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Annales de la faculté des sciences de Toulouse 5e série, tome 12, nº 3 (1991), p. 351-363

<http://www.numdam.org/item?id=AFST_1991_5_12_3_351_0>

Annales de la Faculté des Sciences de Toulouse Vol. XII, n° 3, 1991

Almost periodic solutions of forced second order Hamiltonian systems

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RÉSUMÉ. — Nous étudions des équations différentielles du second ordre forcées par un terme p.p. (presque-périodique) du type $q(t) + V'\big(q(t)\big) = e(t)$. En supposant V et V' lipschitziennes, bornées ou périodiques, nous obtenons que l'ensemble des termes forçants p.p. e pour lesquels l'équation possède une solution p.p. est un sous-ensemble dense de l'espace des fonctions p.p. pour une norme appropriée. Nous appliquons ce résultat à l'équation du pendule forcé.

ABSTRACT. — We study second order differential equations forced by an a.p. (almost periodic) term, like $\ddot{q}(t) + V'\big(q(t)\big) = e(t)$. Under some hypotheses of Lipschitz, boundedness or periodicity on V and V', we obtain that the a.p. forcing terms e for which there exists an a.p. solution of the equation constitute an everywhere dense subset of the space of the a.p. functions relatively at an appropriate norm. We apply this result to the forced pendulum equation.

Introduction

We consider the forced second order equations:

$$\ddot{q}(t) + V'\big(q(t)\big) = e(t) \tag{1}$$

where $q(t) \in \mathbb{R}^N$, V' is the gradient of the function $V \in C^1(\mathbb{R}^N, \mathbb{R})$, and e is an almost periodic forcing term. The homogeneous equation $\ddot{q}(t) + V'\big(q(t)\big) = 0$ is an hamiltonian system; the hamiltonian of this equation is

$$H(q,p) := \frac{1}{2}\,|p|^2 + V(q)\,.$$

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We precise some notations. $AP^0(\mathbb{R}^N)$ denotes the space of the u.a.p. (uniformly almost periodic, i.e. Bohr almost periodic) functions from $\mathbb{R}$ into $\mathbb{R}^N$ ([2] chapter 1, [9]). When $k \in \mathbb{N} \cup \{\infty\}$, $AP^k(\mathbb{R}^N)$ denotes the space of the functions $f \in C^k(\mathbb{R}, \mathbb{R}^N)$ such that the derivative of order j, $f^{(j)} \in AP^0(\mathbb{R}^N)$, for every j between 0 and k. $AP^a(\mathbb{R}^N)$ denotes the set of the $f \in AP^\infty(\mathbb{R}^N)$ that are analytic functions on $\mathbb{R}$. $AP^0(\mathbb{R}^N)$ is a Banach space for the norm of supremum

$$\|f\|_\infty := \sup\{|f(t)|\, ;\ t \in \mathbb{R}\}\ .$$

When $k \in \mathbb{N}$, $AP^k(\mathbb{R}^N)$ is a Banach space for the norm

$$\|f\|_{C^k} := \sum_{\alpha=0}^{k} \|f^{(\alpha)}\|_\infty\,.$$

When $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}_+)$, i.e. f is locally Lebesgue-integrable from $\mathbb{R}$ into $\mathbb{R}_+$, its superior mean value is

$$\overline{\mathcal{M}}\{f\} := \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)\,\mathrm{d}t\,.$$

When $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$, its mean value (when it exists) is

$$\mathcal{M}\{f\} := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)\,\mathrm{d}t\,.$$

For a real $p \geq 1$, $\mathcal{B}^p(\mathbb{R}^N)$ is the closure of $AP^0(\mathbb{R}^N)$ into $L^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$ for the semi-norm $\|f\|_p := \overline{\mathcal{M}}\{|f|^p\}^{1/p}$; it is a space of a.p. (almost periodic) functions in the sense of Besicovitch ([2] chapter 2). If $f \in \mathcal{B}^p(\mathbb{R}^N)$ then $\|f\|_p = \mathcal{M}\{|f|^p\}^{1/p}$ ([2] p. 93), the relation $\|f-g\|_p = 0$ is denoted by $f \underset{p}{\sim} g$, and the quotient-space is $B^p(\mathbb{R}^N) := \mathcal{B}^p(\mathbb{R}^N)/(\cdot \underset{p}{\sim} \cdot)$. In practice, in order to keep light notations, we do not distinguish between an element of $\mathcal{B}^p(\mathbb{R}^N)$ and its class of equivalence in $B^p(\mathbb{R}^N)$. Endowed with the norm $\|\cdot\|_p$, $B^p(\mathbb{R}^N)$ is a Banach space. The norm $\|\cdot\|_2$ of $B^2(\mathbb{R}^N)$ is the euclidean norm associated to the inner $(f \mid g) := \mathcal{M}\{f(t)\cdot g(t)\}_t$, and $B^2(\mathbb{R}^N)$ is a Hilbert space.

When $f \in B^p(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$, we denote $a(f\,;\lambda) := \mathcal{M}\{f(t)e^{-i\lambda t}\}_t$, $\Lambda(f) := \{\lambda \in \mathbb{R}\,;\ a(f\,;\lambda) \neq 0\}$ and $\mathrm{mod}(f)$ the $\mathbb{Z}$-module generated by $\Lambda(f)$ in $\mathbb{R}$. When M is a $\mathbb{Z}$-module in $\mathbb{R}$, and $k \in \mathbb{R} \cup \{\infty\} \cup \{a\}$, we define

$$AP^k(\mathbb{R}^N, M) := \{f \in AP^k(\mathbb{R}^N)\,;\ \mathrm{mod}(f) \subset M\}$$

and when $p \geq 1$ we define

$$B^p(\mathbb{R}^N, M) := \{f \in B^p(\mathbb{R}^N)\,;\ \mathrm{mod}(f) \subset M\}.$$

Now, we recall the definitions of some spaces, like Sobolev spaces, special to the almost periodicity ([7] § 2, 3). Following Vo-Khac, when $f \in B^2(\mathbb{R}^N)$, we denote (when it exists) by ∇f the limit in $B^2(\mathbb{R}^N)$ of the quotients $({}^1/_r)(\mathcal{T}_r f - f)$ when $r \to 0$, $r \neq 0$, where $\mathcal{T}_r f(t) := f(t+r)$. Then $B^{1,2}(\mathbb{R}^N)$ is the space of the $f \in B^2(\mathbb{R}^N)$ such that ∇f exists in $B^2(\mathbb{R}^N)$. Endowed with the inner product

$$\langle f \mid g\rangle := (f \mid g) + (\nabla f \mid \nabla g),$$

$B^{1,2}(\mathbb{R}^N)$ is a Hilbert space, and the Euclidean associated norm is denoted by $\|\cdot\|_{1,2}$.

$B^{2,2}(\mathbb{R}^N)$ is the space of the $f \in B^{1,2}(\mathbb{R}^N)$ such that $\nabla^2 f := \nabla(\nabla f)$ exists in $B^2(\mathbb{R}^N)$. Endowed with the inner product

$$\langle\langle f \mid g\rangle\rangle := \langle f \mid g\rangle + (\nabla^2 f \mid \nabla^2 g),$$

$B^{2,2}(\mathbb{R}^N)$ is a Hilbert space, and the Euclidean associated norm is denoted by $\|\cdot\|_{2,2}$.

When M is a $\mathbb{Z}$-module in $\mathbb{R}$, and $j \in \{1,2\}$,

$$B^{j,2}(\mathbb{R}^N\,;\ M) := B^{j,2}(\mathbb{R}^N) \cap B^2(\mathbb{R}^N, M).$$

When $f \in B^2(\mathbb{R}^N)$ (resp. $B^2(\mathbb{R}^N\,;\ M)$), we also consider the following norm:

$$\|f\|_* := \sup\Big\{\mathcal{M}\{f \cdot h\}\,;\ h \in B^{1,2}(\mathbb{R}^N),\ \|h\|_{1,2} \leq 1\Big\}$$
$$\Big(\text{resp. } \|f\|_{*,M} := \sup\Big\{\mathcal{M}\{f \cdot h\}\,;\ h \in B^{1,2}(\mathbb{R}^N; M),\ \|h\|_{1,2} \leq 1\Big\}\Big).$$

1. Preliminary results

PROPOSITION 1.— *Let M be a real symmetric negative definite $N \times N$ matrix. Then we have:*

i) the unique solution into $B^{2,2}(\mathbb{R}^N)$ of the equation $\nabla^2 h + Mh \underset{2}{\sim} 0$ is $h \underset{2}{\sim} 0$.

ii) for each $k \in B^2(\mathbb{R}^N)$ *there exists a unique* $h \in B^{2,2}(\mathbb{R}^N)$ *such that* $\nabla^2 h + Mh \underset{2}{\sim} k$. *Moreover* $\mathrm{mod}(h) \subset \mathrm{mod}(k)$.

Proof

i) Let $h \in B^{2,2}(\mathbb{R}^N)$ s.t. $\nabla^2 h + Mh \underset{2}{\sim} 0$. Then, for all $\lambda \in \mathbb{R}$, we have

$$a(\nabla^2 h\,;\,\lambda) + a(Mh\,;\,\lambda) = a(0\,;\,\lambda)\,,$$

therefore

$$-\lambda^2 a(h\,;\,\lambda) + Ma(h\,;\,\lambda) = 0 \quad \text{and} \quad (M - \lambda^2)a(h\,;\,\lambda) = 0\,.$$

If $a(h\,;\,\lambda) \neq 0$, then λ^2 is an eigenvalue of M, that is impossible since M is negative definite and $\lambda^2 \geq 0$. And so $a(h\,;\,\lambda) = 0$ for every $\lambda \in \mathbb{R}$, therefore, using the Parseval equality

$$\|h\|_2^2 = \sum_{\lambda \in \mathbb{R}} |a(h\,;\,\lambda)|^2\,,$$

we see that $h \underset{2}{\sim} 0$.

ii) We fix $k \in B^2(\mathbb{R}^N)$. For every $\lambda \in \mathbb{R}$, λ^2 cannot be an eigenvalue of M, and so $(M - \lambda^2)$ is an invertible linear operator from $\mathbb{C}^N$ onto $\mathbb{C}^N$. We denote

$$c_\lambda := (M - \lambda^2)^{-1} a(k\,;\,\lambda) \in \mathbb{C}^N\,.$$

We can endow $\mathbb{C}^N$ with a basis in which the matrix representation of M is a diagonal matrix $\mathrm{diag}[\,m_1, \ldots, m_N\,]$, with $m_\alpha < 0$ for every $\alpha = 1, \ldots, N$. The m_α are the eigenvalues of M and the basis is constituted by eigenvectors of M. In this spectral basis, the component of index α of the vector c_λ is

$$c_{\lambda,\alpha} = \left(m_\alpha - \lambda^2\right)^{-1} \cdot a\big(k\,;\,\lambda\big)_\alpha = \left(m_\alpha - \lambda^2\right)^{-1} \cdot a\big(k_\alpha\,;\,\lambda\big)\,.$$

Consequently

$$\begin{aligned} |c_{\lambda,\alpha}| = \left|\left(m_\alpha - \lambda^2\right)^{-1}\right| \cdot \left|a\big(k_\alpha\,;\,\lambda\big)\right| = \left(\lambda^2 - m_\alpha\right)^{-1} \cdot \left|a\big(k_\alpha\,;\,\lambda\big)\right| \leq \\ \leq \left(-m_\alpha\right)^{-1} \cdot \left|a\big(k_\alpha\,;\,\lambda\big)\right|\,, \end{aligned}$$

and

$$\left|c_{\lambda,\alpha}\right|^2 \leq \left(m_\alpha^2\right)^{-1} \cdot \left|a\big(k_\alpha\,;\,\lambda\big)\right|^2\,.$$

We take $\mu := \max\left\{ \left(m_\alpha^2\right)^{-1} ; \alpha = 1, \ldots, N \right\}$, then $0 < \mu < \infty$ and

$$|c_\lambda|^2 = \sum_{\alpha=1}^{N} |c_{\lambda,\alpha}|^2 \leq \mu \sum_{\alpha=1}^{N} |a(k_\alpha ; \lambda)|^2 = \mu \cdot |a(k ; \lambda)|^2 .$$

Since μ is independant of λ and $\left(a(k ; \lambda)\right)_{\lambda \in \mathbb{R}} \in \ell^2(\mathbb{R}, \mathbb{C}^N)$ ([2] p. 104) the previous inequality implies that $\left(c_\lambda\right)_{\lambda \in \mathbb{R}} \in \ell^2(\mathbb{R} ; \mathbb{C}^N)$. By the theorem of Riesz-Fisher-Besicovitch ([2] p. 109) there exists a unique $h \in B^2(\mathbb{C}^N)$ s.t. $c_\lambda = a(h ; \lambda)$ for every $\lambda \in \mathbb{R}$. The equalities

$$c_{-\lambda} = \left(M - \lambda^2\right)^{-1} \cdot a(k, -\lambda) = \left(M - \lambda^2\right)^{-1} \cdot \overline{a(k, -\lambda)} = \overline{c_\lambda}$$

ensure us that $h \in B^2(\mathbb{R}^N)$.

Ever in the same spectral basis, for the component of index α, we have:

$$i\lambda c_{\lambda,\alpha} = i\lambda\left(m_\alpha - \lambda^2\right)^{-1} a(k_\alpha, \lambda) ,$$

therefore $\left|i\lambda c_{\lambda,\alpha}\right| = |\lambda| \cdot \left|m_\alpha - \lambda^2\right|^{-1} \left|a(k_\alpha ; \lambda)\right|$. We remark that

$$\left|m_\alpha - \lambda^2\right| = |\lambda|^2 - m_\alpha = |\lambda|^2 + \left(\sqrt{-m_\alpha}\right)^2 \geq 2\sqrt{-m_\alpha}\, |\lambda|$$

since $\left(|\lambda| - \sqrt{-m_\alpha}\right)^2 \geq 0$, therefore $|\lambda| \cdot \left|m_\alpha - \lambda^2\right|^{-1} \leq \left(2\sqrt{-m_\alpha}\right)^{-1}$. And so

$$\begin{aligned} \left|i\lambda c_{\lambda,\alpha}\right|^2 &\leq \left(2\sqrt{-m_\alpha}\right)^{-2} \cdot \left|a(k_\alpha ; \lambda)\right|^2 = \\ &= \frac{1}{4} \left(m_\alpha^2\right)^{-1} \cdot \left|a(k_\alpha ; \lambda)\right|^2 \leq \\ &\leq \frac{1}{4}\, \mu \left|a(k_\alpha ; \lambda)\right|^2 , \end{aligned}$$

and, in summing above α, we obtain:

$$\left|i\lambda c_\lambda\right|^2 \leq \frac{1}{4}\, \mu \left|a(k ; \lambda)\right|^2 .$$

We see that $\left(i\lambda c_\lambda\right)_\lambda \in \ell^2(\mathbb{R}, \mathbb{C}^N)$, and consequently, by the theorem of Riesz-Fisher-Besicovitch, there exists $f \in B^2(\mathbb{C}^N)$ s.t. $i\lambda c_\lambda = a(f ; \lambda)$ for every $\lambda \in \mathbb{R}$. Since

$$a(f ; -\lambda) = i(-\lambda) c_{-\lambda} = -i\lambda \overline{c_\lambda} = \overline{i\lambda c_\lambda} = \overline{a(f ; \lambda)} ,$$

we can say that $f \in B^2(\mathbb{R}^N)$. Moreover $a(f\,;\,\lambda) = i\lambda a(h\,;\,\lambda)$, therefore by [7] (prop. 6), we can conclude that $f = \nabla h$, and so $h \in B^{1,2}(\mathbb{R}^N)$.

In using the same spectral basis of $\mathbb{C}^n$, we note that

$$\left|-\lambda^2 c_{\lambda,\alpha}\right|^2 = \lambda^2\left(\lambda^2 - m_\alpha\right)^{-2}\left|a(k_\alpha\,;\,\lambda)\right|^2 \leq \left|a(k_\alpha\,;\,\lambda)\right|^2$$

and consequently

$$\left|-\lambda^2 c_\lambda\right|^2 \leq \left|a(k\,;\,\lambda)\right|^2 .$$

Therefore $\left(-\lambda^2 c_\lambda\right)_\lambda \in \ell^2(\mathbb{R}\,;\,\mathbb{C}^N)$ and by the theorem of Riesz-Fisher-Besicovitch, we know that there exists $g \in B^2(\mathbb{C}^N)$ s.t. $a(g\,;\,\lambda) = -\lambda^2 c_\lambda$ for every $\lambda \in \mathbb{R}$. With the equalities

$$a(g\,;\,-\lambda) = -\lambda^2 c_{-\lambda} = -\lambda^2 \overline{c_\lambda} = \overline{a(g\,;\,\lambda)}$$

we see that $g \in B^2(\mathbb{R}^N)$. Since $a(g\,;\,\lambda) = i\lambda a(\nabla h\,;\,\lambda)$, by [7], prop. 6, we can say that $g = \nabla(\nabla h) = \nabla^2 h$, and so $h \in B^{2,2}(\mathbb{R}^N)$.

Lastly we have

$$\begin{aligned} a(\nabla^2 h + Mh\,;\,\lambda) &= a(\nabla^2 h\,;\,\lambda) + Ma(h\,;\,\lambda) \\ &= -\lambda^2 a(h\,;\,\lambda) + Ma(h\,;\,\lambda) \\ &= (M - \lambda^2)\cdot a(h\,;\,\lambda) = (M - \lambda^2)c_\lambda \\ &= (M - \lambda^2)(M - \lambda^2)^{-1} a(k\,;\,\lambda) = a(k\,;\,\lambda) \end{aligned}$$

for all $\lambda \in \mathbb{R}$, therefore we have $\nabla^2 h + mh \underset{2}{\sim} k$. That justifies the existence. The uniqueness results of i).

After our construction of the solution h, it is clear that

$$\mathrm{mod}(h) \subset \mathrm{mod}(k)\,. \,\square$$

The assertion i) implies that there don't exist any non-zero u.a.p. solution, in $AP^2(\mathbb{R}^N)$, of the ordinary differential equation:

$$\ddot{h}(t) + Mh(t) = 0\,. \tag{2}$$

This non-existence can easily be verifiable in writing (2) as a first order linear system with constant coefficients and in formulating the explicit form of the general solution as in [1] (p. 222). This non-existence is also an easy consequence of the study of the lagrangian systems with convex lagrangians ([5], [6]).

When $k \in AP^0(\mathbb{R}^N)$ we can formulate the following forced ordinary differential equation

$$\ddot{h}(t) + Mh(t) = k(t)\,. \tag{3}$$

The assertion ii) implies that (3) admits at most one solution in $AP^2(\mathbb{R}^N)$. But it is not true that (3) possesses at least a solution in $AP^2(\mathbb{R}^N)$ for each $k \in AP^0(\mathbb{R}^N)$. In keeping our scheme of proof, because of the inequalities

$$\begin{aligned} |c_\lambda| &\le \sqrt{u}\,|a(k\,;\,\lambda)| \\ |i\lambda c_\lambda| &\le \frac{1}{2}\sqrt{u}\,|a(k\,;\,\lambda)| \\ |-\lambda^2 c_\lambda| &\le |a(k\,;\,\lambda)|\,, \end{aligned}$$

we can assert that (3) possesses a solution in $AP^2(\mathbb{R}^N)$ when the Fourier-Bohr series of k is absolutly convergent, since in this case the Fourier-Bohr series of h, $\dot{h}$, $\ddot{h}$ are also absolutely convergent. For example, when k is a trigonometric polynomial or a finite sum of periodic functions, (3) admits a solution in $AP^2(\mathbb{R}^N)$. In this situation, happens a phenomema like that of the theorem of Malkin ([14] p. 222). It is possible to deduce of a theorem of Fink (theorem 5.11 p. 91 in [12]) some more refined sufficient (but not necessary) conditions on $\Lambda(k)$ to ensure the existence of a solution in $AP^2(\mathbb{R}^N)$ of (3). Another way to describe the situation of (3) consists to say that the linear differential operator $h \to \ddot{h} + Mh$, from $AP^2(\mathbb{R}^N)$ in $AP^0(\mathbb{R}^N)$, is not surjective; its range is everywhere dense in $AP^0(\mathbb{R}^N)$ since it contains all the trigonometric polynomials, but its range is not closed into $AP^0(\mathbb{R}^N)$.

Now, we deduce from proposition 1 an explicit formula that relates the norms $\|f\|_*$, $\|f\|_{*,M}$ defined in the introduction and the Fourier-Bohr coefficients of f.

PROPOSITION 2

i) If $f \in B^2(\mathbb{R}^N)$ *then we have:*

$$\|f\|_*^2 = \sum_{\lambda \in \mathbb{R}} (1+\lambda^2)^{-1} |a(f\,;\,\lambda)|^2\,.$$

ii) If M is a $\mathbb{Z}$-module in $\mathbb{R}$, and $f \in B^2(\mathbb{R}^N\,;\, M)$ then we have:

$$\|f\|_{*,M}^2 = \sum_{\lambda \in M} \left(1+\lambda^2\right)^{-1} \left|a(f\,;\,\lambda)\right|^2 .$$

Proof. — We introduce a linear functional on $B^{1,2}(\mathbb{R}^N)$ in taking $U(x) := \mathcal{M}\{f \cdot x\} = (f \mid x)$. Using the Cauchy-Schwarz inequality, we obtain:

$$\left|U(x)\right| \leq \|f\|_2 \|x\|_2 \leq \|f\|_2 \|x\|_{1,2} ,$$

and U is continuous, $U \in B^{1,2}(\mathbb{R}^N)$ (the star indicates the topological dual space). By the definition of $\|f\|_*$, we have $\|U\| = \|f\|_*$. By the theorem of representation of F. Riesz in the Hilbert spaces ([10] p. 81) there exists a unique $u \in B^{1,2}(\mathbb{R}^N)$ such that $U(x) = \langle u \mid x \rangle$ for all $x \in B^{1,2}(\mathbb{R}^N)$ and $\|U\| = \|u\|_{1,2}$. And so we have $\|f\|_* = \|u\|_{1,2}$.

If I denotes the identity matrix $N \times N$, taking $M = -I$ in proposition 1 ii), we can assert that there exists $h \in B^{2,2}(\mathbb{R}^N)$ such that $\nabla^2 h - h \underset{2}{\sim} (-f)$. Then

$$\begin{aligned} U(x) &= \mathcal{M}\{f \cdot x\} = \mathcal{M}\{-(\nabla^2 h - h) \cdot x\} \\ &= \mathcal{M}\{\nabla h \cdot \nabla x\} - \mathcal{M}\{h \cdot x\} \end{aligned}$$

([7] prop. 9), therefore $U(x) = \langle h \mid x \rangle$. The uniqueness of u in the theorem of F. Riesz implies that $h = u$. Therefore, we have $\|f\|_* = \|h\|_{1,2}$.

In using the Parseval equality in $B^2(\mathbb{R}^N)$, we have

$$\begin{aligned} \|h\|_{1,2}^2 &= \|h\|_2^2 + \|\nabla h\|_2^2 = \\ &= \sum_{\lambda \in \mathbb{R}} \left|a(h\,;\,\lambda)\right|^2 + \sum_{\lambda \in \mathbb{R}} \lambda^2 \left|a(h\,;\,\lambda)\right|^2 = \\ &= \sum_{\lambda \in \mathbb{R}} (1+\lambda^2) \left|a(h\,;\,\lambda)\right|^2 . \end{aligned}$$

But, the relation $a(\nabla^2 h - h\,;\,\lambda) = a(-f\,;\,\lambda)$ implies:

$$-\lambda^2 a(h\,;\,\lambda) - a(h\,;\,\lambda) = -a(f\,;\,\lambda) ,$$

therefore

$$(1+\lambda^2) a(h\,;\,\lambda) = a(f\,;\,\lambda) ,$$

and also

$$\left|a(h\,;\,\lambda)\right|^2 = \left(1+\lambda^2\right)^{-2}\left|a(f\,;\,\lambda)\right|^2 .$$

Finally we obtain:

$$\left\|f\right\|_*^2 = \left\|h\right\|_{1,2}^2 = \sum_{\lambda\in\mathbb{R}}(1+\lambda^2)\left(1+\lambda^2\right)^{-2}\left|a(f\,;\,\lambda)\right|^2 =$$
$$= \sum_{\lambda\in\mathbb{R}}\left(1+\lambda^2\right)^{-1}\left|a(f\,;\,\lambda)\right|^2 .$$

That justifies i). The proof of ii) is similar. □

2. Results of Density

THEOREM 1.— *Let $V \in C^1(\mathbb{R}^N, \mathbb{R})$ that satisfy the following conditions:*

i) V is bounded above on $\mathbb{R}^N$;

ii) V' is bounded on $\mathbb{R}^N$;

iii) there exists a constant $a_1 \geq 0$ s.t.

$$\left|V'(x) - V'(y)\right| \leq a_1|x-y| \quad \textit{for all } x,\, y \in \mathbb{R}^N .$$

iv) Then, the following assertion holds: for each $e \in AP^0(\mathbb{R}^N)$ such that $\mathcal{M}\{e\} = 0$, for each $\epsilon > 0$, denoting $M := \mathrm{mod}(e)$, there exists $e_\epsilon \in AP^0(\mathbb{R}^N\,;\,M)$ such that $\left\|e - e_\epsilon\right\|_{*,M} < \epsilon$ and there exists $q_\epsilon \in AP^a(\mathbb{R}^N\,;\,M)$ verifying

$$\ddot{q}_\epsilon(t) + V'\left(q_\epsilon(t)\right) = e_\epsilon(t) \quad \textit{for all } t \in \mathbb{R}.$$

Comments

After proposition 2, the inequality $\left\|e - e_\epsilon\right\|_{*,M} < \epsilon$ means that the Fourier-Bohr coefficients of e_ϵ are near to those of e. In the following proof, we shall see that we can choose q_ϵ as a trigonometric polynomial. The two most important tools of the proof are the Ekeland *variational principle* [11] and the *calculus of variations in mean* ([3], [4], [7]).

Proof.— The condition ii) ant the mean value theorem permits us to write

$$\left|V(x) - V(y)\right| \leq \left\|V'\right\|_\infty |x-y| \quad \text{for all } x,\, y \in \mathbb{R}^N .$$

The Nemytski operator built on V is $\mathcal{N}_V(u) := V \circ u$. By theorem 3 of [7], we know that $\mathcal{N}_V \in C^1(B^2(\mathbb{R}^N), B^1(\mathbb{R}))$, $V'(u) \in B^2(\mathbb{R}^N)$, $(\mathcal{N}_V)'(u) \cdot h = V'(h) \cdot h$, for u, $h \in B^2(\mathbb{R}^N)$. Since the operator $\mathcal{M}$ is linear continuous on $B^1(\mathbb{R})$, the functional $\Phi(u) := \mathcal{M}\{V(u)\}$ is Fréchet C^1 on $B^2(\mathbb{R}^N)$ and $\Phi'(u) \cdot h = \mathcal{M}\{V'(u) \cdot h\}$.

Now, we fix $e \in AP^0(\mathbb{R}^N)$ such that $\mathcal{M}\{e\} = 0$ and we fix $\epsilon > 0$. We denote mod(e) by M.

By the theorem of the uniform approximation of Weierstrass-Bohr ([9] p. 81) we can assert that there exists a trigonometric polynomial

$$P(t) = \sum_{\alpha=1}^{n} c_\alpha \, e^{i\lambda_\alpha t},$$

with $\lambda_\alpha \in \Lambda(e)$ such that $\|e - P\|_\infty < \epsilon/2$. Since $\mathcal{M}\{e\} = 0$, we have $0 \notin \Lambda(e)$ and consequently all the λ_α are different to zero. We take the trigonometric polynomial

$$Q(t) := \sum_{\alpha=1}^{n} \left(- \lambda_\alpha^2 \right)^{-1} c_\alpha \, e^{i\lambda_\alpha t}.$$

We see that $\ddot{Q} = P$.

We introduce the action functional

$$J(u) := \mathcal{M}\left\{ \frac{1}{2} \, |\nabla u|^2 - V(u+Q) \right\}.$$

From the properties above mentioned of the functional Φ, it is easy to see that $J \in C^1\big(B^{1,2}(\mathbb{R}^N\,;\,M), \mathbb{R}\big)$ and that

$$J'(u) \cdot h = \mathcal{M}\{\nabla u \cdot \nabla h - V'(u+Q) \cdot h\} \quad \text{for all } u,\, h \in B^{1,2}(\mathbb{R}^N\,;\,M)$$

(see also theorems 4, 5 in [7] and proposition 1 of [8]). By the hypothesis i), J is bounded from below on $B^{1,2}(\mathbb{R}^N\,;\,M)$. And so we can apply the Ekeland *variational principle* ([11] p. 27) and assert that there exists $u \in B^{1,2}(\mathbb{R}^N)$ such that $\|J'(u)\| < \epsilon/4$ (the norm if that of the dual space $B^{1,2}(\mathbb{R}^N\,;\,M)^*$). By the Weierstrass-Bohr theorem, we can take a trigonometric polynomial v, with $\Lambda(v) \subset \Lambda(u)$, such that v is near to u, and by the continuity of J', such that

$$\|J'(u) - J'(v)\| < \frac{\epsilon}{4}.$$

Consequently we have $J'(v) < \epsilon/2$, with $v \in AP^a(\mathbb{R}^N\,;\,M)$. Therefore, for all $h \in B^{1,2}(\mathbb{R}^N\,;\,M)$ such that $\|h\|_{1,2} < 1$, we have

$$\begin{aligned}\frac{\epsilon}{2} > \left|J'(v)\cdot h\right| &= \left|\mathcal{M}\{\nabla v \cdot \nabla h - V'(v+Q)\cdot h\}\right| \\ &= \left|\mathcal{M}\{\dot v \cdot \nabla h - V'(v+Q)\cdot h\}\right| \\ &= \left|\mathcal{M}\left\{\left(\ddot v + V'(v+Q)\cdot h\right)\right\}\right|\end{aligned}$$

([7] proposition 9). If we denote $z := \ddot v + V'(v+Q)$, we have $z \in AP^0(\mathbb{R}^N\,;\,M)$ ([3] proposition 3) and the previous inequalities mean that $\|z\|_{*,M} < \epsilon/4$. Taking $e_\epsilon := P + z$, $q_\epsilon := v + Q$, we have

$$\begin{gathered} e_\epsilon \in AP^0(\mathbb{R}^N\,;\,M)\,, \quad q_\epsilon \in AP^a(\mathbb{R}^N\,;\,M)\,, \\ \ddot q_\epsilon + V'(q_\epsilon) = \ddot v + \ddot Q + V'(v+Q) = P + z = e_\epsilon \end{gathered}$$

and

$$\begin{aligned}\|e - e_\epsilon\|_{*,M} &= \|z + P - e\|_{*,M} \le \\ &\le \|z\|_{*,M} + \|P - e\|_{*,M} \le \|z\|_{*,M} + \|P - e\|_\infty \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.\ \square\end{aligned}$$

We recall that a mapping $F : \mathbb{R}^N \to \mathbb{R}^d$ is said multiply periodic when there exist real numbers $T_1, T_2, \ldots, T_N$ such that the partial mapping $x_\alpha \mapsto F(x_1, \ldots, x, \ldots, x_N)$ is T_α-periodic for each between 1 and N and for each fixed $x_1, \ldots, x_{\alpha-1}, x_{\alpha+1}, \ldots, x_N$.

THEOREM 2.— *Let $V \in C^2(\mathbb{R}^N, \mathbb{R})$ a multiply periodic potential function. Then the conclusion of theorem 1 holds.*

Proof.— Since the derivative of a periodic function is periodic, we see that the gradient mapping $V' : \mathbb{R}^N \to \mathbb{R}^N$ and the hessian mapping $V'' : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ are also multiply periodic. We denote

$$\Pi := \prod_{\alpha=1}^{N} [\,0\,,\,T_\alpha\,]\,;$$

it is a compact subset of $\mathbb{R}^N$. Because of the continuity and the compactness, the three upper bounds

$$\begin{aligned}\sup V(\mathbb{R}^N) &= \sup V(\Pi)\,, \\ \sup\left|V'(\mathbb{R}^N)\right| &= \sup\left|V'(\Pi)\right| \\ \sup\left|V''(\mathbb{R}^N)\right| &= \sup\left|V''(\Pi)\right|\end{aligned}$$

are finite. Consequently, in using the mean value theorem, the hypotheses i), iii), iv) of theorem 1 are satisfied. □

When V is multiply periodic, for the case of the periodic forcing terms, we dispose of the theorem of Willem ([13] § 1.6). Let us remark that the case of the a.p. forcing terms is not a consequence of the case of the periodic forcing terms, since the subset of the periodic functions is not everywhere dense in the $\left(AP^0(\mathbb{R}^N, M), \|\cdot\|_{*,M}\right)$ when the algebraic dimension of the module M is greater than 1.

3. An application: the forced pendulum

Here, we consider the pendulum equation forced by a u.a.p. function:

$$\ddot{q}(t) + \sin q(t) = e(t). \tag{4}$$

Taking $N = 1$ and $V(q) := -\cos q$, it is a particular case of (1).

THEOREM 3.— *For each $e \in AP^0(\mathbb{R})$ such that $\mathcal{M}\{e\} = 0$, and for each $\epsilon > 0$, denoting $M := \mathrm{mod}(e)$; there exists $e_\epsilon \in AP^0(\mathbb{R}\,;\, M)$ such that $\|e_\epsilon - e\|_{*,M} < \epsilon$ and there exists a trigonometric polynomial q_ϵ (with real values) such that*

$$\ddot{q}_\epsilon(t) + \sin q_\epsilon(t) = e_\epsilon(t) \quad \textit{for all } t \in \mathbb{R}.$$

Proof.— It is a corollary of theorem 2. In the proof of theorem 1, we have seen that we can choose q_ϵ as a trigonometric polynomial, and since q_ϵ, $\ddot{q}_\epsilon$ and the function sin are analytic, we have necessarily the analyticity of e_ϵ. □

This result is different from that of [8] (theorem 2). Here, we do not assume that $e = \ddot{E}$, where $E \in AP^2(\mathbb{R})$ and $\mathrm{osc}(2) < \pi/2$, and so theorem 3 is less restrictive than the theorem of [8]. But here, we only have a density relatively to $\|\cdot\|_{*,M}$, instead of $\|\cdot\|_2$ as in [8], and a $\|\cdot\|_{*,M}$-density is coarser than a $\|\cdot\|_2$-density.

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