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Porosity and continuous, nowhere differentiable functions(*)

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RÉSUMÉ. — Dans l'espace de Banach $C[0,1]$ des fonctions contenues sur $[0,1]$, les fonctions nulle part différentiables sont typiques non seulement dans le sens de la catégorie de Baire, mais aussi dans le sens plus restrictif, en utilisant la porosité. Les résultats sont ensuite généralisés, en remplaçant la différentiabilité par la différentiabilité approximative.

ABSTRACT. — In the Banach space $C[0,1]$ of all continuous functions on $[0,1]$, the nowhere differentiable functions are typical not only in the Baire category sense but also in a stronger sense, using porosity. The results also hold when differentiability is replaced with approximate differentiability.

1. Introduction

S. Banach [1] and S. Mazurkiewicz [6] showed that in the Banach space $C[0,1]$ of all continuous functions on $[0,1]$, endowed with the sup-norm, the set of nowhere differentiable functions is a residual (*i.e.* a complement of a meager set). The first example of a nowhere differentiable function was constructed by Weierstrass in 1873, but even recently, such functions with further properties are exhibited (M. Hata [4], [5]).

The notion of a set of σ-porosity, defined by L. Zajicek [11] in the general context of metric spaces, allows sometimes to improve results concerning meager sets.

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Let (X,d) be a metric space, A a subset of X and $x \in X$. For $R > 0$ we denote $B(x,R)$ the open ball with center x and radius R and define:

$$\gamma(x,R,A) = \sup\{r > 0 \mid B(z,r) \subseteq B(x,R) \setminus A \text{ for some } z \in X\},$$

$$p(x,A) = \limsup_{R \to 0^+} \gamma(x,R,A)/R\,.$$

We say that A is porous at x if $p(x,A) > 0$. It is easily seen that A is porous at x if and only if there exists $\rho > 0$ such that for each $\varepsilon > 0$, there exist R, $0 < R \leq \varepsilon$ and $z \in X$ with

$$B(z,\rho R) \subseteq B(x,R) \setminus A\,.$$

The set A is said to be porous if it is porous at all its points. Finally, A is said to be σ-porous if it is a union of countably many porous sets.

It is proved in [7] that any σ-porous set is of the first category (meager) and if X is an euclidean space, it has Lebesgue measure zero, but in any Banach space there exist meager sets which are not σ-porous.

Let $P(x)$ be a property for the points of X.

We say that P is generic with respect to porosity (or p-generic) if the set of the points that do not possess this property is σ-porous. In this case, we shall also say that a p-typical element of X has the property P.

2. Nowhere differentiable continuous functions

We need the following simple lemma, similar with that of Evans [2].

LEMMA 2.1.— *For each $a > 0$ there exists a function $u_a : \mathbb{R} \to \mathbb{R}$, $1/a$-Lipschitzian and having the period $2a$ such that:*

a) $0 \leq u_a(t) \leq 1 = u_a(a)$, $t \in \mathbb{R}$;

b) for each $t \in \mathbb{R}$, there exists an interval $I_a(t) \subseteq [a/8\,,\,a]$, having the length $a/8$ so that for $h \in I_a(t)$:

$$\left|u_a(t+h) - u_a(t)\right| \geq \frac{h}{3a}\,.$$

Proof.— Define

$$u_a(t) = \begin{cases} \dfrac{t}{a} & \text{for } t \in [\,0\,,\,a) \\ \dfrac{2a-t}{a} & \text{for } t \in [\,a\,,\,2a) \end{cases}$$

and

$$u_a(t+2ka) = u_a(t)\,, \quad \text{for } t \in [\,0\,,\,2a)\,,\ k \in \mathbb{Z}\,.$$

Then u_a is evidently $2a$-periodic, $1/a$-Lipschitzian and satisfies a). It is sufficient to check b) for $t \in [\,0\,,\,2a)$.

If $t \in [\,0\,,\,3a/4) \cup [\,a\,,\,7a/4)$ take $I_a(t) = [\,a/8\,,\,a/4\,]$ and if $t \in [\,3a/4\,,\,a) \cup [\,7a/4\,,\,2a)$ take $I_a(t) = [\,7a/8\,,\,a\,]$.

The conclusion follows after some simple computations. □

THEOREM 2.2.— *In the Banach space $C[\,0\,,\,1\,]$ of all real continuous functions on $[\,0\,,\,1\,]$, endowed with the sup-norm, a p-typical function has at no point finite one-sided derivatives. Moreover, for such a function x, one has*

$$\limsup_{h\to 0^+} \frac{\left|x(t+h)-x(t)\right|}{h} = +\infty\,, \quad t \in [\,0\,,\,1)$$

$$\limsup_{h\to 0^+} \frac{\left|x(t-h)-x(t)\right|}{h} = +\infty\,, \quad t \in (\,0\,,\,1\,]\,.$$

Proof.— For $n \in \mathbb{N}$, $n \geq 2$, denote

$$A_n = \left\{ x \in C[\,0\,,\,1\,] \,\middle|\, \exists\, t \in \left[0\,,\,1-\frac{1}{n}\right], \right.$$
$$\left. \forall\, h \in \left(0\,,\,\frac{1}{n}\right],\ \left|x(t+h)-x(t)\right| \leq nh \right\}.$$

It is clear that $\bigcup_{n=2}^{\infty} A_n$ contains all the functions $x \in C[\,0\,,\,1\,]$ for which there exists $t \in [\,0\,,\,1\,)$ such that

$$\limsup_{h\to 0^+} \frac{\left|x(t+h)-x(t)\right|}{h} < +\infty\,.$$

In particular, if x has a finite right derivative at a point $t \in [\,0\,,\,1\,)$ then $x \in \bigcup_{n=1}^{\infty} A_n$. The case of the left derivatives can be handled in a similar manner, or can be reduced to the previous one using the mapping $t \to 1-t$.

So, it is sufficient to prove that for a fixed $n \geq 2$, the set A_n is porous.

Let x be an arbitrary element in A_n and $\varepsilon > 0$. Using the uniform continuity of x, there exists $\delta > 0$ such that for t', $t'' \in [0, 1]$, $|t' - t''| \leq \delta$ we have

$$\left|x(t') - x(t'')\right| < \varepsilon .$$

Choose $a > 0$ such that $a < 1/n$, $a < \delta$, $a < \varepsilon/n$. For this $a > 0$, take u_a the function constructed in lemma 2.1, restricted to $[0, 1]$.

We have then $u_a \in C[0, 1]$, $||u_a|| = 1$ and

$$\left|u_a(t+h) - u_a(t)\right| \geq \frac{h}{3a}, \quad \text{for } t \in \left[0, 1 - \frac{1}{n}\right], \ h \in I_a(t),$$

where $I_a(t) \subset [a/8, a]$, length $I_a(t) = a/8$.

Denote $u = 75\,\varepsilon u_a$, $z = x + u$. We shall show that

$$B(z, \varepsilon) \cap A_n = \emptyset . \tag{$*$}$$

In fact, for $y \in B(z, \varepsilon)$, $t \in [0, 1 - 1/n]$ and $h \in I_a(t)$, we have:

$$\begin{aligned} \left|y(t+h) - y(t)\right| &\geq \left|u(t+h) - u(t)\right| - \left|(z-u)(t+h) - (z-u)(t)\right| + \\ &\quad - \left|(y-z)(t+h) - (y-z)(t)\right| \\ &\geq 75\,\varepsilon \frac{h}{3a} - \varepsilon - 2||y - z|| \\ &\geq h\left(\frac{25\,\varepsilon}{a} - \frac{24\,\varepsilon}{a}\right) = h\frac{\varepsilon}{a} > nh . \end{aligned}$$

Hence $y \notin A_n$ and $(*)$ holds.

Because $||x - z|| = 75\,\varepsilon$, we have

$$B(z, \varepsilon) \subseteq B(x, 76\,\varepsilon)$$

and so

$$B(z, \varepsilon) \subseteq B(x, 76\,\varepsilon) \setminus A_n \tag{$**$}$$

whence A_n is porous. □

Remark 2.3.— There are many different notions of porosity (*cf.* L. Zajicek [12]). For example, T. Zamfirescu [13] uses a slightly different one: the set A is said to be α-porous $(0 < \alpha < 1)$ if for each $x \in X$ and $\varepsilon > 0$ there exists $z \in B(x, \varepsilon)$ such that

$$B\big(z\,,\, \alpha d(x,z)\big) \cap A = \emptyset\,.$$

It follows from $(**)$ that A_n is 1/76-porous.

Let us mention that, the value of α in our setting is not significant for a σ-porous set, because a theorem of Zajicek [11] states that there exists a countable decomposition which has a uniform porosity arbitrarly closed to 1/2.

Note also that, as the referee pointed out, the proof of the theorem 2.2 also gives that the exceptional set is σ-very porous (in terminology of [12]).

The theorem 2.2 says nothing about the possibility that the typical function x has an infinite, even bilateral, derivative at some points. To treat this aspect, we need the following lemma.

LEMMA 2.4.— *For each $a > 0$, there exists a function $u_a : \mathbb{R} \to \mathbb{R}$, $1/a$-Lipschitzian and having the period $7a$ such that:*

a) $|u_a(t)| \leq 1 = |u_a(a)|$, $t \in \mathbb{R}$;

b) For each $t \in \mathbb{R}$, there exists and interval $I_a(t) \subset [\,a\,,\,6a\,]$, having the length $a/10$ such that for $h \in I_a(t)$:

$$u_a(t+h) - u_a(t-h) \geq \frac{h}{100\,a}\,.$$

Proof.— Define

$$u_a(t) = \begin{cases} -\dfrac{t}{a} & \text{for } t \in [\,0\,,\,a), \\ \dfrac{t-2a}{a} & \text{for } t \in [\,a\,,\,a3), \\ \dfrac{4a-t}{a} & \text{for } t \in [\,3a\,,\,4a), \\ 0 & \text{for } t \in [\,4a\,,\,7a) \end{cases}$$

and

$$u_a(t + 7\,ka) = u(t)\,, \quad \text{for } t \in [\,0\,,\,7a)\,,\ k \in \mathbb{Z}\,.$$

Then a) is obvious and to check b) it is suficient to consider $t \in [0, 7a)$ and take:

$$I_a(t) = \begin{cases} [2.5\,a\,,\,2.6\,a] & \text{for } t \in [0\,,\,a) \cup [3a\,,\,4a), \\ [1.5\,a\,,\,1.6\,a] & \text{for } t \in [a\,,\,3a), \\ [4.5\,a\,,\,4.6\,a] & \text{for } t \in [5a\,,\,6a), \\ [5.5\,a\,,\,5.6\,a] & \text{for } t \in [4a\,,\,5a) \cup [6a\,,\,7a). \ \square \end{cases}$$

THEOREM 2.5.— *In the Banach space $C[0\,,\,1]$, a p-typical function has at no point a finite or infinite two-sided derivative.*

Moreover, for such a function X, on has:

$$\limsup_{h\to 0^+} \frac{x(t+h)-x(t-h)}{2h} = \infty$$

$$\liminf_{h\to 0^+} \frac{x(t+h)-x(t-h)}{2h} = -\infty\,, \quad t \in (0,1)\,.$$

Proof.— For $n \in \mathbb{N}$, $n \geq 3$, denote

$$A_n = \Big\{ x \in C[0\,,\,1] \;\Big|\; \exists\, t \in \Big[\frac{1}{n}\,,\,1-\frac{1}{n}\Big]\,, \\ \forall\, h \in \Big(0\,,\,\frac{1}{n}\Big]\,,\; x(t+h)-x(t-h) \leq 2\,nh \Big\}$$

$$B_n = \Big\{ x \in C[0\,,\,1] \;\Big|\; \exists\, t \in \Big[\frac{1}{n}\,,\,1-\frac{1}{n}\Big]\,, \\ \forall\, h \in \Big(0\,,\,\frac{1}{n}\Big]\,,\; x(t+h)-x(t-h) \geq -2\,nh \Big\}\,.$$

The set $\bigcup_{n=3}^{\infty}(A_n \cup B_n)$ contains all the functions $x \in C[0\,,\,1]$ for which there exists $t \in (0,1)$ such that

$$\limsup_{h\to 0^+} \frac{x(t+h)-x(t-h)}{2h} < \infty$$

or

$$\liminf_{h\to 0^+} \frac{x(t+h)-x(t-h)}{2h} > -\infty\,.$$

It is sufficient to prove that for a fixed $n \geq 3$, the set A_n is porous (the case of B_n being similar).

Let x be an arbitrary element in A_n and $\varepsilon > 0$. If $\delta > 0$ is obtained from the uniform continuity of x, like in the proof of the theorem 2.2, choose $a > 0$ such that

$$6a < \frac{1}{n}, \quad 12a < \delta, \quad a < \frac{\varepsilon}{2n}.$$

Take u_a the function constructed in the lemma 2.4, restricted to $[0, 1]$,

$$u = 400\,\varepsilon u_a, \quad z = x + u.$$

We shall show that:

$$B(z, \varepsilon) \cap A_n = \emptyset.$$

In fact, for $y \in B(z, \varepsilon)$, $t \in [1/n, 1 - 1/n]$ and $h \in I_a(t)$, we have:

$$\begin{aligned} y(t+h) - y(t-h) &\geq u(t+h) - u(t-h) + \\ &\qquad - \big|(z-u)(t+h) - (z-u)(t-h)\big| \\ &\qquad - \big|(y-z)(t+h) - (y-z)(t-h)\big| \\ &\geq \frac{400\,\varepsilon h}{100a} - \varepsilon - 2\varepsilon \\ &\geq 2h\left(\frac{2\varepsilon}{a} - \frac{3\varepsilon}{2h}\right) \geq \frac{h\varepsilon}{a} > 2\,nh. \end{aligned}$$

Hence $y \notin A_n$ and because $||x - z|| = 400\,\varepsilon$, we obtain

$$B(z, \varepsilon) \subseteq B(x, 401\,\epsilon) \setminus A_n\,;$$

therefore A_n is porous. □

Remark 2.6.— The question if a p-typical function in $C[0, 1]$ has at no point a finite or infinite one-sided derivative, has a negative answer. In fact, even if such functions exist, the first example was constructed by Besicovitch (see also [3], [4]), they do not form a residual in $C[0, 1]$, *cf.* S. Saks [8], and *a fortiori* they are not p-typical.

Note however that J. Maly [5] constructed a nonvoid compact subset in $C[0, 1]$ where the functions with no (finite or infinite) one-sided derivatives form a residual. Maly's construction does not use the existence of the Besicovitch-type functions.

3. The case of approximate derivatives

The preceding theorems can be improved, replacing the differentiability, with approximate differentiability, as did Evans [2] in the case of Baire category results.

Recall that if μ denotes the Lebesgue measure in $\mathbb{R}$ and $A \subseteq \mathbb{R}$ is measurable then the right lower density of A at $t \in \mathbb{R}$ is defined (see [9], [10]) as:

$$\underline{D}^+(A,t) = \liminf_{h\to 0^+} \frac{\mu(A \cap [t, t+h])}{h}.$$

If x is a measurable function then:

$$\limsup_{t\to t_0^+} \operatorname{app} x(t) := \inf\left\{r \;\middle|\; \underline{D}^+\left(\{t \mid x(t) < r\}, t_0\right) = 1\right\}.$$

Similarly the approximate right lower limit is defined. Then the approximate right limit is the common value of these two extreme limits should they be the same. The approximate left limit and the approximate derivative are defined in the standard manner.

THEOREM 3.1.— *In the Banach space $C[0,1]$, a p-typical function has at no point a finite one-sided approximate derivative.*

Moreover, for such a function x, one has:

$$\limsup_{h\to 0^+} \operatorname{app} \frac{|x(t+h) - x(t)|}{h} = \infty, \quad t \in [0,1),$$

$$\liminf_{h\to 0^+} \operatorname{app} \frac{|x(t-h) - x(t)|}{h} = \infty, \quad t \in (0,1].$$

Proof.— For $n \in \mathbb{N}$, $n \geq 2$, denote for $7/8 < c < 1$,

$$A_n = \left\{x \in C[0,1] \;\middle|\; \exists\, t \in \left[0, 1 - \frac{1}{n}\right], \forall\, h \in \left(0, \frac{1}{n}\right], \right.$$
$$\left. \mu\left(\left\{s \in [0,h] \;\middle|\; |x(t+s) - x(t)| \leq ns\right\}\right) \geq ch\right\}.$$

If $x \in C[0,1]$ and $t \in [0,1)$ such that

$$\limsup_{h\to 0^+} \operatorname{app} \frac{|x(t+h)-x(t)|}{h} < \infty$$

then $x \in \bigcup_{n=2}^{\infty} A_n$

It is then sufficient to prove that for each $n \geq 2$ (fixed), the set A_n is porous.

Let x be an arbitrary element in A_n and $\varepsilon > 0$. Taking a, u_a, u, z as in the proof of the theorem 2.2, we have for $y \in B(z,\varepsilon)$, $t \in [0, 1-1/n]$, $s \in I_a(t)$:

$$|y(t+s)-y(t)| > ns\,.$$

Therefore

$$\mu\Big(\Big\{s \in (0,a] \;\Big|\; |y(t+s)-y(t)| > ns\Big\}\Big) \geq \mu\big(I_a(t)\big) = \frac{a}{8}$$

and we obtain

$$\mu\Big(\Big\{s \in (0,a] \;\Big|\; |y(t+s)-y(t)| \leq ns\Big\}\Big) \leq \frac{7a}{8} < ca$$

and so, $y \notin A_n$.

Hence, like in the proof of the theorem 2.2:

$$B(z,\varepsilon) \subset B(x, 76\,\varepsilon) \setminus A_n\,. \square$$

Remarks 3.2

a) Using a similar method, the theorem 2.5 may be improved, substituing the derivative with the approximate derivative and the extreme limits with the approximate extreme limits.

b) M. J. Evans [2] proved that in $C[0,1]$ the functions for which

$$\limsup_{h\to 0^+} \operatorname{app} \frac{|x(t+h)-x(t-h)-2x(t)|}{h} = \infty\,, \quad t \in (0,1)\,. \qquad (***)$$

are typical. This result can also be extended to a p-typical one, using the lemma in [2] and the above method.

Note that in [2] a continuous function satisfying (***) is first constructed.

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