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Periodic solutions of quadratic lagrangian systems on p-convex sets

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RÉSUMÉ. — On prouve l'existence d'une infinité de solutions périodiques pour un système lagrangien quadratique sur une certaine classe d'ensembles non réguliers, c'est-à-dire les ensembles p-convexes. On emploie des méthodes variationnelles en analyse non linéaire et non régulière.

ABSTRACT. — We prove the existence of infinitely many periodic solutions for a quadratic Lagrangian system on a certain class of non-smooth sets, namely the p-convex sets. We use variational methods in non-smooth nonlinear analysis.

Introduction

If M is a compact submanifold without boundary in $\mathbb{R}^n$ and $N_x M$ denotes the normal subspace to M at x, the study of the Lagrangian system

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\nabla_v L(s, \gamma, \gamma')\right) - \nabla_q L(s, \gamma, \gamma') \in N_{\gamma(s)} M$$

has been carried out in [1], where the existence of infinitely many periodic solutions γ is proved under quite general assumptions.

The corresponding problem on manifolds with boundary has been treated in [18], where the existence of a periodic solution is proved and in [5], where the existence of infinitely many periodic solutions is shown. The feature of unilateral constraints (cf. [6], [7], [14], [15], [16], [17], [18]) is that, even if the manifold M is of class C^∞, the corresponding variational problem does not have a smooth structure.

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For this reason, it seems to be natural to allow for the set M itself a certain kind of irregularity. The aim of the paper is to treat the case in which M is a p-convex set (see Def. 1.3) and L is quadratic with respect to γ', namely

$$L(s, q, v) = \frac{1}{2}\left(a(s, q)v, v\right) - V(s, q).$$

The particular case $a \equiv \mathrm{Id}$, $V \equiv 0$, which leads to the study of geodesics, was already treated in [2] and [3].

The main tools are the techniques of non-smooth nonlinear analysis developed in [8], [9], [10] and [11]. Actually, the main part of the paper, the second section, is devoted to the proof that these techniques can be applied to our situation.

1. Recalls of non-smooth analysis and the main result

We will recall some notions of non-smooth analysis as developed in [6], [7], [8], [9], [11].

From now on, H will be a real Hilbert space, $|\cdot|$ and $(\cdot, \cdot)$ its norm and scalar product, respectively.

If $u \in H$ and $r > 0$, we set $B(u, r) = \{v \in H \mid |v - u| < r\}$.

DEFINITION 1.1. — (see also [6] and [9]) *Let Ω be an open subset of H and $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ a map.*

We set

$$D(f) = \{u \in \Omega \mid f(u) < +\infty\}.$$

Let u belong to $D(f)$. The function f is said to be subdifferentiable at u if there exists $\alpha \in H$ such that

$$\liminf_{v \to u} \frac{f(v) - f(u) - (\alpha, v - u)}{|v - u|} \geq 0.$$

We denote by $\partial^- f(u)$ the (possibly empty) set of such α's and we set

$$D(\partial^- f) = \{u \in D(f) \mid \partial^- f(u) \neq \emptyset\}.$$

It is easy to check that $\partial^- f(u)$ is convex and closed $\forall u \in D(f)$.

If $u \in D(\partial^- f)$, $\operatorname{grad}^- f(u)$ will denote the element of minimal norm of $\partial^- f(u)$.

Moreover, let M be a subset of H. We denote by I_M the function:

$$I_M(u) = \begin{cases} 0 & u \in M \\ +\infty & u \in H \setminus M. \end{cases}$$

It is easy to check that $\partial^- I_M(u)$ is a closed convex cone $\forall\, u \in M$.

We will call (outward) normal cone to M at u the set $\partial^- I_M(u)$ and tangent cone to M at u its negative polar $\left(\partial^- I_M(u)\right)^-$, i.e.,

$$\left(\partial^- I_M(u)\right)^- = \left\{ v \in H \mid (v, w) \le 0\,,\ \forall\, w \in \partial^- I_M(u) \right\}.$$

Remark 1.2. — Let us suppose that $g : \Omega \to \mathbb{R}$ is Fréchet differentiable at $u \in \Omega$. Then:

$$\partial^-(f + g)(u) \ne \emptyset \text{ if and only if } \partial^- f(u) \ne \emptyset$$

and

$$\partial^-(f + g)(u) = \left\{ \alpha + \operatorname{grad} g(u) \mid \alpha \in \partial^- f(u) \right\}.$$

Let us introduce the class of p-convex sets as defined in [2] and [3]. An other characterization of this class is in [4].

DEFINITION 1.3. — *A subset M of H is said to be a p-convex set if there exists a continuous function $p : M \to \mathbb{R}^+$ such that*

$$(\alpha\,, v - u) \le p(u)|\alpha|\,|v - u|^2$$

whenever $u, v \in M$ and $\alpha \in \partial^- I_M(u)$.

Examples of p-convex sets are the following ones:

(1) The $C^{1,1}_{\text{loc}}$-submanifolds (possibly with boundary) of H;

(2) The convex subsets of H;

(3) The images under a $C^{1,1}_{\text{loc}}$- diffeomorphism of convex sets;

(4) The subset of $\mathbb{R}^n$: $\left\{ x \mid \max |x_i| \le 1,\ \sum x_i^2 \ge 1 \right\}$ (note that it is not included in the classes (1), (2), (3)).

Now, we can state the main result of the paper.

Let M be a p-convex subset of $\mathbb{R}^n$ and let $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Lagrangian of the form

$$L(s, q, v) = \frac{1}{2} \big(a(s, q) v, v \big) - V(s, q)$$

where a, V are of class C^2 on $\mathbb{R} \times \mathbb{R}^n$ and the matrix $a(s, q)$ is symmetric and positive definite, that is there exists a constant $\nu > 0$ such that

$$\big(a(s, q) v, v \big) \geq \nu |v|^2, \quad \forall s \in \mathbb{R}, \ \forall q, v \in \mathbb{R}^n \tag{1.1}$$

Moreover, let us suppose that a and V are 1-periodic in the first variable:

$$a(s + 1, q) = a(s, q); \quad V(s + 1, q) = V(s, q) \tag{1.2}$$

THEOREM 1.4. — *Let us suppose that M is compact, connected and noncontractible in itself and that either*

a) $\pi_1(M)$ *has infinitely many conjugacy classes*

    or

b) $\pi_1(M)$ *has a finite number of elements.*

Then, there exists a sequence $\{\gamma_h\}_h \subset W^{2,\infty}(\mathbb{R} ; \mathbb{R}^n)$ such that $\forall \ h \in \mathbb{N}$

i) γ_h *is 1-periodic and $\gamma_h(s) \in M$*

ii) $\dfrac{\mathrm{d}}{\mathrm{d}s} \big(\nabla_v L(s, \gamma_h, \gamma_h') \big) - \nabla_q L(s, \gamma_h, \gamma_h') \in \partial^- I_M(\gamma_h)$ *a.e. in* $]\,0, 1\,[$

iii) $\lim_{h \to \infty} \int_0^1 L(s, \gamma_h, \gamma_h') \, \mathrm{d}s = +\infty$.

In order to apply the critical point theory for non-smooth functionals some other notions and results have to be recalled.

DEFINITION 1.5. — *Let Ω be an open subset of H and $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ a function. A point $u \in D(f)$ si said to be a lower critical point for f if $0 \in \partial^- f(u)$; $c \in \mathbb{R}$ is said to be a critical value of f if there exists $u \in D(f)$ such that*

$$0 \in \partial^- f(u) \quad and \quad f(u) = c.$$

DEFINITION 1.6. — *(see also [8], [11]) Let Ω be an open subset of H. A function $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ is said to have a φ-monotone subdifferential of order two if there exists a continuous function*

$$\chi : D(f)^2 \times \mathbb{R}^2 \to \mathbb{R}^+$$

such that

$$\big(\alpha - \beta, u - v \big) \geq -\chi \big(u, v, f(u), f(v) \big) \big(1 + |\alpha|^2 + |\beta|^2 \big) |u - v|^2$$

whenever

$$u, v \in D(\partial^- f), \quad \alpha \in \partial^- f(u) \quad and \quad \beta \in \partial^- f(v).$$

The notion of *p*-convex set is actually a particular case of the previous notion. In fact, it turns out (see [2]) that a subset M of H is *p*-convex if and only if I_M has a φ-monotone subdifferential of order two.

THEOREM 1.7. — (see [10]) *Let* $f : H \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous function with a* φ-*monotone subdifferential of order two. We set*

$$d^*(u, v) = |u - v| + \big|f(u) - f(v)\big|, \quad \forall\, u, v \in D(f).$$

Let us suppose that:

i) $\inf_H f > -\infty$

ii) every sequence $\big(u_h\big)_h \subset D(\partial^- f)$ *with* $\sup_h f(u_h) < +\infty$ *and* $\lim_h \operatorname{grad}^- f(u_h) = 0$ *has a subsequence converging in* H.

Then, f *has at least* $\operatorname{cat}\big(D(f), d^*\big) = +\infty$, *then*

$$\sup\big\{f(u) \mid u \in D(\partial^- f),\, 0 \in \partial^- f(u)\big\} = +\infty.$$

Let M be a *p*-convex subset of H.

DEFINITION 1.8. — *Let us denote by* $\widehat{A}$ *the set of* $u's \in H$ *with the two properties:*

i) $\delta_p(u, M) < 1$ *where* $\delta_p(u, M) = \limsup\limits_{\substack{|u-w| \to d(u,M) \\ w \in M}} 2p(w)|u - w|$.

ii) $\exists\, r \geq 0$ *such that* $M \cap \big\{v \in H \mid |v - u| \leq r\big\}$ *is closed in* H *and not empty.*

Obviously, $M \subset \widehat{A}$ and:

PROPOSITION 1.9. — (see prop. 2.9 in [2]) *Let* $M \subset H$ *be p-convex and locally closed. Then* $\widehat{A}$ *is open and* $\forall\, u \in \widehat{A}$ *there exists one and only one* $w \in M$ *such that* $|u - w| = d(u, M)$.

Moreover, if we set $\pi(u) = w$, *then*

i) $\big(u - \pi(u)\big) \in \partial^- I_M\big(\pi(u)\big)$ *and* $2p(\pi(u))\big|u - \pi(u)\big| < 1, \forall\, u \in \widehat{A}$

ii) $\left|\pi(u_1) - \pi(u_2)\right| \leq \left(1 - p\big(\pi(u_1)\big)\left|u_1 - \pi(u_1)\right| +\right.$

$$- p\big(\pi(u_2)\big)\left|u_2 - \pi(u_2)\right|\Big)^{-1}\left|u_1 - u_2\right|,$$

$$\forall\ u_1,\, u_2 \in \widehat{A}$$

iii) $\big(t\pi(u) + (1-t)u\big) \in \widehat{A},\ \forall\ u \in \widehat{A},\ \forall\ t \in [\,0\,,1\,].$

Remark 1.10. — Let us set $A = \left\{u \in \widehat{A} \mid 4p\big(\pi(u)\big)\left|u - \pi(u)\right| < 1\right\}.$
Then A is an open set containing M and, by proposition 1.9 ii), $\pi : A \to M$
is Lipschitz continuous of constant two.

PROPOSITION 1.11. — (see prop. 2.2 in [2]) *Let* $M \subset H$ *be p-convex.*
If $\left\{u_h\right\}_h \subset M$ *is a sequence converging to* $u \in M$ *and* $\left\{\alpha_h\right\}_h \subset H$ *is a*
sequence converging weakly to α *with* $\alpha_h \in \partial^- I_M(u_h)$, *then* $\alpha \in \partial^- I_M(u).$

PROPOSITION 1.12. — (see prop. 2.12 in [2]) *Let* $M \subset H$ *be locally closed*
and p-convex. Then

$$\lim_{t \to 0^+} \frac{\pi(u + tv) - u}{t} = P_u v, \quad \forall\ v \in M \quad and \quad \forall\ v \in H$$

where P_u *is the projection on the tangent cone to* M *at* $u.$

PROPOSITION 1.13. — *Let* $M \subset H$ *be locally closed and p-convex. Let*
$\left\{u_h\right\}_h$ *be a sequence in* M *converging to* $u \in M$ *and let* $\tau \in \big(\partial^- I_M(u)\big)^-.$
Then

$$\lim_h P_{u_h}\tau = \tau.$$

Proof. — Since $\left\{P_{u_h}\tau\right\}_h$ is bounded, up to a subsequence $\left\{P_{u_h}\tau\right\}_h$ is
weakly convergent to some $\xi \in H$. Since $(\tau - P_{u_h}\tau) \in \partial^- I_M(u_h)$, by
proposition 1.11 we have $(\tau - \xi) \in \partial^- I_M(u)$. Therefore $(\tau - \xi, \tau) \leq 0,$
which implies $|\tau| \leq |\xi|$, hence

$$\liminf_h \left|P_{u_h}\tau\right| \geq |\tau|.$$

From the equality

$$|\tau|^2 = \left|P_{u_h}\tau\right|^2 + \left|\tau - P_{u_h}\tau\right|^2$$

the thesis follows. $\square$

2. The variational structure of the problem

In this section, we want to supply our problem, that is the research of periodic orbits of the considered Lagrangian system, with a variational structure. Our aim is to characterize such periodic orbits as lower critical points of the functional

$$f : L^2(0, 1; \mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$$

defined in the following way:

$$f(\gamma) = \begin{cases} \dfrac{1}{2} \displaystyle\int_0^1 (a(s, \gamma)\gamma', \gamma') \, \mathrm{d}s - \displaystyle\int_0^1 V(s, \gamma) \, \mathrm{d}s & \gamma \in X \\ +\infty & \gamma \in L^2(0, 1; \mathbb{R}^n) \backslash X \end{cases}$$

where the space of the admissible paths is:

$$X = \left\{ \gamma \in W^{1,2}(0, 1; \mathbb{R}^n) \mid \gamma(s) \in M, \ \gamma(0) = \gamma(1) \right\}.$$

Since M is compact, we shall assume the function p of definition 1.3 to be constant.

Moreover, if $\gamma \in W^{1,2}(0, 1; \mathbb{R}^n)$ with $\gamma(s) \in M$ and $\delta \in L^2(0, 1; \mathbb{R}^n)$, we set

$$(P_\gamma \delta)(s) = P_{\gamma(s)} \delta(s)$$

where $P_{\gamma(s)}$ is the projection on the tangent cone to M at $\gamma(s)$, according to the scalar product

$$(u, v)_s = (a(s, \gamma(s)) u, v).$$

Let us also denote by π_s the projection on M according to the scalar product $(u, v)_s$. By remark 1.8 and the assumptions on a, there exists an open set A containing M such that each π_s is defined on A and is Lipschitz continuous of constant 2.

Let us begin with a regularity result.

THEOREM 2.1.— *Let us take $\gamma \in X$. If $\partial^- f(\gamma) \neq \emptyset$ then*

$$\gamma \in W^{2,2}(0, 1; \mathbb{R}^n), \quad \gamma'_+(0) = \gamma'_-(1)$$

and

$$\left|\gamma''(s)\right| \leq \text{const}\left(1 + \left|\alpha(s)\right| + \left|\gamma'(s)\right|^2\right) \quad \text{a.e.} \ , \quad \forall \, \alpha \in \partial^- f(\gamma) \, .$$

Moreover, if $0 \in \partial^- f(\gamma)$ *then* $\gamma \in W^{2,\infty}(0, 1; \mathbb{R}^n)$.

For the proof of this theorem, we need some lemmas.

LEMMA 2.2.— *Let us take* $\delta \in W^{1,2}(0, 1; \mathbb{R}^n)$ *and* $\gamma \in W^{1,2}(0, 1; \mathbb{R}^n)$ *such that* $\gamma(s) \in M$, $\forall \, s \in [\,0\,,\,1\,]$.

Then the following facts hold:

a) $\displaystyle \lim_{t \to 0^+} \frac{\pi_s(\gamma + t\delta) - \gamma}{t} = P_\gamma \delta, \quad \forall \, s \in [\,0\,,\,1\,]$

  and

  $\displaystyle \limsup_{t \to 0^+} \left\| \frac{\pi_s(\gamma + t\delta) - \gamma}{t} \right\|_{L^\infty} < \infty$

b) for every sufficiently small $t > 0$, *we have*

  $\pi_s(\gamma + t\delta) \in W^{1,2}(0, 1; \mathbb{R}^n)$

  and a.e. in $]\,0\,,\,1\,[$

  $\left|\left(\pi_s(\gamma + t\delta)\right)'\right|_s \leq \left|\gamma' + t\delta'\right|_s + \text{const}\left|(\gamma + t\delta) - \pi_s(\gamma + t\delta)\right|\left(1 + |\gamma'| + t|\delta'|\right)$

c) $\displaystyle \lim_{t \to 0^+} \left(\pi_s(\gamma + t\delta)\right)' = \gamma'$ *in* $L^2(0, 1; \mathbb{R}^n)$.

Proof.— a) First of all, let us remark that $P_\gamma \delta$ is measurable. By proposition 1.12, we have

$$\lim_{t \to 0^+} \frac{\pi_s(\gamma + t\delta) - \gamma}{t} = P_\gamma \delta$$

and, also

$$\left|\frac{\pi_s(\gamma + t\delta) - \gamma}{t}\right|_s \leq 2 \left|\frac{\gamma + t\delta - \gamma}{t}\right|_s \leq 2\left|\delta\right|_s \qquad (2.2.1)$$

Then, by (2.2.1), we get

$$\left|\frac{\pi_s(\gamma + t\delta) - \gamma}{t}\right| \leq \text{const}\left|\frac{\pi_s(\gamma + t\delta) - \gamma}{t}\right|_s \leq$$

$$\leq \text{const} \, 2\left|\delta\right|_s \leq$$

$$\leq \text{const} \left|\delta\right| \qquad (2.2.2)$$

so, the proof of a) is over.

b) Let us consider the two scalar products

$$\left(u, v\right)_{s_i} = \left(A_{s_i} u, v\right) = \left(a\big(s_i, \gamma(s_i)\big) u, v\right) , \quad i = 1, 2 .$$

For every $u \in A$, let $w_{s_i} = \pi_{s_i} u$ the projection of u according to the scalar product $\left(u, v\right)_{s_i}$, $i = 1, 2$. We want to prove that

$$|w_{s_1} - w_{s_2}| \leq \frac{2}{\nu} |u - w_{s_2}| |A_{s_1} - A_{s_2}| . \qquad (2.2.3)$$

Let us observe that, by proposition 1.9, $u - w_{s_1} \in \partial_{s_1}^- I_M(w_{s_1})$, that is

$$\liminf_{v \to u, \, v \in M} \frac{\left(u - w_{s_1} , \, w_{s_1} - v\right)_{s_1}}{\left|w_{s_1} - v\right|_{s_1}} \geq 0 . \qquad (2.2.4)$$

Passing to the usual metric in $\mathbb{R}^n$, (2.2.4) is equivalent to

$$\liminf_{v \to u, \, v \in M} \frac{\left(A_{s_1}(u) - A_{s_1}(w_{s_1}), \, w_{s_1} - v\right)}{\left|w_{s_1} - v\right|} \geq 0 . \qquad (2.2.5)$$

By (2.2.5), it is easy to deduce that

$$A_{s_1}(u) - A_{s_1}(w_{s_1}) \in \partial^- I_M(w_{s_1}) .$$

Analogously,

$$A_{s_2}(u) - A_{s_2}(w_{s_2}) \in \partial^- I_M(w_{s_2}) .$$

Since M is a p-convex set, we have:

$$\left(A_{s_1}(u) - A_{s_1}(w_{s_1}) - A_{s_2}(u) + A_{s_2}(w_{s_2}), \, w_{s_1} - w_{s_2}\right) \geq$$
$$\geq -p \left(\left|A_{s_1}(u) - A_{s_1}(w_{s_1})\right| + \left|A_{s_2}(u) - A_{s_2}(w_{s_2})\right|\right) \left|w_{s_1} - w_{s_2}\right|^2 . \quad (2.2.6)$$

On the other hand, we have

$$\left(A_{s_1}(u) - A_{s_1}(w_{s_1}) - A_{s_2}(u) + A_{s_2}(w_{s_2}), \, w_{s_1} - w_{s_2}\right) \leq$$
$$\leq \left(A_{s_1}(u) - A_{s_2}(u) + A_{s_2}(w_{s_2}) - A_{s_1}(w_{s_2}), \, w_{s_1} - w_{s_2}\right) +$$
$$+ \left(A_{s_1}(w_{s_2}) - A_{s_1}(w_{s_1}), \, w_{s_1} - w_{s_2}\right) \leq$$
$$\leq \left((A_{s_1} - A_{s_2})(u - w_{s_2}), \, w_{s_1} - w_{s_2}\right) - \left|w_{s_1} - w_{s_2}\right|_{s_1}^2 . \qquad (2.2.7)$$

By (2.2.6) and (2.2.7), we obtain

$$
\begin{aligned}
\left|w_{s_1} - w_{s_2}\right|^2_{s_1} + \\
- p\Big(\left|A_{s_1}(u) - A_{s_1}(w_{s_1})\right| + \left|A_{s_2}(u) - A_{s_2}(w_{s_2})\right|\Big)\left|w_{s_1} - w_{s_2}\right|^2 \le \\
\le \left|u - w_{s_2}\right|\left|A_{s_1} - A_{s_2}\right|\left|w_{s_1} - w_{s_2}\right|.
\end{aligned}
$$

By hypothesis (1.1), it follows

$$
\left|w_{s_1} - w_{s_2}\right|^2_{s_1} \ge \nu\left|w_{s_1} - w_{s_2}\right|^2
$$

and then

$$
\begin{aligned}
\Big[\nu - p\Big(\left|A_{s_1}(u) - A_{s_1}(w_{s_1})\right| + \left|A_{s_2}(u) - A_{s_2}(w_{s_2})\right|\Big)\Big]\left|w_{s_1} - w_{s_2}\right|^2 \le \\
\le \left|u - w_{s_2}\right|\left|A_{s_1} - A_{s_2}\right|\left|w_{s_1} - w_{s_2}\right|. \quad\quad (2.2.8)
\end{aligned}
$$

By substituting A with a smaller open set containing M, we can assume that

$$
\Big[\nu - p\Big(\left|A_{s_1}(u) - A_{s_1}(w_{s_1})\right| + \left|A_{s_2}(u) - A_{s_2}(w_{s_2})\right|\Big)\Big] \ge \frac{\nu}{2},
$$

hence (2.2.3) follows.

Now, let us consider s_1, $s_2 \in \,]\,0\,,1\,[$. We have

$$
\begin{aligned}
\frac{\left|\pi_{s_1}(\gamma + t\delta)(s_1) - \pi_{s_2}(\gamma + t\delta)(s_2)\right|_{s_1}}{\left|s_2 - s_1\right|_{s_1}} \le \\
\le \frac{\left|\pi_{s_1}(\gamma + t\delta)(s_1) - \pi_{s_1}(\gamma + t\delta)(s_2)\right|_{s_1}}{\left|s_2 - s_1\right|_{s_1}} + \\
+ \frac{\left|\pi_{s_1}(\gamma + t\delta)(s_2) - \pi_{s_2}(\gamma + t\delta)(s_2)\right|_{s_1}}{\left|s_2 - s_1\right|_{s_1}}.
\end{aligned}
$$

By i) and ii) of proposition 1.9, we have

$$
\left|\pi_{s_1}(\gamma + t\delta)(s_1) - \pi_{s_1}(\gamma + t\delta)(s_2)\right|_{s_1} \le \frac{\left|(\gamma + t\delta)(s_1) - (\gamma + t\delta)(s_2)\right|_{s_1}}{\mathfrak{D}}
$$

where

$$
\begin{aligned}
\mathfrak{D} = \Big(1 - p\left|(\gamma + t\delta)(s_1) - \pi_{s_1}(\gamma + t\delta)(s_1)\right|_{s_1} + \\
- p\left|(\gamma + t\delta)(s_2) - \pi_{s_1}(\gamma + t\delta)(s_2)\right|_{s_1}\Big).
\end{aligned}
$$

Hence

$$\left|\pi_{s_1}(\gamma + t\delta)(s_1) - \pi_{s_1}(\gamma + t\delta)(s_2)\right|_{s_1} \leq$$
$$\leq \text{const}\left(\left|\gamma(s_1) - \gamma(s_2)\right| + t\left|\delta(s_1) - \gamma(s_2)\right|\right) .$$

Moreover, by applying (2.2.3), we get

$$\left|\pi_{s_1}(\gamma + t\delta)(s_2) - \pi_{s_2}(\gamma + t\delta)(s_2)\right|_{s_1} \leq$$
$$\leq \frac{2}{\nu}\left|(\gamma + t\delta)(s_2) - \pi_{s_2}(\gamma + t\delta)(s_2)\right| \left|a\big(s_1, \gamma(s_1)\big) - a\big(s_2, \gamma(s_2)\big)\right| \leq$$
$$\leq \text{const}\left|(\gamma + t\delta)(s_2) - \pi_{s_2}(\gamma + t\delta)(s_2)\right| \left(\left|s_2 - s_1\right| + \left|\gamma(s_2) - \gamma(s_1)\right|\right) \leq$$
$$\leq \text{const}\left(\left|s_2 - s_1\right| + \left|\gamma(s_2) - \gamma(s_1)\right|\right) .$$

Therefore

$$\pi_s(\gamma + t\delta) \in W^{1,2}(0,1;\mathbb{R}^n)$$

and we have a.e. in $]\,0\,,\,1\,[$

$$\lim_{s_2 \to s_1} \frac{\left|\pi_{s_1}(\gamma + t\delta)(s_1) - \pi_{s_1}(\gamma + t\delta)(s_2)\right|_{s_1}}{\left|s_2 - s_1\right|_{s_1}} \leq$$
$$\leq \frac{\left|(\gamma + t\delta)'(s_1)\right|_{s_1}}{\left(1 - 2p\left|(\gamma + t\delta)(s_1) - \pi_{s_1}(\gamma + t\delta)(s_1)\right|_{s_1}\right)} \leq$$
$$\leq \left|(\gamma + t\delta)'(s_1)\right|_{s_1} +$$
$$+ \text{const}\left|(\gamma + t\delta)(s_1) - \pi_{s_1}(\gamma + t\delta)(s_1)\right|\left|(\gamma + t\delta)'(s_1)\right| \quad (2.2.9)$$

and

$$\lim_{s_2 \to s_1} \frac{\left|\pi_{s_1}(\gamma + t\delta)(s_2) - \pi_{s_2}(\gamma + t\delta)(s_2)\right|_{s_1}}{\left|s_2 - s_1\right|_{s_1}} \leq$$
$$\leq \text{const} \lim_{s_2 \to s_1} \left|(\gamma + t\delta)(s_2) - \pi_{s_2}(\gamma + t\delta)(s_2)\right| \times$$
$$\times \frac{\left(\left|s_2 - s_1\right| + \left|\gamma(s_2) - \gamma(s_1)\right|\right)}{\left|s_2 - s_1\right|} \leq$$
$$\leq \text{const}\left|(\gamma + t\delta)(s_1) - \pi_{s_1}(\gamma + t\delta)(s_1)\right| \left(1 + \left|\gamma'(s_1)\right|\right) . \quad (2.2.10)$$

Hence we have a.e. in $]\,0\,,\,1\,[$

$$\left|\big(\pi_s(\gamma + t\delta)\big)'\right|_s \leq \left|\gamma' + t\delta'\right|_s + \text{const}\left|(\gamma + t\delta) - \pi_s(\gamma + t\delta)\right|\left(1 + \left|\gamma'\right| + t\left|\delta'\right|\right).$$

c) In $L^2(0, 1; \mathbb{R}^n)$ we can consider the following scalar product

$$(\eta, \xi) = \int_0^1 a\big(s, \gamma(s)\big) \eta(s) \xi(s) \, ds = \int_0^1 \big(\eta(s), \, \xi(s)\big)_s \, ds.$$

Since, by a) $\pi_s(\gamma + t\delta) \to \gamma$ in L^∞ and $\big(\pi_s(\gamma + t\delta)\big)'$ is bounded in $L^2(0, 1; \mathbb{R}^n)$ as $t \to 0$, we have that

$$\big(\pi_s(\gamma + t\delta)\big)' \text{ weakly converges to } \gamma' \text{ in } L^2(0, 1; \mathbb{R}^n). \qquad (2.2.11)$$

Moreover

$$\lim_{t \to 0^+} \big|(\gamma + t\delta) - \pi_s(\gamma + t\delta)\big| = 0 \text{ uniformly on } [\, 0 \, , 1\,],$$

and by b) we have

$$\limsup_{t \to 0} \int_0^1 \left|\big(\pi_s(\gamma + t\delta)\big)'\right|_s^2 \, ds \le \int_0^1 |\gamma'|_s^2 \, ds. \qquad (2.2.12)$$

Combining (2.2.11) and (2.2.12), we get

$$\lim_{t \to 0^+} \big(\pi_s(\gamma + t\delta)\big)' = \gamma' \text{ in } L^2(0, 1; \mathbb{R}^n). \ \square$$

LEMMA 2.3. — *Let us take $\delta \in W^{1,2}(0, 1; \mathbb{R}^n)$ and $\gamma \in W^{1,2}(0, 1; \mathbb{R}^n)$ such that $\gamma(s) \in M$, $\forall \, s \in [\, 0 \, , 1\,]$.*

Then

$$\liminf_{t \to 0^+} \frac{1}{t} \left\{ \frac{1}{2} \int_0^1 \big(a(s, \gamma)(\gamma + t\delta)', \, (\gamma + t\delta)'\big) \, ds + \right.$$

$$\left. -\frac{1}{2} \int_0^1 \big(a(s, \gamma)(\pi_s(\gamma + t\delta))', \, (\pi_s(\gamma + t\delta))'\big) \, ds \right\} \ge$$

$$\ge -\text{const} \int_0^1 |\gamma'|\big(1 + |\gamma'|\big)|\delta - P_\gamma \delta| \, ds. \qquad (2.3.1)$$

Proof. — Let us fix $t > 0$ and let us take the path $\gamma + t\delta$. If t is small, $(\gamma + t\delta)(s) \in A$, $\forall \, s \in [\, 0 \, , 1\,]$.

By lemma 2.2 b), we have $\pi_s(\gamma + t\delta) \in W^{1,2}(0, 1; M)$ and

$$\frac{1}{2} \int_0^1 \frac{1}{t} \left\{ |(\gamma + t\delta)'|_s^2 - \left|(\pi_s(\gamma + t\delta))'\right|_s^2 \right\} \, ds \ge$$

$$\ge \frac{1}{2} \int_0^1 \frac{1}{t} \left\{ -\text{const}\big|(\gamma + t\delta) - \pi_s(\gamma + t\delta)\big|^2 \big(1 + |\gamma'| + t|\delta'|\big)^2 + \right.$$

$$\left. -\text{const}\big|(\gamma + t\delta)'\big|\big|(\gamma + t\delta) - \pi_s(\gamma + t\delta)\big|\big(1 + |\gamma'| + t|\delta'|\big) \right\} \, ds.$$

Combining lemma 2.2 c) with Lebesgue theorem, we get:

$$\liminf_{t \to 0^+} \frac{1}{2} \int_0^1 \frac{1}{t} \left\{ \left| (\gamma + t\delta)' \right|_s^2 - \left| (\pi_s(\gamma + t\delta))' \right|_s^2 \right\} ds \geq$$

$$\geq -\operatorname{const} \int_0^1 |\gamma'| (1 + |\gamma'|) |\delta - P_\gamma \delta| \, ds \, . \quad \square$$

LEMMA 2.4.— *Let us take $\delta \in W^{1,2}(0, 1; \mathbb{R}^n)$ with $\delta(0) = \delta(1)$ and $\alpha \in \partial^- f(\gamma)$. Then*

$$\int_0^1 \left(a(s, \gamma)\gamma' \, , \, \delta' \right) ds - \int_0^1 (\alpha, P_\gamma \delta) \, ds \geq$$

$$\geq -\operatorname{const} \int_0^1 |\gamma'| (1 + |\gamma'|) |\delta - P_\gamma \delta| \, ds +$$

$$+ \int_0^1 \left(\nabla_q V(s, \gamma), P_\gamma \delta \right) ds - \frac{1}{2} \int_0^1 \left(\frac{\partial a(s, \gamma)}{\partial q} (P_\gamma \delta)\gamma' \, , \, \gamma' \right) ds \, . \quad (2.4.1)$$

Proof. — Let us take $\delta \in W^{1,2}(0, 1; \mathbb{R}^n)$ with $\delta(0) = \delta(1)$ and $t > 0$ small enough that

$$\pi_s(\gamma + t\delta) \in M \, .$$

Let us observe that, by setting

$$f_1(\gamma) = \frac{1}{2} \int_0^1 \left(a(s, \gamma)\gamma' \, , \, \gamma' \right) ds \, ,$$

from remark 1.2, we deduce that $\alpha \in \partial^- f(\gamma)$ if and only if $\alpha = \widetilde{\alpha} - \nabla_q V(s, \gamma)$ where $\widetilde{\alpha} \in \partial^- f_1(\gamma)$.

Now, let us take $\alpha \in \partial^- f(\gamma)$. By proposition 1.12, we have:

$$\int_0^1 \left(a(s, \gamma)\gamma' \, , \, \delta' \right) ds - \int_0^1 (\alpha, P_\gamma \delta) \, ds =$$

$$= \lim_{t \to 0^+} \frac{1}{t} \int_0^1 \left\{ \frac{1}{2} (a(s, \gamma)(\gamma + t\delta)' \, , \, (\gamma + t\delta)') - \frac{1}{2} (a(s, \gamma)\gamma' \, , \, \gamma') + \right.$$

$$\left. - \widetilde{\alpha} \left(\pi_s(\gamma + t\delta) - \gamma \right) \right\} ds + \int_0^1 \left(\nabla_q V(s, \gamma), P_\gamma \delta \right) ds \geq$$

$$\geq \liminf_{t \to 0^+} \frac{1}{t} \int_0^1 \left\{ \frac{1}{2} \left(a(s, \pi_s(\gamma + t\delta)) (\pi_s(\gamma + t\delta))' \, , \, (\pi_s(\gamma + t\delta))' \right) + \right.$$

$$- \frac{1}{2}(a(s,\gamma)\gamma', \gamma') - \tilde{\alpha}(\pi_s(\gamma + t\delta) - \gamma) \Big\} \, ds +$$

$$+ \liminf_{t \to 0^+} \frac{1}{t} \int_0^1 \Big\{ \frac{1}{2} \big(a(s,\gamma)(\pi_s(\gamma + t\delta))', (\pi_s(\gamma + t\delta))' \big) +$$

$$- \frac{1}{2} \big(a(s, \pi_s(\gamma + t\delta))(\pi_s(\gamma + t\delta))', (\pi_s(\gamma + t\delta))' \big) \Big\} \, ds +$$

$$+ \liminf_{t \to 0^+} \frac{1}{t} \int_0^1 \Big\{ \frac{1}{2}(a(s,\gamma)(\gamma + t\delta)', (\gamma + t\delta)') +$$

$$- \frac{1}{2} \big(a(s,\gamma)(\pi_s(\gamma + t\delta))', (\pi_s(\gamma + t\delta))' \big) \Big\} \, ds +$$

$$+ \int_0^1 (\nabla_q V(s,\gamma), P_\gamma \delta) \, ds.$$

Recalling that $(\pi_s(\gamma + t\delta) - \gamma)/t$ is bounded in $L^2(0, 1; \mathbb{R}^n)$, by lemma 2.2 c), proposition 1.12 and Lebesgue theorem, we get

$$\liminf_{t \to 0^+} \frac{1}{t} \int_0^1 \Big\{ \frac{1}{2} \big(a(s,\gamma)(\pi_s(\gamma + t\delta))', (\pi_s(\gamma + t\delta))' \big) +$$

$$- \frac{1}{2} \big(a(s, \pi_s(\gamma + t\delta))(\pi_s(\gamma + t\delta))', (\pi_s(\gamma + t\delta))' \big) \Big\} \, ds \geq$$

$$\geq - \frac{1}{2} \int_0^1 \left(\frac{\partial a(s,\gamma)}{\partial q} (P_\gamma \delta)\gamma', \gamma' \right) ds.$$

Then, by definition 1.1 and lemma 2.3, we get the thesis. $\square$

LEMMA 2.5. — *Let us take* $\alpha \in L^2(0, 1; \mathbb{R}^n)$ *and* $\gamma \in W^{1,2}(0, 1; \mathbb{R}^n)$ *such that* $\gamma(s) \in M$, $\forall s \in [0, 1]$. *Let us suppose that (2.4.1) holds* $\forall \delta \in W^{1,2}(0, 1; \mathbb{R}^n)$ *with* $\delta(0) = \delta(1)$.

Then

i) $\gamma \in W^{2,2}(0, 1; \mathbb{R}^n)$; $\quad \gamma'_+(0) = \gamma'_-(1)$

ii) $\alpha + \dfrac{d}{ds}(a(s,\gamma)\gamma') + \nabla_q V(s,\gamma) - \dfrac{1}{2} \dfrac{\partial a(s,\gamma)}{\partial q}(\gamma', \gamma') \in \partial^- I_M(\gamma(s))$ a.e.

iii) $|\gamma''(s)| \leq \text{const}\left(1 + |\alpha(s)| + |\gamma'(s)|^2\right)$ a.e.

Proof. — Since $|\delta - P_\gamma \delta| \leq |\delta|$, by applying Cauchy-Schwartz inequality to (2.4.1), we obtain $\forall \, \delta \in W^{1,2}(0,1; \mathbb{R}^n)$ with $\delta(0) = \delta(1)$

$$\int_0^1 (a(s,\gamma)\gamma', \, \delta') \, ds \geq -\|\alpha\|_{L^2} \|\delta\|_{L^2} - \text{const} \int_0^1 |\gamma'| \, |(1 + |\gamma'|)| \, |\delta| \, ds +$$

$$- \int_0^1 |\nabla_q V(s,\gamma)| \, |\delta| \, ds - \frac{1}{2} \int_0^1 \left| \frac{\partial a(s,\gamma)}{\partial q} \right| |\delta| \, |\gamma'|^2 \, ds$$

and then

$$\left| \int_0^1 (a(s,\gamma)\gamma', \, \delta') \, ds \right| \leq \left(\|\alpha\|_{L^2} + \text{const} \int_0^1 |\gamma'| \, ds + \right.$$

$$\left. + \text{const} \int_0^1 |\gamma'|^2 \, ds + \text{const} \right) \|\delta\|_{L^\infty} . \quad (2.5.1)$$

By (2.5.1), we deduce that $a(s,\gamma)\gamma' \in L^\infty$ and

$$\|a(s,\gamma)\gamma'\|_{L^\infty} \leq \|a(s,\gamma)\gamma'\|_{L^1} + \|\alpha\|_{L^2} +$$

$$+ \text{const} \int_0^1 |\gamma'| \, ds + \text{const} \int_0^1 |\gamma'|^2 \, ds + \text{const} . \quad (2.5.2)$$

On the hand, by hypothesis (1.1)

$$\nu|\gamma'|^2 \leq (a(s,\gamma)\gamma', \, \gamma') \leq |a(s,\gamma)\gamma'| \, |\gamma'|$$

which implies

$$\nu\|\gamma'\|_{L^\infty} \leq \|a(s,\gamma)\gamma'\|_{L^\infty} .$$

Thus, $\gamma' \in L^\infty$ and

$$\|\gamma'\|_{L^\infty} \leq \frac{1}{\nu} \left(\|a(s,\gamma)\gamma'\|_{L^1} + \|\alpha\|_{L^2} + \text{const} \int_0^1 |\gamma'| \, ds + \right.$$

$$\left. + \text{const} \int_0^1 |\gamma'|^2 \, ds + \text{const} \right) . \quad (2.5.3)$$

By using (2.5.3), we have

$$\int_0^1 |\gamma'|(1 + |\gamma'|)|\delta| \, ds \leq \|\delta\|_{L^2} \|\gamma'\|_{L^2} + \|\delta\|_{L^2} \|\gamma'\|_{L^4}^2 \leq$$

$$\leq \|\delta\|_{L^2} \|\gamma'\|_{L^2} + \|\delta\|_{L^2} \|\gamma'\|_{L^2} \|\gamma'\|_{L^\infty} \leq$$

$$\leq \|\gamma'\|_{L^2} \left(1 + \|\gamma'\|_{L^\infty} \right) \|\delta\|_{L^2} \leq$$

$$\leq \|\gamma'\|_{L^2} \left(1 + \frac{1}{\nu} \left(\text{const}\|\gamma'\|_{L^1} + \|\alpha\|_{L^2} + \right. \right.$$

$$\left. \left. + \text{const}\|\gamma'\|_{L^2}^2 + \text{const} \right) \right) \|\delta\|_{L^2} . \quad (2.5.4)$$

Since,

$$\frac{1}{2} \int_0^1 \left| \frac{\partial a(s,\gamma)}{\partial q} \right| |\delta| \, |\gamma'|^2 \, ds \le \text{const} \|\delta\|_{L^2} \|\gamma'\|_{L^4}^2$$

by (2.4.1) and (2.5.4), we have

$$\left| \int_0^1 (a(s,\gamma)\gamma', \delta') \, ds \right| \le$$
$$\le \left[\|\alpha\|_{L^2} + \text{const} \|\gamma'\|_{L^2} \left(1 + \|\gamma'\|_{L^1} + \|\alpha\|_{L^2} + \|\gamma'\|_{L^2}^2 \right) + \right.$$
$$\left. + \text{const} \right] \|\delta\|_{L^2} \le$$
$$\le \left[\left(1 + \text{const} \|\gamma'\|_{L^2} \right) \|\alpha\|_{L^2} + \right.$$
$$\left. + \text{const} \|\gamma'\|_{L^2} \left(1 + \|\gamma'\|_{L^1} + \|\gamma'\|_{L^2}^2 \right) + \text{const} \right] \|\delta\|_{L^2}.$$

So, we can conclude that

$$a(s,\gamma)\gamma' \in W^{1,2}(0,1,\mathbb{R}^n).$$

Moreover, by (2.4.1), we deduce that

$$\left| \frac{d}{ds} \left(a(s,\gamma)\gamma' \right) \right| \le \text{const} \left(1 + |\alpha| + |\gamma'|^2 \right) \quad \text{a.e.} \quad \text{in }]0,1[.$$

It remains to prove that $\gamma' \in W^{1,2}(0,1;\mathbb{R}^n)$. By hypothesis (1.1), we have

$$\nu^2 \frac{|\gamma'(s_2) - \gamma'(s_1)|^2}{|s_2 - s_1|} \le$$
$$\le \frac{|a(s_1,\gamma(s_1))(\gamma'(s_2) - \gamma'(s_1))|^2}{|s_2 - s_1|} = \frac{\mathfrak{N}}{|s_2 - s_1|} \le$$
$$\le 2 \frac{|a(s_2,\gamma(s_2))\gamma'(s_2) - a(s_1,\gamma(s_1))\gamma'(s_1)|^2}{|s_2 - s_1|} +$$
$$+ 2 \frac{|a(s_2,\gamma(s_2)) - a(s_1,\gamma(s_1))| \, |\gamma'(s_2)|^2}{|s_2 - s_1|}. \tag{2.5.6}$$

where

$$\mathfrak{N} = \left| a(s_2,\gamma(s_2))\gamma'(s_2) - a(s_1,\gamma(s_1))\gamma'(s_1) + \right.$$
$$\left. - [a(s_2,\gamma(s_2)) - a(s_1,\gamma(s_1))] \, \gamma'(s_2) \right|^2.$$

Since $a(\gamma)$, $a(s,\gamma)\gamma' \in W^{1,2}(0,1;\mathbb{R}^n)$ and $\gamma' \in L^\infty(0,1;\mathbb{R}^n)$, we deduce that

$$\gamma' \in W^{1,2}(0,1;\mathbb{R}^n)$$

and

$$|\gamma''| \leq \frac{1}{\nu}\left(\left|\frac{\mathrm{d}}{\mathrm{d}s}(a(s,\gamma)\gamma')\right| + \left|\frac{\mathrm{d}}{\mathrm{d}s}(a(s,\gamma))\right||\gamma'|\right) \leq$$
$$\leq \mathrm{const}\left(1 + |\alpha| + |\gamma'|^2\right) \quad \text{a.e. in }]0,1[.$$

If we set

$$\widetilde{\gamma}(s) = \begin{cases} \gamma(s + \frac{1}{2}) & 0 \leq s \leq \frac{1}{2} \\ \gamma(s - \frac{1}{2}) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

it turns out that $\widetilde{\gamma}$ also satisfies (2.4.1) with a, V and α substituted by other suitable maps. It follows that

$$\widetilde{\gamma} \in W^{2,2}(0,1;\mathbb{R}^n),$$

hence

$$\gamma'_+(0) = \gamma'_-(1).$$

Since $\partial^- I_M(\gamma(s))$ is a closed convex cone, to prove ii) is equivalent to prove that a.e. $\forall \eta \in \left(\partial^- I_M(\gamma(s))\right)^-$

$$\left(\alpha + \frac{\mathrm{d}}{\mathrm{d}s}(a(s,\gamma)\gamma') + \nabla_q V(s,\gamma) - \frac{1}{2}\frac{\partial a(s,\gamma)}{\partial q}(\gamma',\gamma'), \eta\right) \leq 0.$$

Let us define the following functions:

$$\delta_n(s) = \rho_n(s - s_0)\eta, \quad \forall n \in \mathbb{N}$$

where

$$\begin{cases} s_0 \in \,]0,1[\text{ is a Lebesgue point for } \alpha \text{ and for } \frac{\mathrm{d}}{\mathrm{d}s}(a(s,\gamma)\gamma'); \\ \eta \in \left(\partial^- I_M(\gamma(s_0))\right)^-; \\ \rho_n \in C_0^\infty(\mathbb{R}), \quad \rho_n \geq 0, \quad \int_0^1 \rho_n \, \mathrm{d}s = 1 \text{ and } \mathrm{supt}\,\rho_n \subset \left[-\frac{1}{n}, \frac{1}{n}\right]. \end{cases}$$

Clearly, $\delta_n \in W_0^{1,2}(0, 1; \mathbb{R}^n)$ and then by (2.4.1) we get

$$-\left(\eta, \int_0^1 \rho_n(s_0 - s)\frac{\mathrm{d}}{\mathrm{d}s}\big(a(s, \gamma)\gamma'\big)\,\mathrm{d}s\right) \geq$$

$$\geq \int_0^1 \rho_n(s_0 - s)\big(\alpha(s), P_\gamma\eta\big)(s)\,\mathrm{d}s +$$

$$-\operatorname{const}\int_0^1 \rho_n(s_0 - s)\big|\eta - P_\gamma\eta\big|\,|\gamma'|\,\mathrm{d}s +$$

$$-\operatorname{const}\int_0^1 \rho_n(s_0 - s)\big|\eta - P_\gamma\eta\big|\,|\gamma'|^2\,\mathrm{d}s +$$

$$+\int_0^1 \rho_n(s_0 - s)\big(\nabla_q V(s, \gamma),\, P_\gamma\eta\big)(s)\,\mathrm{d}s +$$

$$-\frac{1}{2}\int_0^1 \rho_n(s_0 - s)\left(\frac{\partial a(s, \gamma)}{\partial q}(P_\gamma\eta)\gamma',\, \gamma'\right)\,\mathrm{d}s. \qquad (2.5.7)$$

By proposition 1.13, $P_\gamma\delta$ is continuous at s_0, hence passing to the limit as $s \to s_0$ in (2.5.7), we obtain

$$-\left(\eta, \frac{\mathrm{d}}{\mathrm{d}s}\big(a\big(s_0, \gamma(s_0)\big)\gamma'(s_0)\big)\right) \geq$$

$$\geq \big(\alpha(s_0),\, (P_\gamma\eta)(s_0)\big) - \operatorname{const}|\eta - P_\gamma\eta|(s_0)|\gamma'|^2(s_0) +$$

$$-\operatorname{const}|\eta - P_\gamma\eta|(s_0)|\gamma'|(s_0) + \big(\nabla_q V\big(s_0, \gamma(s_0)\big),\, P_\gamma\eta\big)(s_0) +$$

$$-\frac{1}{2}\left(\frac{\partial a\big(s_0, \gamma(s_0)\big)}{\partial q}(P_\gamma\eta)(s_0)\gamma'(s_0),\, \gamma'(s_0)\right) =$$

$$= \big(\alpha(s_0),\, \eta(s_0)\big) + \big(\nabla_q V\big(s_0, \gamma(s_0)\big),\, P_\gamma\eta\big)(s_0) +$$

$$-\frac{1}{2}\left(\frac{\partial a\big(s_0, \gamma(s_0)\big)}{\partial q}(P_\gamma\eta)(s_0)\gamma'(s_0),\, \gamma'(s_0)\right) \quad \text{a.e. in }]0, 1[. \ \square$$

Finally, we are able to prove theorem 2.1.

Proof of theorem 2.1

As a direct consequence of lemmas 2.4 and 2.5, we get

$$\gamma \in W^{2,2}(0, 1; \mathbb{R}^n), \quad \gamma'_+(0) = \gamma'_-(1)$$

and

$$\big|\gamma''(s)\big| \leq \operatorname{const}\left(1 + |\alpha(s)| + \big|\gamma'(s)\big|^2\right) \quad \text{a.e. in }]0, 1[.$$

If $0 \in \partial^- f(\gamma)$, it is evident that

$$\gamma'' \in L^\infty(0, 1; \mathbb{R}^n) \, . \, \square$$

Now, let us prove two properties of f.

THEOREM 2.6. — *The functional $f : L^2(0, 1; \mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and there exists a continuous function $\theta : \mathbb{R} \to \mathbb{R}$ such that*

$$f(\gamma + \delta) - f(\gamma) - \int_0^1 (\alpha, \delta) \, \mathrm{d}s \geq -\theta\big(f(\gamma)\big) \left(1 + \|\alpha\|_{L^2}^2\right) \|\delta\|_{L^2}^2$$

whenever $\gamma, \, \gamma + \delta \in X$ and $\alpha \in \partial^- f(\gamma)$.

In particular, f has a φ-monotone subdifferential of order two.

Proof. — Let us take a sequence $\{\gamma_n\}_n \subset X$ such that

$$\lim_n \gamma_n = \gamma \text{ in } L^2(0, 1; \mathbb{R}^n) \text{ and } f(\gamma_n) \leq c \, .$$

In order to prove that f is lower semicontinuous, it is enough to prove that $f(\gamma) \leq c$. Since,

$$\frac{1}{2} \int_0^1 (a(s, \gamma_n)\gamma_n', \, \gamma_n') \, \mathrm{d}s - \int_0^1 V(s, \gamma_n) \, \mathrm{d}s \leq c$$

and by hypothesis (1.1)

$$\int_0^1 (a(s, \gamma_n)\gamma_n', \, \gamma_n') \, \mathrm{d}s \geq \nu \int_0^1 |\gamma_n'|^2 \, \mathrm{d}s$$

we can deduce that $\{\gamma_n'\}_n$ is bounded in $L^2(0, 1; \mathbb{R}^n)$ and thus, by the compactness of M, γ_n weakly converges to γ in $W^{1,2}(0, 1; \mathbb{R}^n)$. Besides, L is continuous in the three variables and convex in the third one, so it is weakly lower semicontinuous in $W^{1,2}(0, 1; \mathbb{R}^n)$. This implies

$$\frac{1}{2} \int_0^1 (a(s, \gamma)\gamma', \, \gamma') \, \mathrm{d}s - \int_0^1 V(s, \gamma) \, \mathrm{d}s \leq$$

$$\leq \liminf_n \frac{1}{2} \int_0^1 (a(s, \gamma_n)\gamma_n', \, \gamma_n') \, \mathrm{d}s - \int_0^1 V(s, \gamma_n) \, \mathrm{d}s \leq c \, .$$

Moreover, $\{\gamma_n\}_n$ converges uniformly to γ. Since M is closed, then $\gamma \in X$.

Now, let us take $\gamma \in X \cap W^{2,2}(0,1;\mathbb{R}^n)$, with $\gamma'_+(0) = \gamma'_-(1)$ and $\alpha \in \partial^- f(\gamma)$. Let $\delta \in W^{1,2}(0,1;\mathbb{R}^n)$ with $\delta(0) = \delta(1)$ be such that $\gamma + \delta \in X$. Then, by Taylor's formula, we have:

$$f(\gamma + \delta) - f(\gamma) - \int_0^1 (\alpha, \delta)\, ds =$$

$$= \frac{1}{2} \int_0^1 (a(s, \gamma + \delta)(\gamma' + \delta'), (\gamma' + \delta'))\, ds - \frac{1}{2} \int_0^1 (a(s, \gamma)\gamma', \gamma')\, ds +$$

$$- \int_0^1 V(s, \gamma + \delta)\, ds + \int_0^1 V(s, \gamma)\, ds - \int_0^1 (\alpha, \delta)\, ds =$$

$$= \frac{1}{2} \int_0^1 \left(\frac{\partial a(s,\gamma)}{\partial q}(\delta)\gamma', \gamma' \right) ds - \int_0^1 (\nabla_q V(s, \gamma), \delta)\, ds +$$

$$+ \int_0^1 (a(s,\gamma)\gamma', \delta')\, ds - \int_0^1 (\alpha, \delta)\, ds +$$

$$+ \frac{1}{4} \int_0^1 \left(\frac{\partial^2 a(s,\widetilde\gamma)}{\partial q^2}(\delta)^2 \gamma', \gamma' \right) ds - \frac{1}{2} \int_0^1 (\nabla^2_{qq} V(s,\gamma)\delta, \delta)\, ds +$$

$$+ \frac{1}{4} \int_0^1 \left(\frac{\partial a(s,\widetilde\gamma)}{\partial q}(\delta), \delta' \right) ds + \frac{1}{4} \int_0^1 (a(s,\gamma)\delta', \delta')\, ds$$

where $\widetilde\gamma = \gamma + t\delta$ for some $t = t(s) \in\,]0, 1[$. Reordering terms, by hypothesis (1.1), we get:

$$f(\gamma + \delta) - f(\gamma) - \int_0^1 (\alpha, \delta)\, ds \geq$$

$$\geq \frac{1}{2} \int_0^1 \left(\frac{\partial a(s,\gamma)}{\partial q}(\gamma', \gamma') - \nabla_q V(s,\gamma) - \frac{d}{ds}(a(s,\gamma)\gamma') - \alpha, \delta \right) ds +$$

$$- \left(\frac{1}{4} \left\| \frac{\delta^2 a(s,\gamma)}{\partial q^2} \right\|_{L^\infty} \|\gamma'\|^2_{L^2} + \frac{1}{2} \|\nabla_{qq} V(s,\gamma)\|_{L^\infty} \right) \|\delta\|^2_{L^\infty} +$$

$$- \frac{1}{4} \left\| \frac{\delta a(s,\gamma)}{\partial q} \right\|_{L^\infty} \|\delta\|_{L^2} \|\delta'\|_{L^2} + \frac{\nu}{2} \|\delta'\|^2_{L^2}\,.$$

Thus, by p-convexity of M, we get:

$$f(\gamma + \delta) - f(\gamma) - \int_0^1 (\alpha, \delta)\, ds \geq$$

$$\geq - \left(\frac{1}{2} \left\| \frac{\partial a(s,\gamma)}{\partial q} \right\|_{L^\infty} \|\gamma'\|_{L^2} + \|\nabla_q V(s,\gamma)\| + \right.$$

$$+ \left. \left\| \frac{d}{ds}(a(s,\gamma))\gamma' + a(s,\gamma)\gamma'' \right\|_{L^1} + \|\alpha\|_{L^1} \right) \|\delta\|_{L^\infty} +$$

$$- \text{const} \|\gamma'\|_{L^2}^2 \|\delta\|_{L^\infty}^2 - \text{const} \|\delta\|_{L^\infty} \|\delta'\|_{L^2} + \frac{\nu}{2} \|\delta'\|_{L^2}^2 \geq$$

$$\geq - \text{const} \left(\|\gamma''\|_{L^1} + \|\alpha\|_{L^1} \right) \|\delta\|_{L^\infty} - \text{const} \|\gamma'\|_{L^2}^2 \|\delta\|_{L^\infty}^2 +$$

$$- \text{const} \|\delta\|_{L^\infty} \|\delta'\|_{L^2} + \frac{\nu}{2} \|\delta'\|_{L^2}^2 .$$

Using the interpolation inequality:

$$\|\delta\|_{L^\infty}^2 \leq \|\delta\|_{L^2}^2 + 2 \|\delta\|_{L^2} \|\delta'\|_{L^2}$$

we obtain:

$$f(\gamma + \delta) - f(\gamma) - \int_0^1 (\alpha, \delta) \, ds \geq$$

$$\geq - \text{const} \left(\|\gamma''\|_{L^1} + \|\alpha\|_{L^1} \right) \left(\|\delta\|_{L^2}^2 + 2 \|\delta\|_{L^2} \|\delta'\|_{L^2} \right) +$$

$$- \text{const} \|\gamma'\|_{L^2} \left(\|\delta\|_{L^2}^2 + 2 \|\delta\|_{L^2} \|\delta'\|_{L^2} \right) +$$

$$- \text{const} \left(\|\delta\|_{L^2} \|\delta'\|_{L^2} + \|\delta\|_{L^2}^{1/2} \|\delta'\|_{L^2}^{3/2} \right) + \frac{\nu}{2} \|\delta'\|_{L^2}^2 .$$

Thus, by theorem 2.1 and hypothesis 1.1, we are able to conclude that:

$$f(\gamma + \delta) - f(\gamma) - \int_0^1 (\alpha, \delta) \, ds \geq -\theta \big(f(\gamma) \big) \left(1 + \|\alpha\|^2 \right) \|\delta\|_{L^2}^2 . \ \square$$

THEOREM 2.7. — *Let us consider* $\alpha \in L^2(0,1; \mathbb{R}^n)$ *and* $\gamma \in X \cap W^{2,2}(0,1; \mathbb{R}^n)$ *with* $\gamma^+(0) = \gamma^-(1)$. *Then* $\alpha \in \partial^- f(\gamma)$ *if and only if*

$$\alpha(s) + \frac{d}{ds} \big(a(s, \gamma) \gamma' \big) + \nabla_q V(s, \gamma) - \frac{1}{2} \frac{\partial a(s, \gamma)}{\partial q} (\gamma', \gamma') \in \partial^- I_M \big(\gamma(s) \big) \quad \text{a.e.}$$

Proof. — If $\alpha \in \partial^- f(\gamma)$, we have the thesis by lemmas 2.4 and 2.5. Viceversa, if

$$\alpha(s) + \frac{d}{ds} \big(a(s, \gamma) \gamma' \big) + \nabla_q V(s, \gamma) - \frac{1}{2} \frac{\partial a(s, \gamma)}{\partial q} (\gamma', \gamma') \in \partial^- I_M \big(\gamma(s) \big) \quad \text{a.e.}$$

the proof of theorem 2.6 shows that $\alpha \in \partial^- f(\gamma)$. $\square$

Finally, we can state the already quoted characterization.

THEOREM 2.8. — *Let us consider* $\gamma \in X$. *Then* $0 \in \partial^- f(\gamma)$ *if and only if* $\gamma \in W^{2,\infty}$, $\gamma'_+(0) = \gamma'_-(1)$ *and*

$$\alpha(s) + \frac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\gamma'\big) + \nabla_q V(s,\gamma) - \frac{1}{2}\frac{\partial a(s,\gamma)}{\partial q}\big(\gamma',\gamma'\big) \in \partial^- I_M\big(\gamma(s)\big) \text{ a.e.}$$

Proof. — If $0 \in \partial^- f(\gamma)$, from theorem 2.1 $\gamma \in W^{2,\infty}(0,1;\mathbb{R}^n)$ and $\gamma^+(0) = \gamma^-(1)$. Moreover from theorem 2.7

$$\alpha(s) + \frac{\mathrm{d}}{\mathrm{d}s}\big(a(s,\gamma)\gamma'\big) + \nabla_q V(s,\gamma) - \frac{1}{2}\frac{\partial a(s,\gamma)}{\partial q}\big(\gamma',\gamma'\big) \in \partial^- I_M\big(\gamma(s)\big) \text{ a.e.}$$

Viceversa, it is enough to apply theorem 2.7 with $\alpha = 0$. $\square$

3. The category of the space of the admissible paths

After theorem 2.8, our goal is to prove the existence of infinitely many lower critical points for f on X by means to theorem 1.7. Therefore, let us investigate the topological properties of X.

If Y is a topological space, we will denote by $\Lambda(Y)$ the free loop space of Y.

Let us recall that in [5], using results contained in [12], [13] and in [19], it is proved the following theorem.

THEOREM 3.1. — (see theorem 3.3 in [5]) *Let A be an open subset of* $\mathbb{R}^n$, *connected and non-contractible in itself. Moreover, let us suppose that either*

i) $\pi_1(A)$ *has infinitely many conjugacy classes*

or

ii) $\pi_1(A)$ *has a finite number of elements.*

Then $\operatorname{cat} \Lambda(A) = +\infty$.

Now, let us consider X endowed with the $W^{1,2}$-topology and the space

$$\Lambda(M) = \big\{\gamma \mid [\,0\,,\,1\,] \to M, \gamma \text{ is continuous and } \gamma(0) = \gamma(1)\big\}$$

endowed with the uniform topology.

THEOREM 3.2. — (see theorem 4.5 in [4]) *The inclusion map* $i : X \to \Lambda(M)$ *is a homotopy equivalence.*

Now, we are to able evaluate the category of X.

THEOREM 3.3. — *Let $M \subset \mathbb{R}^n$ be a connected, non-contractible in itself, compact p-convex set. Let us suppose that either*

i) $\pi_1(M)$ *has infinitely many conjugacy classes*

or

ii) $\pi_1(M)$ *has a finite number of elements.*

Then $\text{cat}(X) = +\infty$.

Proof. — Let us consider A, the open subset of $\mathbb{R}^n$ defined in Remark 1.10. Clearly, M is a deformation retract of it. Then, A is homotopically equivalent to M and $\Lambda(A)$ is homotopically equivalent to $\Lambda(M)$. By applying theorems 3.1 and 3.2, the proof is over. □

Finally we are able to prove the main theorem.

Proof of theorem 1.4

We want to apply theorem 1.7. Let us consider the functional f defined in section 2. By hypothesis (1.1) and theorem 2.6, f is a lower semicontinuous function, bounded below and it has a φ-monotone subdifferential of order two.

Moreover, let us observe that $D(f) = X$ and that d^* induces the $W^{1,2}$-topology on X. By theorem 3.3, $\text{cat}(D(f), d^*) = +\infty$.

Now, we will consider a sequence $\{\gamma_h\}_h \subset D(\partial^- f)$ with $\sup_h f(\gamma_h) < +\infty$ and $\lim_h \text{grad}^- f(\gamma_h) = 0$. Since M is compact, $\{\gamma_h\}_h$ is bounded in $L^2(0, 1; \mathbb{R}^n)$. But also $\{\gamma_h'\}_h$ is bounded in $L^2(0, 1; \mathbb{R}^n)$. Thus, by Rellich's theorem, $\{\gamma_h\}_h$ has a subsequence converging in $L^2(0, 1; \mathbb{R}^n)$.

So, applying theorem 1.7 and theorem 2.8, the thesis follows. □

References

[1] BENCI (V.) . — *Periodic solutions of Lagrangian systems on a compact manifold,* J. Differential Equations **63** (1986), pp. 135-161.

[2] CANINO (A.) . — *On p-convex sets and geodesics,* J. Differential Equations **75** (1988), pp. 118-157.

[3] CANINO (A.) . — *Existence of a closed geodesic on p-convex sets,*
Ann. Inst. H. Poincaré, Anal. Non Linéaire 5 (1988), pp. 501-518.

[4] CANINO (A.) . — *Local properties of geodesics on p-convex sets,*
Ann. Mat. Pura Appl., (in press).

[5] CANINO (A.) . — *Periodic solutions of Lagrangian systems on manifolds with boundary,*
Nonlinear Anal., 16 (1991), pp. 567-586.

[6] CHOBANOV (G.), MARINO (A.) and SCOLOZZI (D.) . — *Evolution equation for the eigenvalue problem for the Laplace operator with respect to an obstacle,*
Rend. Accad. Naz. Sci. XL Mem. Mat. 5, (in press).

[7] CHOBANOV (G.), MARINO (A.) and SCOLOZZI (D.) . — *Multiplicity of eigenvalues for the Laplace operator with respect to an obstacle and non tangency conditions,*
Nonlinear Anal. 15 (1990), pp. 199-215.

[8] DE GIORGI (E.), DEGIOVANNI (M.), MARINO (A.) and TOSQUES (M.) . — *Evolution equations for a class of nonlinear operators,*
Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 75 (1983), pp. 1-8.

[9] DE GIORGI (E.), MARINO (A.) and TOSQUES (M.) . — *Problemi di evoluzione in spazi metrici e curve di massima pendenza,*
Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 68 (1980), pp. 180-187.

[10] DEGIOVANNI (M.) . — *Homotopical properties of a class of nonsmooth functions*
Ann. Mat. Pura Appl. (4) 156 (1990), pp. 37-71.

[11] DEGIOVANNI (M.), MARINO (A.) and TOSQUES (M.) . — *Evolution equations with lack of convexity,*
Nonlinear Anal. 9 (1985), pp. 1401-1443.

[12] FADELL (E.) and HUSSEINI (S.) . — *A note on the category of the free loop space,*
Proc. Amer. Math. Soc. 107 (1989), pp. 527-536.

[13] FADELL (E.) and HUSSEINI (S.) . — *Extending Serre's theorem on the category of loop spaces,*
Preprint, 1990.

[14] MARINO (A.) and SCOLOZZI (D.) . — *Geodetiche con ostacolo,*
Boll. Un. Mat. Ital. B(6) 2 (1983), pp. 1-31.

[15] SCOLOZZI (D.) . — *Esistenza e molteplicità di geodetiche con ostacolo e con estremi variabili,*
Ricerche Mat. 33 (1984), pp. 171-201.

[16] SCOLOZZI (D.) . — *Un teorema di esistenza di una geodetica chiusa su variatà con bordo,*
Boll. Un. Mat. Ital. A(6) 4 (1985), pp. 451-547.

[17] SCOLOZZI (D.) . — *Molteplicità di curve con ostacolo e stazionarie per una classe di funzionali non regolari,*
Preprint n° 69, Dip. Mat. Pisa, 1984.

[18] SCOLOZZI (D.) . — *Esistenza di una curva chiusa stazionaria e con ostacolo,*
Preprint n° 70, Dip. Mat. Pisa, 1984.

[19] SERRE (J.P.) . — *Homologie singulière des espaces fibrés,*
Ann. of Math. 54 (1951), pp. 425-505.