ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 6e série, tome 1, nº 1 (1992), p. 53-94

<http://www.numdam.org/item?id=AFST_1992_6_1_1_53_0>

Annales de la Faculté des Sciences de Toulouse Série 6, Vol. I, n° 1, 1992

About the growth of entire functions solutions of linear algebraic q-difference equations

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Qu'un cul de dame damascène.

GUILLAUME APOLLINAIRE, La chanson du mal-aimé.

RÉSUMÉ. — Nous prouvons un q-analogue d'un résultat de Valiron [V1, 2]. Une fonction entière f, solution d'une équation différentielle linéaire algébrique $Df = b$, a une croissance exponentielle d'ordre fini $k > 0$ et de type fini. De plus les valeurs optimales de k sont des nombres rationnels appartenant à un ensemble fini explicitement calculable à partir du polygone de Newton de D. Ce resultat a d'abord été prouvé par Valiron (1926).

Notre principal résultat est un q-analogue (q étant un nombre complexe, avec $|q| > 1$) : une fonction entière f, solution d'une équation aux q-différences linéaire algébrique $Sf = b$, a une croissance q-exponentielle d'ordre fini $k > 0$ et de type fini. De plus les valeurs optimales de k sont des nombres rationnels appartenant à un ensemble fini explicitement calculable à partir du polygone de Newton de S.

Ce résultat se déduit facilement, par "dualité et résidus" d'un résultat récent de Bezivin sur des estimations q-Gevrey pour les solutions séries formelles d'équations aux q-différences linéaires analytiques. Mutatis mutandis la preuve est semblable à celle du théorème de Valiron que nous avons donnée dans [Ra1].

Nous en déduisons que si une fonction entière f est solution commune d'une équation différentielle linéaire algébrique et d'une équation aux q-différences linéaires algébrique (avec $|q| \neq 0, 1$), alors f est un polynôme. Plus généralement, si une série formelle $\widehat{f}$ est solution commune d'une équation différentielle linéaire algébrique et d'une équation aux q-différences linéaire algébrique (avec $|q| \neq 0, 1$, ou $|q| = 1$ et q transcendant), alors $\widehat{f}$ est le développement à l'origine d'une fraction rationnelle $f \in \mathbb{C}(x)$.

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ABSTRACT. — We prove a q-analog of a result of Valiron [V1, 2].

If f is an entire function solution of a linear algebraic differential equation $Df = b$, then f has an exponential growth of finite order $k > 0$ and of finite type. Moreover the optimal values of k are rational numbers belonging to a finite set explicitely computable from the Newton polygon of D. This result was first proved by Valiron (1926).

Our main result is the following q-analog (q being a complex number, with $|q| > 1$): if f is an entire function solution of a linear algebraic q-differential equation $Sf = b$, then f has a $|q|$-exponential growth of finite order $k > 0$ and of finite type. Moreover the optimal values of k are rational numbers belonging to a finite set explicitely computable from the Newton polygon of S.

This result follows easily by "duality and residues" from a recent theorem of Bezivin about q-Gevrey estimates of formal power series solutions of linear analytic q-difference equations. Mutatis mutandis the proof is similar to the proof of Valiron's theorem given in [Ra1].

As an application we prove that if an entire function f is a common solution of a linear algebraic differential equation *and* of a linear algebraic q-difference equation (with $|q| \neq 0, 1$), then f is a ***polynomial***. More generally, if a formal power series expansion $\hat{f}$ is a common solution of a linear algebraic differential equation and of a linear algebraic q-difference equation (with $|q| \neq 0, 1$, or with $|q| = 1$ and q transcendental), then $\hat{f}$ is the expansion at the origin of a rational function $f \in \mathbb{C}(x)$.

1. Preliminaries

In all this paper q is a non zero complex number, with $|q| \neq 1$; and we set $p = q^{-1}$.

We recall the notations (for $a \in \mathbb{C}$, $n \in \mathbb{N}$):

$$(a;q)_0 = 1\,,$$

and, if $n \in \mathbb{N}^*$:

$$(a;q)_n = \prod_{m=0}^{n-1} (1 - aq^m)$$

$$(q;q)_n = \prod_{m=1}^{n} (1 - q^m)$$

and, if $|q| < 1$:

$$(a;q)_\infty = \prod_{n=0}^{+\infty} (1 - aq^n)\,.$$

We have:

$$\left(q;q\right)_{\infty} = \prod_{n=1}^{+\infty} (1-q^n).$$

We set:

$$\left[n\right]_q = \frac{q^n - 1}{q - 1}$$

$$\left[n\right]_q! = \prod_{k=1}^{n} \left[k\right]_q.$$

We have:

$$\left[n\right]_q! = \frac{\left(q;q\right)_n}{(1-q)^n}.$$

We have the following identities:

$$(-1)^n \frac{q^{\frac{n(n-1)}{2}}}{\left(q;q\right)_n} = \frac{p^n}{\left(p;p\right)_n}$$

$$\frac{q^{\frac{n(n-1)}{2}}}{\left(q;q\right)_n}(1-q)^n = \frac{(1-p)^n}{\left(p;p\right)_n}$$

$$q^{-\frac{n(n-1)}{2}} \left[n\right]_q! = \left[n\right]_p!.$$

If $|q| > 1$, $\left[n\right]_q!$ is equivalent to

$$\left(p;p\right)_{\infty} q^{\frac{n(n+1)}{2}}(1-p)^{-n} = \left(p;p\right)_{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(1-q^{-1})^n}, \quad \text{when } n \to +\infty.$$

There are some analogies between q-difference equations on one side and difference or differential equations on the other side.

We set

$$\sigma_q f(x) = f(qx)$$
$$\delta_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$$

and we denote by $\mathbb{C}[x][\sigma_q]$ the $\mathbb{C}$-algebra of q-difference operators. We have the commutation relation

$$\sigma_q\, x = qx\, \sigma_q\,.$$

We set

$$Tf(x) = f(x+1)\,.$$

There are analogies between $\mathbb{C}[x][\sigma_q]$ and the algebras $\mathbb{C}[x]\left[{}^{\mathrm{d}}/_{\mathrm{d}x}\right]$ and $\mathbb{C}[x][T]$, with respectively commutation relations

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}, x\right] = 1\,,$$

and

$$[T, x] = T\,.$$

It is well known that it is possible to identify the study of linear q-difference equations with polynomial coefficients with the study of linear difference equations with coefficients polynomial in an exponential. We identify the algebra $\mathbb{C}[x][\sigma_q]$ and the algebra $\mathbb{C}[q^t][T]$ (with $Tf(t) = f(t+1)$), by the identification $x = q^t$. Then $T : t \mapsto t+1$ corresponds to $\sigma_q : x \mapsto qx$. The x-complex plane $\mathbb{C}^*$ without the origin is identified with the cylinder $\mathbb{C}/\tau^{-1}\mathbb{Z}$, where $\mathbb{C}$ is the t-complex plane and where τ is defined by $\tau = {}^{\mathrm{Log}\, q}/_{2i\pi}$.

So we identify

$$\mathbb{C}^*/\{q^n / n \in \mathbb{Z}\}$$

with the *elliptic curve*

$$\mathbb{C}/(\tau^{-1}\mathbb{Z} \oplus \mathbb{Z})\,.$$

We set $z = \tau t$. Then $x = e^{2i\pi z} = q^t = e^{t\,\mathrm{Log}\, q}$.

To the lattice $\tau^{-1}\mathbb{Z} \oplus \mathbb{Z}$ in the t-plane corresponds the lattice $\mathbb{Z} \oplus \tau\mathbb{Z}$ in the z-plane. Below we will relate some elliptic and θ functions associated to the periods 1 and τ and q-difference equations.

There is a *first* set of analogies between q-difference equations and difference or differential equations based upon the analogy between an *arithmetic* progression and a *geometric* progression:

$\alpha,\ \alpha+1,\ \ldots,\ \alpha+n$	$a,\ aq,\ \ldots,\ aq^n$
n	$a = q^n$
α	$a = q^\alpha$
$n!$	$q^{\frac{n(n+1)}{2}}$
T	σ_q
$x\dfrac{\mathrm{d}}{\mathrm{d}x}\, x^a = ax^a$	$\sigma_q\, x^\alpha = q^\alpha\, x^\alpha$
$x\dfrac{\mathrm{d}}{\mathrm{d}x}$	$\sigma_q\,.$

A *second* set of analogies between q-difference equations and differential equations is based upon the remark

$$\lim_{q\to 1} \frac{q^\alpha - 1}{q-1} = \alpha\,.$$

(Cf. Heine [He]):

n	$[n]_q = \dfrac{q^n-1}{q-1}$
α	$[\alpha]_q = \dfrac{q^\alpha-1}{q-1}$
$n!$	$[n]_q!$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\, x^\alpha = \alpha x^{\alpha-1}$	$\delta_q x^\alpha = [\alpha]_q x^{\alpha-1}$
$\dfrac{\mathrm{d}}{\mathrm{d}x}$	δ_q
$\delta = x\dfrac{\mathrm{d}}{\mathrm{d}x}$	$[\partial]_q = x\delta_q$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^n}{n!} = \dfrac{x^{n-1}}{(n-1)!}$	$\delta_q \dfrac{x^n}{[n]_q!} = \dfrac{x^{n-1}}{[n-1]_q!}\,.$

We have $T = e^{\mathrm{d}/\mathrm{d}x} - 1$ and $[\partial]_q = (q^\delta - 1)/(q-1)$.

2. Entire functions whose rate of growth is q-exponential of finite order

In all this section q is a real number, with $q > 1$.

LEMMA .— *Let k be a non zero real number. For an infinite sequence $\{a_n\}_{n\in\mathbb{N}}$ of complex numbers, the following conditions are equivalent:*

i) there exist real numbers C, $A > 0$, such that

$$|a_n| < Cq^{\frac{n(n+1)}{2k}} A^n \,;$$

ii) there exist real numbers C', $A' > 0$ such that

$$|a_n| < C' \left([n]_q!\right)^{\frac{1}{k}} A'^n \,;$$

iii) we set $\mu = {}^k/_{\mathrm{Log}\, q}$; there exist real numbers C'', $A'' > 0$, such that

$$|a_n| < C'' e^{\frac{n(n+1)}{2\mu}} A''^n \,;$$

iv) there exist real numbers C_0, A_0, such that

$$|a_n| < C_0 q^{\frac{n^2}{2k}} A_0^n \,;$$

v) we set $\mu = {}^k/_{\mathrm{Log}\, q}$; there exist real numbers C_0', A_0', such that

$$|a_n| < C_0' e^{\frac{n^2}{2\mu}} A_0'^n \,.$$

We remark that we get the same conditions for (q,k) and (q',k') if ${}^k/_{\mathrm{Log}\, q} = {}^{k'}/_{\mathrm{Log}\, q'}$.

DEFINITION .— *Let k be a non zero real number. A formal power series expansion $\widehat{f} = \sum_{n=0}^{+\infty} a_n x^n \in \mathbb{C}[[x]]$ is*

i) q-Gevrey of order $s = {}^1/_k$ (or of level k), if there exist real numbers C, $A > 0$, such that

$$|a_n| < C q^{\frac{n(n+1)}{2k}} A^n \,;$$

ii) q-Gevrey-Beurling of order $s = 1/k$ (or of level k), if for every real number $A > 0$, there exists a real number $C_A > 0$ such that

$$|a_n| < C_A\, q^{\frac{n(n+1)}{2k}}\, A^n .$$

For $k > 0$ this definition was introduced by J.-P. Bezivin [Be1]. (Be careful; what Bezivin denotes q is our p...)

If $\widehat{f} \in \mathbb{C}[[x]]$ is q-Gevrey (resp. q-Gevrey-Beurling) of order s, we will denote $\widehat{f} \in \mathbb{C}[[x]]_{q,s}$ (resp. $\widehat{f} \in \mathbb{C}[[x]]_{q,(s)}$). We will say that a convergent power series expansion (resp. an entire function) is q-Gevrey (resp. q-Gevrey-Beurling) of order $s = 0$, and will denote $\mathbb{C}[[x]]_{q,0} = \mathbb{C}\{x\}$ (resp. $\mathbb{C}[[x]]_{q,(0)} = \mathcal{O}(\mathbb{C})$. We will say that $\widehat{f}$ is *exactly* of order s, if $\widehat{f} \in \mathbb{C}[[x]]_{q,s}$, and if, for any $s' < s$, $\widehat{f} \notin \mathbb{C}[[x]]_{q,s'}$.

We remark that, if $s \operatorname{Log} q = s' \operatorname{Log} q'$, then $\mathbb{C}[[x]]_{q,s} = \mathbb{C}[[x]]_{q',s'}$.

DEFINITION . — *Let k be a non zero real number. Let q be a real number, with $|q| > 1$. An entire function f has a q-exponentiel growth of order k and a finite type, if there exist real numbers $K > 0$, α, such that*

$$\big|f(x)\big| < K\, q^{\frac{k}{2}\left(\frac{\operatorname{Log}|x|}{\operatorname{Log} q}\right)^2} |x|^{\alpha} = K\, q^{\left(\frac{\operatorname{Log}|x^k|}{\operatorname{Log} q}\right)^2} |x|^{\alpha} ,$$

or equivalently

$$\big|f(x)\big| < K\, q^{\frac{k \operatorname{Log}|x|(\operatorname{Log}|x| + 2\alpha \operatorname{Log} q)}{2(\operatorname{Log} q)^2}} ,$$

or equivalently

$$\big|f(x)\big| < K\, e^{\frac{k(\operatorname{Log}|x|)^2}{2 \operatorname{Log} q} + \alpha \operatorname{Log}|x|} = K\, e^{(\operatorname{Log}|x^{\mu}|)^2 + \alpha \operatorname{Log}|x|} .$$

PROPOSITION 2.1.— *Let k be a non zero real number. Let q be a real number, with $|q| > 1$. Let $\widehat{f} = \sum_{n=0}^{+\infty} a_n x^n$ be a formal power series expansion. Then the following conditions are equivalent:*

i) the series $\widehat{f}$ is q-Gevrey of order $-k$;

ii) the series $\widehat{f}$ is the power series expansion at the origin of an entire function f having a q-exponential growth of order k and a finite type.

Using this result we can found q-exponential growth estimates of entire functions if we know q-Gevrey estimates on their expansions at the origin.

Proposition 2.1 follows from

LEMMA 2.2. — *Let $k > 0$ be a real number. Let f be an entire function, with the expansion $\widehat{f} = \sum_{n=0}^{+\infty} a_n x^n$ at the origin. Then ($|x| = r$):*

i) if there exist real numbers $C > 0$, α such that

$$|a_n| \leq C\, e^{-\frac{(n-\alpha)^2}{2k}},$$

then:

a) for every $\lambda > 1$ there exists a real number $C_\lambda > 0$ such that

$$\left|f(x)\right| \leq C_\lambda\, e^{\frac{k}{2}(\operatorname{Log}\lambda r)^2 + \alpha \operatorname{Log}\lambda r};$$

b) for every $\epsilon > 0$ there exists a real number $K_\epsilon > 0$ such that

$$\left|f(x)\right| \leq K_\epsilon\, e^{\frac{k}{2}(\operatorname{Log} r)^2 + (\alpha+\epsilon)\operatorname{Log} r};$$

ii) if there exists real numbers $K > 0$, α such that

$$\left|f(x)\right| \leq K\, e^{\frac{k}{2}(\operatorname{Log} r)^2 + \alpha \operatorname{Log} r},$$

then

$$|a_n| \leq K\, e^{-\frac{(n-\alpha)^2}{2k}}.$$

If q is a given real number, with $q > 1$, we can reformulate this lemma with $k = 1/\operatorname{Log} q$, $e^{-\frac{(n-\alpha)^2}{2k}} = q^{-\frac{(n-\alpha)^2}{2}}$ and $e^{\frac{k}{2}(\operatorname{Log} r)^2 + \alpha \operatorname{Log} r} = q^{\frac{1}{2}\left(\frac{\operatorname{Log} r}{\operatorname{Log} q}\right)^2} r^\alpha$.

Example

Let q be a real number, with $q > 1$. We set

$$f(x) = \exp_q(x) = \sum_{n=0}^{+\infty} \frac{x^n}{[n]_q!}.$$

Then $a_n = {}^1/[n]_q!$ is equivalent to

$$\frac{1}{(p;p)_\infty} q^{-\frac{n(n-1)}{2}} (1-p)^n = \frac{1}{(p;p)_\infty} q^{-\frac{n^2}{2}+(-\frac{1}{2}+\frac{\operatorname{Log}(q-1)}{\operatorname{Log} q})n}$$

and, for every $\epsilon > 0$ there exists $K_\epsilon > 0$ such that

$$\left|\exp_q(x)\right| < K_\epsilon\, q^{-\frac{1}{2}+(\frac{\operatorname{Log} r}{\operatorname{Log} q})^2+(-\frac{1}{2}+\frac{\operatorname{Log}(q-1)}{\operatorname{Log} q}+\epsilon)\frac{\operatorname{Log} r}{\operatorname{Log} q}} .$$

We will improve this result later (cf. proposition 5.5 below).

It remains to prove lemma 2.2.

We will first prove that

$$\inf_{r>0} e^{\frac{k}{2}(\operatorname{Log} r)^2+(\alpha-n)\operatorname{Log} r} = e^{-\frac{(n-\alpha)^2}{2k}} .$$

We set $\varphi_n(r) = ({}^k/2)(\operatorname{Log} r)^2 + (\alpha - n)\operatorname{Log} r$. We have

$$\varphi_n'(r) = \frac{k \operatorname{Log} r + \alpha - n}{r} .$$

Then $\varphi_n'(r) = 0$ for $\operatorname{Log} r = -(\alpha - n)/_k$, $r = r_0 = e^{\frac{n-\alpha}{k}}$. We have

$$e^{\frac{k}{2}(\operatorname{Log} r_0)^2+(\alpha-n)\operatorname{Log} r_0} = e^{-\frac{(n-\alpha)^2}{2k}} .$$

Proof of ii)

We suppose that

$$\left|f(x)\right| \le K\, e^{\frac{k}{2}(\operatorname{Log} r)^2+\alpha \operatorname{Log} r} .$$

We set $M(f;r) = \sup_{|x|=r}\left|f(x)\right|$. Then, using Cauchy inequalities, we get

$$|a_n| \le \frac{M(f;r)}{r^n} = K\, e^{\frac{1}{2}(\operatorname{Log} r)^2+(\alpha-n)\operatorname{Log} r} ,$$

for every $r > 0$.

Then

$$|a_n| \le K \inf_{r>0} e^{\frac{k}{2}(\operatorname{Log} r)^2+(\alpha-n)\operatorname{Log} r} = K\, e^{-\frac{(n-\alpha)^2}{2k}} .$$

Proof of i)

We suppose that

$$|a_n| \leq C\, e^{-\frac{(n-\alpha)^2}{2k}}.$$

Then

$$|a_n x^n| \leq C\, e^{-\frac{(n-\alpha)^2}{2k}}\, r^n \leq C\, e^{\frac{k}{2}(\mathrm{Log}\,\rho)^2+(\alpha-n)\,\mathrm{Log}\,\rho+n\,\mathrm{Log}\,r},$$

for every $\rho > 0$.

We choose $\rho = \lambda r$, with $\lambda > 1$:

$$\begin{aligned}|a_n x^n| &\leq C\, e^{\frac{k}{2}(\mathrm{Log}\,\lambda r)^2+\alpha\,\mathrm{Log}\,\lambda r-n\,\mathrm{Log}\,\lambda}\\ &\leq C\, e^{\frac{k}{2}(\mathrm{Log}\,\lambda r)^2+\alpha\,\mathrm{Log}\,\lambda r}\,\frac{1}{\lambda^n}.\end{aligned}$$

We have

$$\big|f(x)\big| \leq \sum_{n=0}^{+\infty} |a_n x^n| \leq C\, e^{\frac{k}{2}(\mathrm{Log}\,\lambda r)^2+\alpha\,\mathrm{Log}\,\lambda r}\sum_{n=0}^{+\infty}\frac{1}{\lambda^n}$$

$$\big|f(x)\big| < C_\lambda\, e^{\frac{k}{2}(\mathrm{Log}\,\lambda r)^2+\alpha\,\mathrm{Log}\,\lambda r},$$

with $C_\lambda = C \sum_{n=0}^{+\infty} \frac{1}{\lambda_n}$.

Such results are well known (cf. [N]).

3. Newton polygon of a linear q-difference equation

In this section we recall (and complete) the definition of the Newton polygon of a linear q-difference equation (first introduced by Adams [A1]), and we give some useful notations.

We are studying linear analytic q-difference operators

$$S = a_m(x)(\sigma_q)^m + \cdots + a_i(x)(\sigma_q)^i + \cdots + a_0,$$

with $a_i \in \mathbb{C}\{x\}[x^{-1}]$. We set

$$a_i(x) = \sum_{j=j_0}^{+\infty} a_{i,j} x^j.$$

We denote by $\mathcal{J}_r$ the closed quadrants of $\mathbb{R}^2$ $(r = 1, 2, 3, 4)$:

$$\mathcal{J}_1 = \{(u, v) \in \mathbb{R}^2 \mid u\,,\ v \geq 0\}\,, \quad \mathcal{J}_2 = \{(u, v) \in \mathbb{R}^2 \mid u \leq 0\,,\ v \geq 0\}\,,$$

$$\mathcal{J}_3 = \{(u, v) \in \mathbb{R}^2 \mid u\,,\ v \leq 0\}\,, \quad \mathcal{J}_4 = \{(u, v) \in \mathbb{R}^2 \mid u \geq 0\,,\ v \leq 0\}\,.$$

For $(a, b) \in \mathbb{R}^2$, we set $\mathcal{J}_r(a, b) = (a, b) + \mathcal{J}_r$, and we denote by $M_r(S)$ the union of the quadrants $\mathcal{J}_r(i, j)$ for $i \in [\,0\,,\, m\,]$ and $j \in \mathbb{Z}$ such that $a_{i,j} \neq 0$. We denote by $P_r(S)$ the *convex hull* in $\mathbb{R}^2$ of $M_r(S)$, and we set:

$$N_l(S) = P_1(S) \cap P_2(S)\,,$$
$$N_u(S) = P_3(S) \cap P_4(S)\,,$$
$$N(S) = N_l(S) \cap N_l(S)\,,$$

$N_l(S)$ is the lower Newton polygon of S, $N_u(S)$ is the upper Newton polygon of S, and $N(S)$ is the Newton polygon of S.

If $a_i \in \mathbb{C}[\,x\,,\, x^{-1}\,]$ $(i = 1, \ldots, m)$, then the Newton polygon $N(S)$ is the convex hull in $\mathbb{R}^2$ of the set $\{(i, j) \in \mathbb{Z}^2 \mid a_{i,j} \neq 0\}$. If one of the a_i's $\notin \mathbb{C}[\,x\,,\, x^{-1}\,]$, then $N(S) = N_l(S)$.

We will introduce now some notations. The parameter k will take the following values: $k \in \mathbb{R}^*$, $k = \infty$, $k = 0_l$ or 0_u.

First we suppose that $|q| > 1$.

If $k \in \mathbb{R}^*$ and if the intersection of $N_l(S)$ and its contact line with slope k is reduced to only one point, we will denote by $\big(i(k), j(k)\big)$ the coordinate of this point. If $k \in \mathbb{R}^*$ and if the intersection of $N_l(S)$ and its contact line with slope k is not reduced to only one point, then this intersection is a segment $\big[\big(i_1(k), j_1(k)\big),\ \big(i_2(k), j_2(k)\big)\big]$ (with $\big(i_1(k), j_1(k)\big) \neq \big(i_2(k), j_2(k)\big)$, $i_1(k) < i_2(k)$). Then $j_1(k) < j_2(k)$ if $k > 0$ and $j_1(k) > j_2(k)$ if $k < 0$. In that case we will say that k is a lower exceptionnal value ($k \in \mathbb{Q}$). We will call the positive lower exceptional values the *lower irregular slopes* of $N(S)$, and the negative lower exceptional values the *lower regular slopes* of $N(S)$.

The horizontal side of $N_l(S)$ is denoted by

$$\big[\big(i_1(0_l), j_1(0_l)\big)\,,\ \big(i_2(0_l), j_2(0_l)\big)\big]$$

(with $i_1(0_l) < i_2(0_l)$, $j_1(0_l) = j_2(0_l)$).

The right vertical side of $N_l(S)$ is a segment or a half line, denoted by $\big[\big(i(\infty) = n, j_1(\infty)\big)\,,\ \big(i(\infty) = n, j_2(\infty)\big)\big]$ (with $j_1(\infty) < j_2(\infty)$).

We suppose now that $a_i \in \mathbb{C}[x, x^{-1}]$ $(i = 1, \ldots, m)$. If $k \in \mathbb{R}^*$ and if the intersection of $N_u(S)$ and its contact line with slope k is reduced to only one point, we will denote by $\big(i(k), j(k)\big)$ the coordinate of this point. If $k \in \mathbb{R}^*$ and if the intersection of $N_l(S)$ and its contact line with slope k is not reduced to only one point, then this intersection is a segment $\big[\big(i_1(k), j_1(k)\big)\,,\,\big(i_2(k), j_2(k)\big)\big]$ (with $\big(i_1(k), j_1(k)\big) \neq \big(i_2(k), j_2(k)\big)$, $i_1(k) > i_2(k)$). Then $j_1(k) < j_2(k)$ if $k < 0$ and $j_1(k) > j_2(k)$ if $k > 0$. In that case we will say that k is an upper exceptionnal value ($k \in \mathbb{Q}$). We will call the upper positive exceptional values the *upper irregular slopes* of $N(S)$, and the upper negative exceptional values the *upper regular slopes* of $N(S)$.

The horizontal side of $N_u(S)$ is denoted by

$$\big[\big(i_1(0_u), j_1(0_u)\big)\,,\,\big(i_2(0_u), j_2(0_u)\big)\big]$$

(with $i_1(0_l) > i_2(0_l)$).

Now we suppose that $|q| < 1$.

If $k \in \mathbb{R}^*$ and if the intersection of $N_l(S)$ and its contact line with slope $-k$ is reduced to only one point, we will denote by $\big(i(k), j(k)\big)$ the coordinate of this point. If $k \in \mathbb{R}^*$ and if the intersection of $N_l(S)$ and its contact line with slope $-k$ is not reduced to only one point, then this intersection is a segment $\big[\big(i_1(k), j_1(k)\big)\,,\,\big(i_2(k), j_2(k)\big)\big]$ (with $\big(i_1(k), j_1(k)\big) \neq \big(i_2(k), j_2(k)\big)$, $i_1(k) > i_2(k)$). Then $j_1(k) < j_2(k)$ if $k > 0$ and $j_1(k) > j_2(k)$ if $k < 0$. In that case we will say that k is a lower exceptionnal value ($k \in \mathbb{Q}$). We will call the positive lower exceptional values the *lower irregular slopes* of $N(S)$, and the negative lower exceptional values the *lower regular slopes* of $N(S)$.

The horizontal side of $N_l(S)$ is denoted by

$$\big[\big(i_1(0_l), j_1(0_l)\big)\,,\,\big(i_2(0_l), j_2(0_l)\big)\big]$$

(with $i_1(0_l) > i_2(0_l)$).

The left vertical side of $N_l(S)$ is a segment or a half line, denoted by $\big[\big(i(\infty), j_1(\infty)\big)\,,\,\big(i(\infty), j_2(\infty)\big)\big]$ (with $j_1(\infty) < j_2(\infty)$).

We suppose now that $a_i \in \mathbb{C}[\,x\,,\,x^{-1}\,]$ $(i = 1, \ldots, m)$. If $k \in \mathbb{R}^*$ and if the intersection of $N_u(S)$ and its contact line with slope $-k$ is reduced to only one point, we will denote by $\big(i(k), j(k)\big)$ the coordinate of this point. If $k \in \mathbb{R}^*$ and if the intersection of $N_l(S)$ and its contact line with slope $-k$ is not reduced to only one point, then this intersection is a segment $[(i_1(k),j_1(k))\,,\,(i_2(k),j_2(k))]$ (with $(i_1(k),j_1(k))\neq(i_2(k),j_2(k))$, $i_1(k)<i_2(k)$). Then $j_1(k)>j_2(k)$ if $k<0$ and $j_1(k)<j_2(k)$ if $k>0$. In that case we will say that k is an upper exceptionnal value ($k\in\mathbb{Q}$). We will call the upper positive exceptional values the *upper irregular slopes* of $N(S)$, and the upper negative exceptional values the *lower regular slopes* of $N(S)$ (be careful, in that case the corresponding sides of the Newton polygon have slopes *opposite* to these slopes!).

The horizontal side of $N_u(S)$ is denoted by

$$[(i_1(0_u),j_1(0_u))\,,\,(i_2(0_u),j_2(0_u))]$$

(with $i_1(0_l)>i_2(0_l)$).

If $N(S)$ has none lower (resp. upper) irregular slope, we will say that S is fuchsian (or regular singular) at zero (resp. infinity). If S is fuchsian at zero and infinity, we will say that it is Fuchsian.

4. Index theorems for linear analytic q-difference equations acting on entire functions spaces

In this section we prove some dualities between some spaces of q-Gevrey power series expansions (extending a classical result of Silva [Si], [G]). Then we get index theorems for spaces of entire functions with q-exponential growth from Bezivin's index theorems.

If

$$\widehat{f}(x)=\sum_{n=0}^{+\infty}a_nx^n\quad\text{and}\quad\widehat{g}(x)=\sum_{n=0}^{+\infty}b_nx^n$$

are formal power series expansions, we denote

$$\widehat{h}=\widehat{f}\circ\widehat{g}$$

their Hadamard product:

$$\widehat{h}(x)=\sum_{n=0}^{+\infty}a_nb_nx^n\,.$$

We set

$$\eta_{q,s}(x)=\sum_{n=0}^{+\infty}q^{s\frac{n(n+1)}{2}}\,x^n\,.$$

We have $\eta_{q,s}\circ\eta_{q,-s}=1$.

The map

$$\psi_{q,s} : \widehat{f} \to \eta_{q,-s} \circ \widehat{f}$$

$$\sum_{n=0}^{+\infty} a_n x^n \to \sum_{n=0}^{+\infty} \frac{a_n}{q^{s\frac{n(n+1)}{2}}} x^n$$

induces isomorphisms of complex vector spaces

$$\psi_{q,s} : \mathbb{C}[[x]]_{q,s} \to \mathbb{C}\{x\}$$

and

$$\psi_{q,s} : \mathbb{C}[[x]]_{q,(s)} \to \mathcal{O}(\mathbb{C}).$$

Using these isomorphism we transport the DFN topology of $\mathbb{C}\{x\}$ on $\mathbb{C}[[x]]_{q,s}$, and the FN topology of $\mathcal{O}(\mathbb{C})$ on $\mathbb{C}[[x]]_{q,(s)}$.

If

$$\widehat{f}(x) = \sum_{n=0}^{+\infty} a_n x^n \in \mathbb{C}[[x]] \quad \text{and} \quad \widehat{g}(x) = \sum_{n=1}^{+\infty} b_n x^{-n} \in x^{-1}\mathbb{C}[[x^{-1}]],$$

and if the series $\sum_{n=0}^{+\infty} a_n b_{n+1}$ converges, then we set

$$\langle \widehat{f}, \widehat{g} \rangle = \sum_{n=0}^{+\infty} a_n b_{n+1}.$$

We denote by $P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ the Riemann sphere. Let K be a closed disc centered at the origin in $\mathbb{C}$. We will set $U = P^1(\mathbb{C}) - K$. Let f be a function holomorphic on an open disc V centered at zero, containing K. We denote by $\widehat{f}(x) = \sum_{n=0}^{+\infty} a_n x^n$ its expansion at 0. Let g be a function holomorphic on the open disc U. We denote by $\widehat{g}(x) = \sum_{n=1}^{+\infty} b_n x^{-n}$ its expansion at infinity. If γ is a simple closed curve (positively oriented) in $V - K = U \cap V$, then (by Cauchy's residue formula) we get

$$\langle \widehat{f}, \widehat{g} \rangle = \frac{1}{2i\pi} \int_\gamma f(x) g(x)\, \mathrm{d}x.$$

So we get a pairing (that is a $\mathbb{C}$-bilinear map)

$$\mathbb{C}\{x\} \times x^{-1}\mathcal{O}\big(P^1(\mathbb{C}) - \{0\}\big) \to \mathbb{C}$$

$$(\widehat{f}, \widehat{g}) \to \langle \widehat{f}\,,\, \widehat{g}\rangle\,.$$

We have the classical result [G]:

PROPOSITIONS 4.1

i) If the vector space $\mathbb{C}\{x\}$ is endowed with its natural DFN topology, and the vector space $x^{-1}\mathcal{O}\big(P^1(\mathbb{C})-\{0\}\big)$ with its natural FN topology, then the pairing

$$\mathbb{C}\{x\} \times x^{-1}\mathcal{O}\big(P^1(\mathbb{C}) - \{0\}\big) \to \mathbb{C}$$

$$(\widehat{f}, \widehat{g}) \to \langle \widehat{f}\,,\, \widehat{g}\rangle$$

induces a topological duality.

ii) If the vector space $\mathbb{C}[[x]]$ is endowed with its natural FN topology (product topology), and the vector space $x^{-1}\mathbb{C}[x^{-1}]$ with its natural DFN topology (direct limit of the classical topology on finite dimensional subspace), then the pairing

$$\mathbb{C}[[x]] \times x^{-1}\mathbb{C}[x^{-1}] \to \mathbb{C}$$

$$(\widehat{f}, \widehat{g}) \to \langle \widehat{f}\,,\, \widehat{g}\rangle$$

induces a topological duality.

From this result we get easily

PROPOSITION 4.2.— *Let k be a real number, $k > 0$. Then :*

i) the pairing

$$\mathbb{C}[[x]]_{q;\frac{1}{k}} \times x^{-1}\mathbb{C}[[x^{-1}]]_{q;(-\frac{1}{k})} \to \mathbb{C}$$

$$(\widehat{f}, \widehat{g}) \to \langle \widehat{f}\,,\, \widehat{g}\rangle$$

induces a topological duality between the DFN-space $\mathbb{C}[[x]]_{q;\frac{1}{k}}$ and the FN-space $x^{-1}\mathbb{C}[[x^{-1}]]_{q;(-\frac{1}{k})}$;

ii) the pairing

$$\mathbb{C}[[x]]_{q;(\frac{1}{k})} \times x^{-1}\mathbb{C}[[x^{-1}]]_{q;-\frac{1}{k}} \to \mathbb{C}$$

$$(\hat{f},\hat{g}) \to < \hat{f},\hat{g} >$$

induces a topological duality between the FN-space $\mathbb{C}[[x]]_{q;(\frac{1}{k})}$ *and the DFN-space* $x^{-1}\mathbb{C}[[x^{-1}]]_{q;-\frac{1}{k}}$*;*

iii) the pairing

$$\mathbb{C}[[x]] \times x^{-1}\mathbb{C}[x^{-1}] \to \mathbb{C}$$

$$(\hat{f},\hat{g}) \to \langle \hat{f}, \hat{g}\rangle$$

induces a topological duality between the FN-space $\mathbb{C}[[x]]$ *and the DFN-space* $x^{-1}\mathbb{C}[x^{-1}]$.

LEMMA 4.3.— *For the pairing*

$$\mathbb{C}\{x\} \times x^{-1}\mathcal{O}\big(P^1(\mathbb{C}) - \{0\}\big) \to \mathbb{C}$$

$$(\hat{f},\hat{g}) \to \langle \hat{f}, \hat{g}\rangle = \sum_{n=0}^{+\infty} a_n b_{n+1}$$

$\left(\hat{f}(x) = \sum_{n=0}^{+\infty} a_n x^n ,\ \hat{g}(x) = \sum_{n=1}^{+\infty} b_n x^{-n}\right)$ *the adjoints of the operators* x *and* σ_q *are respectively the operators* x *and* $p\sigma_p$*. The adjoint of the operator* $x^j\sigma_q^i$ *(*$i, j \in \mathbb{Z}$*) is* $p^i\sigma_p^i x^j = p^{i+j}x^j\sigma_p^i$.

Using Newton polygons, we can reformulate Bezivin's index theorems [Be1]:

THEOREM 4.4.— *Let* k, q *be real numbers,* $k > 0$, $|q| \neq 0, 1$. *Let* $S = \sum_{i=0}^{n} a_i\sigma_q^i$, *with* $a_i \in \mathbb{C}\{x\}$ $(i = 1, \ldots, n)$. *Then:*

i) the operator

$$S : \mathbb{C}[[x]]_{q;\frac{1}{k}} \to \mathbb{C}[[x]]_{q;\frac{1}{k}}$$

is a Fredholm operator with index $\chi\left(S; \mathbb{C}[[x]]_{q;\frac{1}{k}}\right) = -j_1(k)$*;*

ii) *the operator*

$$S : \mathbb{C}[[x]]_{q;(\frac{1}{k})} \to \mathbb{C}[[x]]_{q;(\frac{1}{k})}$$

is a Fredholm operator with index $\chi\left(S; \mathbb{C}[[x]]_{q;(\frac{1}{k})}\right) = -j_2(k)$;

iii) *the operator*

$$S : \mathbb{C}[[x]] \to \mathbb{C}[[x]]$$

is a Fredholm operator with index $\chi\left(S; \mathbb{C}[[x]]\right) = -j_1(0_l)$.

Using duality (S^* operates on $\mathbb{C}[[x^{-1}]]$ as $\mathbb{C}[[x^{-1}]][x]/x\mathbb{C}[x]$, by definition) we get

COROLLARY 4.5.— *Let k, q be real numbers, $k < 0$, $|q| \neq 0, 1$. Let $S = \sum_{i=0}^{n} a_i \sigma_q^i$, with $a_i \in \mathbb{C}[x]$ $(i = 1, \ldots, n)$, and $S^* = \sum_{i=0}^{n} p^i \sigma_p^i a_i$ its adjoint. Then:*

i) *the operator*

$$S^* : \mathbb{C}[[x^{-1}]]_{q;\frac{1}{k}} \to \mathbb{C}[[x^{-1}]]_{q;\frac{1}{k}}$$

is a Fredholm operator with index

$$\chi\left(S^*; \mathbb{C}[[x^{-1}]]_{q;\frac{1}{k}}\right) = -\chi\left(S; \mathbb{C}[[x]]_{q;(-\frac{1}{k})}\right) = j_2(-k)\,;$$

ii) *the operator*

$$S^* : \mathbb{C}[[x^{-1}]]_{q;(\frac{1}{k})} \to \mathbb{C}[[x^{-1}]]_{q;(\frac{1}{k})}$$

is a Fredholm operator with index

$$\chi\left(S^*; \mathbb{C}[[x^{-1}]]_{q;(\frac{1}{k})}\right) = -\chi\left(S; \mathbb{C}[[x]]_{q;-\frac{1}{k}}\right) = j_1(-k)\,;$$

iii) *the operator*

$$S^* : \mathbb{C}[x^{-1}] \to \mathbb{C}[x^{-1}]$$

is a Fredholm operator with index

$$\chi(S^*; \mathbb{C}[x^{-1}]) = -\chi\left(S; \mathbb{C}[[x^{-1}]]\right) = j_1(0_l)\,.$$

If we replace x by $1/z$ and σ_q by σ'_p ($\sigma'_p g(z) = g(pz)$) in $S = \sum_{i=0}^n a_i(x)\sigma_q^i$, we get $S' = \sum_{i=0}^n a_i(z^{-1})\sigma'^i_p$. A monomial $x^j\sigma_q^i$ is replaced by $z^{-j}\sigma'^i_p$. Therefore the Newton polygons $N(S)$ and $N(S')$ are *symmetric* relatively to the horizontal axis. (Be careful: $N(S)$ is a q-polygon and $N(S^*)$ is a p-polygon.)

The adjoint of $S = \sum_{i=0}^n a_i\sigma_q^i$ (with $a_i \in \mathbb{C}\{x\}$ $(i = 1, \ldots, n)$) is $S^* = \sum_{i=0}^n p^i\sigma_p^i a_i$. Then $N(S) = N(S^*)$ (however $N(S)$ is a q-polygon and $N(S^*)$ is a p-polygon).

Therefore $N(S'^*)$ and $N(S)$ are *symmetric* relatively to the horizontal axis. (These two polygons are q-polygons.) Then the *lower irregular slopes* of $N(S'^*)$ correspond by symmetry (that is with the opposite sign) to the *upper regular slopes* of $N(S)$.

Now replacing x by z^{-1} and S by S'^* we apply 4.5 (as in [Ra1]). We get an extension of 4.4 to *negative* values of k ($\mathbb{C}[[x]]_{q;\frac{1}{(0,u)}} = \mathbb{C}[x]$, $\mathbb{C}[[x]]_{q;\frac{1}{(0,l)}} = \mathbb{C}[[x]]$, and $\mathbb{C}[[x]]_{q;\frac{1}{\infty}} = \mathbb{C}[[x]]_{q;0} = \mathbb{C}\{x\}$).

THEOREM 4.6.— *Let $q \in \mathbb{R}$, $|q| \neq 0, 1$. Let $k \in \mathbb{R}^* \cup \{(0,l), (0,u), \infty\}$. Let $S = \sum_{i=0}^{n} a_i\sigma_q^i$, with $a_i \in \mathbb{C}\{x\}$ $(i = 1, \ldots, n)$. Then:*

i) the operator

$$S : \mathbb{C}[[x]]_{q;\frac{1}{k}} \to \mathbb{C}[[x]]_{q;\frac{1}{k}}$$

is a Fredholm operator with index $\chi\left(S; \mathbb{C}[[x]]_{q;\frac{1}{k}}\right) = -j_1(k)$;

ii) the operator

$$S : \mathbb{C}[[x]]_{q;(\frac{1}{k})} \to \mathbb{C}[[x]]_{q;(\frac{1}{k})}$$

is a Fredholm operator with index $\chi\left(S; \mathbb{C}[[x]]_{q;(\frac{1}{k})}\right) = -j_2(k)$.

Using [Ra1, lemme 0.13, p. 5], we get

PROPOSITION 4.7.— *Let $q \in \mathbb{R}$, $|q| \neq 0, 1$. Let $k \in \mathbb{R}^* \cup \{(0_l), (0_u), \infty\}$. Let $S = \sum_{i=0}^{n} a_i\sigma_q^i$, with $a_i \in \mathbb{C}\{x\}$ $(i = 1, \ldots, n)$. Then:*

i) *the operator*

$$S : \mathbb{C}[x] \to \mathbb{C}[x]$$

is a Fredholm operator with index $\chi\big(S; \mathbb{C}[x]\big) = -j_1(0_u)$*;*

ii) *the operator*

$$S : \mathbb{C}[\![x]\!]_{q;\frac{1}{k}}/\mathbb{C}[x] \to \mathbb{C}[\![x]\!]_{q;\frac{1}{k}}/\mathbb{C}[x]$$

is onto and the complex dimension of its kernel is finite and equal to $\chi\Big(S; \mathbb{C}[\![x]\!]_{q;\frac{1}{k}}\Big) - \chi\big(S; \mathbb{C}[x]\big) = j_1(0_u) - j_1(k)$*;*

iii) *the operator*

$$S : \mathbb{C}[\![x]\!]_{q;(\frac{1}{k})}/\mathbb{C}[x] \to \mathbb{C}[\![x]\!]_{q;(\frac{1}{k})}/\mathbb{C}[x]$$

is onto and the complex dimension of its kernel is finite and equal to $\chi\Big(S; \mathbb{C}[\![x]\!]_{q;(\frac{1}{k})}\Big) - \chi\big(S; \mathbb{C}[x]\big) = j_1(0_u) - j_2(k)$*.*

When k varies from 0_u to ∞ (in $\pm\mathbb{R}^*$) the dimension of the space $\operatorname{Ker}\Big(S : \mathbb{C}[\![x]\!]_{q;(\frac{1}{k})}/\mathbb{C}[x] \to \mathbb{C}[\![x]\!]_{q;(\frac{1}{k})}/\mathbb{C}[x]\Big)$ increases. It remains constant between critical values of k (that is regular upper slopes) and jumps for such critical values. Then mimicking a method of [Ra1] we get

THEOREM 4.8.— *Let q be a non zero complex number, with $|q| \neq 1$. Let f be an entire function satisfying a linear analytic q-difference equation*

$$S\,f(x) = a_m f(q^m x) + \cdots + a_0 f(x) = b\,,$$

($a_i, b \in \mathbb{C}[x]$). We denote by $k'_1, \ldots, k'_r$, the absolute values of the upper regular slopes of the Newton polygon $N(S)$. We set $q_0 = \operatorname{Sup}\big(|q|, |q^{-1}|\big)$. Then the entire function f is a polynomial or there exists $k > 0$ such that f has a q_0-exponential growth of exact order ${}^1/_k$ and of finite type: there exist real numbers K, $\alpha > 0$, such that

$$\big|f(x)\big| < K\, q_0^{\frac{k}{2}\left(\frac{\operatorname{Log}|x|}{\operatorname{Log} q_0}\right)^2} |x|^\alpha = K\, e^{\frac{k}{2\operatorname{Log} q_0}\operatorname{Log}^2|x|}\, |x|^\alpha\,,$$

and there exists no real $k' > 0$, $k' < k$, such that f has a q_0-exponential growth of order ${}^1/_{k'}$ and of finite type. Moreover k is equal to one of the k'_i's.

This result is in general *false* for solutions of *non-linear* analytic q-difference equations: the exponential e^x is a solution of the non-linear equation $f(2x) = \big(f(x)\big)^2$, and its growth is clearly not 2-exponential!

5. q-difference equations of order one, q-analogs of exponential function and Jacobi θ-functions

In this section we solve some elementary linear algebraic q-difference equations of order one, and introduce various q-analogs of exponential. Then, following an idea of G. W. Starcher [St](1), we get some relations with Jacobi θ-functions. From Jacobi triple product formula [J], we derive important asymptotic estimates. We will use notations of [GR, p. 9](2).

It is possible to read sections 6 and 7 without reading section 5.

An algebraic linear q-difference equation $f(qx) = R(x)f(x)$ (where $R \in \mathbb{C}(x)$ is a rational function) can be solved "formally" by formal infinite products:

$$f(x) = \frac{1}{\prod_{n=0}^{+\infty} R(q^n x)}$$

and

$$f(x) = \prod_{n=1}^{+\infty} R(p^n x)$$

If one of these products converges, we get an actual solution.

PROPOSITION 5.1.— *Let q be a non zero complex number, with $|q| \neq 1$ $(p = q^{-1})$. We consider the q-difference equation*

$$(1+x)f(qx) - f(x) = 0\,. \qquad (I_q)$$

i) This equation admits a unique formal power series solution $\widehat{E}_q$ such that $\widehat{E}_q(0) = 1$:

$$\widehat{E}_q(x) = \sum_{n=0}^{+\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} x^n\,.$$

(1) Be careful there is a small error in Starcher's formulae [St, p. 578]: replace q^n by q^{n^2} in the second member of the identity (4).

(2) Be careful there is a small difference with Hahn's notations : cf. [Ha, p. 342].

If $|q| < 1$ this series converges in all the complex plane and defines an entire function E_q. If $|q| > 1$ the radius of convergence of this series is $|q|$ and its sum defines a holomorphic function E_q in the open disc of convergence.

ii) If $|q| < 1$

$$E_q(x) = \sum_{n=0}^{+\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} x^n = \prod_{n=0}^{+\infty} (1 + q^n x) = (-x;q)_\infty \,.$$

The function E_q has zeros only at $-1, -p, \ldots, -p^n, \ldots$ (these zeroes are simple).

iii) If $|q| > 1$

$$E_q(x) = \sum_{n=0}^{+\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} x^n = \frac{1}{\prod_{n=1}^{+\infty} (1 + p^n x)} = \frac{1}{(-px;p)_\infty},$$

for $|x| > 1$. The infinite product $(-px;p)_\infty$ converges in all the complex plane and the function E_q admits a meromorphic extension to all $\mathbb{C}$ (that we denote always by E_q). The function E_q has no zeros and poles only at $-q, -q^2, \ldots, -q^n, \ldots$ (these poles are simple).

If f is a solution of the equation

$$(1 + x)f(qx) - f(x) = 0\,, \qquad (I_q)$$

then $g(x) = {}^1/_{f(x)}$ is a solution of the equation

$$g(qx) - (1 + x)g(x) = 0\,. \qquad (1_q)$$

If we replace x by px in (I_q), we get $f(px) - (1 + px)f(x) = 0$

If we replace x by x^{-1} in (I_q), we get $(1 + x)h(px) - xh(x) = 0$ $(h(x) = f(x^{-1}))$.

PROPOSITION 5.2.— *Let q be a non zero complex number, with $|q| \neq 1$ $(p = q^{-1})$. Consider the q-difference equation*

$$f(qx) - (1 - x)f(x) = 0\,. \qquad (1_q)$$

i) *This equation admits a unique formal power series solution* $\widehat{e}_q$ *such that* $\widehat{e}_q(0) = 1$*:*

$$\widehat{e}_q(x) = \sum_{n=0}^{+\infty} \frac{x^n}{(q;q)_n}.$$

If $|q| > 1$ *this series converges in all the complex plane and defines an entire function* e_q. *If* $|q| < 1$ *the radius of convergence of this series is one and its sum defines a holomorphic function* e_q *in the open unit disc.*

ii) *If* $|q| > 1$

$$e_q(x) = \sum_{n=0}^{+\infty} \frac{x^n}{(q;q)_n} = \prod_{n=1}^{+\infty}(1 - p^n x) = (px;p)_\infty .$$

The function e_q *has zeros only at* $q, q^2, \ldots, q^n, \ldots$ *(these zeroes are simple).*

iii) *If* $|q| < 1$

$$e_q(x) = \sum_{n=0}^{+\infty} \frac{x^n}{(q;q)_n} = \frac{1}{\prod_{n=0}^{+\infty}(1 - q^n x)} = \frac{1}{(x;q)_\infty},$$

for $|x| > 1$. *The infinite product* $(x;q)_\infty$ *converges in all the complex plane and* e_q *admits a meromorphic extension to all* $\mathbb{C}$ *(that we denote also* e_q*). The function* e_q *has no zeros and poles only at* $1, q, \ldots, q^n, \ldots$ *(these poles are simple).*

iv) *We have* $e_q(x)E_q(-x) = 1$, $e_p(px) = E_q(-x)$, *and* $e_q(x)\ e_p(px) = 1$.

We will denote

$$\exp_q(x) = e_q((1-q)x) = \sum_{n=0}^{+\infty} \frac{x^n}{[n]_q!}$$

and

$$\mathrm{Exp}_q(x) = E_q\big((1-q)x\big) = \sum_{n=0}^{+\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n}(1-q)^n x^n .$$

We have

$$\exp_q(x) = \mathrm{Exp}_p(x),$$

and $\exp_q$ is a solution of the q-difference equation

$$\delta_q f = f\,.$$

We have

$$\lim_{q\to 1} e_q(x) = \lim_{q\to 1} E_q(x) = e^x\,.$$

Using our *second* set of analogies we have got *two* q-analogs of the exponential function (two because the invariant, up to conjugation, of an homography with two distincts fixed points is a pair $(q,p)\ldots$). Later, using our *first* set of analogies we will get two new q-analogs of the exponential Θ_q ($|q|<1$) and Ξ_q; Θ_q is strongly related to Jacobi's θ_1 function and Ξ_q is "highly ramified".

If the function f is a meromorphic solution of

$$(1+x)f(qx) = f(x) \tag{1}$$

and the function g a meromorphic solution of

$$xg(qx) = (1+x)g(x)\,, \tag{2}$$

then the function $h = fg$ is clearly a meromorphic solution of the equation

$$(x\sigma_q - 1)h(x) = xh(qx) - h(x) = 0\,. \tag{0_q}$$

Let q be a complex number, with $|q|<1$. Using propositions 5.1 and 5.2 we verify that the entire function $f(x) = E_q(x) = \left(-x;q\right)_\infty$ is a solution of (1), and that the entire function $g(x) = e_p(-x^{-1}) = \left(-qx^{-1};q\right)_\infty$ is a solution of (2). Therefore the entire function

$$h(x) = f(x)g(x) = E_q(x)e_p(-x^{-1}) = \left(-x;q\right)_\infty\left(-qx^{-1};q\right)_\infty$$

is a solution of the q-difference equation (0_q).

If q is a complex number with $|q|>1$, the meromorphic function

$$E_q(x)e_p(-x^{-1}) = \frac{1}{\left(-px;p\right)_\infty\left(-x^{-1};p\right)_\infty}$$

is a solution of the q-difference equation (0_q).

PROPOSITION 5.3.— *Let q be a non zero complex number, with $|q| \neq 1$ ($p = q^{-1}$). Consider the q-difference equation*

$$(x\sigma_q - 1)h(x) = xh(qx) - h(x) = 0\,. \qquad (0_q)$$

i) This equation admits a unique formal Laurent series solution f_q such that $f_q(0) = 1$:

$$\widehat{f_q}(x) = \sum_{n=-\infty}^{+\infty} q^{\frac{n(n-1)}{2}} x^n\,.$$

ii) If $|q| < 1$, the Laurent series $\widehat{f_q}$ converges on $\mathbb{C}^*$. Its sum f_q is an holomorphic solution of (0_q) on $\mathbb{C}^*$.

iii) If $|q| < 1$, we have the Jacobi triple product formula

$$f_q(-x) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{\frac{n(n-1)}{2}} x^n = (q;q)_\infty (x;q)_\infty (qx^{-1};q)_\infty\,,$$

for $x \in \mathbb{C}^*$.

A nice application of Jacobi triple product formula [J] is a formula due to Euler ($|q| < 1$):

$$(q;q)_\infty = \prod_{n=1}^{+\infty}(1-q^n) \sum_{n=-\infty}^{+\infty} (-1)^n q^{\frac{n(3n-1)}{2}}\,.$$

(We change q in q^3 in the triple product formula and set $x = q$.)

Assertions i) and ii) are easy to prove. In order to establish iii) we will first prove a preliminary result [St, p. 578].

LEMMA 5.4.— *Let q be a non zero complex number, with $|q| < 1$. We have the identities*

i) $e_q(qx) = \dfrac{1}{(qx;q)_\infty} \dfrac{1}{\prod_{n=1}^{+\infty}(1-q^n x)} = \sum_{n=0}^{+\infty} \dfrac{q^{n^2}}{(q;q)_n} \dfrac{x^n}{(qx;q)_n}$, *for $x \in \mathbb{C}^*$;*

ii) $\dfrac{1}{(q;q)_\infty} = \sum_{n=0}^{+\infty} \dfrac{q^{n^2}}{(q;q)_n^2}$.

The second identity follows from the first one (setting $x = 1$). In order to prove i) we remark that the meromorphic function $f(x) = e_q(qx)$ is a solution of the q-difference equation

$$f(qx) = (1 - qx)f(x)\,.$$

It is meromorphic on $\mathbb{C}$ and holomorphic on $\mathbb{C} - \{p, p^2, \ldots, p^n, \ldots\}$, with $f(0) = 1$.

We search for a formal q-factorial series solution of this equation

$$\widehat{g}(x) = \sum_{n=0}^{+\infty} \frac{c_n x^n}{(qx;q)_n} = \sum_{n=0}^{+\infty} \frac{c_n x^n}{(1-qx)(1-q^2x)\cdots(1-q^nx)}\,.$$

We verify easily that there exists a unique such solution with $c_0 = 1$:

$$\widehat{g}(x) = \sum_{n=0}^{+\infty} \frac{q^{n^2}}{(q;q)_n} \frac{x^n}{(qx;q)_n}\,.$$

The series $\widehat{g}$ converges uniformly on every compact of the open set $\mathbb{C} - \{p, p^2, \ldots, p^n, \ldots\}$, and defines a solution g of the equation $g(qx) = (1 - qx)g(x)$ holomorphic on $\mathbb{C} - \{p, p^2, \ldots, p^n, \ldots\}$. This solution is holomorphic at the origin and we have $g(0) = 1$. But the function f has the same properties. Then using the unicity in proposition 5.2 we get $f = g$.

Now we can go back to the proof of assertion iii) of proposition 5.3. The function $h(x) = (-x;q)_\infty(-qx^{-1};q)_\infty$ is a solution of the q-difference equation

$$xh(qx) = h(x) \qquad (0_q)$$

holomorphic on $\mathbb{C}^*$. From "unicity" of Laurent expansions at the origin of solutions of (0_q) ($f_q(0) = 1$), we get $h(x) = h(0)f_q(x)$. It remains only to prove that $h(0) = {}^1/(q;q)_\infty$.

We have the identities

$$E_q(x) = (-x;q)_\infty = \prod_{n=0}^{+\infty}(1 + q^n x) = \sum_{n=0}^{+\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} x^n$$

$$E_q(qx^{-1}) = \prod_{n=1}^{+\infty}(1 + q^n x^{-1}) = (-qx^{-1};q)_\infty = \sum_{n=0}^{+\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n} x^{-n}\,.$$

Then $f_q(0)$ is the sum of the products of terms involving x^n in the first expansion by the terms involving x^{-n} in the second expansion:

$$f_q(0) = \sum_{n=0}^{+\infty} \frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty}.$$

Finally in order to get the triple product formula, we have only to change x in $-x$.

When $|q| < 1$, the Laurent series

$$\Theta_q(x) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{\frac{n(n+1)}{2}} x^{n+1} = -f_q(-x)$$

appearing (up to the sign) in the triple product formula is simply related to Jacobi θ function. It is a solution of the q-difference equation

$$(x\sigma_q + 1)f = xf(qx) + f(x) = 0,$$

holomorphic on $\mathbb{C}^*$. This q-difference equation is an analog of the differential equation

$$\left(x^2 \frac{\mathrm{d}}{\mathrm{d}x} + 1\right) y = x^2 y'' + y = 0.$$

So $\Theta_q(x)$ appears as a q-analog of the exponential $e^{\frac{1}{x}}$.

We set

$$t = \frac{\operatorname{Log} x}{\operatorname{Log} q}$$

$$\Xi_q(x) = q^{-\frac{t(t-1-\frac{2i\pi}{\operatorname{Log} q})}{2}} = q^{-\frac{t(t-1)}{2}} e^{i\pi t}$$

$$\Xi_q(x) = q^{-\frac{t(t-1)}{2}} x^{\frac{i\pi}{\operatorname{Log} q}}.$$

The function Ξ_q is ramified holomorphic without zero "on" $\mathbb{C}^*$. It is a solution of the q-difference equation

$$(x\sigma_q + 1)f = xf(qx) + f(x) = 0.$$

It also in some sense a q-analog of $e^{\frac{1}{x}}$.

The function

$$\mho_q(x) = \Theta_q(x)\,\Xi_q^{-1}(x)$$

satisfies the q-difference equation

$$(\sigma_q - 1)f(x) = f(qx) - f(x) = 0\,. \qquad (C_q)$$

It is a "*constant*" for the theory of q-difference equations. We will call q-constants the solutions of the equation (C_q); the field of q-constants will be denoted by $\mathcal{C}_q$

Be careful, the notation Ξ_q is abusive; this function depends of the choice of a Logarithm $\operatorname{Log} q$ of q. If we change the determination of this Logarihm we replace $\Xi_q(x)$ by $x^{(2i\pi m)/\operatorname{Log} q}\,\Xi_q(x) = x^{m\tau^{-1}}\Xi_q(x)$.

The function $x^{\tau^{-1}}$ is also a solution of the equation (C_q). It is a q-constant ($q^{\tau^{-1}} = 1$). For $a \in \mathbb{C}$ we set $\Xi_{q;a}(x) = \Xi_q(a^{-1}x)$. There is also an ambiguity with this notation: we have to choose α such that $q^\alpha = a$. Then

$$\begin{aligned}\Xi_{q;a}(x) &= q^{-\frac{\alpha(\alpha-1)}{2}}\,x^{-\alpha}\,\Xi_q(x)\\ &= \Xi_q(-a)\,x^{-\alpha}\,\Xi_q(x)\,.\end{aligned}$$

The function $\Xi_{q;a}$ is a solution of the q-difference equation

$$(x\sigma_q + a)f = xf(qx) + af(x) = 0\,.$$

For $a = -1$ we will (in general) choose $\alpha = {}^{i\pi}/_{\operatorname{Log} q}$. Then

$$\Xi_q(-x) = q^{-\frac{t(t-1)}{2}}$$

is a solution of the q-difference equation

$$(x\sigma_q - 1)f = xf(qx) - f(x) = 0\,.$$

The field $\mathcal{C}'_{q,F}$ generated by the function $x^{\tau^{-1}}$, and the functions $\mho_{q;a}$ ($a \in \mathbb{C}^*$; $\mho_{q,a}(x) = \mho_q(a^{-1}x)$) is a "field of constants" for the theory of q-difference equations. I think that it is in some sense the *minimal* field of constants if one wants to deal with "q-difference equations reducible up to elementary transformations to fuchsian q-difference equations" (cf. [B]). Anyway such a field cannot be *smaller*:

If $|q| > 1$, it is easy to check that the function

$$\Xi_q(x)^{-1}\, e_p(x^{-1}) = q^{\frac{t(t-1)}{2}}\, e^{-i\pi t}\, \frac{1}{\prod_{n=0}^{+\infty}(1 - p^n x^{-1})}$$

is a solution of the q-difference equation

$$f(qx) - (1 - x)f(x) = 0\,. \qquad (1_q)$$

But the function

$$e_q(x) = \prod_{n=1}^{+\infty}(1 - p^n x) = (px;p)_\infty$$

is also a solution of the same equation. Then

$$e_q(x) = \mathfrak{p}(x)\, \Xi_q(x)^{-1}\, e_p(x^{-1})\,,$$

where $\mathfrak{p}$ is a q-constant (cf. [B], p. 560). We have

$$\mathfrak{p}(x)^{-1} = \Xi_q(x)^{-1}\, e_p(x^{-1})^{-1}\, e_q(x)\,.$$

We have

$$f_p(-x^{-1}) = \sum_{n=-\infty}^{+\infty} (-1)^n p^{\frac{n(n-1)}{2}} x^{-n} = f_p(-x)$$

and

$$f_p(-x^{-1}) = f_p(-x) = (p;p)_\infty (x^{-1};p)_\infty (px;p)_\infty$$

$$f_p(-x) = (p;p)_\infty\, e_p(x^{-1}) e_q(x)^{-1}$$

$$\mathfrak{p}(x)^{-1} = (p;p)_\infty^{-1} \Xi_q(x)^{-1} f_p(-x)$$

$$= -(p;p)_\infty^{-1} \mathfrak{D}_p(x)\,.$$

The equation (1_q) is not fuchsian (it is fuchsian at zero but irregular at infinity). If we want to deal *only* with *fuchsian* q-difference equations we can try to use a smaller field of q-constants, that is the field $C_{q,F}$ generated by the function $x^{\tau^{-1}}$, and the functions $\frac{\mathfrak{D}_{q,a}}{\mathfrak{D}_{q,b}}$ $(a,\ b \in C^*)$[(1)] (we have $\mathfrak{D}_{q;a;b} = \mathfrak{D}_{q;a}/\mathfrak{D}_{q;b} = (\Theta_{q;a}/\Theta_{q;b})x^{\alpha-\beta}$; $a = q^\alpha$, $b = q^\beta$).

[(1)] This works for "generic" linear algebraic fuchsian q-difference equations. A rational linear q-difference system of order one and of rank n, up to rational equivalence, is associated with a representation of a free non abelian group with two generators in $GL\ (n; C_{q,F})$ (the q-holonomy representation) [Ra2].

We remark that

$$\frac{\mathfrak{D}_{q;a_1}\mathfrak{D}_{q;a_2}\cdots\mathfrak{D}_{q;a_m}}{\mathfrak{D}_{q;b_1}\mathfrak{D}_{q;b_2}\cdots\mathfrak{D}_{q;b_m}} = \mathfrak{D}_{q;\,a_1,a_2,\ldots,a_m;\,b_1,b_2,\ldots,b_m}$$

is an elliptic function (in the variable t) if and only if

$$a_1, a_2, \ldots, a_m = b_1, b_2, \ldots, b_m\,.$$

If this condition is satisfied, we have

$$\frac{\mathfrak{D}_{q;a_1}\mathfrak{D}_{q;a_2}\cdots\mathfrak{D}_{q;a_m}}{\mathfrak{D}_{q;b_1}\mathfrak{D}_{q;b_2}\cdots\mathfrak{D}_{q;b_m}} = \frac{\Theta_{q,a_1}\Theta_{q;a_2}\cdots\Theta_{q;a_m}}{\Theta_{q;b_1}\Theta_{q;b_2}\cdots\Theta_{q;b_m}}$$

We will prove below that the field of elliptic functions is a subfield of $\mathcal{C}_{q,F}$.

Now we will get asymptotic estimates of Θ_q and E_q. Some asymptotic estimates on E_q are well known [H], [Li], [Wa]. Usually they are derived by quite complicated methods. Here we will see that they follow easily from the fact that $\mathfrak{D}_q$ is a q-constant and from the Jacobi triple product formula.

For $|q| < 1$, we write

$$\Theta_q = \mathfrak{D}_q \Xi_q\,.$$

We use this formula for $t = u + v\tau^{-1}$, with $v \in [\,0\,,\,1\,[$. In the corresponding strip in the t-plane the *periodic* function $\mathfrak{D}_q$ is *bounded*. Then we get an inequality:

$$\left|\Theta_q(x)\right| \leq C\left|\Xi_q(x)\right|,$$

for some $C > 0$ (independant of x) and $\operatorname{Arg} x - u \operatorname{Arg} q \in [\,0\,,\,2\pi\,[$.

We have $\Theta_q(x) = -\left(q;q\right)_\infty E_q(x) e_p(-x^{-1})$, and $e_p^{-1}(-x^{-1})$ is holomorphic on a neighborhood of infinity. Then we get an inequality

$$\left|E_q(x)\right| \leq C'\left|\Xi_q(x)\right|,$$

for some $C' > 0$ and x in a neighborhood of infinity and such that $\operatorname{Arg} x - u \operatorname{Arg} q \in [\,0\,,\,2\pi\,[$, and an inequality

$$\left|\exp_p(x)\right| \leq C'\left|\Xi_q\big((1-q)x\big)\right|\,,$$

in the same conditions.

The only zeros of the functions Θ_q and $\mathfrak{D}_q$ on the strip

$$\{t = u + v\tau^{-1} \mid u \in \mathbb{R},\ v \in [0\,,\,1[\,\},$$

in the t-plane, are the points $t = u + \left(\tau^{-1}/2\right) = u + \left(i\pi/\mathrm{Log}\, q\right)$, $u \in \mathbb{Z}$ ($x = -q^n$, $n \in \mathbb{Z}$). These zeros are simple. The function

$$\sin\left(\pi \frac{\mathrm{Log}(-x)}{\mathrm{Log}\, q}\right) = \sin \pi \left(t - \frac{\tau^{-1}}{2}\right)$$

has the same zeros with the same order, and is periodic with period 2 in the variable t. Then, if we set

$$G_q(x) = \frac{\mathfrak{D}_q(x)}{\sin\left(\pi \frac{\mathrm{Log}(-x)}{\mathrm{Log}\, q}\right)},$$

the functions G_q and G_q^{-1} are bounded on the strip

$$\{t = u + v\tau^{-1} \mid u \in \mathbb{R},\ v \in [0\,,\,1[\,\},$$

in the t-plane.

If we set

$$H_q(x) = G_q(x) e_p^{-1}(-x^{-1}),$$

we get

$$E_q(x) = H_q(x)\Xi_q(x) \sin\left(\pi \frac{\mathrm{Log}(-x)}{\mathrm{Log}\, q}\right).$$

There exists ρ, K_1, $K_2 > 0$ such that

$$0 < K_1 < \left|H_q(x)\right| < K_2,$$

for $\left|x > \rho\right|$ and such that $\mathrm{Arg}\, x - u\, \mathrm{Arg}\, q \in [\,0\,,\,2\pi\,[$.

For an entire function f, we set $M(f; r) = \sup_{|x|=r} \left|f(x)\right|$.

PROPOSITION 5.5.— *We set*

$$G_q(x) = \frac{\mathfrak{D}_q(x)}{\sin\left(\pi \frac{\mathrm{Log}(-x)}{\mathrm{Log}\, q}\right)}$$

and

$$E_q(x) = \prod_{n=0}^{+\infty} (1 + q^n x) = H_q(x) \Xi_q(x) \sin\left(\pi \frac{\mathrm{Log}(-x)}{\mathrm{Log}\, q}\right) .$$

Then:

i) *there exists* ρ, K_1, $K_2 > 0$ *such that* $0 < K_1 < |H_q(x)| < K_2$, *for* x *such that* $|x| > \rho$ *and* $\mathrm{Arg}\, x - u\ \ \mathrm{Arg}\, q \in [0\,,\, 2\pi[$;

ii) *if* q *is real,* $0 < q < 1$, *we have*

$$E_q(x) = H_q(x) \sqrt{-x}\, e^{-\frac{\mathrm{Log}^2(-x)}{2\,\mathrm{Log}\, q}} \sin\left(\pi \frac{\mathrm{Log}(-x)}{\mathrm{Log}\, q}\right) ,$$

with $0 < K_1 < |H_q(x)| < K_2$, *for* $x = -re^{i\varphi}$ *such that* $r > \rho$, *and* $\varphi \in [-\pi\,,\, \pi[$;

iii) *if* q *is real,* $0 < q < 1$, *we have*

$$\mathrm{Log}\, M(E_q; r) = -\frac{\mathrm{Log}^2 r}{2\ \mathrm{Log}\, q} + \frac{\mathrm{Log}\, r}{2} + O(1) ;$$

iv) *if* p *is real,* $p > 1$, *we have*

$$\mathrm{Log}\, M(e_p; r) = \frac{Log^2 r}{2\ \mathrm{Log}\, p} + \left(-\frac{1}{2} + \frac{\mathrm{Log}(p-1)}{\mathrm{Log}\, p}\right) \mathrm{Log}\, r + O(1) .$$

The assertion ii) is due to G. H. Hardy [H, (54), p. 172]. The assertion iv) seems "well known" (cf. [Wa, p. 331]). The assertion i) implies some results of Mellin [M] and Littlewood [Li].

We can now reformulate the definition of the q-exponential growth:

PROPOSITION 5.6. — *Let q and k be real numbers, with $q > 1$ and $k > 0$. Let f be an entire function, with*

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n .$$

Then the following conditions are equivalent:

i) the series $\widehat{f} = \sum_{n=0}^{+\infty} a_n x^n$ is q-Gevrey of order $-k$;

ii) the entire function f has a q-exponential growth of order $1/k$ and a finite type:

there exist real numbers $K > 0$, α, such that

$$\left|f(x)\right| < K\, q^{\frac{k}{2}\left(\frac{\operatorname{Log}|x|}{\operatorname{Log} q}\right)^2} |x|^\alpha = K\, e^{\frac{k}{2\operatorname{Log} q}\operatorname{Log}^2|x|}|x|^\alpha .$$

iii) there exist real numbers K_0, $A_0 > 0$, such that

$$\begin{aligned}\left|f(x)\right| = \left|f(re^{i\theta})\right| &\le M(f;r)\\ &< K_0 M\big(\exp_q;(A_0 r)^k\big) = K_0 \exp_q\big((A_0 r)^k\big) .\end{aligned}$$

Proposition 5.6 follows easily from lemma 2.2 and proposition 5.5.

To end this section we will explicit a simple relation between the function Θ_q and Jacobi θ-function θ_1: $\theta_1(z,\tau)$, with z, $\tau \in \mathbb{C}$ and $\operatorname{Im}\tau > 0$.

Usually $q = e^{i\pi\tau}$. We will use this relation at the beginning. At the end we will set $q = e^{2i\pi\tau}$. We set also $x = e^{2i\pi z}$.

We have $|q| = e^{-\pi \operatorname{Im}\tau} < 1$.

By definition

$$\begin{aligned}\theta_1(z,\tau) &= 2\sum_{n=0}^{+\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}\sin(2n+1)\pi z\\ &= -i\sum_{n=0}^{+\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}\left[e^{i(2n+1)\pi z} - e^{-i(2n+1)\pi z}\right]\\ &= -iq^{\frac{1}{4}}\sum_{n=0}^{+\infty}(-1)^n q^{n(n+1)}\left(e^{i\pi z}\right)^{2n+1}\end{aligned}$$

$$- iq^{\frac{1}{4}} \sum_{n=0}^{+\infty} (-1)^n \, q^{-n(-n-1)} \left(e^{i\pi z}\right)^{-2n-1}$$

$$= -iq^{\frac{1}{4}} \, e^{i\pi z} \sum_{n=0}^{+\infty} (-1)^n \, q^{n(n+1)} \left(e^{2i\pi z}\right)^n$$

$$- iq^{\frac{1}{4}} \, e^{i\pi z} \sum_{n=-1}^{-\infty} (-1)^n q^{n(n+1)} \left(e^{2i\pi z}\right)^n$$

$$= -iq^{\frac{1}{4}} \, e^{i\pi z} \sum_{n=-\infty}^{+\infty} (-1)^n \, q^{n(n+1)} \left(e^{2i\pi z}\right)^n .$$

So using the variable x:

$$\theta_1(z, \tau) = -iq^{\frac{1}{4}} \, x^{\frac{1}{2}} \sum_{n=-\infty}^{+\infty} (-1)^n \, q^{n(n+1)} \, x^n$$

$$= iq^{\frac{1}{4}} \, x^{-\frac{1}{2}} \sum_{n=-\infty}^{+\infty} (-1)^{n+1} \, q^{n(n+1)} \, x^{n+1}$$

Changing now our notation, we set $q = e^{2i\pi\tau}$ (in place of $q = e^{i\pi\tau}$). We get:

$$\theta_1(z, \tau) = -iq^{\frac{1}{8}} \, x^{-\frac{1}{2}} \sum_{n=-\infty}^{+\infty} (-1)^n \, q^{\frac{n(n+1)}{2}} \, x^{n+1} ,$$

that is

$$\theta_1(z, \tau) = -iq^{\frac{1}{8}} \, x^{-\frac{1}{2}} \, \Theta_q(x)$$

with $x = e^{2i\pi z}$, $q = e^{i\pi\tau}$.

There are similar relations for the other Jacobi θ-functions θ_2, θ_3, θ_0 (cf. [St (11), (12), (13), p. 580]).

It is easy to check that the field $\mathcal{C}_{q,F}$ (generated by the function $x^{\tau^{-1}}$ and the functions $\mathfrak{O}_{q,a}/\mathfrak{O}_{q,b}$ ($a, b \in \mathbb{C}^*$)) contains the field of elliptic functions associated to the lattice $\mathbb{Z} \oplus \tau\mathbb{Z}$.

We express Jacobi θ-functions θ_2, θ_0, θ_3 with $\mathfrak{O}_a(x)$ for some values of the parameter a: $a = e^{i\pi} = -1$, $a = e^{i\pi\tau}$, $a = e^{i\pi(1+\tau)} = -e^{i\pi\tau}$. Then we use classical relations between Jacobi θ-functions and Weierstrass elliptic functions $\wp$ and $\wp'$ (cf. [MOS, p. 389]) and we prove that $\wp$, $\wp' \in \mathcal{C}_{q,F}$. That ends the proof (the field of elliptic functions is generated on $\mathbb{C}$ by $\wp$ and $\wp'$).

6. An application: common solutions of differential and q-difference equations

If $f(x) = x^m$, $x(\mathrm{d}/\mathrm{d}x)f(x) = mx^m$ and $\sigma_q f(x) = q^m x^m$, so the function $f(x) = x^m$ satisfy the linear algebraic differential and q-difference equations:

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x} - m\right) f = 0\,,$$

and

$$(\sigma_q - q^m)f = 0\,.$$

More generally it is easy to check that every polynomial $P \in \mathbb{C}[x]$ is a common solution of a linear algebraic differential equation and of a linear algebraic q-difference equation:

If the degree of P is m, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m+1} P = (\sigma_q - q^m)(\sigma_q - q^{m-1}) \dots (\sigma_q - 1)P = 0\,.$$

Conversely the only entire functions satisfying such a condition are polynomials:

THEOREM 6.1.— *Let q be a non zero complex number, with $|q| \neq 1$. Then, if an entire function f satisfies a linear algebraic differential equation*

$$a_n(x)f^{(n)}(x) + \cdots + a_0(x)f(x) = c\,, \tag{1}$$

and a linear algebraic q-difference equation

$$b_m(x)f(q^m x) + \cdots + b_0(x)f(x) = d\,, \tag{2}$$

(a_i, b_j, c and $d \in \mathbb{C}[x]$), then f is a polynomial.

We can suppose $|q| > 1$. If an entire function satisfies equations (2) it admits a q-exponential growth (of some order $k > 0$), then it admits an exponential growth of order *zero*. But if an entire function is solution of a linear algebraic differential equations and admits an exponential growth of order zero it is a polynomial [Ra1, th. 3.2.8, p. 80].

There is also a similar result for formal power series expansions:

THEOREM 6.2.— *Let q be a non zero complex number, with $|q| \neq 1$. Then, if a formal power series expansion (resp. a Laurent series with a finite number of negative terms) $\widehat{f}$ satisfies a linear analytic differential equation*

$$a_n(x)f^{(n)}(x) + \cdots + a_0(x)f(x) = c\,, \tag{1}$$

and a linear analytic q-difference equation

$$b_m(x)f(q^m x) + \cdots + b_0(x)f(x) = d\,, \tag{2}$$

(a_i, b_j, c and $d \in \mathbb{C}\{x\}$), then $\widehat{f}$ is convergent.

Moreover if $|q| > 1$, if a_i, b_j, c and $d \in \mathbb{C}[x]$, and if a_n is a non zero complex constant, then $\widehat{f}$ is a polynomial (resp. $\widehat{f} \in \mathbb{C}[x\,,\,x^{-1}]$).

The proof is easy. We can suppose $|q| > 1$. First we have inclusions $\mathbb{C}[[x]]_s \subset \mathbb{C}[[x]]_{q,s'}$, for every $s > 0$ and every $s' > 0$. If $\widehat{f}$ satisfies the equation (1), then $\widehat{f}$ is Gevrey of some order: $\widehat{f} \in \mathbb{C}[[x]]_s$, for some $s > 0$. Then $\widehat{f} \in \mathbb{C}[[x]]_{q,s'}$ for *every* $s' > 0$. The convergence of $\widehat{f}$ follows then from Bezivin's comparison theorem. ([Be1, proposition 3.4]).

If a_i, b_j, c and $d \in \mathbb{C}[x]$, and if a_n is a non zero complex constant, then using equation (2) we can extend the sum f of $\widehat{f}$ to an entire function satisfying equations (1) and (2) and thence f is a polynomial (theorem 6.1). The case of Laurent series is similar.

7. Problems and conjectures

We consider the entire function $f(z) = 2^z = e^{z \operatorname{Log} 2}$. For $n \in \mathbb{N}$, $f(n)$ is an *integer*: the entire function f takes entire values on the *arithmetic progession* $0, 1, \ldots, n, \ldots$ The following result is well known (cf. [Bo, th. 9.12.1, p. 175]):

THEOREM 7.1.— *Let f be an entire function of exponential order one and type less (strictly) than* $\operatorname{Log} 2$. *If $f(n)$ is an integer for $n \in \mathbb{N}$, then f is a polynomial.*

There is a similar result if we replace the *arithmetic* progression $0, 1, \ldots, n, \ldots$ by a *geometric* progression $1, q, \ldots, q^n, \ldots$ (with q an integer, $q > 1$) and the exponential growth by the q-exponential growth. This result is due to A. O. Guelfond ([Ge1], [Ge2, th. VIII, p. 179]):

THEOREM 7.2. — *Let q be an integer, with $q > 1$. Let f be an entire function satisfying*

$$\operatorname{Log}|f(x)| \leq \operatorname{Log} M(f;r) < \frac{\operatorname{Log}^2 r}{4 \operatorname{Log} q} - \frac{1}{2} \operatorname{Log} r - \omega(r),$$

with $\lim_{r \to +\infty} \omega(r) = +\infty$.

Then, if $f(q^n)$ is an integer for $n \in \mathbb{N}$, f is a polynomial.

So we see that it is interesting to get some information on the exponential (resp. q-exponential) order of an entire function, but even more interesting, if the type (resp. q-type) is finite, to get estimates of this type (resp. q-type). For entire functions solutions of linear algebraic differential equations it is possible to compute explicitely a finite list for the possible types using the characteristic equations of the negative slopes of the Newton polygon [Ra1]. First one get *precised* Gevrey estimates on the formal solutions using the characteristic equations of the positive slopes of the Newton polygon. I conjecture that mimicking the methods of [Ra1], it is possible to get *precised* q-Gevrey estimates for the formal solutions of a linear analytic q-difference equation, using the characteristic equations of the slopes of its Newton polygon (introduced by Adams [A1], [A2]), and after, by duality, to compute a finite list for the possible q-types for entire functions solutions of a linear algebraic q-difference equation.

For a fixed q-exponential order k, the q-type is defined mimicking the usual definition of the (exponential) type (cf. [Wa]):

DEFINITION . — *Let q, k be real numbers, $q > 1$, $k > 0$. Let f be an entire function of q-exponential order k and of finite type. Then the q-type of f (the q-order k being fixed) is*

$\lambda_0 = \operatorname{Inf}\{\lambda \in \mathbb{R}$ *such that there exists an inequality* $M(f;r) \leq C_\lambda M(e_q; \lambda r^k)$, *with* $C_\lambda > 0$ *(independant of r)*$\}$.

Theorem 6.1 is in general false for $|q| = 1$ (that is $q = e^{2i\pi\lambda}$, $\lambda \in \mathbb{R}$): if $q = -1$, then an even entire function (like $\cos x$) satisfies the q-difference equation $f(qx) - f(x)$.

PROBLEM 7.3

Does Theorem 6.1 remain true with $q = e^{2i\pi\lambda}$, with $\lambda \in \mathbb{R} - \mathbb{Q}$?

PROBLEM 7.4

Let q be a non zero complex number, with $|q| \neq 1$ (or $q = e^{2i\pi\lambda}$, with $\lambda \in \mathbb{R} - \mathbb{Q}$).

- *i) What can be said of an entire function which is a common solution of a linear algebraic difference equation and of a linear algebraic q-difference equation ?*
- *ii) What can be said of an entire function which is a common solution of a linear algebraic differential equation and of a linear algebraic difference equation ?*

Some Praagman's results [Pr] could be of some use in these problems. Notice that the entire function $\cos 2\pi x$ is a common solution of the differential equation $y'' + y = 0$ and of the difference equation $Ty - y = 0$.

It is well known that the entire function ${}^1/\Gamma(x)$ (which is a solution of the difference equation $xTy - y = 0$) cannot be a solution of a linear algebraic differential equation. Using "Stirling formula" (for $\operatorname{Arg} x = {}^{3\pi}/4$) we see that the entire function ${}^1/\Gamma(x)$ cannot have a $|q|$-exponential growth. Then, using our theorem 4.8, we get:

PROPOSITION 7.5.— *Let q be a non zero complex number, with $|q| \neq 1$. Then the entire function ${}^1/\Gamma(x)$ cannot be a solution of a linear algebraic q-difference equation.*

Using G. K. Immink's results [I] about asymptotic expansions at infinity of solutions of analytic difference equations it is possible to derive large generalizations of this last result (and of the similar result for differential equations). We conjecture that an entire function, which is a common solution of a linear algebraic difference equation and of a linear algebraic q-difference equation (with $|q| \neq 0, 1$), is necessarily a polynomial. We conjecture that an entire function, which is a common solution of a "purely irregular" linear algebraic difference equation and of a linear algebraic differential equation, is necessarily a polynomial.

The following precised version of theorem 6.2 is due to J.-P. Bezivin:

THEOREM 7.6.— *Let q be a non zero complex number with $|q| \neq 1$, or with $|q| = 1$, and q transcendental. Then, if a formal power series expansion (or more generally a Laurent series with a finite number of negative terms) $\widehat{f}$ satisfies a linear algebraic differential equation*

$$a_n(x)f^{(n)}(x) + \cdots + a_0(x)f(x) = c\,, \tag{1}$$

and a linear algebraic q-difference equation

$$b_m(x)f(q^m x) + \cdots + b_0(x)f(x) = d \tag{2}$$

(a_i, b_j, c and $d \in \mathbb{C}[x]$), then $\widehat{f}$ is a rational function: $f \in \mathbb{C}(x)$.

We first suppose that $|q| \neq 1$. From theorem 6.2, we get that $\widehat{f}$ is convergent. We denote by f its sum. Then f is a solution of the linear algebraic q-difference equation (2); it follows that it can be extended in a meromorphic function f on $\mathbb{C}$. This function f is a solution of the linear algebraic differential equation (1), so it admits only a finite number of poles, and there exists a polynomial h such that hf is an entire function. This entire function is also a common solution of a linear algebraic differential equation and of a linear algebraic q-difference equation. Then (using theorem 6.1) $hf = g$ is a polynomial and $f = g/h \in \mathbb{C}(x)$.

J.-P. Bezivin also pointed out the possibility to get the same result when $|q| = 1$ and q is *transcendental.* Everything takes place in a finitely generated extension of $\mathbb{Q}$ in $\mathbb{C}$. We set $\widehat{f}(x) = \sum_{n\geq 0} c_n x^n$. We denote by K_N the subfield of $\mathbb{C}$ generated on $\mathbb{Q}$ by q, the coefficients of the polynomials $a_0, \ldots, a_n$, $b_0, \ldots, b_m$, and $c_0, \ldots c_N$. If N is sufficiently big the field $K_N = K$ is independent of N and contains all the coefficients c_n ($n \in \mathbb{N}$): $\widehat{f}$ is a solution of (1) and we can compute the c_n's from the $c_0, \ldots, c_N$ by a recursion formula. We choose a transcendence basis of K over $\mathbb{Q}$ containing q and we fix an arbitrary transcendental complex number q'. Then there exists an isomorphism φ of the field K on a subfield L of $\mathbb{C}$, containing q', such that $\varphi(q) = q'$. We can choose q' such that $|q'| < 1$, and apply the preceding result.

Acknowledgments

I thank Bernard Malgrange who read a first version of this paper and suggested some improvements.

REFERENCES

[A1] ADAMS (C. R.) .— *Linear q-difference Equations,* Bull. A.M.S., 1931, pp. 361-399.

[A2] ADAMS (C. R.) .— *On the linear ordinary q-difference equations,* Ann. Math., Ser. II, **30** (1929), pp. 95-205.

[Be1] Bezivin (J.-P.) .— *Sur les équations fonctionnelles aux q-différences*, preprint Paris VI (1990).

[Be2] Bezivin (J.-P.) .— *Letter to J. P. Ramis*, february (1992).

[B] Birkhoff (G. D.) .— *The Generalised Riemann Problem for Linear Differential Equations and the Allied Problems for Linear Differential and q-Difference Equations*, Proc. Am. Acad., **49** (1913) pp. 521-568.

[Bo] Boas (R. P.) Jr .— *Entire Functions*, Academic Press Inc., New-York (1954).

[GR] Gasper (G.) and Rahman (M.) .— *Basic Hypergeometric Series, Encyclopedia of Mathematics and its Applications*, G.C. Rota ed., Vol. (1990).

[Ge1] Gelfond (A. O.) .— *Sur les fonctions entières, qui prennent des valeurs entières dans les points β^n, β est un nombre entier positif et $n = 0, 1, 2, \ldots$*, Rec. Math. Moscou, **40** (1933) pp. 42-47.

[Ge2] Gelfond (A. O.) .— *Calcul des différences finies*, Dunod, Paris (1963).

[G] Grothendieck (A.) .— *Sur certains espaces de fonctions holomorphes* I, II, Journal de Crelle, **192** (1953) pp. 35-64 et 77-95.

[H] Hardy (G. H.) .— *On the function $P_\rho(x)$*, Quarterly Journal of Math., **3**, n° 37 (1906) pp. 146-172.

[Ha] Hahn (W.) .— *Beiträge zur Theorie der Heinischen Reihen. Die 24 Integrale der hypergeometrischen q-Differenzengleichung, das q-Analogon der Laplace-Transformation*, Math. Nach., **2** (1940) pp. 340-379.

[He] Heine (E.) .— *Über die Reihe...*, J. Reine Angew. Math., **32** (1846) pp. 210-212.

[I] Immink (G. K.) .— *Asymptotics of analytic difference equations*, Springer Lecture Notes in Math., **1082** (1984).

[J] Jacobi (C. G. J.) .— *Fundamenta Nova Theoriae Functionum Ellepticarum, Regiomonti.* Sumptibus fratrum Bornträger (1829); reprinted in *Gesammelte Werke* 1, Reiner 1881, pp. 49-239; reprinted by Chelsea, New York (1969).

[Li] Littlewood (J. E.) .— *On the asymptotic approximation to integral functions of zero order*, Proc. London Math. Soc., Ser. 2, **5** (1907), pp. 361-410.

[MOS] Magnus (W.), Oberhettinger (F.) and Soni (R. P.) .— *Formulas and Theorems for the Special Functions of Mathematical Physics*, Grund. Math. Wiss., **52**, Springer Verlag New York (1966).

[M] Mellin (M.) Acta Soc. Fennicae, t. XXIX, n° 4, p. 15.

[N] Nachbin (L.) .— *An extension of the notion of integral functions of the finite exponential type*, Anais Acad. Brasil. Ciencias, **16** (1944) pp. 143-47.

[Pr] PRAAGMAN (C.) . — *Stokes and Gevrey Phenomena in Relation to Index Theorems in the Theory of Meromorphic Linear Difference Equations*, Funkcialaj Ekvacioj, **29** (1986) pp. 259-279.

[Ra1] RAMIS (J.-P.) . — *Théorèmes d'indices Gevrey pour les équations différentielles ordinaires*, Memoirs of the American Mathematical Society, **296** (1984) pp. 1-95.

[Ra2] RAMIS (J.-P.) . — *Analytic classification of fuchsian q-difference equations*, in preparation (1992).

[Si] SILVA (J.) . — *As Funções Analiticas e a Anãlise Funcional*, Port. Math., **9** (1950) pp. 1-130.

[St] STARCHER (G. W.) . — *A solution of a simple functional equation as a basis for readily obtaining certain fundamental formulas in the theory of elliptic functions*, Bulletin of American Math. Society, **36** (1930) pp. 577- 581.

[V1] VALIRON (G.) . — *Sur les fonctions entières vérifiant une classe d'équations différentielles*, Bull. Soc. Math de france, **51** (1926), pp. 33-45.

[V2] VALIRON (G.) . — *Fonctions analytiques et équations différentielles*, Journal de Mathématiques, **31** (1952) pp. 293-303.

[Wa] WALLISER (R.) . — *Über ganze Funktionen, die in einer geometrischen Folge ganze Werte annehmen*, Monatsh. für Math., **100** (1985), pp. 329-335.

Appendix[1]

This paper was already accepted when I discovered some old and new related works. I give here some new references and I make some comments.

In the following q is a complex number, with $|q| > 1$.

In a set of papers ([P], [Pi1, 2, 3, 4, 5]) H. Poincaré and E. Picard have studied rational systems of q-difference equations (linear and non-linear cases), and more precisely entire solutions of such systems:

$$Y(qx) = R\big(Y(x)\big)$$

(where Y is a unknown function with values in $\mathbb{C}^m$, and R a rational function of n scalar variables).

They said that, "generally speaking", entire functions solutions of such equations are "*new transcendental functions*" (Poincaré: *classe nouvelle de transcendantes*, Picard: *transcendantes nouvelles*). It is interesting to try

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to give a precise meaning to this assertion. In order to do that the idea of G. Valiron was to try to get some information about the *growth* of such entire functions: if that growth is "unusual", then the corresponding function is a "*new* transcendental function" (... *déterminer leur type de croissance, ce qui permettra de reconnaître si ces fonctions sont d'une espèce nouvelle.* [V3, introduction, p. 2]). He began to apply this program for simple cases of linear algebraic q-difference equations; he studied equations (using our notations)

$$f(q^m x) + b_{m-1}(x) f(q^{m-1} x) + \cdots + b_0(x) f(x) = d(x)\,,$$

(where $b_{m-1}, \ldots, b_0,\ d \in \mathbb{C}[x]$), and proved that, for an entire function solution f:

$$\operatorname{Log} M(f;r) < K'(\operatorname{Log} r)^2\,,$$

for some positive constant K'. And more precisely that:

$$\operatorname{Log} M(f;r) \sim \frac{\mu}{2\operatorname{Log}|q|}(\operatorname{Log} r)^2\,,$$

if the degree μ of b_{m-1} is greater than the degree of the b_j's ($j = 0, \ldots, m-2$) and d.

Such entire functions are "new transcendental functions": they have an exponential growth of order zero and "classical transcendental functions" have an exponential growth of *finite non zero* order.

So some simple particular cases of our results are already in [V3]— and our work give a complete answer to the problem of "new transcendental functions" raised by Poincaré's and Picard's works for the *linear* case. Unfortunately the non-linear case remains open and seems difficult to handle: Valiron remarked [V3] that, if q and α are positive integers, with $q > 1$, then the entire function e^{x^α} is a solution of the non-linear q-difference equation

$$f(qx) = \big(f(x)\big)^\beta\,,$$

with $\beta = q^\alpha$. More generally he proved exponential growth estimates (of finite non zero order) for entire solutions of some classes of non linear q-difference equations.

In a recent paper [Gr], F. Gramain presented an attempt to prove Gelfond's theorem [Gel] by transcendental methods (as M. Waldschmidt for Polya's theorem). In that direction he proved a converse version of one of our results for entire functions taking entire values on a geometric progression: proposition 4.4, p. 133. I think that such a result could be improved using the improvements of our results (that is estimates on the q-type given by roots of characteristic equations associated to a Newton polygon) conjectured above.

References

[Gr] GRAMAIN (F.) .— *Fonctions entières d'une ou plusieurs variables complexes prenant des valeurs entières sur une progression géométrique,* Cinquante ans de Polynômes — Fifty years of Polynomials, M. Laurent and M. Waldschmidt ed., Springer Lecture Notes, **1418** (1991), pp. 123-137.

[Pi1] PICARD (E.) .— *Leçons sur quelques équations fonctionnelles,* Gauthier-Villars, Paris (1928).

[Pi2] PICARD (E.) .— *Sur une classe de transcendantes nouvelles,* Premier mémoire, Acta mathematica, **18** (1894) pp. 133-154.

[Pi3] PICARD (E.) .— *Sur une classe de transcendantes nouvelles,* Deuxième mémoire, Acta mathematica, **23** (1900) pp. 323-338.

[Pi4] PICARD (E.) .— *Sur une classe de transcendantes généralisant les fonctions elliptiques et abéliennes,* Annales de l'Ecole Normale, 3ème série, **30** (1913), pp. 247-253.

[Pi5] PICARD (E.) .— *Sur une classe de surfaces algébriques dont les coordonnées s'expriment par des fonctions uniformes de deux paramètres,* Bulletin de la Soc. Math. de France, **28** (1900).

[P] POINCARÉ (H.) .— *Sur une classe nouvelle de transcendantes uniformes,* Journal de Liouville, 4e série, **VI** (1890) pp. 539-582; Œuvres Complètes, t. IV.

[V3] VALIRON (G.) .— *Fonctions analytiques,* Presses Universitaires de France, Paris (1954).