^1_{sn}$ and $\overline{\mu} \in B(\{X_\mu\})$ such that $X_{\overline{\mu}}$ has a saddle-node periodic orbit θ. Let $\alpha_{\overline{\mu}}$ be a hyperbolic critical element of $X_{\overline{\mu}}$ such that $\mathcal{W}^u(\alpha_{\overline{\mu}})$ has a generic s-criticality with $\mathcal{F}^{ss}(q)/\mathcal{W}^s(q)$. We say the s-criticality between $\mathcal{W}^u(\alpha_{\overline{\mu}})$ and $\mathcal{F}^{ss}(q)/\mathcal{W}^s(q)$ is of codimension p, $0 \leq p \leq \dim \mathcal{W}^{cs}_{\overline{\mu}}(q)$, if $\mathcal{W}^u(\alpha_{\overline{\mu}}) \times \{\overline{\mu}\} \cap \mathcal{W}^{cs}_{\overline{\mu}}(q)$ is a codimension p submanifold of $\mathcal{W}^{cs}_{\overline{\mu}}(q)$. If $p = \dim \mathcal{W}^{cs}_{\overline{\mu}}(q)$, we say the s-criticality is of zero dimension.*

An analogous definition is given for u-criticalities.

Remark 1.— If the s-criticality is of zero dimension, without loss of generality, we may suppose that $\alpha_{\overline{\mu}} = \alpha_{k,\overline{\mu}}$ in the order given above.

Remark 2.— If the s-criticality of codimension $p \geq 1$, since we are assuming a unique criticality in a fundamental domain for $\mathcal{P}/\mathcal{W}^{cs}_{\overline{\mu}}(q)$, we have that $\mathcal{W}^u(\alpha_{\overline{\mu}}) \cap \mathcal{W}^{cs}_{\overline{\mu}}(q)$ is non-compact. Thus, there exists a hyperbolic critical element ρ ($\rho \neq \alpha_{\overline{\mu}}$) such that $\mathcal{W}^u(\rho) \cap \mathcal{W}^{cs}_{\overline{\mu}}(q)$ has no criticalities and $\mathcal{W}^u(\alpha_{\overline{\mu}}) \cap \mathcal{W}^s(\rho)$ is non-empty. Without loss of generality, we suppose $\rho = \alpha_{k,\overline{\mu}}$ and $\alpha_{\overline{\mu}} = \alpha_{k-1,\overline{\mu}}$ in the order given above.

We will make a construction which is easily adapted to the general case.

Let $\{X_\mu\} \in \mathcal{X}^\infty_1(M)$ and α be a hyperbolic singularity with real weakest contraction of $X_{\overline{\mu}}$. Let I_1 and U be neighborhoods of $\overline{\mu}$ in I and of α in M, respectively. Then there exists a C^r ($r \geq 1$) center-unstable manifold $\mathcal{W}^{cu}(\alpha_{\overline{\mu}})$ (not unique) for $X(x,\mu) = \big(X_\mu(x)\,,\,0\big)$ at $(\alpha_{\overline{\mu}},\overline{\mu})$ such that for each $\mu \in I_1$,

$$\mathcal{W}^{cu}(\alpha_\mu) = \mathcal{W}^{cu}(\alpha) \cap M \times \{\mu\}$$

is a C^r center-manifold for X_μ at α_μ. A C^r center-manifold for X at $(\alpha,\overline{\mu})$ (resp. for X_μ at α_μ) is given by

$$\mathcal{W}^c(\alpha) = \mathcal{W}^s(\alpha) \cap \mathcal{W}^{cu}(\alpha)\,, \quad (\text{resp. } \mathcal{W}^c(\alpha_\mu) = \big(\mathcal{W}^s(\alpha_\mu) \cap \mathcal{W}^{cu}(\alpha_\mu)\big) \times \{\mu\}).$$

Throughout, we will use C^r ($r \geq 1$) center-unstable foliation $\mathcal{F}^{cu}(\alpha)$ and $\mathcal{F}^{cu}_1(\alpha)$ as constructed in [5].

LEMMA 2.— *Under the above conditions, for* $\{X_\mu\} \in \Gamma^1_{sn}$, *we have:*

(a) if $\mathcal{W}^u(\alpha_{k,\overline{\mu}})$ *has a zero dimensional s-criticality, then there exists a compatible system of foliations*

$$\mathcal{F}^u(\alpha_1),\,\ldots,\,\mathcal{F}^u(\alpha_{k-1}),\,\mathcal{F}^{cu}(\alpha_k)\big(\mathcal{F}^{cu}_1(\alpha_k)\big),\,\mathcal{F}^{uu}(q)$$

for $\{X_\mu\}$, μ *near* $\overline{\mu}$;

(b) if $\mathcal{W}^u(\alpha_{k-1,\overline{\mu}})$ *has a codimension* $p \geq 1$ *s-criticality, then there exists a compatible system of foliations*

$$\mathcal{F}^u(\alpha_1),\,\ldots,\,\mathcal{F}^u(\alpha_{k-2}),\,\mathcal{F}^{cu}(\alpha_{k-1})\big(\mathcal{F}^{cu}_1(\alpha_{k-1})\big),\,\mathcal{F}^{cu}(\alpha_k),\,\mathcal{F}^{uu}(q)$$

for $\{X_\mu\}$, μ *near* $\overline{\mu}$.

Remark.— **(Important) Due to a result of F. Takens [10], we have that if we construct a conjugacy $H_q : \Sigma_q \times I_1 \to \Sigma_{\widetilde{q}} \times \widetilde{I}_1$ between $\mathcal{P} = \{\mathcal{P}_\mu\}$ and $\widetilde{\mathcal{P}} = \{\widetilde{\mathcal{P}}_\mu\}$ as above so that:**

- **the homeomorphism H_q sends the intersections of the foliations $\mathcal{F}^{cu}(\alpha)$, $\mathcal{F}_1^{cu}(\alpha)$, $\mathcal{F}^{cs}(\beta)$ and $\mathcal{F}_1^{cs}(\beta)$ with $\Sigma_q \times I_1$ for $\{X_\mu\}$, into the corresponding intersections of the foliations $\mathcal{F}^{cu}(\widetilde{\alpha})$, $\mathcal{F}_1^{cu}(\widetilde{\alpha})$, $\mathcal{F}^{cs}(\widetilde{\beta})$ and $\mathcal{F}_1^{cs}(\widetilde{\beta})$ with $\Sigma_{\widetilde{q}} \times \widetilde{I}_1$ for $\{\widetilde{X}_\mu\}$ continuously on the parameter;**
- **H_q is compatible with the homeomorphisms H^s, H^u, and $h^c : \mathcal{W}^c(q) \to \mathcal{W}^c(\widetilde{q})$ which will be constructed shortly;**
- **H_q is C^1-close to the inclusion map.**

Therefore, using Lyapunov functions, we extend the homeomorphism H_q to a neighborhood of α or β, depending on the case, in such a way that, using the methods developed in [9] or [4]. We construct a global equivalence $H : M \times I_1 \to M \times \widetilde{I}_1$ between $\{X_\mu\}$ at $\overline{\mu}$ and $\{\widetilde{X}_\mu\}$ at $\widetilde{\mu}$.

Construction of equivalences in $\mathcal{W}^c$ for arcs in Γ^1_{sn}

Let $\{X_\mu\} \in \Gamma^1_{sn}$ and $\overline{\mu} \in B(\{X_\mu\})$ such that $X_{\overline{\mu}}$ has a saddle-node periodic orbit θ. We let $\mathcal{P} = \{\mathcal{P}_\mu\}$ denote the Poincaré map associated to $X(x,\mu) = (X_\mu(x), 0)$ at $(q,\overline{\mu})$. Let $\{\widetilde{X}_\mu\} \in \Gamma^1_{sn}$ be near $\{X_\mu\}$ with first bifurcation value $\widetilde{\mu}$ near $\overline{\mu}$, and such that $\widetilde{X}_{\widetilde{\mu}}$ has a saddle-node periodic orbit $\widetilde{\theta}$ near θ. We let $\widetilde{\mathcal{P}} = \{\widetilde{\mathcal{P}}_\mu\}$ denote the Poincaré map associated to $\widetilde{X}(x,\mu) = (\widetilde{X}_\mu(x), 0)$ at $(\widetilde{q},\widetilde{\mu})$. Then $\widetilde{\mathcal{P}}$ and $\mathcal{P}$ are close diffeomorphism arcs.

Let $\alpha_{\overline{\mu}}$ and $\beta_{\overline{\mu}}$ be critical elements (hyperbolic) of $X_{\overline{\mu}}$ such that $\mathcal{W}^u(\alpha_{\overline{\mu}})$ is s-critical and $\mathcal{W}^s(\beta_{\overline{\mu}})$ is u-critical. Let $\mathcal{P}^c_\mu = \mathcal{P}/\mathcal{W}^c_\mu(q)$ and

$$\pi_1^{uu} : \Sigma_q \times I_1 \to \mathcal{W}^{cs}(q)\,, \quad \pi_1^{ss} : \Sigma_q \times I_1 \to \mathcal{W}^{cu}(q)$$

be the projections via leaves of $\mathcal{F}^{uu}(q)$ and $\mathcal{F}^{ss}(q)$, respectively, and let

$$\pi^{ss} : \mathcal{W}^{cs}(q) \to \mathcal{W}^c(q)\,, \quad \pi^{uu} : \mathcal{W}^{cu}(q) \to \mathcal{W}^c(q)$$

be the projections via leaves of $\mathcal{F}^{ss}(q)/\mathcal{W}^{cs}(q)$ and $\mathcal{F}^{uu}(q)/\mathcal{W}^{cu}(q)$, respectively. We have

$$\pi^{ss} \circ \pi_1^{uu} \circ \mathcal{P}_\mu = \mathcal{P}^c_\mu \circ \pi^{uu} \circ \pi_1^{ss}\,.$$

We let $x(\overline{\mu})$ denote the point of s-criticality between $\mathcal{W}^u(\alpha_{\overline{\mu}})$ and $\mathcal{F}^{ss}(q)$ in a fundamental domain $D^{cs}_{\overline{\mu}}(q)$ of $\mathcal{P}/\mathcal{W}^{cs}_{\overline{\mu}}$. Similarly, $y(\overline{\mu})$ is the point of u-criticality between $\mathcal{W}^s(\beta_{\overline{\mu}})$ and $\mathcal{F}^{uu}(q)$ in a fundamental domain $D^{cu}_{\overline{\mu}}(q)$ of $\mathcal{P}/\mathcal{W}^{cu}(q)$. Set

$$x^c(\overline{\mu}) = \pi^{ss}\big(x(\overline{\mu})\big) \quad \text{and} \quad y^c(\overline{\mu}) = \pi^{uu}\big(y(\overline{\mu})\big)\,.$$

We have the following lemma.

LEMMA 3.— *Under the above conditions, there are $\varepsilon > 0$ and a strictly monotone sequence $(\overline{\mu}_n)$ of parameter values such that:*

(a) $\overline{\mu} < \overline{\mu}_n < \varepsilon$, $\overline{\mu}_n \to \overline{\mu}$,

(b) for each $\mu \in \,]\overline{\mu} - \varepsilon\,,\, \overline{\mu} + \varepsilon[$, we have

- *if $\mu \neq \overline{\mu}_n$ for all $n \in \mathbb{N}$, then X_μ has no quasi-transversal intersection orbits,*
- *for each $n \in \mathbb{N}$, $\overline{\mu}_n \in B\big(\{X_\mu\}\big)$ and $X_{\overline{\mu}_n}$ has a unique quasi-transversal intersection orbit between $\mathcal{W}^u(\alpha_{\overline{\mu}_n})$ and $\mathcal{W}^s(\beta_{\overline{\mu}_n})$.*

Proof.— We use the equation $\pi^{ss} \circ \pi^{uu}_1 \circ \mathcal{P}_\mu = \mathcal{P}^c_\mu \circ \pi^{uu} \circ \pi^{ss}_1$ and Lemma 1 of [6, p. 21] to obtain the result.

In what follows, we choose $I_1 = [\,\overline{\mu} - \varepsilon\,,\, \overline{\mu} + \varepsilon\,[$ as a neighborhood of $\overline{\mu}$ in I, ε as in the lemma.

LEMMA 4.— (I. Malta and J. Palis [6]) *Given $\{X_\mu\} \in \Gamma^1_{sn}$ and $\overline{\mu} \in B\big(\{X_\mu\}\big)$ as above. Let $x_1, y_1 \in \mathcal{W}^c_{\overline{\mu}}(q)$ be near q such that*

$$x_1 < x^c(\overline{\mu}) < \mathcal{P}^c_{\overline{\mu}}(x_1) < q < y_1 < y^c(\overline{\mu}) < \mathcal{P}^c_{\overline{\mu}}(y_1)\,.$$

Let $\mathcal{Z}$ be a C^5 saddle-node vector field adapted to $\{\mathcal{P}^c_\mu\}$ defined in a neighborhood of $(q, \overline{\mu})$ in $\mathcal{W}^c(q)$. Then there exists a local conjugacy, $H = (h, \eta)$, between $\{\mathcal{P}^c_\mu\}$ and $\mathcal{Z}_{t=1}$ such that:

(a) $h_\mu\big(x^c(\mu)\big) = x^c(\mu)$, $h_\mu\big(y^c(\mu)\big) = y^c(\mu)$;

(b) $(\mathcal{P}^c_\mu)^n\big(x_c(\mu)\big) = y^c(\mu)$ if and only if $\mathcal{Z}^n_1\big(x^c(\mu)\,,\, \eta(\mu)\big) = y^c(\mu)$, where $x^c(\mu) = \pi^{ss}\big(x(\mu)\big)$, $y^c(\mu) = \pi^{uu}\big(y(\mu)\big)$ and $x(\mu)$ (resp. $y(\mu)$) is the point of s-criticality (resp. u-criticality) between $\mathcal{W}^u(\alpha_\mu)$ and $\mathcal{F}^{ss}(q)$ (resp. $\mathcal{W}^s(\beta_\mu)$ and $\mathcal{F}^{uu}(q)$) in a fundamental domain $D^{cs}_\mu(q)$ (resp. $D^{cu}_\mu(q)$) for $\mathcal{P}/\mathcal{W}^{cs}_\mu(q)$ (resp. for $\mathcal{P}/\mathcal{W}^{cu}_\mu(q)$) for μ near $\overline{\mu}$.

For the definition of a saddle-node vector field adapted to saddle-node diffeomorphism arcs (see [8]).

Let $\{X_\mu\}$ $\{\widetilde{X}_\mu\} \in \Gamma^1_{sn}$ be close families, and $\overline{\mu}$, $\widetilde{\mu} \in I$ be their first bifurcation values. If we let $\mathcal{Z}$, $\widetilde{\mathcal{Z}}$ denote the C^5 saddle-node vector fields adapted to the saddle-node diffeomorphism arcs $\{\mathcal{P}^c_\mu\}$ and $\{\widetilde{\mathcal{P}}^c_\mu\}$ respectively, then:

(a) there exists a conjugacy $h^c_2 : \mathcal{W}^c(q) \to \mathcal{W}^c(\widetilde{q})$ between $\mathcal{Z}$ and $\widetilde{\mathcal{Z}}$; this conjugacy induces a conjugacy between $\mathcal{Z}_{t=1}$ and $\widetilde{\mathcal{Z}}_{t=1}$, which we will denote by $h_2 = (h_{2,\mu}, k)$ where $k : (I_1, \overline{\mu}) \to (\widetilde{I}_1, \widetilde{\mu})$ is a reparametrization and $h_{2,\mu} : \mathcal{W}^c_\mu(q) \to \mathcal{W}^c_{k(\mu)}(\widetilde{q})$ is a conjugacy between $\mathcal{Z}_{t=1}/\mathcal{W}^c_\mu(q)$ and $\widetilde{\mathcal{Z}}_{t=1}/\mathcal{W}^c_{k(\mu)}(\widetilde{q})$ (see [8]);

(b) there are conjugacies:

- $h_1 : \mathcal{W}^c(q) \to \mathcal{W}^c(q)$ between $\{\mathcal{P}^c_\mu\}$ and $\mathcal{Z}_{t=1}$,

$$h_1 = (h_{1,\mu}, \eta)\,, \quad \eta : (I_1, \overline{\mu}) \to (I_1, \widetilde{\mu})$$

reparametrization;

- $h_3 : \mathcal{W}^c(\widetilde{q}) \to \mathcal{W}^c(\widetilde{q})$ between $\{\widetilde{\mathcal{P}}^c_\mu\}$ and $\widetilde{\mathcal{Z}}_{t=1}$,

$$h_3 = (h_{3,\mu}, \widetilde{\eta})\,, \quad \widetilde{\eta} : (\widetilde{I}_1, \widetilde{\mu}) \to (\widetilde{I}_1, \widetilde{\mu})$$

reparametrization.

Thus $h^c : \mathcal{W}^c(q) \to \mathcal{W}^c(\widetilde{q})$ given by

$$h^c(x, \mu) = \left(h^{-1}_{3,\mu} \circ h_{2,\mu} \circ h_{1,\mu}(x)\,,\, \widetilde{\eta}^{-1} \circ k \circ \eta(\mu)\right)$$

is a conjugacy between $\{\mathcal{P}^c_\mu\}$ and $\{\widetilde{\mathcal{P}}^c_\mu\}$ such that the reparametrization $\rho = \widetilde{\eta}^{-1} \circ k \circ \eta$ sends the parameter values $\overline{\mu}_n$, for which there exists a quasi-transversal intersection orbit between $\mathcal{W}^u(\alpha_{\overline{\mu}_n})$ and $\mathcal{W}^s(\beta_{\overline{\mu}_n})$, into the corresponding parameter values $\widetilde{\mu}_{m(n)}$ for which there exists a quasi-transversal intersection orbit between $\mathcal{W}^u\big(\widetilde{\alpha}_{\widetilde{\mu}_{m(n)}}\big)$ and $\mathcal{W}^s\big(\widetilde{\beta}_{\widetilde{\mu}_{m(n)}}\big)$.

Construction of equivalences in $\mathcal{W}^{cs}$ ($\mathcal{W}^{cu}$)

Let $\alpha = \alpha_{\overline{\mu}}$ be a singularity (hyperbolic) of $X_{\overline{\mu}}$ such that $\mathcal{W}^u(\alpha)$ has a generic s-criticality with $\mathcal{F}^{ss}(q)/\mathcal{W}^s(q)$ in a fundamental domain $D^{cs}_{\overline{\mu}}(q)$ for $\mathcal{P}_{\overline{\mu}}/\mathcal{W}^{cs}_{\overline{\mu}}(q)$.

LEMMA 5. — *Under the above conditions, there exists a C^r $(1 \le r \le \infty)$ vector field Y defined in a neighborhood of $\mathcal{W}^u(\alpha) \times \{\overline{\mu}\} \cap \mathcal{W}^{cs}_{\overline{\mu}}(q)$ in $D^{cs}_{\overline{\mu}}(q)$ which satisfies:*

(a) *the only singularities of Y are the tangency points between $\mathcal{W}^u(\alpha)$ and $\mathcal{F}^{ss}(q)$ in $D^{cs}_{\overline{\mu}}(q)$;*

(b) *if the s-criticality between $\mathcal{W}^u(\alpha)$ and $\mathcal{F}^{ss}(q)$ is of codimension p $(p \ge 1)$, then Y is of the saddle-node type, where a center-manifold is the C^1 curve given by the tangencies among the leaves of $\mathcal{F}^u(\alpha)$ and $\mathcal{F}^{ss}(q)$;*

(c) *Y is transversal to the leaves of $\mathcal{F}^{ss}(q)$ except at the singularity;*

(d) *Y is tangent at the leaves of $\mathcal{F}^{cu}(\alpha)$ and $\mathcal{F}^{cu}_1(\alpha)$ in a neighborhood of the s-criticality.*

Proof. — We let p denote the codimension of the s- criticality.

Case 1. $p = 1$

Let $(y_1, \ldots, y_s, x)$ be a coordinate in $\mathcal{W}^{cs}_{\overline{\mu}}(q)$, where $(y_1, \ldots, y_s)$ are the coordinates in $\mathcal{W}^{ss}(q)$, and x is the coordinate in $\mathcal{W}^{c}_{\overline{\mu}}(q)$. In these coordinates, $\mathcal{F}^{ss}(q)$ is given by a C^r $(r \ge 2)$ projection $\pi^{ss} : \mathcal{W}^{cs}_{\overline{\mu}}(q) \to \mathcal{W}^{c}_{\overline{\mu}}(q)$,

$$\pi^{ss}(y_1, \ldots, y_s, x) = x + Q(y_1, \ldots, y_s, x)$$

where Q only has terms of degree ≥ 2. In $D^{cs}_{\overline{\mu}}(q)$, $\mathcal{W}^u(\alpha) \cap D^{cs}_{\overline{\mu}}(q)$ is given by

$$x = F(y)\,, \quad y = (y_1, \ldots, y_s)\,,$$

where F is a Morse function with a unique critical point x_0 which corresponds to the tangency between $\mathcal{W}^{cu}(\alpha) \cap D^{cs}_{\overline{\mu}}(q)$ and $\mathcal{F}^{ss}(q)/\mathcal{W}^{cs}_{\overline{\mu}}(q)$. Let γ be a C^1 curve given by the tangencies among the leaves of $\mathcal{F}^u(\alpha)$ and the ones of $\mathcal{F}^{ss}(q)/\mathcal{W}^{cs}_{\overline{\mu}}(q)$. Restricting π^{ss} to $\mathcal{W}^u(\alpha) \cap D^{cs}_{\overline{\mu}}(q)$, the tangency condition between $\mathcal{F}^u(\alpha)$ and $\mathcal{F}^{ss}(q)/\mathcal{W}^{cs}_{\overline{\mu}}(q)$ is given by the differential equation

$$\frac{\partial F}{\partial y} + \frac{\partial Q}{\partial y} + \frac{\partial Q}{\partial x} \cdot \frac{\partial F}{\partial y} = 0\,.$$

Therefore, in a neighborhood of $\mathcal{W}^u(\alpha) \cap D^{cs}_{\overline{\mu}}(q)$ in $\mathcal{W}^{cs}_{\overline{\mu}}(q)$, we consider the vector field Y:

$$Y : \begin{cases} \dot{x} = \big(x - F(y)\big)^2 + \dfrac{\partial F}{\partial y}\,\dot{y} \\ \dot{y}_i = \dfrac{\partial \pi^{ss}}{\partial x}\,(x, y) \cdot \dfrac{\partial F}{\partial y_i}\,(y) + \dfrac{\partial \pi^{ss}}{\partial y_i}\,(y, x)\,, \quad i = 1, \ldots, s. \end{cases}$$

The singularities of Y are given by $\overset{\circ}{y} = (\overset{\circ}{y}_1, \ldots, \overset{\circ}{y}_s)$ and $F(y) - x = 0$: that is the tangency point x_0. On the other hand, $Y/\big(\mathcal{W}^u(\alpha) \cap D^{cs}_{\overline{\mu}}(q)\big)$ has the form

$$\begin{cases} \dot{x} = \dfrac{\partial F}{\partial y}(y)\,\dot{y} \\ \dot{y}_i = \dfrac{\partial \pi^{ss}}{\partial x}\,(y, x)\dfrac{\partial F}{\partial y_i}\,(y) + \dfrac{\partial \pi^{ss}}{\partial y_i}\,(y, x)\,, \quad i = 1, \ldots, s. \end{cases}$$

Thus, Y is tangent to $\mathcal{W}^u(\alpha) \cap D^{cs}_{\overline{\mu}}(q)$ and x_0 is a hyperbolic singularity of $Y/\big(\mathcal{W}^u(\alpha) \cap D^{cs}_{\overline{\mu}}(q)\big)$. In addition

$$Y/\gamma\,(y, x) = \Big(\big(x - F(y)\big)^2, 0\Big)\ ;$$

thus x_0 is a saddle-node singularity for Y/γ.

We have

$$\begin{aligned} \frac{\partial \pi^{ss}}{\partial x}\,\dot{x} + \frac{\partial \pi^{ss}}{\partial y}\,\dot{y} &= \frac{\partial \pi^{ss}}{\partial x}\,\big(x - F(y)\big)^2 + \frac{\partial \pi^{ss}}{\partial x}\,\frac{\partial F^{ss}}{\partial y}\,\dot{y} + \frac{\partial \pi^{ss}}{\partial y}\,\dot{y} \\ &= \frac{\partial \pi^{ss}}{\partial x}\,\big(x - F(y)\big)^2 + \left(\frac{\partial \pi^{ss}}{\partial x}\,\frac{\partial F^{ss}}{\partial y} + \frac{\partial \pi^{ss}}{\partial y}\right)\dot{y} \\ &= \frac{\partial \pi^{ss}}{\partial x}\,\big(x - F(y)\big)^2 + \big\|\dot{y}\big\| > 0\,, \quad \text{if } x \neq x_0\,. \end{aligned}$$

Then Y is transversal to the leaves of $\mathcal{F}^{ss}(q)/\mathcal{W}^{cs}_{\overline{\mu}}(q)$.

Case 2. $1 < p \leq \dim \mathcal{W}^{cs}_{\overline{\mu}}(q)$

Let $\mathcal{W}^{cu}(\alpha)$ be a C^r $(r \geq 2)$ center-unstable manifold of α. Since $\mathcal{W}^u(\alpha)$ is a codimension 1 submanifold of $\mathcal{W}^{cu}(\alpha)$, by Case 1, there exists a C^1 vector field Y_1 defined in a neighborhood of $\mathcal{W}^u(\alpha) \cap D^{cs}_{\overline{\mu}}(q)$ in $\mathcal{W}^{cu}(\alpha) \cap D^{cs}_{\overline{\mu}}(q)$. Now as in [11], we extend Y_1 to a neighborhood of $\mathcal{W}^u(\alpha) \cap D^{cs}_{\overline{\mu}}(q)$ in $\mathcal{W}^{cs}_{\overline{\mu}}(q)$. The only condition we require for this extension is that the extended vector field be tangent to the leaves of $\mathcal{F}^{cu}(\alpha)$ and $\mathcal{F}_1^{cu}(\alpha)$.

As above: let $\{X_\mu\} \in \Gamma^1_{sn}$; let $\overline{\mu} \in B(\{X_\mu\})$; and let $\mathcal{P} = \{\mathcal{P}_\mu\}$. Let

$$x^c(\mu) \in (\mathcal{W}^c(q) \cap \Sigma_q \times \{\mu\})$$

vary continuously with $\mu \in I_1$ such that the continuous curve $\mu \to (x^c(\mu), \mu)$ is transversal to $\mathcal{W}^c_\mu(q)$. Let $A^s_\mu \subseteq \mathcal{F}^{ss}(x^c(\mu))$ be a continuous family of closed discs centered at $x^c(\mu)$. We now let $C \subseteq \mathcal{W}^{cs}_{\overline{\mu}}(q)$ be a transversal cylinder to $\mathcal{W}^{ss}(q)$ such that

$$C \times \{\mu\} \cap F^{ss}(x^c(\mu)) = \partial A^c_\mu$$

and, for each $\mu \in [\overline{\mu} - \varepsilon, \overline{\mu}[$, $C \times \{\mu\}$ is transversal to $\mathcal{W}^{ss}(q_{1,\mu})$ and to $\mathcal{W}^{ss}(q_{2,\mu})$, and $C \cap \mathcal{W}^{cu}(\alpha) = \emptyset$ for each critical element α of $X_{\overline{\mu}}$ such that $\mathcal{W}^u(\alpha) \cap \mathcal{W}^{ss}(q) = \emptyset$. Now we define a fundamental domain $D^{cs}_{\overline{\mu}}$ for $(\mathcal{P}_{\overline{\mu}}, \overline{\mu})$ in $\mathcal{W}^{cs}_{\overline{\mu}}(q)$ as: the external boundary of $D^{cs}_{\overline{\mu}}$ is $C \cup A^c_{\overline{\mu}}$ and the internal boundary is $\mathcal{P}_{\overline{\mu}}(C \cup A^c_{\overline{\mu}})$.

A fundamental domain for $\mathcal{P}/W^{cs}(q)$ is defined as: the external boundary is $A^s \cup C \times I_1$ and the internal boundary is $\mathcal{P}(A^s \cup C \times I_1)$, $A^s = \bigcup_{\mu \in I_1} A^s_\mu$.

If $\{\widetilde{X}_\mu\} \in \Gamma^1_{sn}$ is close to $\{X_\mu\}$, we assume the same hypotheses and constructions for $\{\widetilde{X}_\mu\}$ as have been made for $\{X_\mu\}$.

Construction of h^{cs} (h^{cu})

The construction of the homeomorphism h^{cs} (h^{cu}) is carried out in several steps which depend on the $s(u)$-criticality.

Step 1. Codimension one critical

We assume $\mathcal{W}^u(\alpha_{k-1,\overline{\mu}})$ has a codimension one s-criticality. Let $x_0(\mu) \in D^{cs}_\mu(q)$ be the point of s-criticality. Since $x_0(\overline{\mu})$ is generic, $\mathcal{W}^u(\alpha_{k-1,\overline{\mu}}) \cap D^{cs}_{\overline{\mu}}(q)$ may be expressed as the graph of Morse function

$$\mathcal{F}_{\overline{\mu}} : \mathcal{W}^{ss}(q) \to \mathcal{W}^c_{\overline{\mu}}(q)\,.$$

Case 1. $\mathcal{F}_{\overline{\mu}}$ has a minimum (or maximum) critical point at $x_0(\overline{\mu})$

Let $\{Y_{2,\mu}\}$ be family of C^1 vector fields defined in a neighborhood of

$$\bigcup_{\mu\in I_1} \left(\mathcal{W}^u(\alpha_{k-1,\mu}) \times \{\mu\}\right) \cap D^{cs}(q)$$

as in Lemma 5. We may assume the continuous curve $\mu \to (x_0(\mu), \mu)$ is transversal to $\mathcal{W}^{cs}_\mu(q)$. Since $\mathcal{W}^u(\alpha_{i,\mu})\times\{\mu\}$ is transversal to $\mathcal{F}^{ss}(q)/\mathcal{W}^{cs}_\mu(q)$, $i \neq k-1$, we may construct a family of C^1 vector fields $\{Y_{1,\mu}\}$, such that restricted to leaves of $\mathcal{F}^u(\alpha_i)$, $i \neq k-1$, it is also C^1, $Y_{1,\mu}$ has no singularities in $D^{cs}_\mu(q)$, its trajectories are transversal to the leaves of $\mathcal{F}^{ss}(q)/\mathcal{W}^{cs}_\mu(q)$, and the cylinder $C \times \{\mu\}$ is $Y_{1,\mu}$-invariant. We use the families $\{Y_{1,\mu}\}$ and $\{Y_{2,\mu}\}$, and construct a family of C^1 vector fields $\{Y_\mu\}$ such that, in a neighborhood of $\bigcup_{\mu\in I_1}\left(\mathcal{W}^u(\alpha_{k-1,\mu}) \times \{\mu\}\right) \cap D^{cs}(q)$ it coincides with $\{Y_{2,\mu}\}$, and on the outside it does so with $\{Y_{1,\mu}\}$.

Now let $h^c : \mathcal{W}^c(q) \to \mathcal{W}^c(\widetilde{q})$ be a conjugacy between $\{\mathcal{P}^c_\mu\}$ and $\{\widetilde{\mathcal{P}}^c_\mu\}$, and let $\rho(I_1, \overline{\mu}) \to (\widetilde{I}_1, \widetilde{\mu})$ be the reparametrization determined by h^c, and

$$H^s : \bigcup_{\mu\in I_1} \left(\bigcup_{i=1}^{k} \mathcal{W}^s(\alpha_{i,\mu}) \times \{\mu\}\right) \to \bigcup_{\mu\in \widetilde{I}_1} \left(\bigcup_{i=1}^{k} \mathcal{W}^s(\widetilde{\alpha}_{i,\mu}) \times \{\mu\}\right)$$

be the conjugacy constructed as before. We now consider a family $\{D_{1,\mu}\}$ (resp. $\{\widetilde{D}_{1,\mu}\}$) of s-dimensional discs $D_{1,\mu} \subseteq A^s_\mu$ (resp. $\widetilde{D}_{1,\mu} \subseteq \widetilde{A}^s_\mu$) which varies continuously with the parameter as in figure 3.1. For each $z \in D_{1,\mu}$, we have that the ω-limit of z is $x_0(\mu)$. Since $\{X_\mu\}$ and $\{\widetilde{X}_\mu\}$ are close, for each μ near $\overline{\mu}$, there exists a C^1 diffeomorphism

$$h_{1,\mu} : D_{1,\mu} \to \widetilde{D}_{1,\rho(\mu)}$$

C^1-close to the inclusion map which varies continuously with μ, and such that $h_{1,\mu}(\partial D_{1,\mu}) = \partial\widetilde{D}_{1,\rho(\mu)}$, and that $h_{1,\mu}(r_\mu) = \widetilde{r}_{\rho(\mu)}$ where $r_\mu = \mathcal{W}^c_{Y_\mu}(x_0(\mu)) \cap D_{1,\mu}$ (resp. $\widetilde{r}_\mu = \mathcal{W}^c_{\widetilde{Y}_\mu}(\widetilde{x}_0(\mu)) \cap \widetilde{D}_{1,\rho(\mu)}$).

Let $D_{2,\mu} = \overline{\bigcup_{t\geq 0} Y_{t,\mu}(D_{1,\mu})}$ and let $D_2 = \bigcup_{\mu\in I_1} D_{2,\mu} \times \{\mu\}$; similar definitions for $\widetilde{D}_{2,\mu}$ and $\widetilde{D}_2$. Thus, for each $z \in D_{2,\mu} \setminus \{x_0(\mu)\}$, there exists a unique $t(z) < 0$ such that $Y_{\mu,t(z)}(z) \in D_{1,\mu}$. Since we know

$$h_{1,\mu}\left(Y_{\mu,t(z)}(z)\right) \in \widetilde{D}_{1,\rho(\mu)}$$

and

$$h^c\big(\pi^{ss}(z)\big) \in \mathcal{W}^c(\widetilde{q}) \cap \Sigma_{\widetilde{q}} \times \{\rho(\mu)\},$$

we define

$$h_{1,\mu}(z) = \left(\bigcup_{t\geq 0} \widetilde{Y}_{\rho(\mu),t}\left(h_{1,\mu}\big(Y_{\mu,t(z)}(z)\big)\right)\right) \cap F^{ss}_{\widetilde{q}}\left(h^c\big(\pi^{ss}(z)\big)\right)$$

and

$$h_{1,\mu}\big(x_0(\mu)\big) = \widetilde{x}_0\big(\rho(\mu)\big).$$

This defines a homeomorphism $h_1 : D_2 \to \widetilde{D}_2$ compatible with h^c and the reparametrization ρ. Now, let $\{T_\mu\}$ be a continuous family of s-dimensional open discs centered at r_μ such that $T_\mu \supseteq D_{1,\mu}$ and that $T_\mu \setminus \overline{D}_{1,\mu}$ is an open annulus. On $\partial D_{1,\mu}$ we raise a radial one-dimensional foliation S^c_μ whose leaves are contained in $T_\mu \setminus D_{1,\mu}$. The construction is performed continuously with μ and in a compatible way with the leaves of $\mathcal{F}^u(\alpha_i)$, $i \neq k-1$. Given $x \in \partial D_{1,\mu}$, let

$$\overset{\circ}{S}{}^c_\mu(x) = S^c_\mu(x) \setminus \{x\},$$

and let

$$F^c(x) = \bigcup_{t\geq 0} Y_{\mu,t}\big(\overset{\circ}{S}{}^c_\mu(x)\big).$$

Clearly $F^c(x)$ is a two-dimensional surface. Let I^c be the one-dimensional foliation induced by $\mathcal{F}^c$ in a neighborhood of $i_\mu = \mathcal{W}^c_{Y_\mu}\big(x_0(\mu)\big) \cap \mathcal{P}(A^s_\mu)$; we extend I^c to a singular foliation by including the point i_μ. Using H^s, we extend h_1 to a homeomorphism

$$H_1 : T = \bigcup_{\mu\in I_1} T_\mu \times \{\mu\} \longrightarrow \widetilde{T} = \bigcup_{\mu\in\widetilde{I}_1} \widetilde{T}_\mu \times \{\mu\}$$

such that, for each $\mu \in I_1$, $H_{1,\mu} = H_1/T_\mu \times \{\mu\}$:

$$T_\mu \times \{\mu\} \longrightarrow \widetilde{T}_{\rho(\mu)} \times \{\rho(\mu)\}$$

is a diffeomorphism C^1-close to the inclusion map; this H_1 induces a homeomorphism $H_1 : \mathcal{P}(T \setminus D_1) \longrightarrow \widetilde{\mathcal{P}}(\widetilde{T} \setminus \widetilde{D}_1)$ where

$$D_1 = \bigcup_{\mu\in I_1} D_{1,\mu} \times \{\mu\}, \quad \widetilde{D}_1 = \bigcup_{\mu\in\widetilde{I}_1} \widetilde{D}_{1,\mu} \times \{\mu\},$$

such that, for each $\mu \in I_1$,

$$H_{1,\mu} : \mathcal{P}_\mu(T_\mu \setminus D_{1,\mu}) \longrightarrow \widetilde{\mathcal{P}}_{\rho(\mu)}\big(\widetilde{T}_{\rho(\mu)} \setminus \widetilde{D}_{1,\rho(\mu)}\big) \quad \text{and} \quad H_{1,\mu}(i_\mu) = \widetilde{\imath}_{\rho(\mu)} .$$

To extend H_1 to a homeomorphism

$$H_1 : \bigcup_{\mu \in I_1} \left(\bigcup_{t \geq 0} (Y_{\mu,t}(T_\mu \setminus D_{1,\mu}) \times \{\mu\}) \right) \longrightarrow \bigcup_{\mu \in \widetilde{I}_1} \left(\bigcup_{t \geq 0} (\widetilde{Y}_{\mu,t}(\widetilde{T}_\mu \setminus \widetilde{D}_{1,\mu}) \times \{\mu\}) \right) ,$$

we use the homeomorphisms h^c and H_1, the foliations $\mathcal{F}^{ss}(q)$ and $\mathcal{F}^{ss}(\widetilde{q})$, and the trajectories of Y_μ and $\widetilde{Y}_\mu$ (fig. 3.2). This defines a homeomorphism $H_1 : V \to \widetilde{V}$, where V (resp. $\widetilde{V}$) is a neighborhood of

$$\bigcup_{\mu \in I_1} (\mathcal{W}^u(\alpha_{k-1,\mu}) \times \{\mu\}) \cap D^{cs}(q)$$

(resp. $\bigcup_{\mu \in \widetilde{I}_1} (\mathcal{W}^u(\widetilde{\alpha}_{k-1,\mu}) \times \{\mu\}) \cap D^{cs}(\widetilde{q})$) in $D^{cs}(q)$ (resp. $D^{cs}(\widetilde{q})$), which satisfies:

- it is compatible with h^c, H^s and ρ,
- it conjugates $\mathcal{P}$ and $\widetilde{\mathcal{P}}$.

To obtain a homeomorphism $h^{cs} : D^{cs}(q) \to D^{cs}(\widetilde{q})$, we extend the definition of H_1 to $C \times I_1 \cup A^s$ using the leaves of $\mathcal{F}^{ss}(q)$ and of $\mathcal{F}^{ss}(\widetilde{q})$ as well as the trajectories of Y and of $\widetilde{Y}$, the definition of H^s and the Isotopy Extension Theorem as is done in [2]. Using H_1 we define a homeomorphism

$$H_1 : \mathcal{P}\big(C \times I_1 \cup A^s\big) \longrightarrow \widetilde{\mathcal{P}}\big(\widetilde{C} \times \widetilde{I}_1 \cup \widetilde{A}^s\big) .$$

Finally, we extend these homeomorphisms to D^{cs} using the trajectories of Y_μ and of $\widetilde{Y}_\mu$, the leaves of $\mathcal{F}^{ss}(q)$ and of $\mathcal{F}^{ss}(\widetilde{q})$, and the homeomorphisms h^c and the one defined in $\big(C \times I_1 \cup A^s\big) \cup \mathcal{P}\big(C \times I_1 \cup A^s\big)$. We denote by $h^{cs} : \mathcal{W}^{cs}(q) \to \mathcal{W}^{cs}(\widetilde{q})$ the resulting homeomorphism. By construction it is clear that h^{cs} verifies

- it is compablible with h^c, H^c and ρ,
- it sends leaves of $\mathcal{F}^{ss}$(q) into leaves of $\mathcal{F}^{ss}(\widetilde{q})$,
- $h^{cs}(x_0(\mu)) = \widetilde{x}(\rho(\mu))$, $h^{cs}_{\overline{\mu}}(q) = \widetilde{q}$, $h^{cs}_{\mu}(q_{j,\mu}) = \widetilde{q}_{j,\rho(\mu)}$, $j = 1, 2$.

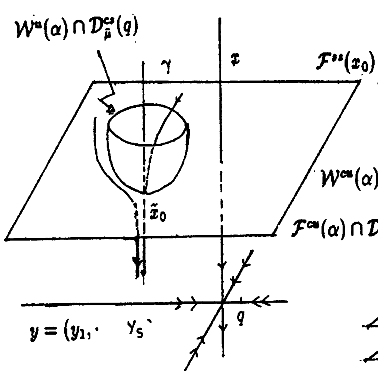

Fig. 3.1

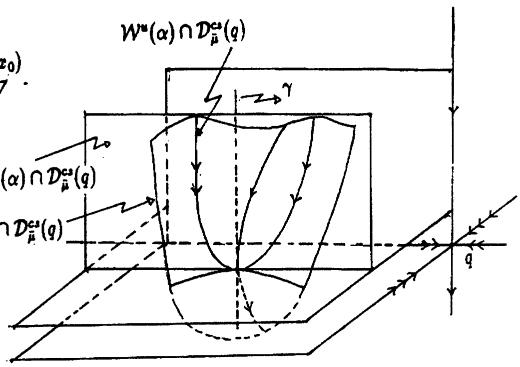

Fig. 3.2

Case 2. $\mathcal{F}_{\overline{\mu}}$ has a saddle type singularity at $x_0(\overline{\mu})$

In this case the family $\{Y_{2,\mu}\}$ of C^1 vector fields given by Lemma 5 has a center-unstable and a center-stable manifolds; we construct homeomorphisms on these manifolds as in Case 1. The extension of these homeomorphisms to $h^{cs} : \mathcal{W}^{cs}(q) \to \mathcal{W}^{cs}(\widetilde{q})$ is also done as in Case 1.

Step 2. Codimension $p > 1$ criticality

In this case, we take a center-unstable manifold $\mathcal{W}^{cu}(\alpha_{k-1,\overline{\mu}})$ and, in

$$\bigcup_{\mu \in I_1} (\mathcal{W}^{cu}(\alpha_{k-1,\mu}) \times \{\mu\}) \cap D^{cs}(q),$$

we construct a homeomorphism as in Step 1, the extension to a homeomorphism $h^{cs} : \mathcal{W}^{cs}(q) \to \mathcal{W}^{cs}(\widetilde{q})$ is done as above.

Global stability of families in Γ^1_{sn}

Let $\{X_\mu\} \in \Gamma^1_{sn}$ such that, in its first bifurcation value $\overline{\mu} \in I$, $X_{\overline{\mu}}$ has a saddle-node periodic orbit θ. In what follows we will suppose the critical elements of $\{X_\mu\}$ ordered as follows:

$$\begin{cases} \mu < \overline{\mu}: & \alpha_{1,\mu} \le \cdots \le \alpha_{k,\mu} \le \theta_{1,\mu} \le \theta_{1,\mu} \le \beta_{1,\mu} \le \cdots \le \beta_{\ell,\mu} \\ \mu = \overline{\mu}: & \alpha_{1,\overline{\mu}} \le \cdots \le \alpha_{k,\overline{\mu}} \le \theta \le \beta_{1,\overline{\mu}} \le \cdots \le \beta_{\ell,\overline{\mu}} \\ \mu > \overline{\mu}: & \alpha_{1,\mu} \le \cdots \le \alpha_{k,\mu} \le \beta_{1,\mu} \le \cdots \le \beta_{\ell,\mu} \end{cases}$$

where for $\mu < \overline{\mu}$, $\theta_{1,\mu}$ and $\theta_{2,\mu}$ are the periodic orbits of X_μ which produce the saddle-node periodic orbit θ.

Let $\{\widetilde{X}_\mu\} \in \Gamma^1_{sn}$ be close to $\{X_\mu\}$ which has its first bifurcation value $\widetilde{\mu}$ near $\overline{\mu}$ at which $\widetilde{X}_{\widetilde{\mu}}$ has a saddle-node periodic orbit $\widetilde{\theta}$ near θ. We let $\widetilde{\alpha}_{i,\mu}$, $\widetilde{\beta}_{j,\mu}$, $\widetilde{\theta}_{1,\mu}$, $\widetilde{\theta}_{2,\mu}$, $\widetilde{\theta}$, $i = 1, \ldots, k$, $j = 1, \ldots, \ell$, denote the critical elements of $\widetilde{X}_\mu$ near the respective critical elements of $\widetilde{X}_\mu$. As before, each time we make an assumption for $\{X_\mu\}$, we will suppose it done for $\{\widetilde{X}_\mu\}$.

The proof of the global stability of families in Γ^1_{sn} will be done in several steps according to the types of s-criticalities. We will prove the stability in the codimension one s-criticality, u-criticality case; the remaining cases may be proved with similar arguments.

In what follows we will assume that $\mathcal{W}^u(\alpha_{k-1,\overline{\mu}})$ as a codimension one s-criticality and that $\mathcal{W}^s(\beta_{2,\overline{\mu}})$ has a codimension one u-criticality. Let $x_0(\mu)$ (resp. $y_0(\mu)$) be the point of s-criticality (resp. u-criticality) in $D^{cs}_\mu(q)$ (resp. $D^{cu}_\mu(q)$), where $D^{cs}(q)$ (resp. $D^{cu}(q)$) is constructed in such a way that

$$\pi^{ss}\big(x_0(\mu)\big) = x_0^c(\mu) \quad (\text{resp. } \pi^{uu}\big(y_0(\mu)\big) = y_0^c(\mu))$$

is in the interior of $\mathcal{W}^c(q) \cap D^{cs}_\mu(q)$ (resp. $\mathcal{W}^c(q) \cap D^{cu}_\mu(q)$). In addition we suppose the continuous curves $\mu \to \big(x_0(\mu)\,,\,\mu\big)$ and $\mu \to \big(y_0(\mu)\,,\,\mu\big)$ are transversal to $\mathcal{W}^{cs}_\mu(q)$, $\mathcal{W}^{cu}_\mu(q)$, respectively. Recall that we already have the homeomorphisms

$$h^{cs} : \mathcal{W}^{cs}(q) \longrightarrow \mathcal{W}^{cs}(\widetilde{q}) \tag{1}$$

$$h^{cu} : \mathcal{W}^{cu}(q) \longrightarrow \mathcal{W}^{cu}(\widetilde{q}) \tag{2}$$

$$H^s : \bigcup_{\mu \in I_1} \left(\bigcup_{i=1}^{k} \mathcal{W}^s(\alpha_{i,\mu}) \times \{\mu\} \right) \longrightarrow \bigcup_{\mu \in \widetilde{I}_1} \left(\bigcup_{i=1}^{k} \mathcal{W}^s(\widetilde{\alpha}_{i,\mu}) \times \{\mu\} \right) \tag{3}$$

$$H^u : \bigcup_{\mu\in I_1} \left(\bigcup_{j=1}^{\ell} \mathcal{W}^u(\beta_{j,\mu}) \times \{\mu\} \right) \longrightarrow \bigcup_{\mu\in \widetilde{I}_1} \left(\bigcup_{j=1}^{\ell} \mathcal{W}^u(\widetilde{\beta}_{j,\mu}) \times \{\mu\} \right) \quad (4)$$

$$h^c : \mathcal{W}^c(q) \longrightarrow \mathcal{W}^c(\widetilde{q})\,. \quad (5)$$

Now, by the transversality of

(a) $\mathcal{W}^u(\alpha_i)$, $\mathcal{W}^s(\beta_j)$, $i = 1, \dots, k$, $j = 1, \dots, \ell$, with $\mathcal{W}^{cs}(q)$, $\mathcal{W}^{cu}(q)$, respectively,

(b) $\mathcal{W}^u(\alpha_i)$, $\mathcal{W}^s(\beta_j)$, $i \neq k-1$, $j \neq 2$, with $\mathcal{F}^{ss}(q)/\mathcal{W}^{cs}(q)$, $\mathcal{F}^{uu}(q)/\mathcal{W}^{cu}(q)$, respectively,

in $\Sigma_q \times I$, we may construct C^1, $\mathcal{P}$-invariant foliations $\mathcal{F}^{uu}(q)$ and $\mathcal{F}^{ss}(q)$ compatible with the systems of foliations $\mathcal{F}^u(\alpha_1), \dots, \mathcal{F}^u(\alpha_k)$, $\mathcal{F}^s(\beta_1), \dots, \mathcal{F}^s(\beta_\ell)$. Moreover, we may construct a codimension two C^1, $\mathcal{P}$-invariant foliation $\mathcal{F}^{su}(q)$ in $\Sigma_q \times I_1$ which is compatible with the above foliations. We let $\{Y_\mu^{cs}\}$ and $\{Y_\mu^{cu}\}$ denote the families of C^1 vector fields constructed in $\mathcal{W}^{cs}(q)$ and $\mathcal{W}^{cu}(q)$ as in Lemme 5, respectively. Using $\{Y_\mu^{cs}\}$ and $\{Y_\mu^{cu}\}$, we construct the singular foliations $\mathcal{F}^{cs}(q)$ and $\mathcal{F}^{cu}(q)$ in $\Sigma_q \times I_1$ as follows:

Each leaf F^{cs} of $\mathcal{F}^{cs}(q)$ is a union of leaves of $\mathcal{F}^{uu}(q)$, where the union is taken over a trajectory of Y_μ^{cs}. Note that each leaf of $\mathcal{F}^{cs}(q)$ intersects transversally $\mathcal{W}^{cu}(q)$ along a trajectory of $\{Y_\mu^{cs}\}$, and that $\mathcal{F}^{cs}(q)$ has distinguished leaves:

- one of them is $F^{cs}(x_0(\mu))$ which is raised over $\mathcal{W}^c_{Y_\mu^{cs}}(x_0(\mu))$;
- the others are those raised over the trajectories of

$$Y_\mu^{cs} / \left((\mathcal{W}^s(\alpha_{k-1,\mu}) \cap \Sigma_q) \times \{\mu\} \right) .$$

Analogously we construct $\mathcal{F}^{cu}(q)$. Without loss of generality, we will suppose that, for each $\mu > \overline{\mu}$,

$$L_\mu = F^{cs}(x_0(\mu)) \cap F^{cu}(y_0(\mu))$$

contains the tangencies between the leaves of $\mathcal{F}^u(\alpha_{k-1})$ and $\mathcal{F}^s(\beta_2)$. We set $L = \bigcup_{\mu\in I_1} L_\mu \times \{\mu\}$ ($\dim L = 2$).

In $\bigcup_{\mu\in I_1} \left(F^{cs}(x_0(\mu)) \times \{\mu\}\right)$ (resp. $\bigcup_{\mu\in I_1} \left(F^{cu}(y_0(\mu)) \times \{\mu\}\right)$), we consider the induced foliations:

- $\mathcal{F}^{uu} = \mathcal{F}^{su}(q) \cap F^{cs}(x_0(\mu))$ (resp. $\mathcal{F}^{ss} = \mathcal{F}^{su}(q) \cap F^{cu}(y_0(\mu))$) whose space of leaves is L;

- $\mathcal{F}_1 = \mathcal{F}^{cu}(q) \cap F^{cs}(x_0(\mu))$ (resp. $\mathcal{F}_2 = \mathcal{F}^{cs}(q) \cap F^{cu}(y_0(\mu))$),

and the homeomorphism $h^c_L : L \to \widetilde{L}$ induced by $h^c : \mathcal{W}^c(q) \to \mathcal{W}^c(\widetilde{q})$ via leaves of $\mathcal{F}^{su}(q)$. Note that, for each $\mu \in I_1$,

$$h^c_{L,\mu} = h^c_L / L_\mu \times \{\mu\} : L_\mu \times \{\mu\} \to \widetilde{L}_{\rho(\mu)} \times \{\rho(\mu)\}.$$

We now use the homeomorphisms $h^c_L : L \to \widetilde{L}$, $h^{cs} : \mathcal{W}^{cs}(q) \to \mathcal{W}^{cs}(\widetilde{q})$, and $h^{cu} : \mathcal{W}^{cu}(q) \to \mathcal{W}^{cu}(\widetilde{q})$ to construct homeomorphisms

$$H_1 : \bigcup_{\mu \in I_1} \left(F^{cs}(x_0(\mu)) \times \{\mu\}\right) \longrightarrow \bigcup_{\mu \in \widetilde{I}_1} \left(F^{cs}(\widetilde{x}_0(\mu)) \times \{\mu\}\right)$$

and

$$H_2 : \bigcup_{\mu \in I_1} \left(F^{cu}(y_0(\mu)) \times \{\mu\}\right) \longrightarrow \bigcup_{\mu \in \widetilde{I}_1} \left(\widetilde{F}^{cu}(\widetilde{y}_0(\mu)) \times \{\mu\}\right)$$

such that:

- H_1 and H_2 are compatible with h^c_L;
- for each $\mu \in I_1$,

$$H_{1,\mu} = H_1 / F^{cs}(x_0(\mu)) \times \{\mu\} : \\ F^{cs}(x_0(\mu)) \times \{\mu\} \longrightarrow \widetilde{F}^{cs}\left(\widetilde{x}_0(\rho(\mu))\right) \times \{\rho(\mu)\}$$

  and

$$H_{2,\mu} = H_2 / F^{cu}(y_0(\mu)) \times \{\mu\} : \\ F^{cu}(y_0(\mu)) \times \{\mu\} \longrightarrow \widetilde{F}^{cu}\left(\widetilde{y}_0(\rho(\mu))\right) \times \{\rho(\mu)\};$$

- H_1 (resp. H_2) sends leaves of $\mathcal{F}^{uu}$ and $\mathcal{F}_1$ (resp. $\mathcal{F}^{ss}$ and $\mathcal{F}_2$) into the respective ones of $\widetilde{\mathcal{F}}^{uu}$ and $\widetilde{\mathcal{F}}_1$ (resp. $\widetilde{\mathcal{F}}^{ss}$ and $\widetilde{\mathcal{F}}_2$); clearly, this construction is compatible with the ones above.

To construct the homeomorphism $H_q : \Sigma_q \times I_1 \to \Sigma_{\widetilde{q}} \times \widetilde{I}_1$ which conjugates the Poincaré maps $\mathcal{P}$ and $\widetilde{\mathcal{P}}$, we use the homeomorphisms h^{cs}, h^{cu}, H_1, H_2, h, h^c_L, H^s, H^u and the foliations $\mathcal{F}^{ss}(q)$, $\mathcal{F}^{uu}(q)$, $\mathcal{F}^{su}(q)$, $\mathcal{F}^{cs}(q)$, $\mathcal{F}^{cu}(q)$ for $\{X_\mu\}$, and the respective ones for $\{\widetilde{X}_\mu\}$. The construction is carried out preserving the later foliations (see [8]). Finally, we extend the homeomorphism H^s to neighborhoods of the critical elements $\beta_1, \beta_2, \ldots, \beta_\ell$ thus obtaining a global equivalence $H = (H_\mu, \rho) : M \times I_1 \to M \times \widetilde{I}_1$ between $\{X_\mu\}$, and $\{\widetilde{X}_\mu\}$.

Remark 1.— Since $\mathcal{W}^s(\beta_{2,\overline{\mu}})$ is u-critical of codimension one, we have that $\dim \mathcal{W}^u(\beta_{2,\overline{\mu}}) = 1$; therefore, to extend the definition of the homeomorphism H^s to a neighborhood of β_2 via Lyapunov functions, the preservation of $\mathcal{F}^s(\beta_2)$ is not required.

Remark 2.— The construction in

$$\bigcup_{\mu \in I_1} \left(\left(F^{cs}(x_0(\mu)) \cup F^{cu}(y_0(\mu))\right) \times \{\mu\}\right)$$

is essential to guarantee that, for each $\overline{\mu}_n \in I_1$ given by Lemma 3, we are sending the orbits of quasi-transversal intersection between

$$\left(\mathcal{W}^u(\alpha_{k-1,\overline{\mu}_n}) \cap \Sigma_q\right) \times \{\overline{\mu}_n\} \quad \text{and} \quad \left(\mathcal{W}^s(\beta_{2,\overline{\mu}_n}) \cap \Sigma_q\right) \times \{\overline{\mu}_n\}$$

for $X_{\overline{\mu}_n}$ into the respective orbits of quasi-transversal intersection between

$$\left(\mathcal{W}^u(\widetilde{\alpha}_{k-1,\rho(\overline{\mu}_n)}) \cap \Sigma_{\widetilde{q}}\right) \times \{\rho(\overline{\mu}_n)\} \quad \text{and} \quad \left(\mathcal{W}^s(\widetilde{\beta}_{2,\rho(\overline{\mu}_n)}) \cap \Sigma_{\widetilde{q}}\right) \times \{\rho(\overline{\mu}_n)\}$$

for $\widetilde{X}_{\rho(\overline{\mu}_n)}$.

Remark 3.— Only on the surface L, the foliations $\mathcal{F}^{cs}(q)$ and $\mathcal{F}^{cu}(q)$ are not transverse, that is L contains the tangency points (quasi-transversal) between

$$\bigcup_{\mu \in I_1} \left(\mathcal{W}^u(\alpha_{k-1,\mu}) \times \{\mu\}\right) \cap \Sigma_q \times I_1 \quad \text{and} \quad \bigcup_{\mu \in I_1} \left(\mathcal{W}^s(\beta_{2,\mu}) \times \{\mu\}\right) \cap \Sigma_q \times I_1$$

at the parameter values $\mu = \overline{\mu}_n$.

Acknowlegments

This work is part of my Doctoral Thesis at I.M.P.A. under the guidance of Professor J. Palis. I wish to thank him for his guidance and friendly atmosphere. Also, I whish to thank Professors M. J. Días Carneiro and R. Labarca for many helpful conversations.

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