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Slant surfaces of codimension two

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RÉSUMÉ. — Une immersion isométrique d'une variété riemannienne dans une variété presque hermitienne est dite oblique si l'angle de Wirtinger est constant [1]. Le but de cet article est d'étudier et de caractériser les surfaces obliques dans le plan complexe $\mathbb{C}^2$ utilisant l'application de Gauss. Nos démontrons que chaque surface n'ayant aucun point tangent complexe dans une variété presque hermitienne $\widetilde{M}$ de dimension 4 est oblique par rapport à une structure presque complexe bien choisie sur $\widetilde{M}$.

ABSTRACT. — A slant immersion is an isometric immersion from a Riemannian manifold into an almost Hermitian manifold with constant Wirtinger angle [1]. In this article we study and characterize slant surfaces in the complex 2-plane $\mathbb{C}^2$ via the Gauss map. We also prove that every surface without complex tangent points in a 4-dimensional almost Hermitian manifold $\widetilde{M}$ is slant with respect to a suitably chosen almost complex structure on $\widetilde{M}$.

1. Introduction

Let $x : M \to \widetilde{M}$ be an isometric immersion of a Riemannian manifold M with Riemannian metric g into an almost Hermitian manifold $\widetilde{M}$ with an almost complex structure $\widetilde{J}$ and an almost hermitian metric $\widetilde{g}$. For each nonzero vector X tangent to M at $p \in M$, the angle $\theta(X)$ between $\widetilde{J}X$ and the tangent space T_pM of M at p is called the Wirtinger angle of X. The immersion x is said to be *general slant* if the Wirtinger angle $\theta(X)$ is constant (which is independent of the choice of $p \in M$ and $X \in T_pM$). In this case the angle θ is called the *slant angle* of the slant immersion.

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If x is a totally real (or Lagrangian) immersion, then $T_p^{\perp}M \supseteq \widetilde{J}(T_pM)$ for any $p \in M$ where $T_p^{\perp}M$ is the normal space of M in $\widetilde{M}$ at p. Thus a totally real immersion is a general slant immersion with Wirtinger angle $\theta \equiv \pi/2$. If M is also an almost Hermitian manifold with almost complex structure J, then the immersion $x : M \to \widetilde{M}$ is called *holomorphic* (respectily, *anti-holomorphic*) if we have

$$x_*(JX) = \widetilde{J}(x_*X) \quad (\text{respectively, } x_*(JX) = -\widetilde{J}(x_*X)) \tag{1.1}$$

for any $X \in T_pM$. It is clear that holomorphic and anti-holomorphic immersions are general slant immersions with $\theta \equiv 0$. A general slant immersion which is neither holomorphic nor anti-holomorphic is simply called a *slant immersion* [1]. In paragraph 2 we review the geometry of the Grassmannian $G(2,4)$ for later use. In paragraph 3 we investigate the relationship between 2-planes in the Euclidean 4-space E^4 and the complex structures on E^4. By applying the relationship we obtain a pointwise observation concerning slant surfaces. In paragraph 4 we study slant surfaces via their Gauss map. In particular we obtain in this section a new characterization of slant surfaces and we also prove that a non-minimal surface in E^4 can be slant with respect to at most four compatible complex structures on E^4. In paragraph 5 we prove that every Hermitian manifold is a proper slant surface with any prescribed slant angle with respect to a suitable almost complex structure. In the last section we define doubly slant surfaces and show that their Gaussian and the normal curvatures vanish identically.

2. Geometry of $G(2,4)$

In the section we recall some results concerning the geometry of the Grassmannian $G(2,4)$ of oriented 2-planes in E^4 (for details, see [2, 3, 7, 8]).

Let $E^m = (\mathbb{R}^m, \langle\cdot,\cdot\rangle)$ be the Euclidean m-space with the canonical inner product $\langle\cdot,\cdot\rangle$. Let $\{\epsilon_1, \ldots, \epsilon_m\}$ be the canonical basis of E^m. Then $\Psi := \epsilon_1 \wedge \cdots \wedge \epsilon_m$ gives the canonical orientation of E^m. For each $n \in \{1, \ldots, m\}$, the space $\bigwedge^n E^m$ is an $\binom{m}{n}$-dimensional real vector space with the inner product, also denote by $\langle\cdot,\cdot\rangle$, defined by

$$\langle X_1 \wedge \cdots \wedge X_n, Y_1 \wedge \cdots \wedge Y_n\rangle = \det(\langle X_i, X_j\rangle) \tag{2.1}$$

and be extended linearly. The two vector spaces $\bigwedge^n(E^m)^*$ and $(\bigwedge^n E^m)^*$ are identified in a natural way by

$$\Phi(X_1 \wedge \cdots \wedge X_n) = \Phi(X_1, \ldots, X_n) \tag{2.2}$$

for any $\Phi \in \bigwedge^n(E^m)^*$ and any $X_1, \ldots, X_n \in E^m$. The Grassmannian $G(n,m)$ of oriented n-planes in E^m was identified with the set $D_1(n,m)$ of unit decomposable n-vectors in $\bigwedge^n E^m$. The identification $\varphi : G(n,m) \to D_1(n,m)$ is given by $\varphi(V) = X_1 \wedge \cdots \wedge X_n$ for any positive orthonormal basis $\{X_i\}$ of $V \in G(n,m)$.

The star operator $* : \bigwedge^2 E^4 \to \bigwedge^2 E^4$ is defined by

$$\langle *\xi \,,\, \eta \rangle \Psi = \xi \bigwedge \eta \quad \text{for } \xi \,,\eta \in \bigwedge{}^2 E^4 . \tag{2.3}$$

If we regard a $V \in G(2,4)$ as an element in $D_1(2,4)$ via φ, we have $*V = V^\perp$, where $V^\perp$ is the oriented orthogonal complement of V in E^4. Since $*$ is a symmetric involution, $\bigwedge^2 E^4$ is decomposed into the following orthogonal direct sum :

$$\bigwedge{}^2 E^4 = \bigwedge{}^2_+ E^4 \oplus \bigwedge{}^2_- E^4 \tag{2.4}$$

of eigenspaces of $*$ with eigenvalues 1 and -1, respectively. Denote by π_+ and π_- the natural projections from $\bigwedge^2 E^4$ into $\bigwedge^2_+ E^4$ and $\bigwedge^2_- E^4$, respectively.

Given a positive orthonormal basis $\{e_1, \ldots, e_4\}$ of E^4, we put

$$\begin{aligned} \eta_1 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4), & \eta_2 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_3 - e_2 \wedge e_4), \\ \eta_3 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_4 + e_2 \wedge e_3), & \eta_4 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_2 - e_3 \wedge e_4), \\ \eta_5 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_3 + e_2 \wedge e_4), & \eta_6 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_4 - e_2 \wedge e_3). \end{aligned} \tag{2.5}$$

Then $\{\eta_1, \eta_2, \eta_3\}$ and $\{\eta_4, \eta_5, \eta_6\}$ are orthonormal bases of $\bigwedge^2_+ E^4$ and $\bigwedge^2_- E^4$ respectively. We shall orient $\bigwedge^2_+ E^4$ and $\bigwedge^2_- E^4$ so that these two bases are positive.

For any $\xi \in D_1(2,4)$ we have

$$\pi_+(\xi) = \frac{1}{2}(\xi + *\xi) \quad \text{and} \quad \pi_-(\xi) = \frac{1}{2}(\xi - *\xi). \tag{2.6}$$

If we denote by S_+^2 and S_-^2 the 2-spheres centered at the origin with radius $1/\sqrt{2}$ in $\bigwedge_+^2 E^4$ and $\bigwedge_-^2 E^4$, respectively, then we have

$$\pi_+ : D_1(2,4) \to S_+^2 \quad \text{and} \quad \pi_- : D_1(2,4) \to S_-^2 \tag{2.7}$$

and

$$D_1(2,4) = S_+^2 \times S_-^2 \,. \tag{2.8}$$

3. Complex structures on E^4

Let $\mathbb{C}^2 = (\mathbb{R}^2, \langle \cdot , \cdot \rangle , J_0)$ be the complex plane with the canonical (almost) complex structure J_0 defined by $J_0(a,b,c,d) = (-b,a,-d,c)$. J_0 is an orientation preserving isomorphism. We denote by $\mathcal{G}$ the set of all (almost) complex structures on E^4 which are compatible with the inner product $\langle \cdot , \cdot \rangle$, i.e.,

$$\begin{aligned} \mathcal{G} = \{ J : E^4 \to E^4 \mid & J \text{ is linear, } J^2 = -\operatorname{Id}, \\ & \langle JX, JY \rangle = \langle X, Y \rangle , \text{ for any } X,\ X \in E^4 \} . \end{aligned}$$

For each $J \in \mathcal{G}$, we choose an (orthonormal) J-basis $\{e_1, \ldots, e_4\}$ so that $Je_1 = e_2$, $Je_3 = e_4$. Two J-bases of the same complex structure J have the same orientation. By using the canonical orientation $\Psi = \epsilon_1 \wedge \cdots \wedge \epsilon_4$, we divide $\mathcal{G}$ into two disjoint subsets :

$$\mathcal{G}^+ = \{ J \in \mathcal{G} \mid J\text{-bases are positive,} \}$$

and

$$\mathcal{G}^- = \{ J \in \mathcal{G} \mid J\text{-bases are negative.} \}$$

For each $J \in \mathcal{G}$, there is a unique 2-vector $\zeta_J \in \bigwedge^2 E^4$ defined as follows.

Let Ω_J be the Kaehler form of J, i.e.,

$$\Omega_J(X,Y) = \langle X, JY \rangle , \tag{3.1}$$

for any X, $Y \in E^4$. The 2-vector ζ_J associated with J is defined to be the unique 2-vector satisfying

$$\langle \zeta_J \,,\, X \wedge Y \rangle = -\Omega_J(X,Y) , \quad \text{for any } X,\ Y \in E^4 \tag{3.2}$$

LEMMA 3.1.. — *The mapping*

$$\zeta : \mathcal{G} \to \bigwedge^2 E^4 \tag{3.3}$$

defined by $\zeta(J) = \zeta_J$ *gives rise to two bijections :*

$$\zeta : \mathcal{G}^+ \to S^2_+(\sqrt{2}) \quad \text{and} \quad \zeta : \mathcal{G}^- \to S^2_-(\sqrt{2})\,, \tag{3.4}$$

where $S^2_+(\sqrt{2})$ *and* $S^2_-(\sqrt{2})$ *are the 2-spheres centered at the origin with radius* $\sqrt{2}$ *in* $\bigwedge^2_+ E^4$ *and* $\bigwedge^2_- E^4$*, respectively.*

Proof.— Let $J \in \mathcal{G}$ and $\{e_1, \ldots, e_4\}$ be a J-basis. If $J \in \mathcal{G}^+$ (respectively, $J \in \mathcal{G}^-$), then $\{e_1, \ldots, e_4\}$ is positive (respectively, negative). Since $\zeta_J = e_1 \wedge e_2 + e_3 \wedge e_4$ by (3.1), ζ maps $\mathcal{G}^+$ into $S^2_+(\sqrt{2})$ and maps $\mathcal{G}^-$ into $S^2_-(\sqrt{2})$. Their injectivity are clear.

Conservely, for each $\xi \in S^2_+(\sqrt{2})$, we have $\frac{1}{2}\zeta \in S^2_+$. Hence, we can pick an oriented 2-plane V such that $V \in \pi_+^{-1}(\frac{1}{2}\zeta)$. Now we choose a positive orthonormal basis $\{e_1, \ldots, e_4\}$ such that $\{e_1, e_2\}$ is a positive basis of V. Let J be the complex structure on E^4 such that $Je_1 = e_2$, $Je_2 = -e_1$, $Je_3 = e_4$, $Je_4 = -e_3$. Then $J \in \mathcal{G}^+$ and $\zeta_J = \xi$.

If $\xi \in S^2_-(\sqrt{2})$, we pick $V \in \pi_-^{-1}(\frac{1}{2}\zeta)$ and define J by $Je_1 = e_2$, $Je_2 = -e_1$, $Je_3 = -e_4$, $Je_4 = e_3$. Then we have a similar result. □

In the following we identify $\mathcal{G}$, $\mathcal{G}^+$ and $\mathcal{G}^-$ with $S^2_+(\sqrt{2}) \cup S^2_-(\sqrt{2})$, $S^2_+(\sqrt{2})$, and $S^2_-(\sqrt{2})$, respectively, via ζ.

For each $V \in G(2,4)$ and each $J \in \mathcal{G}$, we define

$$\alpha_J(V) = \cos^{-1}\big(-\Omega_J(V)\big)\,. \tag{3.5}$$

Then $\alpha_J(V) \in [\,0\,,\,\pi\,]$. A 2-plane V is said to be *a-slant* if $\alpha_J(V) = a$.

The relation between $\theta(X)$ defined in paragraph 1 and $\alpha_J(V)$ is as follows. Let x be an isometric immersion of M into $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. If we regard $(T_p\widetilde{M}, \widetilde{g}, \widetilde{J})$ as a complex plane with the induced inner product $\langle\,\cdot\,,\,\cdot\,\rangle$, then we have

$$\theta(X) = \min\big\{\alpha_{\widetilde{J}}(T_pM)\,,\ \pi - \alpha_{\widetilde{J}}(T_pM)\big\} \tag{3.6}$$

for any non-zero vector $X \in T_pM$.

If M is oriented, then M has a unique complex structure J determined by its orientation and its metric induced from $\tilde{g}$. With respect to α_J, we have:

$$x \text{ is holomorphic} \Leftrightarrow \alpha_{\tilde{J}}(T_pM) \equiv 0\,, \tag{3.7}$$

$$x \text{ is anti-holomorphic} \Leftrightarrow \alpha_{\tilde{J}}(T_pM) \equiv \pi\,, \tag{3.8}$$

$$x \text{ is totally real} \Leftrightarrow \alpha_{\tilde{J}}(T_pM) \equiv \frac{\pi}{2}\,. \tag{3.9}$$

The argument above also hold for the case $\dim \widetilde{M} > 4$. We note that α_J coincides with the angle defined in [6].

LEMMA 3.2

(i) If $J \in \mathcal{G}^+$, then $\alpha_J(V)$ is the angle between $\pi_+(V)$ and ζ_J.

(ii) If $J \in \mathcal{G}^-$, then $\alpha_J(V)$ is the angle between $\pi_-(V)$ and ζ_J.

Proof.— If $J \in \mathcal{G}^+$, then, by (3.2), (3.5) and lemma 3.1, we have

$$\cos\big(\alpha_J(V)\big) = -\Omega_J(V) = \langle \zeta_J, V\rangle = \big\langle \zeta_J\,,\, \pi_+(V) + \pi_-(V)\big\rangle = \big\langle \zeta_J\,,\, \pi_+(V)\big\rangle$$

which is the cosine of the angle between $\pi_+(V)$ and ζ_J, since $||\zeta_J|| = \sqrt{2}$ and $\big\|\pi_+(V)\big\| = 1/\sqrt{2}$. Similar argument applies to the case $J \in \mathcal{G}^-$. □

For each $a \in [\,0\,,\,\pi\,]$ and $J \in \mathcal{G}$, we define

$$G_{J,a} = \big\{V \in G(2,4) \mid \alpha_J(V) = a\big\}\,, \tag{3.10}$$

i.e., $G_{J,a}$ is the set of all oriented 2-planes in E^4 which are a-slant with respect to J. Also for each $a \in [\,0\,,\,\pi\,]$ and $V \in G(2,4)$ we define

$$\mathcal{G}_{V,a} = \big\{J \in \mathcal{G} \mid \alpha_J(V) = a\big\}\,, \tag{3.11}$$

i.e., $\mathcal{G}_{V,a}$ is the set of all compatible complex structure on E^4 with respect to which V is a-slant. We put

$$\mathcal{G}^+_{V,a} = \mathcal{G}_{V,a} \cap \mathcal{G}^+ \quad \text{and} \quad \mathcal{G}^-_{V,a} = \mathcal{G}_{V,a} \cap \mathcal{G}^-\,. \tag{3.12}$$

By applying lemma 3.2 we obtain the following generalization of proposition 2 of [3].

PROPOSITION 3.3

(i) *If $J \in \mathcal{G}^+$, then $G_{J,a} = S^+_{J,a} \times S^2_-$ where $S^+_{J,a}$ is the circle on S^2_+ consisting of 2-vectors which make constant angle a with ζ_J.*

(ii) *If $J \in \mathcal{G}^-$, then $G_{J,a} = S^2_+ \times S^-_{J,a}$ where $S^-_{J,a}$ is the circle on S^2_- consisting of 2-vectors which make constant angle a with ζ_J.*

(iii) *Via the identification ζ given in (3.4), $\mathcal{G}^+_{V,a}$ is a circle on $S^2_+(\sqrt{2})$ consisting of 2-vectors in $S^2_+(\sqrt{2})$ which make constant angle a with $\pi_+(V)$. Similarly, $\mathcal{G}^-_{V,a}$ is a circle on $S^2_-(\sqrt{2})$ obtained in a similar way.*

For simplicity, for each $V \in G(2,4)$, we define J^+_V and J^-_V as the complex structures given by

$$J^+_V = \zeta^{-1}\big(\pi_+(V)\big) \quad \text{and} \quad J^-_V = \zeta^{-1}\big(\pi_-(V)\big)\,, \tag{3.13}$$

where ζ is the bijection given in lemma 3.1. It is clear that $J^+_V \in \mathcal{G}^+$ and $J^-_V \in \mathcal{G}^-$.

4. Slant surfaces and Gauss map

Let $x : M \to E^4$ be an isometric immersion from an oriented surface M into E^4. We denote by ∇ and $\tilde{\nabla}$ the Riemannian connections of M and E^4, respectively. We choose a positive orthonormal local frame field $\{e_1, \ldots, e_4\}$ in E^4 such that, restricted to M, the vectors e_1, e_2 give a positive frame field tangent to M (and hence e_3, e_4 are normal to M). Let $\omega^1, \ldots, \omega^4$ be the field of dual frame field. The structure equations of E^4 are then given by

$$\tilde{\nabla} e_A = \sum \omega^B_A \otimes e_B\,, \quad \omega^B_A + \omega^A_B = 0\,, \tag{4.1}$$

$$d\omega^B = \sum \omega^B_A \otimes \omega^A\,, \tag{4.2}$$

$$d\omega^A_B = -\sum \omega^A_C \wedge \omega^C_B\,, \quad A,\, B,\, C = 1,\, 2,\, 3,\, 4\,. \tag{4.3}$$

If we restrict these forms to M, then $\omega^3 = \omega^4 = 0$. From $d\omega^3 = d\omega^4 = 0$, (4.2), and Cartan's lemma, we have

$$\omega^r_i = \sum h^r_{ij}\omega^j\,, \quad h^r_{ij} = h^r_{ji}\,, \quad r = 3,\, 4\,; \quad i,\, j = 1,\, 2\,. \tag{4.4}$$

The Gauss curvature G and the normal curvature G^D of M in E^4 are given respectively by

$$G = \sum (h^r_{11}h^r_{22} - h^r_{12}h^r_{12}), \tag{4.5}$$

$$G^D = (h^3_{11} - h^3_{22})h^4_{12} - (h^4_{11} - h^4_{22})h^3_{12}. \tag{4.6}$$

In the following we denote by $\nu : M \to G(2,4)$ the *Gauss map* of the immersion x defined by

$$\nu(p) = (e_1 \wedge e_2)(p), \tag{4.7}$$

where we identify $G(2,4)$ with $D_1(2,4)$ consisting of unit decomposable 2-vectors in $\bigwedge^2 E^4$. We define two maps ν_+ and ν_- by

$$\nu_+ = \pi_+ \circ \nu \quad \text{and} \quad \nu_- = \pi_- \circ \nu, \tag{4.8}$$

where π_+ and π_- are the projections given in paragraph 2. Then ν_+ and ν_- map M into S^2_+ and S^2_-, respectively.

In terms of ν_+ and ν_-, we have the following characterization of slant surfaces in $\mathbb{C}^2$.

PROPOSITION 4.1.— *Let $x : M \to E^4$ be an isometric immersion from an oriented surface M into E^4. Then x is a slant immersion with respect to a complex structure $J \in \mathcal{G}^+$ (respectively, $J \in \mathcal{G}^-$) if and only if $\nu_+(M)$ is contained in a circle on S^2_+ (respectively, $\nu_-(M)$ is contained in a circle on S^2_-).*

Moreover, x is a-slant with respect to $J \in \mathcal{G}^+$ (respectively, $J \in \mathcal{G}^-$) if and only if $\nu_+(M)$ is contained in a circle $S^+_{J,a}$ on S^2_+ (respectively, $\nu_-(M)$ is contained in a circle $S^-_{J,a}$ on S^2_-), where $S^+_{J,a}$ and $S^-_{J,a}$ are the circles defined in proposition 3.3.

Proof.— If $x : M \to E^4$ is a-slant with respect to $J \in \mathcal{G}^+$, then by (3.11) and proposition 3.3, we have $\nu(T_pM) \in S^+_{J,a} \times S^2_-$ for any $p \in M$. Thus $\nu_+(M)$ is contained in a circle $S^+_{J,a}$ on S^2_+ consisting of 2-vectors in S^2_+ which make angle a with ζ_J.

Conversely, if $x : M \to E^4$ is an immersion such that $\nu_+(M)$ is contained in a circle S^1 on S^2_+. Let η be a vector in $\bigwedge^2 E^4$ with length $\sqrt{2}$ which is normal to the 2-plane in $\bigwedge^2 E^4$ containing S^1. Then $\eta \in S^2_+(\sqrt{2})$. By lemma 3.1, there is a unique $J \in \mathcal{G}^+$ such that $\zeta_J = \eta$. It is clear that S^1 is a $S^+_{J,a}$ for some constant angle a. Therefore, by proposition 3.3, the immersion x is a-slant with respect to $J \in \mathcal{G}^+$. Similar argument applies to the other case. □

The following lemma was obtained in [1] (given in the proof of theorem 1 of [1]). Here we reprove it by using Gauss map.

LEMMA 4.2.— *Let $x : M \to E^4$ be an isometric immersion from an surface M into E^4. Then M is minimal and slant with respect to some $J \in \mathcal{G}^+$ (respectively, $J \in \mathcal{G}^-$) if and only if $\nu_+(M)$ (respectively, $\nu_-(M)$) is a singleton.*

Proof.— If x is minimal, then both ν_+ and ν_- are anti-holomorphic [5]. In particular, ν_+ and ν_- are open maps if they are not constant maps. However, if x is slant with respect to $J \in \mathcal{G}^+$ (respectively, $J \in \mathcal{G}^-$), then, by proposition 4.1, ν_+ (respectively, ν_-) cannot be an open map. Hence, $\nu_+(M)$ (respectively, $\nu_-(M)$) is a singleton.

Conversely, if $\nu_+(M)$ (respectively, $\nu_-(M)$) is a singleton, say $\{\xi\}$. Then $2\xi \in S^2_+(\sqrt{2})$. Thus, by lemma 3.1, there is a unique $J = \zeta^{-1}(2\xi) \in \mathcal{G}^+$ such that $\nu_+(M)$ is contained in $S^+_{J,0}$. Thus x is holomorphic with respect to J (see proposition 2 of [3]), in particular, x is a minimal immersion. Because a singleton $\{\xi\}$ lies in every circle S^1 on S^2_+ through ξ, proposition 4.1 implies that for any $a \in [0, \pi]$, there exists a $J_a \in \mathcal{G}^+$ (respectively, $J_a \in \mathcal{G}^-$), such that x is a-slant with respect to J_a. □

By applying proposition 4.1 and lemma 4.2 we have the following result concerning non-minimal surfaces.

THEOREM 4.3.— *If $x : M \to E^4$ is not miminal, then there exist at most two complex structures $\pm J \in \mathcal{G}^+$ and at most two complex structures $\pm J' \in \mathcal{G}^-$ such that x is slant with respect them.*

Proof.— If x is a non-minimal, a-slant immersion with respect to a complex structure $J \in \mathcal{G}^+$ (respectively, $J' \in \mathcal{G}^-$), then, by proposition 4.1 and lemma 4.2, $\nu_+(M)$ (respectively, $\nu_-(M)$) contains an arc of the circle $S^+_{J,a}$ (respectively, $S^+_{J',a}$).

Thus, $\pm J$ and $\pm J'$ are the only possible complex structures which make x to be slant according to proposition 4.1. □

From proposition 4.1, lemma 4.2 and theorem 4.3 we have the following.

PROPOSITION 4.4.— *Let $x : M \to E^4$ be an isometric immersion from a surface M into E^4. Then the following statements are equivalent:*

(i) x in minimal and slant with respect to some complex structure $J \in \mathcal{G}^+$ (respectively, $J \in \mathcal{G}^-$);

(ii) $\nu_+(M)$ (respectively, $\nu_-(M)$) is a singleton;

(iii) x is holomorphic with respect some complex structure $J \in G^+$ (respectively, $J \in G^-$);

(iv) for each $a \in [0, \pi]$, there exist $J_a \in \mathcal{G}^+$ (respectively, $J_a \in \mathcal{G}^-$) such that x is a-slant with respect to J_a.

Theorem 4.3 and proposition 4.4 provide us a clear geometric understanding of lemmas 5 and 6 and theorems 1, 3 and 4 of [1].

5. Slant surfaces in 4-dimensional almost Hermitian manifolds

Let $x : M \to (\widetilde{M}, J)$ be an immersion of a manifold M into an almost complex manifold $(\widetilde{M}, J)$. Then a point $p \in M$ is called a *complex tangent point* if the tangent plane of M at p is invariant under the action of J. The purpose of this section is to prove the following.

THEOREM 5.1.— *Let $x : M \to (\widetilde{M}, g, J)$ be an imbedding from an oriented surface M into an almost Hermitian manifold $(\widetilde{M}, g, J)$. If x has no complex tangent point, for any prescribed angle $a \in (0, \pi)$, there exists an almost complex structure $\widetilde{J}$ on $\widetilde{M}$ satisfying the following conditions:*

(i) $(\widetilde{M}, g, \widetilde{J})$ is an almost Hermitian manifold, and

(ii) x is a-slant with respect to $\widetilde{J}$.

Proof.— $\widetilde{M}$ has a natural orientation determined by J. At each point $p \in \widetilde{M}$, $(T_p\widetilde{M}, g_p)$ is a Euclidean 4-space and so we can apply the argument given in paragraphs 2 and 3.

According to (2.4) the vector bundle $\bigwedge^2(\widetilde{M})$ of 2-vectors on $\widetilde{M}$ is a direct sum of two vector subbundles.

$$\bigwedge\nolimits^2(\widetilde{M}) = \bigwedge\nolimits^2_+(\widetilde{M}) \oplus \bigwedge\nolimits^2_-(\widetilde{M}). \tag{5.1}$$

We define two sphere-bundles over $\widetilde{M}$ by

$$\begin{aligned} S^2_+(\widetilde{M}) &= \left\{\xi \in \bigwedge\nolimits^2_+(\widetilde{M}) \mid \|\xi\| = \frac{1}{\sqrt{2}}\right\}, \\ \overline{S}^2_+(\widetilde{M}) &= \left\{\xi \in \bigwedge\nolimits^2_-(\widetilde{M}) \mid \|\xi\| = \sqrt{2}\right\}. \end{aligned} \tag{5.2}$$

By applying lemma 3.1 we can identify a cross-section

$$\gamma : \widetilde{M} \to \overline{S}^2_+(\widetilde{M}) \tag{5.3}$$

with an almost complex structure J_γ on $\widetilde{M}$ such that $(\widetilde{M}, g, J_\gamma)$ is an almost Hermitian manifold. In the following we denote by ρ the cross-section corresponding to J and we want to construct another cross-section $\tilde{\sigma}$ to obtain the desired almost complex structure $\tilde{J}$.

We consider the pull-backs of these bundles via the immersion x, i.e.,

$$\begin{aligned} \textstyle\bigwedge^2_+(M) &= x^*(\textstyle\bigwedge^2_+(\widetilde{M}))\,, \\ S^2_+(M) &= x^*(S^2_+(\widetilde{M}))\,, \\ \overline{S}^2_+(M) &= x^*(\overline{S}^2_+(\widetilde{M}))\,. \end{aligned} \tag{5.4}$$

The tangent bundle TM determines a cross-section $\tau : M \to S^2_+(M)$ given by

$$\tau(p) = \pi_+(T_pM)\,, \quad \text{for any } p \in M\,, \tag{5.5}$$

where π_+ is the projection of $\bigwedge^2(T_p\widetilde{M})$ onto $\bigwedge^2_+(T_p\widetilde{M})$. Note that 2τ is a cross-section of $\overline{S}^2_+(M)$;

$$2\tau : M \to \overline{S}^2_+(M)\,. \tag{5.6}$$

We denote $x^*\rho$ also by ρ, which gives us another cross-section:

$$\rho = x^*\rho : M \to \overline{S}^2_+(M)\,. \tag{5.7}$$

Since x has no complex tangent point,

$$\rho(p) \neq \pm 2\tau(p)\,, \quad \text{for any } p \in M\,. \tag{5.8}$$

Therefore, $\rho(p)$ and $2\tau(p)$ determine a 2-plane in $\bigwedge^2_+(T_p\widetilde{M})$ which intersects the circle $(\mathcal{G}^+_{\tau,a})_p$ at two points, where $(\mathcal{G}^+_{\tau,a})_p$ is the circle on $(\overline{S}^2_+(M))_p$ determined in proposition 3.3 with $V = T_pM$. Let $\sigma(p)$ be one of the two points which lies on the half-great-circle on $(\overline{S}^2_+(M))_p$ starting from $2\tau(p)$ and passing through $\rho(p)$. Since ρ and τ are differentiable, so is σ. Thus we get the third cross-section:

$$\sigma : M \to \overline{S}^2_+(M) \tag{5.9}$$

and we want to extend σ to a cross-section $\tilde{\sigma}$ of $\overline{S}^2_+(\widetilde{M})$.

For each $p \in M$, we choose an open neighborhood U_p of p in $\widetilde{M}$ such that $\sigma|_{U_p \cap M}$ can be extended to a cross-section of $\overline{S}^2_+(\widetilde{M})$ on U_p:

$$\sigma_p : U_p \to \overline{S}^2_+(\widetilde{M})|_{U_p} . \tag{5.10}$$

We identify here the manifold M with its image $x(M)$ via the imbedding. We put

$$U = \bigcup_{p \in M} U_p \tag{5.11}$$

and pick a locally finitely countable refinement $\{U_i\}_{i=1}^{\infty}$ of the open covering $\{U_p\}_{p \in M}$ of U. For each i we pick a point $p \in M$ such that U_i is contained in U_p and put

$$\sigma_i = \sigma_p|_{U_i} . \tag{5.12}$$

Let $\{\phi_i\}$ be a differentiable partition of unity on U subordinated to the covering $\{U_i\}$. We define a cross-section $\overline{\sigma}$ of $\bigwedge^2_+(\widetilde{M})|_U$ by

$$\overline{\sigma} : U \to \Lambda^2_+(\widetilde{M})|_U \quad ; \quad \overline{\sigma} = \sum \phi_i \sigma_i . \tag{5.13}$$

By the construction of σ_i and $\overline{\sigma}$ we have

$$\overline{\sigma}|_M = \sigma . \tag{5.14}$$

Since the angle $\angle(\overline{\sigma}(p), \rho(p)) < \pi$ for any $p \in M$, we have

$$\overline{\sigma}(p) \neq 0 , \quad \angle(\overline{\sigma}(p), \rho(p)) < \pi \text{ for any } p \in M . \tag{5.15}$$

By continuity of $\overline{\sigma}$ we can pick an open neighborhood W of M contained in U such that

$$\overline{\sigma}(q) \neq 0 , \quad \angle(\overline{\sigma}(q), \rho(q)) < \pi \text{ for any } q \in W . \tag{5.16}$$

We define a cross-section $\widehat{\sigma}$ of $\overline{S}^2_+(\widetilde{M})|_W$ by

$$\widehat{\sigma} : W \to \overline{S}^2_+(\widetilde{M})|_W ; \quad \widehat{\sigma} = \frac{\overline{\sigma}}{\sqrt{2}||\overline{\sigma}||} . \tag{5.17}$$

Then we have $\angle(\widehat{\sigma}(q), \rho(q)) < \pi$ for any $q \in M$ too. Finally, we consider the open covering $\{W, \widetilde{M} - M\}$ of $\widetilde{M}$ and local cross-section $\widehat{\sigma}$ and $\rho|_{\widetilde{M}-M}$ and repeat the same argument using a partition of unity subordiante to $\{W, \widetilde{M} - M\}$ to get a cross-section $\widetilde{\sigma} : \widetilde{M} \to \overline{S}^2_+(\widetilde{M})$ satisfying $\widetilde{\sigma}|_M = \sigma$. Now, it is clear that the almost complex structure $\widetilde{J}$ corresponding to $\widetilde{\sigma}$ is the desired almost complex structure. □

6. Appendix : double slant surfaces in $\mathbb{C}^2$

An immersion $x : M^2 \to E^4$ is called *doubly slant* if it is slant with respect to a complex structure $J \in \mathcal{G}^+$ and at the same time it is slant with respect to another complex structure $\widehat{J} \in \mathcal{G}^-$.

Equivalently, the immersion x is doubly slant if and only if there exists a $V \in G(2,4)$ such that x is slant with respect to both J^+_V and J^-_V, where J^+_V and J^-_V are defined by (3.13).

PROPOSITION 6.1.— *If $x : M^2 \to E^4$ is a doubly slant immersion, then* $G = G^D = 0$.

Proof.— If x is doubly slant, then, by proposition 4.1, we know that $\nu_+(M)$ and $\nu_-(M)$ both lie in some circles on S^2_+ and S^2_-, respectively. Thus, both $(\nu_+)_*$ and $(\nu_-)_*$ are singular at every point $p \in M$. The result then follows from the following lemma of [7].

LEMMA 6.2.— *For an isometric immersion $x : M^2 \to E^4$, we have*

$$\textit{Jacobian of } \nu_+ = \frac{G^D + G}{2} \quad ; \quad \textit{Jacobian of } \nu_- = \frac{G^D - D}{2}.$$

This lemma can be proved in our notation as follows.

From (4.8) we have $\nu_* X = (\widetilde{\nabla}_X e_1) \wedge e_2 + e_1 \wedge (\widetilde{\nabla}_X e_2)$ for any X tangent to M. Thus, by applying (2.5), (4.1) and (4.4), we find (see [3])

$$\begin{aligned} \nu_* X = \frac{1}{\sqrt{2}} \Big\{ &(\omega_1^4 + \omega_2^3)(X)\eta_2 + (-\omega_1^3 + \omega_2^4)(X)\eta_3 + \\ &+ (-\omega_1^4 + \omega_2^3)(X)\eta_5 + (\omega_1^3 + \omega_2^4)(X)\eta_6 \Big\}. \end{aligned} \tag{6.1}$$

Since $\{\eta_2, \eta_3\}$ is a positive orthonormal basis of $T_{\nu_+(p)}S^2_+$ and $\{\eta_5, \eta_6\}$ is a positive orthonormal basis of $T_{\nu_-(p)}S^2_-$, we obtain

$$\begin{aligned}(\nu_+)_* X &= \frac{1}{\sqrt{2}}\left\{(\omega_1^4+\omega_2^3)(X)\eta_2 + (-\omega_1^3+\omega_2^4)(X)\eta_3\right\}, \\ (\nu_-)_* X &= \frac{1}{\sqrt{2}}\left\{(-\omega_1^4+\omega_2^3)(X)\eta_5 + (\omega_1^3+\omega_2^4)(X)\eta_6\right\}.\end{aligned} \tag{6.2}$$

By applying (4.4), (4.6), (4.7) and (6.2) we obtain the lemma.

Remark 6.3.— Examples 1 through 6 of [1] are examples of doubly slant surfaces in $\mathbb{C}^2$. Here we give another example of doubly slant surfaces.

Example.— For any two nonzero real numbers p and q, we consider the following immersion from $\mathbb{R} \times (0, \infty)$ into $\mathbb{C}^2$ defined by

$$x(u, \nu) = (p\nu \sin u\,,\ p\nu \cos u\,,\ \nu \sin qu\,,\ \nu \cos qu)\,. \tag{6.3}$$

The slant angles and the ranks of ν, ν_+ and ν_- of those examples can be determined by direct compution. We list them as the following table.

	Slant angles	rank ν	rank ν_+	rank ν_-
Example 1	$a = b = \frac{\pi}{2}$	2	1	1
Example 2	$a = b = \frac{\pi}{2}$	1	1	1
Example 3	$a = b = \frac{\pi}{2}$	2	1	1
Example 4	$a = b \in \left[0\,, \frac{\pi}{2}\right]$	0	0	0
Example 5	$a \in \left[0\,, \frac{\pi}{2}\right], b = \frac{\pi}{2}$	2	1	1
Example 6	$a = b \in \left(0\,, \frac{\pi}{2}\right)$	non-constant	1	1
Example A	$a, b \in \left(0\,, \frac{\pi}{2}\right)$	1	1	1

Except for example 4, a and b above are the slant angles with respect to $J_0 = J^+_{\epsilon_1\wedge\epsilon_2}$ and $J_1 = J^-_{\epsilon_1\wedge\epsilon_2}$ (cf. (3.13)). For example 4, a is the slant angle with respect to J_0 and b is the slant angle with respect to $J^-_{\epsilon_1\wedge\epsilon_3}$.

In [4] further results on slant immersions have been obtained.

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