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*Annales de la faculté des sciences de Toulouse 5<sup>e</sup> série*, tome 11,  
n° 1 (1990), p. 39-66

[http://www.numdam.org/item?id=AFST\\_1990\\_5\\_11\\_1\\_39\\_0](http://www.numdam.org/item?id=AFST_1990_5_11_1_39_0)

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## Bifurcation For Odd Nonlinear Elliptic Variational Inequalities

MARCO DEGIOVANNI<sup>(1)(2)</sup>

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**RÉSUMÉ.** — On étudie un problème de bifurcation variationnelle associé à des fonctionnelles paires non régulières. On prouve un théorème de multiplicité pour les branches de bifurcation. On montre une application aux équations de von Kármán avec contraintes symétriques.

**ABSTRACT.** — A problem of variational bifurcation associated with non-smooth even functionals is studied. A multiplicity theorem for bifurcation branches is proved. An application to von Kármán's equations with symmetric constraints is shown.

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### 1. Introduction

The study of eigenvalue problems for variational inequalities of the form:

$$\begin{cases} (\lambda, u) \in \mathbf{R} \times \mathbf{K} \\ \langle A(u), v - u \rangle \geq \lambda \langle Lu, v - u \rangle, \quad \forall v \in \mathbf{K}, \end{cases} \quad (1.1)$$

where  $\mathbf{K}$  is a closed convex subset of a Hilbert space,  $A$  is a nonlinear operator and  $L$  is a symmetric bounded linear operator, has been the object of several papers in the recent years (see [2, 3, 7, 8, 12, 13, 14, 15, 17, 18, 24, 25, 26, 27, 30, 31, 32, 33, 34, 35, 38, 39, 41, 42, 43] and references therein).

Some of them concern the study of bifurcation, under the assumptions that  $0 \in \mathbf{K}$  and  $A(0) = 0$ . The main problem is to characterize the values  $\bar{\lambda} \in \mathbf{R}$  (bifurcation values) such that the pair  $(\bar{\lambda}, 0)$  accumulates solutions  $(\lambda, u)$  of (1.1) with  $u \neq 0$ .

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<sup>(2)</sup> This work was partially supported by a national research grant financed by the Ministero della Pubblica Istruzione (40 % - 1986)

Under reasonable assumptions, it is quite easy to see that every bifurcation value  $\lambda$  of (1.1) is an eigenvalue of the “linearized” problem:

$$\begin{cases} (\lambda, u) \in \mathbf{R} \times \mathbf{K}_0 \\ \langle A'(0)u, v - u \rangle \geq \lambda \langle Lu, v - u \rangle, \quad \forall v \in \mathbf{K}_0, \end{cases} \quad (1.2)$$

namely there exists a solution  $(\lambda, u)$  of (1.2) with  $u \neq 0$ , where  $\mathbf{K}_0$  is the closed convex cone defined as the closure of  $\bigcup_{t>0} t\mathbf{K}$ .

Now let us assume that  $A$  is a potential operator and let  $\mathcal{A}$  be the potential of  $A$  such that  $\mathcal{A}(0) = 0$ . Then the converse is known to be true for equations [23]. For variational inequalities only partial results are available in the direction.

If  $\mathbf{K}$  is a cone (so that  $\mathbf{K}_0 = \mathbf{K}$ ) and  $\lambda$  is the minimum of  $\mathcal{A}$  on  $M^+ = \{u : \frac{1}{2}\langle Lu | u \rangle = 1\}$  or  $M^- = \{u : \frac{1}{2}\langle Lu | u \rangle = -1\}$ , then  $\lambda$  is an eigenvalue of (1.2) and a bifurcation value of (1.1) (see [17, 39]). The same result is generalized in [30], where  $\mathbf{K}$  is supposed to verify a suitable intersection condition with  $M^\pm$ .

In [12, 13, 14, 15] we obtain the same conclusion without assuming any relation between the convex set  $\mathbf{K}$  and  $M^\pm$ . Moreover it is shown that every eigenvalue of (1.2) is a bifurcation value of (1.1) in the case in which  $\mathbf{K}_0$  is a linear space.

On the other hand multiplicity results are also known for equations (see [6, 21, 28, 36]). Some of them concern the case in which the operator  $A$  is odd.

The purpose of the present paper is just to extend a multiplicity result of [6, 28] to variational inequalities. More precisely, we assume that  $-\mathbf{K} = \mathbf{K}$  and  $A$  is odd. In this situation  $\mathbf{K}_0$  turns out to be a linear space, so that (1.2) becomes a linear problem and we can define the multiplicity of an eigenvalue  $\lambda$  of (1.2). Our main results (Theorems 3.23 and 4.8) assert that, if  $\lambda$  is an eigenvalue of (1.2) of multiplicity  $m$ , then there exist  $\rho_0 > 0$ ,

$$\begin{cases} \{\lambda_\rho^{(i)} : 0 < \rho \leq \rho_0, 1 \leq i \leq 2m\} \subset \mathbf{R}, \\ \{u_\rho^{(i)} : 0 < \rho \leq \rho_0, 1 \leq i \leq 2m\} \subset \mathbf{K}, \end{cases}$$

such that  $(\lambda_\rho^{(i)}, u_\rho^{(i)})$  satisfies (1.1),  $\frac{1}{2} | \langle Lu_\rho^{(i)} | u_\rho^{(i)} \rangle | = \rho^2$ ,  $i \neq j \Rightarrow u_\rho^i \neq u_\rho^j$  and  $\lim_{\rho \rightarrow 0^+} (\lambda_\rho^{(i)}, u_\rho^{(i)}) = (\lambda, 0)$ .

In the corresponding result for equations [6, 28, 37], the main tool was constituted by a finite-dimensional reduction based on the implicit function theorem. This technique is not extendable to variational inequalities, because of the loss of regularity caused by the unilateral constraint  $\mathbf{K}$ . Our approach relies on the techniques of critical point theory for nonsmooth functionals developed in [7, 9, 11, 16]. To obtain the multiplicity result, we take advantage of the theory of the relative cohomological index of [19, 20].

In the next section we recall some notions and results from [7, 9, 11, 16]. In §3 we prove the abstract bifurcation result (Theorem 3.23) and in §4 we show an application to elasticity (Theorem 4.8).

## 2. Some recalls of nonsmooth analysis

In this section we recall some notions and results of nonsmooth analysis [7, 9, 11, 16] which will be used later. For the notions of topology involved here, the reader is referred to [40].

Throughout this section  $H$  will denote a real Hilbert space. The scalar product, norm and metric of  $H$  will be denoted by  $(\cdot | \cdot)$ ,  $|\cdot|$  and  $d_H$  respectively, while  $B(u, r)$  will denote the open ball of center  $u$  and radius  $r$ .

Let  $W$  be an open subset of  $H$  and:

$$f : W \rightarrow \mathbf{R} \cup \{+\infty\}$$

a function. We set:

$$D(f) = \{u \in W : f(u) < +\infty\}$$

$$\forall c \in \mathbf{R} \cup \{+\infty\}, \quad f^c = \{u \in D(f) : f(u) \leq c\}.$$

For every  $u$  in  $D(f)$  let us denote by  $\partial^- f(u)$  the (possibly empty) set of  $\alpha$ 's in  $H$  such that:

$$\liminf_{v \rightarrow u} \frac{f(v) - f(u) - (\alpha | v - u)}{|v - u|} \geq 0.$$

We set also  $\partial^- f(u) = \emptyset, \forall u \in W \setminus D(f)$  and:

$$D(\partial^- f) = \{u \in W : \partial^- f(u) \neq \emptyset\}.$$

Since  $\partial^- f(u)$  is convex and closed, for every  $u$  in  $D(\partial^- f)$  we can denote by  $\text{grad}^- f(u)$  the element of  $\partial^- f(u)$  having minimal norm.

If  $W = H$  and  $f$  is convex, the notion of  $\partial^- f$  coincides with the usual notion of subdifferential in convex analysis.

If  $g : W \rightarrow \mathbf{R}$  is Fréchet differentiable at  $u \in W$ , then  $\partial^-(f + g) = \partial^- f(u) + \text{grad } g(u)$ .

DEFINITION 2.1. — *A point  $u \in W$  is said to be critical from below for  $f$ , if  $0 \in \partial^- f(u)$ . A value  $c \in \mathbf{R}$  is said to be critical from below for  $f$ , if there exists  $u \in W$  such that  $0 \in \partial^- f(u)$ ,  $f(u) = c$ .*

DEFINITION 2.2. — *The function  $f$  is said to have a  $\varphi$ -monotone subdifferential of order two, if there exists a continuous function  $\chi : (D(f))^2 \times \mathbf{R}^2 \rightarrow \mathbf{R}^+$  such that:*

$$(\alpha - \beta \mid u - v) \geq -\chi(u, v, f(u), f(v)) \left(1 + |\alpha|^2 + |\beta|^2\right) |u - v|^2$$

whenever  $u, v \in D(\partial^- f)$ ,  $\alpha \in \partial^- f(u)$ ,  $\beta \in \partial^- f(v)$ .

DEFINITION 2.3. — *Let  $c$  be a real number. The function  $f$  is said to verify the Palais-Smale condition at level  $c$  (or, briefly,  $(PS)_c$ ), if for every sequence  $(u_h)$  in  $D(\partial^- f)$  with  $\lim_h \text{grad}^- f(u_h) = 0$ ,  $\lim_h f(u_h) = c$ , there exists a subsequence  $(u_{h_k})$  converging to an element of  $W$ .*

Besides the metric  $d_H$  induced by  $H$ , it is convenient to consider on  $D(f)$  also the graph metric  $d^*$  defined by:

$$d^*(u, v) = |u - v| + |f(u) - f(v)|. \quad (2.4)$$

However, when the metric is not specified, we mean that  $D(f)$  is endowed with the metric  $d_H$ .

In the following we shall be concerned with metric spaces on which the group  $\mathbf{Z}_2$  acts. If  $X$  is a symmetric (with respect to the origin) subset of some normed space,  $X$  will be considered as a  $\mathbf{Z}_2$ -space endowed with the usual action.

THEOREM 2.5. — *Let us suppose that  $f$  is lower semicontinuous and has a  $\varphi$ -monotone subdifferential of order two. Moreover let us assume that  $W$  is symmetric and  $f$  is even. Let  $-\infty < a \leq b < +\infty$ .*

*Then the pair  $(f^b, f^a)$  endowed with the metric  $d^*$  is equivariantly homotopically equivalent to the pair  $(f^b, f^a)$  endowed with the metric  $d_H$ .*

*Proof.* — See [11, Theorem 3.18].  $\square$

Now let us recall the relative cohomological index of [19, 20]. More precisely, we shall consider a very special case.

Let  $S^\infty$  be the unit sphere in a real normed space of infinite dimension,  $\mathbf{RP}^\infty = S^\infty/\mathbf{Z}_2$  the corresponding real projective space and  $\alpha \in H^1(\mathbf{RP}^\infty; \mathbf{Z}_2) \setminus \{0\}$ , where the functor  $H^*$  denotes Alexander-Spanier cohomology.

Let  $X$  be a metric space on which  $\mathbf{Z}_2$  acts freely and let  $A$  be a closed invariant subset of  $X$ . Let  $\tilde{X} = X/\mathbf{Z}_2$  and  $\tilde{A} = A/\mathbf{Z}_2$  denote the corresponding orbit spaces. It is always possible to define an equivariant continuous map  $X \rightarrow S^\infty$ . Let  $q_X : \tilde{X} \rightarrow \mathbf{RP}^\infty$  be the corresponding map between the orbit spaces, which induces a homomorphism  $q_X^* : H^*(\mathbf{RP}^\infty; \mathbf{Z}_2) \rightarrow H^*(\tilde{X}; \mathbf{Z}_2)$  in cohomology. According to [19, Definition 7.4] and [20, Definition 2.3] (see also [21, 22] if  $A = \emptyset$ ), we set:

$$\text{Index}(X, A) = \inf \left\{ k \in \mathbf{N} : q_X^*(\alpha^k) \cup \gamma = 0, \forall \gamma \in H^*(\tilde{X}, \tilde{A}; \mathbf{Z}_2) \right\}$$

with the conventions  $\alpha^0 = 1 \in H^0(\mathbf{RP}^\infty; \mathbf{Z}_2)$ ,  $\alpha^{k+1} = \alpha^k \cup \alpha$ ,  $\inf \emptyset = +\infty$ .

The index turns out to be well defined. Let us recall the properties that will be used in the following.

**THEOREM 2.6.** — *The following facts hold :*

i) *if  $(X', A')$  is another pair which is equivariantly homotopically equivalent to  $(X, A)$ , then:*

$$\text{Index}(X', A') = \text{Index}(X, A);$$

ii) *if  $S^n$  ( $n \geq 0$ ) is the unit sphere in  $\mathbf{R}^{n+1}$ , then:*

$$\begin{aligned} \text{Index}(X \circ S^n, S^n) &= \text{Index}(X, \emptyset), \\ \text{Index}(S^n, \emptyset) &= n + 1, \end{aligned}$$

*where  $X \circ S^n$  denotes the join of  $X$  and  $S^n$ .*

*Proof.* — Property i) is obvious. By [20, Proposition 4.1] and [21, Proposition 3.13 ], also ii) follows.  $\square$

The relative index has applications to critical point theory (see [19, 20] where functions of class  $C^1$  are considered). We are interested in a case involving a class of nonsmooth functions.

**THEOREM 2.7.** — *Let us assume that  $f$  is lower semicontinuous and has a  $\varphi$ -monotone subdifferential of order two. Moreover let  $W$  be symmetric and  $f$  even. Let  $-\infty < a < b < +\infty$  and let us suppose that  $a$  and  $b$  are not critical from below for  $f$ ,  $0 \notin f^b$  and that  $\forall c \in [a, b[$  the function  $f$  verifies  $(PS)_c$  and  $f^c$  is closed in  $H$ .*

*Then there exist at least  $\text{Index}(f^b, f^a)$  pairs of antipodal points in  $f^b \setminus f^a$  which are critical from below for  $f$ .*

*Proof.* — See [11, Theorem 4.13].  $\square$

If  $A$  is a subset of  $H$ , we define a function  $I_A : H \rightarrow \mathbf{R} \cup \{+\infty\}$  by:

$$I_A(u) = \begin{cases} 0 & u \in A \\ +\infty & u \in H \setminus A. \end{cases}$$

For every  $u$  in  $A$ ,  $\partial^- I_A(u)$  is a closed convex cone (in some sense, the outward normal cone to  $A$  at  $u$ ).

**Remark 2.8.** — If  $M$  is a hypersurface in  $H$  of class  $C^1$ , we have for every  $u$  in  $M$ :

$$\partial^- I_M(u) = \{\lambda \nu(u) : \lambda \in \mathbf{R}\}$$

where  $\nu(u)$  is a normal unit vector to  $M$  at  $u$ .

**DEFINITION 2.9.** — *Let  $A$  and  $B$  be two subsets of  $H$  and  $u \in A \cap B$ . Then  $A$  and  $B$  are said to be (outwardly) tangent at  $u$ , if:*

$$\partial^- I_A(u) \cap (-\partial^- I_B(u)) \neq \{0\}.$$

**THEOREM 2.10.** — *Let  $M$  be a hypersurface in  $W$  of class  $C^1$  and let us assume that  $f$  is lower semicontinuous and, for some continuous function  $q : D(f) \rightarrow \mathbf{R}$ ,*

$$f(v) \geq f(u) + (\alpha | v - u) - q(u)|v - u|^2$$

*whenever  $v \in W$ ,  $u \in D(\partial^- f)$ ,  $\alpha \in \partial^- f(u)$ .*

*Let  $u_0 \in D(f) \cap M$  and let us suppose that  $D(f)$  and  $M$  are not tangent at  $u_0$ .*

*Then we have:*

$$\partial^-(f + I_M)(u_0) = \partial^- f(u_0) + \partial^- I_M(u_0).$$

*Proof.* — See [7, Theorem 1.13 and Remark 1.12<sub>b</sub>].  $\square$

Finally, let us recall the notion of variational convergence from [1, 10].

DEFINITION 2.11. — *Let  $X$  be a topological space and:*

$$g_h : X \rightarrow \mathbf{R} \cup \{+\infty\} \quad (h \in \overline{\mathbf{N}} := \mathbf{N} \cup \{\infty\})$$

*a sequence of functions. We say that:*

$$g_\infty = \Gamma^-(X) \lim_h g_h$$

*if the following facts hold:*

i) *for every  $u$  in  $X$  and for every sequence  $(u_h)$  in  $X$  converging to  $u$ , we have:*

$$g_\infty(u) \leq \liminf_h g_h(u_h);$$

ii) *for every  $u$  in  $X$  there exists a sequence  $(u_h)$  in  $X$  converging to  $u$  such that:*

$$g_\infty(u) = \lim_h g_h(u_h).$$

*From now on in this section we shall consider a sequence of functions:*

$$f_h : H \rightarrow \mathbf{R} \cup \{+\infty\}, \quad (h \in \overline{\mathbf{N}}).$$

DEFINITION 2.12. — *The sequence  $(f_h)$  is said to be equicoercive, if for every real number  $c$  the closure of the set  $\bigcup_{h \in \mathbf{N}} (f_h)^c$  is compact.*

*Remark 2.13.* — Let us suppose that  $(f_h)$  is equicoercive and that  $f_\infty = \Gamma^-(H) \lim_h f_h$ . Then for every real number  $c$  the set  $(f_\infty)^c$  is compact. Therefore the closure of the set  $\bigcup_{h \in \overline{\mathbf{N}}} (f_h)^c$  is compact.

THEOREM 2.14. — *Let us suppose that:*

i) *for every  $h$  in  $\overline{\mathbf{N}}$ ,  $f_h$  is lower semicontinuous and even;*

ii) *there exists a continuous function:*

$$\chi : \left( \bigcup_{h \in \overline{\mathbf{N}}} D(f_h) \right)^2 \times \mathbf{R}^2 \rightarrow \mathbf{R}^+$$



such that:

$$\begin{aligned} (\alpha - \beta | u - v) &\geq \\ &\geq -\chi(u, v, f_h(u), f_h(v)) \left(1 + |\alpha|^2 + |\beta|^2\right) |u - v|^2 \end{aligned}$$

whenever  $h \in \overline{\mathbf{N}}$ ,  $u, v \in D(\partial^- f_h)$ ,  $\alpha \in \partial^- f_h(u)$ ,  $\beta \in \partial^- f_h(v)$ ;

iii)  $f_\infty = \Gamma^-(H) \lim_h f_h$ ;

iv) the sequence  $(f_h)$  is equicoercive.

Let  $-\infty < a \leq b < +\infty$  and let us assume that  $a$  and  $b$  are not critical from below for  $f_\infty$ .

Then there exists  $h_0 \in \mathbf{N}$  such that for every  $h \geq h_0$ ,  $a$  and  $b$  are not critical from below for  $f_h$  and the pair  $(f_h^b, f_h^a)$  is equivariantly homotopically equivalent to the pair  $(f_\infty^b, f_\infty^a)$ .

*Proof.* — By Remark 2.13 we can apply [11, Theorem 5.12 and Remark 5.13].  $\square$

*Remark 2.15.* — In the previous theorem it is understood that all the pairs  $(f_h^b, f_h^a)$  are endowed with the metric  $d_H$ . However, by Theorem 2.5 nothing changes if some of these pairs is endowed with the corresponding graph metric  $d_h^*$ .

**THEOREM 2.16.** — *Let  $M$  be a symmetric hypersurface in  $H$  of class  $C_{\text{loc}}^{1,1}$ . Let us suppose that:*

a)  $M$  is a closed subset of  $H$  and for every  $h$  in  $\overline{\mathbf{N}}$ ,  $f_h$  is lower semicontinuous and even;

b) there exists  $q$  in  $\mathbf{R}^+$  such that for every  $h$  in  $\overline{\mathbf{N}}$  the function  $\{y \mapsto f_h(u) + q|u|^2\}$  is convex;

c)  $f_\infty = \Gamma^-(H) \lim_h f_h$ ;

d) the sequence  $(f_h + I_M)$  is equicoercive;

e) for every  $u$  in  $D(f_\infty) \cap M$ ,  $D(f_\infty)$  and  $M$  are not tangent at  $u$ .

Then there exists a sequence  $(\widehat{f}_h)$  of functions  $\widehat{f}_h : H \rightarrow \mathbf{R} \cup \{+\infty\}$  ( $h \in \overline{\mathbf{N}}$ ) with the following properties.

i)  $\widehat{f}_\infty = f_\infty$ ;  $\forall h \in \mathbf{N}$ ,  $\widehat{f}_h \geq f_h$ ;

ii)  $\forall h \in \overline{\mathbf{N}}$ ,  $\widehat{f}_h$  is lower semicontinuous and even;

- iii)  $\forall h \in \overline{\mathbf{N}}$ , the function  $\{u \mapsto \widehat{f}_h(u) + q|u|^2\}$  is convex;
- iv)  $\widehat{f}_\infty = \Gamma^-(H) \lim_h \widehat{f}_h$ ;
- v)  $\forall h \in \overline{\mathbf{N}}, \forall u \in D(\widehat{f}_h) \cap M$ ,  $D(\widehat{f}_h)$  and  $M$  are not tangent at  $u$ ;
- vi) if  $(u_h)$  is a sequence in  $M$  with  $\limsup_h \widehat{f}_h(u_h) < +\infty$ , we have:

$$\widehat{f}_h(u_n) = f_h(u_h), \quad \partial^- \widehat{f}_h(u_n) = \partial^- f_h(u_h)$$

eventually as  $h \rightarrow \infty$ .

Moreover, if we set  $\widetilde{f}_h = \widehat{f}_h + I_M$ ,  $\forall h \in \overline{\mathbf{N}}$ , the following facts hold:

- vii)  $\forall h \in \overline{\mathbf{N}}$ ,  $\widetilde{f}_h$  is lower semicontinuous and even;
- viii) there exists a continuous function:

$$\chi : \left( \bigcup_{h \in \overline{\mathbf{N}}} D(\widetilde{f}_h) \right)^2 \times \mathbf{R}^2 \rightarrow \mathbf{R}^+$$

such that:

$$\begin{aligned} (\alpha - \beta | u - v) &\geq \\ &\geq -\chi(u, v, \widetilde{f}_h(u), \widetilde{f}_h(v)) \left(1 + |\alpha|^2 + |\beta|^2\right) |u - v|^2; \end{aligned}$$

whenever  $h \in \overline{\mathbf{N}}, u, v \in D(\partial^- \widetilde{f}_h)$ ,  $\alpha \in \partial^- \widetilde{f}_h(u)$ ,  $\beta \in \partial^- \widetilde{f}_h(v)$ ;

- ix)  $\widetilde{f}_\infty = \Gamma^-(H) \lim_h \widetilde{f}_h$ ;
- x) the sequence  $(\widetilde{f}_h)$  is equicoercive.

*Proof.* — See [12, Theorem 2.17].  $\square$

### 3. Variational bifurcation for even nonsmooth functions

Throughout this section we keep the notations of §2. We shall consider a real Hilbert space  $H$ , a convex open subset  $W$  of  $H$  with  $0 \in W$ ,  $-W = W$ , an even function:

$$f : \overline{W} \rightarrow \mathbf{R} \cup \{+\infty\}$$

such that:

$$f(0) = 0, \quad 0 \in \partial^- f(0) \tag{3.1}$$

and a symmetric bounded linear operator:

$$L : H \rightarrow H .$$

Our purpose is to study the set of the pairs  $(\lambda, u)$  such that:

$$\begin{cases} (\lambda, u) \in \mathbf{R} \times W \\ \lambda Lu \in \partial^- f(u) . \end{cases} \quad (3.2)$$

Because of (3.1), for every  $\lambda$  in  $\mathbf{R}$  the pair  $(\lambda, 0)$  satisfies (3.2). Moreover, since  $f$  is even, solutions of (3.2) always occur in pairs  $(\lambda, u)$  and  $(\lambda, -u)$ .

**DEFINITION 3.3.** — *A real number  $\lambda$  is said to be of bifurcation for (3.2), if there exists a sequence  $((\lambda_h, u_h))$  of solutions of (3.2) with  $u_h \neq 0$  and:*

$$\lim_h (\lambda_h, u_h) = (\lambda, 0) .$$

As in the case of smooth functions  $f$  (see [23]), a first problem is to compare the bifurcation values with the eigenvalues of some “linearized” problem. This question has been the object of [12, 13, 14, 15]. Here we are interested in a multiplicity result, corresponding to that of [6, 28], which is connected with the evenness hypothesis on  $f$ .

Let us make the following further assumptions of  $f$ :

- (A1) the function  $f$  is lower semicontinuous on  $\overline{W}$  and there exists  $q \in \mathbf{R}^+$  such that the function  $\{u \mapsto f(u) + q|u|^2\}$  is convex on  $\overline{W}$ ;
- (A2) there exists a function  $f_0 : H \rightarrow \mathbf{R} \cup \{+\infty\}$  such that for every sequence  $(\rho_h)$  in  $]0, 1]$  converging to zero, we have:

$$f_0 = \Gamma^-(H) \lim_h f_{\rho_h}$$

where  $\forall \rho \in ]0, 1]$ ,  $f_\rho : H \rightarrow \mathbf{R} \cup \{+\infty\}$  is defined by:

$$f_\rho(u) = \begin{cases} \rho^{-2} f(\rho u) & \rho u \in \overline{W} \\ +\infty & \rho u \in H \setminus \overline{W} . \end{cases}$$

Let us summarize the main properties of  $f_\rho$  ( $0 \leq \rho \leq 1$ ).

PROPOSITION 3.4. — *Let  $\rho \in [0, 1]$ . Then the following facts hold:*

i)  $f_\rho$  is lower semicontinuous on  $H$ , even and the function:

$$\left\{ u \mapsto f_\rho(u) + q|u|^2 \right\}$$

is convex on  $H$ ;

ii)  $\forall v \in H, \forall u \in D(\partial^- f_\rho), \forall \alpha \in \partial^- f_\rho(u)$ , we have:

$$f_\rho(v) \geq f_\rho(u) + (\alpha | v - u) - q|v - u|^2;$$

iii)  $f_\rho(0) = 0, 0 \in \partial^- f_\rho(0)$ ;

iv) for every sequence  $(\rho_h)$  in  $[0, 1]$  converging to  $\rho$ , we have:

$$f_\rho = \Gamma^-(H) \lim_h f_{\rho_h}.$$

v) if  $\rho > 0, \forall u, \alpha \in H$  we have:

$$u \in D(\partial^- f_\rho) \text{ and } \alpha \in \partial^- f_\rho(u) \Leftrightarrow \rho u \in D(\partial^- f) \text{ and } \rho \alpha \in \partial^- f(\rho u);$$

vi)  $\forall s > 0, \forall u \in H, f_0(su) = s^2 f_0(u)$ ;

vii)  $\forall s > 0, \forall u \in D(\partial^- f_0), \forall \alpha \in \partial^- f_0(u), s\alpha \in \partial^- f_0(su)$ ;

viii)  $\forall u \in D(\partial^- f_0), \forall \alpha \in \partial^- f_0(u), (\alpha | u) = 2f_0(u)$ .

*Proof.* — See [12, Propositions 3.4 and 3.6].  $\square$

The function  $f_0(u)$  introduced in (A2) plays the role of the quadratic form  $\frac{1}{2}f''(0)(u, u)$  in the smooth case. In the following it will be convenient to consider also the “linearized” problem:

$$\begin{cases} (\lambda, u) \in \mathbf{R} \times H \\ \lambda Lu \in \partial^- f_0(u). \end{cases} \quad (3.5)$$

*Remark 3.6.* — For every  $\lambda$  in  $\mathbf{R}$  the set:

$$\{u \in H : \lambda Lu \in \partial^- f_0(u)\}$$

is a closed cone.

*Proof.* — It is a consequence of (i), (ii) and (vii) of Proposition 3.4.  $\square$

DEFINITION 3.7. — *A real number  $\lambda$  is said to be an eigenvalues of (3.5), if the pair  $(\lambda, u)$  satisfies (3.5) for some  $u \neq 0$ .*

*Remark 3.8.* — If  $(\lambda, u)$  satisfies (3.5), then:

$$f_0(u) = \frac{1}{2} \lambda (Lu | u).$$

*Proof.* — It follows from Proposition 3.4<sub>viii</sub>.  $\square$

In order to give a variational characterization of the eigenvalues of (3.5), let us introduce the sets:

$$M^+ = \left\{ u \in H : \frac{1}{2} (Lu | u) = 1 \right\},$$

$$M^- = \left\{ u \in H : \frac{1}{2} (Lu | u) = -1 \right\},$$

which are symmetric hypersurfaces in  $H$  of class  $C^\infty$  and closed subsets of  $H$ .

According to Remark 2.8, we have for every  $u$  in  $M^\pm$ ,

$$\partial^- I_{M^\pm}(u) = \{ \lambda Lu : \lambda \in \mathbf{R} \}.$$

**PROPOSITION 3.9.** — *For every  $u \in D(f_0) \cap M^\pm$ ,  $D(f_0)$  and  $M^\pm$  are not tangent at  $u$ .*

*Proof.* — See [12, Proposition 3.11].  $\square$

**PROPOSITION 3.10.** — *Let  $\lambda \in \mathbf{R}$  and let us consider the following facts:*

- i)  $(\lambda, u)$  satisfies (3.5) for some  $u$  with  $(Lu | u) > 0$ ;*
- ii)  $\lambda$  is critical from below for  $(f_0 + I_{M^+})$ ;*
- iii)  $(\lambda, u)$  satisfies (3.5) for some  $u$  with  $(Lu | u) < 0$ ;*
- iv)  $-\lambda$  is critical from below for  $(f_0 + I_{M^-})$ .*

*Then we have:*

$$i) \Leftrightarrow ii);$$

$$iii) \Leftrightarrow iv).$$

*Proof.* — See [12, Proposition 3.12].  $\square$

**THEOREM 3.11.** — *Let us assume that for every sequence  $(u_h)$  in  $W \setminus \{0\}$  with:*

$$\lim_h u_h = 0, \quad \sup_h |u_h|^{-2} f(u_h) < +\infty,$$

the sequence  $(u_h/|u_h|)$  has a convergent subsequence.

Then every  $\lambda$  of bifurcation for (3.2) is an eigenvalue of (3.5).

*Proof.* — See [12, Theorem 3.14].  $\square$

The converse is not true, in general. Let us consider  $H = \mathbf{R}^2$ ,  $W = B(0, 1)$ ,  $f : \overline{W} \rightarrow \mathbf{R} \cup \{+\infty\}$  defined by:

$$f(x, y) = \begin{cases} x^3 y^3 (x^2 + y^2)^{-2} + xy(x^2 + y^2) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

and  $L$  the identity map. Then all the assumptions of Theorem 3.11 are satisfied with:

$$f_0(x, y) = \begin{cases} x^3 y^3 (x^2 + y^2)^{-2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

However  $\lambda = 0$  is an isolated eigenvalue of (3.5) which is not of bifurcation for (3.2). The feature of this example is that  $f_0$  does not behave like a quadratic form. On the contrary, if  $f_0$  is a quadratic form in a generalized sense that will be precised later, then the converse of Theorem 3.11 holds [12, Theorem 3.29]. Our purpose is to show that the evenness of  $f$  allows to prove also a multiplicity result, as it is done in [6, 28] for smooth functions. Actually, we shall consider only the eigenvalues  $\lambda$  of (3.5) such that there exists a solution  $(\lambda, u)$  of (3.5) with  $(Lu | u) \neq 0$ . Since we can change  $L$  in  $-L$ , this is equivalent to consider the eigenvalues  $\lambda$  of (3.5) such that there exists a solution  $(\lambda, u)$  of (3.5) with  $(Lu | u) > 0$ .

First of all let us consider the following compactness assumptions:

(A3<sup>+</sup>) for every sequence  $(u_h)$  in  $\overline{W}$  with  $\frac{1}{2}(Lu_h | u_h) \in ]0, 1]$  and:

$$\sup_h |(Lu_h | u_h)|^{-1} f(u_h) < +\infty,$$

the sequence  $\left(|(Lu_h | u_h)|^{-1/2} u_h\right)$  has a convergent subsequence;

(A3<sup>-</sup>) for every sequence  $(u_h)$  in  $\overline{W}$  with  $\frac{1}{2}(Lu_h | u_h) \in [-1, 0[$  and:

$$\sup_h |(Lu_h | u_h)|^{-1} f(u_h) < +\infty,$$

the sequence  $\left(|(Lu_h | u_h)|^{-1/2} u_h\right)$  has a convergent subsequence.

If  $L$  is the identity map of  $H$ , hypothesis  $(A3^+)$  implies the compactness assumption made in Theorem 3.11.

PROPOSITION 3.12. — *Condition  $(A3^+)$  (resp.  $(A3^-)$ ) holds if and only if for every sequence  $(\rho_h)$  in  $]0, 1]$  the sequence  $(f_{\rho_h} + I_{M^+})$  (resp.  $(f_{\rho_h} + I_{M^-})$ ) is equicoercive.*

*Proof.* — See [12, Proposition 3.15].  $\square$

PROPOSITION 3.13. — *Let  $(A3^+)$  (resp.  $(A3^-)$ ) hold. Then for every  $c$  in  $\mathbf{R}$  the set  $(f_0 + I_{M^+})^c$  (resp.  $(f_0 + I_{M^-})^c$ ) is compact.*

*Proof.* — See [12, Proposition 3.16].  $\square$

Let us denote by  $\mathcal{C}$  the set of the  $c$ 's in  $\mathbf{R}$  such that the function  $\left\{ u \mapsto f_0(u) + (c - \epsilon/2)|u|^2 \right\}$  is convex for some  $\epsilon > 0$  (in particular,  $f_0(u) + (c - \epsilon/2)|u|^2 \geq 0, \forall u \in H$  by Proposition 3.4<sub>iii</sub>).

The set  $\mathcal{C}$  is not empty. In fact  $]2q, +\infty[ \subset \mathcal{C}$  by Proposition 3.4<sub>i</sub>.

If  $c \in \mathcal{C}$  and  $u \in H$ , let us denote by  $i_c^*u$  the minimum point of the function  $\left\{ v \mapsto f_0(v) + c/2|v|^2 - (v | u) \right\}$ . Evidently:

$$\forall u \in H, \quad (u - ci_c^*u) \in \partial^- f_0(i_c^*u); \quad (3.14)$$

$$\forall u \in D(\partial^- f_0), \quad \forall \alpha \in \partial^- f_0(u), \quad u = i_c^*(\alpha + cu). \quad (3.15)$$

LEMMA 3.16. — *Let  $X$  be a real Hilbert space and  $B, K : X \rightarrow X$  two symmetric bounded linear operators such that  $BK = KB$ . Let:*

$$M = \left\{ u \in X : \frac{1}{2}(Bu | u) = 1 \right\},$$

$$Q_0(u) = \frac{1}{2}(Ku | u), \quad Q(u) = \frac{1}{2}|u|^2 - Q_0(u)$$

*and let us assume that  $M$  is nonempty and sequentially weakly closed in  $X$ ,  $Q_0|_M$  is sequentially weakly continuous and:*

$$\lim_{|u| \rightarrow +\infty} Q|_M(u) = +\infty,$$

*if  $M$  is not bounded. Let  $\lambda$  be a critical value of  $Q|_M$ .*

Then the following facts hold:

- i) the linear space  $\{u \in X : u - Ku = \lambda Bu\}$  has finite dimension, say  $m$ ;
- ii) there exists  $\epsilon_0 > 0$  such that  $Q|_M$  has no critical values in  $[\lambda - \epsilon_0, \lambda + \epsilon_0] \setminus \{\lambda\}$ ;
- iii) for every  $\epsilon \in ]0, \epsilon_0]$ , the pair  $\left((Q|_M)^{\lambda+\epsilon}, (Q|_M)^{\lambda-\epsilon}\right)$  is equivalently homotopically equivalent to the pair  $(S^{m+n-1}, S^{n-1})$  for some integer  $n \geq 0$  (with the convention  $S^{-1} = \emptyset$ ).

*Proof.* — For properties i) and ii), we refer the reader to [12, Lemma 3.22]. In order to prove iii), let us recall some steps in the proof of [12, Lemma 3.22].

Let  $A = I - K$  and let

$$\Lambda = \{\mu \in \mathbf{R} : Au = \mu Bu \text{ for some } u \in X \text{ with } (Bu | u) > 0\},$$

which is also the set of the critical values of  $Q|_M$ . Then  $\Lambda \cap ]-\infty, \lambda]$  is finite, say  $\Lambda \cap ]-\infty, \lambda] = \{\lambda_1, \dots, \lambda_h\}$  with  $\lambda_1 < \dots < \lambda_h = \lambda$ . If we set  $V_i = \{u \in X : Au = \lambda_i Bu\}$  for  $i = 1, \dots, h$ , then the subspaces  $V_i$  are finite-dimensional and pairwise orthogonal. Let  $V' = V_1 \oplus \dots \oplus V_{h-1}$ ,  $V = V' \oplus V_h$ ,  $Y = V^\perp$ ,  $m = \dim(V_h)$  and  $n = \dim(V')$ , ( $n = 0$  if  $h = 1$ ). The subspaces  $V'$ ,  $V$  and  $Y$  are invariant for  $A$  and  $B$ . Moreover  $(Bu | u) > 0$ ,  $\forall u \in V \setminus \{0\}$  and:

$$\begin{aligned} \lambda_h &= \max\{Q(u) : u \in M \cap V\} = \\ &= \min\{Q(u) : u \in M \cap (V_h \oplus Y)\} < \min\{Q(u) : u \in M \cap Y\}, \\ \lambda_{h-1} &= \max\{Q(u) : u \in M \cap V'\} \end{aligned}$$

Finally let  $\epsilon_0$  be such that:

$$\lambda_{h-1} < \lambda - \epsilon_0 < \lambda + \epsilon_0 < \min_{M \cap Y} Q.$$

To prove iii), let us take  $\epsilon \in ]0, \epsilon_0]$ .

Let  $P_{V'}$  be the orthogonal projection on  $V'$ ,  $\theta = \mathbf{R} \rightarrow [0, 1]$  a continuous function such that  $\theta(s) = 1$  for  $s \leq \lambda - \epsilon$ ,  $\theta(s) = 0$  for  $s \geq \lambda - \epsilon/2$  and:

$$\mathcal{H}_1 : \left((Q|_M)^{\lambda+\epsilon}, (Q|_M)^{\lambda-\epsilon}\right) \times [0, 1] \rightarrow \left((Q|_M)^{\lambda+\epsilon}, (Q|_M)^{\lambda-\epsilon}\right)$$

the map defined by:

$$\mathcal{H}_1(u, t) = \frac{\psi_1(u, t)}{\left\{\frac{1}{2} (B\psi_1(u, t) | \psi_1(u, t))\right\}^{1/2}},$$

where  $\psi_1(u, t) = P_{V'} u + (1 - t\theta(Q(u)))(u - P_{V'} u)$ .



It is readily seen that  $\mathcal{H}_1(-u, t) = -\mathcal{H}_1(u, t)$ ,  $\mathcal{H}_1(u, 0) = u$ ,

$$\begin{aligned} \forall u \in (Q|_M)^{\lambda-\epsilon}, \quad \mathcal{H}_1(u, 1) \in M \cap V', \\ \forall u \in M \cap V', \forall t \in [0, 1], \quad \mathcal{H}_1(u, t) = u. \end{aligned}$$

Therefore the pair  $\left((Q|_M)^{\lambda+\epsilon}, (Q|_M)^{\lambda-\epsilon}\right)$  is equivariantly homotopically equivalent to the pair  $\left((Q|_M)^{\lambda+\epsilon}, M \cap V'\right)$ .

Now let us consider the map:

$$\mathcal{H}_2 : \left((Q|_M)^{\lambda+\epsilon}, M \cap V'\right) \times [0, 1] \rightarrow \left((Q|_M)^{\lambda+\epsilon}, M \cap V'\right)$$

defined by:

$$\mathcal{H}_2(u, t) = \frac{\psi_2(u, t)}{\left\{ \frac{1}{2} (B\psi_2(u, t) | \psi_2(u, t)) \right\}^{1/2}},$$

where  $\psi_2(u, t) = P_V u + (1-t)(u - P_V u)$  and  $P_V$  is the orthogonal projection on  $V$ .

Then  $\mathcal{H}_2(-u, t) = -\mathcal{H}_2(u, t)$ ,  $\mathcal{H}_2(u, 0) = u$ ,

$$\begin{aligned} \forall u \in M \cap V, \forall t \in [0, 1], \quad \mathcal{H}_2(u, t) = u, \\ \forall u \in (Q|_M)^{\lambda+\epsilon}, \quad \mathcal{H}_2(u, 1) \in M \cap V. \end{aligned}$$

Therefore the pair  $\left((Q|_M)^{\lambda+\epsilon}, M \cap V'\right)$  is equivariantly homotopically equivalent to the pair  $(M \cap V, M \cap V')$ .

Finally, since  $B$  is positive definite on  $V$ , it is readily seen that the pair  $(M \cap V, M \cap V')$  is equivariantly homeomorphic to the pair  $(S^{m+n-1}, S^{n-1})$ .  $\square$

**THEOREM 3.17.** — *Let  $(A\mathcal{G}^+)$  hold and let us assume that:*

$$\forall u, v \in H, \quad f_0(u+v) + f_0(u-v) = 2f_0(u) + 2f_0(v) \quad (3.18)$$

*and that:*

$$ci_c^* L = Lci_c^* \quad (3.19)$$

*for some  $c \in \mathcal{C}$ . Let  $(\lambda, u)$  be a solution of (3.5) with  $(Lu | u) > 0$ .*

Then the following facts hold:

- i) the set  $E_\lambda := \{v \in H : \lambda Lv \in \partial^- f_0(u)\}$  is a linear subspace of  $H$  of finite dimension;
- ii) there exists  $\epsilon_0 > 0$  such that every  $\mu \in [\lambda - \epsilon_0, \lambda + \epsilon_0] \setminus \{\lambda\}$  is not critical from below for  $(f_0 + I_{M^+})$ ;
- iii) for every  $\epsilon \in ]0, \epsilon_0]$ , the pair  $\left((f_0 + I_{M^+})^{\lambda+\epsilon}, (f_0 + I_{M^+})^{\lambda-\epsilon}\right)$ , endowed with the graph metric  $d^*$ , is equivariantly homotopically equivalent to the pair  $\left(S^{m+n-1}, S^{n-1}\right)$ , where  $m = \dim(E_\lambda)$  and  $n \geq 0$  a suitable integer;
- iv) for every  $\epsilon \in ]0, \epsilon_0]$ , we have:

$$\text{Index} \left( (f_0 + I_{M^+})^{\lambda+\epsilon}, (f_0 + I_{M^+})^{\lambda-\epsilon} \right) = \dim(E_\lambda).$$

*Proof.* — The proof follows the lines of the proof of [12, Theorem 3.26]. We sketch it for reader's convenience.

Let  $c \in \mathcal{C}$  be such that (3.19) holds. Combining Proposition 3.4 with (3.18), we deduce that  $D(f_0)$  is a linear subspace of  $H$ . Moreover:

$$((v | w)) = \frac{1}{4} \left( 2f_0(v+w) + c|v+w|^2 - 2f_0(v-w) - c|v-w|^2 \right)$$

defines a scalar product on  $D(f_0)$  such that the embedding

$$i : (D(f_0), ((\cdot | \cdot))) \rightarrow (H, (\cdot | \cdot))$$

is continuous. The space  $(D(f_0), ((\cdot | \cdot)))$  is complete, hence a Hilbert space, and  $i_c^* : H \rightarrow D(f_0)$  is just the adjoint map of the embedding  $i$ .

We can define two symmetric bounded linear operators  $K, B : D(f_0) \rightarrow D(f_0)$  by  $K = ci_c^*i$ ,  $B = i_c^*Li$ . We have  $BK = KB$  by (3.19) and:

$$\forall v \in D(f_0), \quad \frac{1}{2} (((I - K)v | v)) = f_0(v), \quad ((Bv | v)) = (Lv | v).$$

If we set:

$$\begin{aligned} M &:= \left\{ v \in D(f_0) : \frac{1}{2} ((Bv | v)) = 1 \right\} = M^+ \cap D(f_0), \\ Q_0(v) &:= \frac{1}{2} ((Kv | v)) = \frac{1}{2} |v|^2, \\ Q(v) &= \frac{1}{2} ((v | v)) - Q_0(v), \end{aligned}$$

the assumptions of Lemma 3.16 are satisfied. Moreover:

$$\forall (\mu, v) \in \mathbf{R} \times D(f_0), \quad \mu Lv \in \partial^- f_0(v) \Leftrightarrow v - Kv = \mu Bv.$$

Therefore  $\lambda$  is a critical value of  $Q|_M$  so that i) and ii) follows from Proposition 3.10 and the corresponding i) and ii) of Lemma 3.16.

On the other hand it is readily seen that the scalar product  $((\cdot | \cdot))$  and the graph metric  $d^*$  associated with  $(f_0 + I_{M+})$  induce the same topology on  $D(f_0 + I_{M+})$  and  $\forall b \in \mathbf{R}$ ,  $(f_0 + I_{M+})^b = (Q|_M)^b$ . Then iii) follows from iii) of Lemma 3.16.

Finally, if  $n = 0$  iv) is an immediate consequence of iii) and Theorem 2.6. If  $n \geq 1$ , the pair  $(S^{m+n-1}, S^{n-1})$  is equivariantly homeomorphic to the pair  $(S^{m-1} \circ S^{n-1}, S^{n-1})$ . Combining again iii) with Theorem 2.6, iv) follows also in this case.  $\square$

*Remark 3.20.* — Assumption (3.19) is satisfied in each of the following situations:

- i)  $L$  is the identity map of  $H$ ;
- ii) the map  $\{v \mapsto f_0(v) - \epsilon|v|^2\}$  is convex for some  $\epsilon > 0$  (so that we can choose  $c = 0$ ).

*Remark 3.21.* — We shall see in the proof of Theorem 3.23 that the function  $(f_0 + I_{M+})$  satisfies the assumptions of Theorem 2.5. Therefore the precisation that  $((f_0 + I_{M+})^{\lambda+\epsilon}, (f_0 + I_{M+})^{\lambda-\epsilon})$  is endowed with the metric  $d^*$  is not essential.

**LEMMA 3.22.** — *Let  $(A\mathcal{G}^+)$  holds and let  $\lambda \in \mathbf{R}$ ,  $m \in \mathbf{N}$   $m \geq 1$ . Then the following facts are equivalent:*

- i) *there exists  $\rho_0 > 0$ ,  $\{\lambda_\rho^{(i)} : 0 < \rho \leq \rho_0, 1 \leq i \leq 2m\} \subset \mathbf{R}$ ,  $\{u_\rho^{(i)} : 0 < \rho \leq \rho_0, 1 \leq i \leq 2m\} \subset W$  such that:*

$$\lambda_\rho^{(i)} Lu_\rho^{(i)} \in \partial^- f(u_\rho^{(i)}), \quad \frac{1}{2} (Lu_\rho^{(i)} | u_\rho^{(i)}) = \rho^2;$$

$$i \neq j \Rightarrow u_\rho^{(i)} \neq u_\rho^{(j)};$$

$$\lim_{\rho \rightarrow 0^+} (\lambda_\rho^{(i)}, u_\rho^{(i)}) = (\lambda, 0), \quad \lim_{\rho \rightarrow 0^+} \frac{f(u_\rho^{(i)})}{\rho^2} = \lambda;$$

ii) for every  $\epsilon > 0$  and for every sequence  $(\rho_h)$  in  $]0, 1]$  converging to zero, there exists  $h_0 \in \mathbf{N}$ ,  $\{\lambda_h^{(i)} : h \geq h_0, 1 \leq i \leq 2m\} \subset \mathbf{R}$ ,  $\{u_h^{(i)} : h \geq h_0, 1 \leq i \leq 2m\} \subset M^+$  such that:

$$\begin{aligned} \lambda_h^{(i)} Lu_h^{(i)} &\in \partial^- f_{\rho_h}(u_h^{(i)}), \\ i \neq j &\Rightarrow u_h^{(i)} \neq u_h^{(j)}; \\ \lambda - \epsilon &\leq f_{\rho_h}(u_h^{(i)}) \leq \lambda + \epsilon. \end{aligned}$$

*Proof.* — The case in which  $2m$  is substituted by  $1$  is proved in [12, Lemma 3.19]. The adaptation to the present case is straightforward.  $\square$

Now we can prove the main abstract result.

**THEOREM 3.23.** — *Let us assume that  $(A\mathcal{G}^+)$ , (3.18) and (3.19) hold. Let  $(\lambda, u)$  be a solution of (3.5) with  $(Lu | u) > 0$  and let  $m = \dim(\{v \in H : \lambda Lv \in \partial^- f_0(v)\})$ , which is a linear subspace of  $H$  of finite dimension by Theorem 3.17.*

*Then there exist  $\rho_0 > 0$ ,*

$$\begin{aligned} \{\lambda_\rho^{(i)} : 0 < \rho \leq \rho_0, 1 \leq i \leq 2m\} &\subset \mathbf{R}, \\ \{u_\rho^{(i)} : 0 < \rho \leq \rho_0, 1 \leq i \leq 2m\} &\subset W \end{aligned}$$

*such that:*

$$\begin{aligned} \lambda_\rho^{(i)} Lu_\rho^{(i)} &\in \partial^- f(u_\rho^{(i)}), \quad \frac{1}{2} (Lu_\rho^{(i)} | u_\rho^{(i)}) = \rho^2; \\ i \neq j &\Rightarrow u_\rho^{(i)} \neq u_\rho^{(j)}; \\ \lim_{\rho \rightarrow 0^+} (\lambda_\rho^{(i)}, u_\rho^{(i)}) &= (\lambda, 0), \quad \lim_{\rho \rightarrow 0^+} \frac{f(u_\rho^{(i)})}{\rho^2} = \lambda; \end{aligned}$$

*Proof.* — Let  $\epsilon_0$  be as in Theorem 3.17. To prove the thesis, it is sufficient to verify condition ii) of Lemma 3.22. Therefore let  $\epsilon > 0$  and let  $(\rho_h)$  be a sequence in  $]0, 1]$  converging to zero. Without loss of generality, we can assume  $\epsilon \leq \epsilon_0$ .

If we set  $\rho_\infty = 0$ , by Propositions 3.4, 3.9, 3.12 and (A2) we can apply Theorem 2.16 to the sequence  $(f_{\rho_h})$  and the hypersurface  $M^+$ . Let  $(\tilde{f}_h)$  and  $(\tilde{u}_h)$  be the sequences given by Theorem 2.16.

Since  $\tilde{f}_\infty = \hat{f}_\infty + I_{M^+} = f_0 + I_{M^+}$ , by Theorem 3.17,  $\lambda - \epsilon$  and  $\lambda + \epsilon$  are not critical from below for  $\tilde{f}_\infty$ . Therefore Theorem 2.16 allows to apply Theorem 2.14 to the sequence  $(\tilde{f}_h)$  with  $a = \lambda - \epsilon$  and  $b = \lambda + \epsilon$ . Let  $h_0 \in \mathbb{N}$  be such that  $\forall h \geq h_0$   $\lambda - \epsilon$  and  $\lambda + \epsilon$  are not critical from below for  $\tilde{f}_h$  and the pair  $(\tilde{f}_h^{\lambda+\epsilon}, \tilde{f}_h^{\lambda-\epsilon})$  is equivariantly homotopically equivalent to the pair  $((f_0 + I_{M^+})^{\lambda+\epsilon}, (f_0 + I_{M^+})^{\lambda-\epsilon})$ .

Combining Theorem 2.6 with Theorem 3.17, we deduce that:

$$\forall h \geq h_0, \quad \text{Index}(\tilde{f}_h^{\lambda+\epsilon}, \tilde{f}_h^{\lambda-\epsilon}) = m.$$

By Theorem 2.16<sub>x</sub> it is readily seen that  $\tilde{f}_h$  satisfies  $(PS)_c$  for every  $c \in \mathbb{R}$ . Therefore we can apply Theorem 2.7 to  $\tilde{f}_h$ , obtaining the existence of  $u_h^{(i)} \in M^+$  ( $1 \leq i \leq 2m$ ) with:

$$\begin{aligned} 0 &\in \partial^- \tilde{f}_h(u_h^{(i)}), \\ i \neq j &\Rightarrow u_h^{(i)} \neq u_h^{(j)}, \\ \lambda - \epsilon &< \tilde{f}_h(u_h^{(i)}) < \lambda + \epsilon. \end{aligned}$$

By Theorem 2.16<sub>iii</sub> we have:

$$\hat{f}_h(w) \geq \hat{f}_h(v) + (\alpha | w - v) - q|w - v|^2$$

whenever  $h \in \mathbb{N}$ ,  $w \in H$ ,  $v \in D(\partial^- \hat{f}_h)$ ,  $\alpha \in \partial^- \hat{f}_h(v)$ . Therefore we can apply Theorem 2.10 to  $\hat{f}_h$  and  $M^+$ , obtaining  $\lambda_h^{(i)} Lu_h^{(i)} \in \partial^- \hat{f}_h(u_h^{(i)})$  for some  $\lambda_h^{(i)} \in \mathbb{R}$ . Since  $\hat{f}_h(u_h^{(i)}) < \lambda - \epsilon$ , we conclude that:

$$\hat{f}_h(u_h^{(i)}) = f_{\rho_h}(u_h^{(i)}), \quad \partial^- \hat{f}_h(u_h^{(i)}) = \partial^- f_{\rho_h}(u_h^{(i)})$$

eventually as  $h \rightarrow \infty$ , hence the thesis.  $\square$

#### 4. On the buckling of a thin elastic plate subjected to symmetric constraints

We wish to show an application of the results of the previous section to the buckling of a thin elastic plate subjected to symmetric unilateral constraints.

Let  $\Omega$  be a bounded open subset of  $\mathbf{R}^2$ ,  $F_0$  (the initial Airy stress function) an element of  $W^{2,2}(\Omega)$  and  $\varphi : \Omega \rightarrow [0, +\infty]$  a lower semicontinuous function. If we set:

$$\mathbf{K} = \left\{ u \in W_0^{2,2}(\Omega) : -\varphi \leq u \leq \varphi \right\},$$

we have to study, according to [4, 5, 29, 30, 39], the problem:

$$\begin{cases} (\lambda, u) \in \mathbf{R} \times \mathbf{K} \\ \langle \Delta^2 u, v - u \rangle + \langle [J[u, u], u], v - u \rangle \geq \lambda \langle [F_0, u], v - u \rangle, \forall v \in \mathbf{K} \end{cases} \quad (4.1)$$

where:

$$[v, w] := D_{xx}^2 v D_{yy}^2 w - 2D_{xy}^2 v D_{xy}^2 w + D_{yy}^2 v D_{xx}^2 w$$

and for every  $\eta \in W^{-2,2}(\Omega)$ ,  $J\eta \in W_0^{2,2}(\Omega)$  is characterized by  $\Delta^2(J\eta) = \eta$ . Let us remark that for every  $v, w \in W^{2,2}(\Omega)$ ,

$$[v, w] \in L^1(\Omega) \hookrightarrow W^{-2,2}(\Omega), \text{ as } W_0^{2,2}(\Omega) \hookrightarrow C(\bar{\Omega}) \hookrightarrow L^\infty(\Omega).$$

Therefore the problem is meaningful.

It is readily seen that for every  $\lambda$  in  $\mathbf{R}$ , the pair  $(\lambda, 0)$  is a solution of (4.1).

**DEFINITION 4.2.** — *A real number  $\lambda$  is said to be of bifurcation for (4.1), if there exists a sequence  $((\lambda_h, u_h))$  of solutions of (4.1) with  $u_h \neq 0$ ,*

$$\lim_h \lambda_h = \lambda, \quad \lim_h u_h = 0 \quad \text{in } W_0^{2,2}(\Omega).$$

Let  $\mathbf{K}_0$  be the closure in  $W_0^{2,2}(\Omega)$  of the set  $(\bigcup_{t>0} t\mathbf{K})$ .

Since  $-\mathbf{K} = \mathbf{K}$ ,  $\mathbf{K}_0$  is a closed linear subspace of  $W_0^{2,2}(\Omega)$  and:

$$\begin{cases} (\lambda, u) \in \mathbf{R} \times \mathbf{K}_0 \\ \langle \Delta^2 u, v \rangle = \lambda \langle [F_0, u], v \rangle, \forall v \in \mathbf{K}_0 \end{cases} \quad (4.3)$$

can be regarded as the linearized problem associated with (4.1).

**DEFINITION 4.4.** — *A real number  $\lambda$  is said to be an eigenvalue of (4.3), if the pair  $(\lambda, u)$  satisfies (4.3) for some  $u \neq 0$ .*

**THEOREM 4.5.** — *Let  $\lambda$  be a real number. Then  $\lambda$  is of bifurcation for (4.1) if and only if  $\lambda$  is an eigenvalue of (4.3).*

*Proof.* — Since  $\mathbf{K}_0$  is a linear space,  $(\lambda, u) \in \mathbf{R} \times \mathbf{K}_0$  satisfies (4.3) if and only if:

$$\langle \Delta^2 u, v - u \rangle \geq \lambda \langle [F_0, u], v - u \rangle, \quad \forall v \in \mathbf{K}_0.$$

Then the result is contained in [13, Theorem 3.5 and 3.13].  $\square$

Our purpose is to show a multiplicity result, taking advantage of the symmetry of  $\mathbf{K}$ .

For every  $v \in W^{2,2}(\Omega)$ ,  $w \in W^{1,4}(\Omega)$ , let us set:

$$\begin{aligned} [v, w]_1 &= D_x \left( D_{yy}^2 v D_x w - D_{xy}^2 v D_y w \right) + D_y \left( D_{xx}^2 v D_y w - D_{xy}^2 v D_x w \right) \\ [w]_2 &= - \left( D_{xx}^2 (D_y w)^2 - 2D_{xy}^2 (D_x w D_y w) + D_{yy}^2 (D_x w)^2 \right). \end{aligned}$$

It is readily seen that the maps:

$$\begin{aligned} [\cdot, \cdot]_1 &: W^{2,2}(\Omega) \times W^{1,4}(\Omega) \rightarrow W^{-1,4/3}(\Omega), \\ [\cdot]_2 &: W^{1,4}(\Omega) \rightarrow W^{-2,2}(\Omega) \end{aligned}$$

are of class  $C^\infty$ . Moreover (see [4, 29]) we have for every  $v, w \in W_0^{2,2}(\Omega)$ ,

$$[v, w]_1 = [v, w], \quad [w]_2 = [w, w]$$

as elements of  $W^{-2,2}(\Omega)$ .

Let  $3/2 < s < 2$ ,  $H = W_0^{s,2}(\Omega)$  and let us define  $A, f_0 : H \rightarrow \mathbf{R} \cup \{+\infty\}$ ,  $B, C : H \rightarrow \mathbf{R}$  by:

$$\begin{aligned} A(u) &= \begin{cases} \frac{1}{2} \int_{\Omega} |\Delta u|^2 dx & u \in \mathbf{K} \\ +\infty & u \in H \setminus \mathbf{K}, \end{cases} \\ f_0(u) &= \begin{cases} \frac{1}{2} \int_{\Omega} |\Delta u|^2 dx & u \in \mathbf{K}_0 \\ +\infty & u \in H \setminus \mathbf{K}_0, \end{cases} \\ B(u) &= \frac{1}{2} \langle [F_0, u]_1, u \rangle, \\ C(u) &= \frac{1}{4} \langle [J[u]_2, u]_1, u \rangle, \end{aligned}$$

which is meaningful, as  $W_0^{s,2}(\Omega) \hookrightarrow W_0^{1,4}(\Omega)$ .

The functionals  $B$  and  $C$  are of class  $C^\infty$  and homogenous of degree 2 and 4, respectively. The functionals  $A$  and  $f_0$  are convex and lower semicontinuous. Let us denote by  $L : H \rightarrow H$  the gradient of  $B$  and let us set  $f = A + C$ . Finally, let  $W$  be the open unit ball centred at the origin in  $H$ .

PROPOSITION 4.6. — *The following facts hold:*

- i)  $f$  is lower semicontinuous, even and  $\left\{ u \mapsto f(u) + q|u|_H^2 \right\}$  is convex on  $\overline{W}$  for some  $q \in \mathbf{R}^+$ ;
- ii)  $f(0) = 0, 0 \in \partial^- f(0)$ ;
- iii)  $L$  linear, bounded and symmetric;
- iv) for every sequence  $(\rho_h)$  in  $]0, +\infty[$  converging to zero, we have:

$$f_0 = \Gamma^-(H) \lim_h f_{\rho_h}$$

where  $f_\rho(u) = \rho^{-2}f(\rho u)$  if  $\rho u \in \overline{W}$ ,  $f_\rho(u) = +\infty$  elsewhere;

- v) for every sequence  $(\rho_h)$  in  $]0, +\infty[$ , the sequence  $(f_{\rho_h})$  is equicoercive;
- vi)  $\forall u, v \in H, f_0(u+v) + f_0(u-v) = 2f_0(u) + 2f_0(v)$ ;
- vii) the function  $\left\{ u \mapsto f_0(u) - \epsilon|u|_H^2 \right\}$  is convex for some  $\epsilon > 0$ ;
- viii) for every  $\lambda$  in  $\mathbf{R}$ ,  $u$  in  $\mathbf{K}$ , we have  $\lambda Lu \in \partial^- f(u)$  if and only if:
 
$$\left\langle \Delta^2 u, v - u \right\rangle + \langle [J[u, u], u], v - u \rangle \geq \lambda \langle [F_0, u], v - u \rangle, \quad \forall v \in \mathbf{K};$$
- ix) for every  $\lambda$  in  $\mathbf{R}$ ,  $u$  in  $\mathbf{K}_0$ , we have  $\lambda Lu \in \partial^- f_0(u)$  if and only if:

$$\left\langle \Delta^2 u, v \right\rangle = \lambda \langle [F_0, u], v \rangle, \quad \forall v \in \mathbf{K}_0.$$

*Proof.* — The evenness of  $f$  is obvious. Then i), ii), iii), iv), v), viii) and ix) are contained in [13, Proposition 3.7]. Since  $\mathbf{K}_0$  is a linear subspace of  $H$ , vi) follows immediately. Finally, vii) is a consequence of the continuous embedding  $W_0^{2,2}(\Omega) \hookrightarrow H$ .  $\square$

*Remark 4.7.* — Let  $(\lambda, u)$  be a solution of (4.3) with  $u \neq 0$ . Then  $\lambda \langle [F_0, u], u \rangle > 0$  and the set  $\{u : (\lambda, u) \text{ satisfies (4.3)}\}$  is a linear subspace of  $W_0^{2,2}(\Omega)$  of finite dimension.



Now we restrict our attention to positive eigenvalues of (4.3). By changing  $F_0$  in  $-F_0$ , we can always reduce ourselves to this case. Let us state our main result.

**THEOREM 4.8.** — *Let  $\lambda$  be a positive eigenvalue of (4.3) and let  $m = \dim\{u : (\lambda, u) \text{ satisfies (4.3)}\}$ .*

*Then there exist  $\rho_0 > 0$ ,*

$$\begin{aligned} \left\{ \lambda_0^{(i)} : 0 < \rho \leq \rho_0, 1 \leq i \leq 2m \right\} &\subset \mathbf{R}, \\ \left\{ u_\rho^{(i)} : 0 < \rho \leq \rho_0, 1 \leq i \leq 2m \right\} &\subset \mathbf{K} \end{aligned}$$

*such that  $(\lambda_\rho^{(i)}, u_\rho^{(i)})$  satisfies (4.1) and:*

$$\begin{aligned} \frac{1}{2} \int_{\Omega} [F_0, u_\rho^{(i)}] u_\rho^{(i)} dx &= \rho^2, \\ i \neq j &\Rightarrow u_\rho^{(i)} \neq u_\rho^{(j)}, \\ \lim_{\rho \rightarrow 0^+} (\lambda_\rho^{(i)}, u_\rho^{(i)}) &= (\lambda, 0) \text{ in } \mathbf{R} \times W_0^{2,2}(\Omega), \\ \lim_{\rho \rightarrow 0^+} \frac{1}{2\rho^2} \int_{\Omega} |\Delta u_\rho^{(i)}|^2 dx &= \lambda. \end{aligned}$$

*Proof.* — Combining Proposition 4.6<sub>v</sub> with Proposition 3.12, we deduce that assumption (A3<sup>+</sup>) is satisfied. By Proposition 4.6 and Remark 3.20 we can apply Theorem 3.23. To conclude the proof, we have only to show that:

$$\lim_{\rho \rightarrow 0^+} \frac{1}{2\rho^2} \int_{\Omega} |\Delta u_\rho^{(i)}|^2 dx = \lambda,$$

so that  $\lim_{\rho \rightarrow 0^+} u_\rho^{(i)} = 0$  in  $W_0^{2,2}(\Omega)$  and not only in  $H$ .

Combining the inequality:

$$\forall u \in W_0^{2,2}(\Omega), \quad f_\rho(u) \geq \frac{1}{2} \int_{\Omega} |\Delta u|^2 dx$$

with:

$$\lim_{\rho \rightarrow 0^+} \frac{f(u_\rho^{(i)})}{\rho^2} = \lambda,$$

we deduce that the set  $\left\{ u_\rho^{(i)} / \rho : 0 < \rho \leq \rho_0, 1 \leq i \leq 2m \right\}$  is bounded in  $W_0^{2,2}(\Omega)$ . Therefore:

$$\lim_{\rho \rightarrow 0^+} \rho^{-2} C(u_\rho^{(i)}) = \lim_{\rho \rightarrow 0^+} \rho^2 C\left(\frac{u_\rho^{(i)}}{\rho}\right) = 0,$$

so that:

$$\lim_{\rho \rightarrow 0^+} \frac{1}{2\rho^2} \int_{\Omega} \left| \Delta u_\rho^{(i)} \right|^2 dx = \lambda. \square$$

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