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Jump functions of a real interval to a Banach space

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RÉSUMÉ. — Les fonctions à variation localement bornée dans un intervalle réel I, à valeurs dans un espace de Banach X, forment l'espace vectoriel $lbv(I,X)$, sur lequel une topologie est définie par la famille de semi-normes $f \mapsto \mathrm{var}(f;a,b)$, $a \in I$, $b \in I$. Par définition, les fonction de saut sont les éléments de l'adhérence de l'espace des fonctions localement en escalier. On construit la décomposition de toute $f \in lbv(I,X)$ en somme d'une fonction de saut et d'une fonction continue. Entre autres propriétés, on étudie comment cette décomposition se réflète sur la fonction variation indéfinie V_f de f. Le lien avec la décomposition d'une mesure réelle sur I en somme d'une mesure diffuse et d'une mesure atomique est explicité.

ABSTRACT. — On the space $lbv(I,X)$ of the functions of a real interval I to a Banach space X, with locally bounded variation, a topology is defined by the semi-norms $f \mapsto \mathrm{var}(f;a,b)$, $a \in I$, $b \in I$. By definition, jump functions are the elements of the closure of the space of local step functions. The decomposition of every $f \in lbv(I,X)$ into the sum of a jump function and a continuous element of $lbv(I,X)$ is constructed. Among other properties, it is studied how this decomposition is reflected on the real function V_f, the indefinite variation of f. The connection of what precedes with the decomposition of a real measure into the sum of a diffuse measure and an atomic measure is investigated.

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1. Introduction

Real functions of a real variable with *bounded variation* are a standard topic. Elementarily, a function belongs to this class if and only if it equals the difference of two nondecreasing real functions. In that context, the decomposition of any element of the class into the sum of a continuous component and an element of the special sort called a *jump function* is easily derived.

From the real valued case, the concept of bounded variation may be extended to functions with values in $\mathbf{R}^n$ by referring to their components. Actually, many properties discovered in that way are found, through the use of different techniques, to hold also for functions, with values in an arbitrary Banach space X. This is the setting of the present paper, devoted to extending the above decomposition property and to developing some of its topological features.

The author's primary motivation lies in *Nonsmooth Dynamics*, i.e. the study of Mechanical Systems whose motion is not regular enough for accelerations to exist. The velocity is only assumed to be a vector function of time with locally bounded variation; this allows for a vector measure, on the considered time interval, to play the role of acceleration. Accordingly, the system dynamics is governed by a *mesure differential equation* or, if such mechanical effect as unilateral contact, possibly with Coulomb friction, are taken into account, a *measure differential inclusion* [6]. On this purpose, a chapter of a recent book [5] has been devoted to exposing some old and new properties of the (locally) b.v. functions of a real interval to a Banach space X and of the associated vector measures. The present paper is intended to complement this text.

Let I denote a real interval, not empty nor reduced to a singleton. By $f \in lbv(I,X)$ it is meant that f is a function of I to X with *locally bounded variation*, i.e. it has bounded variation on every compact subinterval of I (in [2], this is called a function with *finite variation*). Trivial inequalities concerning variations imply that $lbv(I,X)$ is a vector space and that, for every compact subinterval [a, b] of I, the mapping $f \mapsto var(f;a,b)$ is a semi-norm on this space. If [a, b] ranges through the totality of the compact subintervals of I, or equivalently through some increasing sequence of such subintervals with union equal to I, the collection of the corresponding semi-norms defines on $lbv(I,X)$ a (non Hausdorff) locally convex topology that we shall call the ***variation topology***.

The closure of $\{O\}$ in this topology equals the subspace of constants; by going to the quotient, one obtains a metrizable locally convex Hausdorff space. Equivalently, we shall once for all choose in I *a reference point*, say ρ, and restrict ourselves to $lbv_0(I,X)$, namely the subspace of $lbv(I,X)$ consisting of functions which vanish at point ρ. This linear space will be shown in Sec. 2 to be complete in the variation topology.

Denoting by $|.|_X$ the Banach norm in X, one obviously has, for every $f \in lbv_0(I,X)$ and every $[a,b] \subset I$ containing ρ,

$$\sup_{t\in[a,b]} |f(t)|_X \leq \text{ var } (f;a,b). \tag{1.1}$$

Therefore, in the space $lbv_0(I,X)$, the variation topology is stronger than the topology of the uniform convergence on compact subsets of I.

One denotes by $bv(I,X)$ the subspace of $lbv(I,X)$ constituted by the functions whose total variation on I is finite. Its subspace $bv_0(I,X)$, consisting of the elements which vanish at point ρ, is a Banach space in the norm $f \mapsto \text{ var}(f;I)$.

By writing $s \in lst(I,X)$, we shall mean that s is a *local step function* in the following sense : there exists a locally finite partition of the interval I into subintervals of any sort (some of them possibly reduced to singletons), on each member of which s equals a constant. Such functions make a linear subspace of $lbv(I,X)$, easily proved to be dense relative to the topology of uniform convergence on the compact subsets of I. But, in the variation topology, the closure of $lst(I,X)$ is a proper subspace of $lbv(I,X)$, whose elements, by definition, are the *jump functions*. A series of propositions established in Sec. 3 below yield the proof of :

THEOREM 1.— *Every $f \in lbv(I,X)$ possesses a unique decomposition into the sum of a jump function vanishing at point ρ, say $J_0(f)$, and a continuous element of $lbv(I,X)$, say $C(f)$. The mappings J_0 and C are linear projectors of $lbv(I,X)$ into itself, continuous in the variation topology.*

On may call $J_0(f)$ the *jump component* of f in $lbv_0(I,X)$. Changing the reference point ρ results in adding constants to $J_0(f)$ and $C(f)$.

With every $f \in lbv(I,X)$, there is associated its *variation function*, vanishing at point ρ, namely the nondecreasing element V_f of $lbv_0(I,\mathbf{R})$ defined as

$$V_f(t) = \begin{cases} \text{var}(f;\rho,t) & \text{if } t \geq \rho \\ -\text{var}(f;t,\rho) & \text{if } t \leq \rho. \end{cases}$$

We shall establish in Sec. 4 :

THEOREM 2.— *The nonlinear mapping $f \mapsto V_f$ is continuous of $lbv(I,X)$ to $lbv_0(I,\mathbf{R})$ in the respective variation topologies of these spaces.*

THEOREM 3.— *If the notations C and J_0 are also applied to elements of $lbv(I,\mathbf{R})$ one has $V_{C(f)} = C(V_f)$ and $V_{J_0(f)} = J_0(V_f)$.*

Theorems 1 and 3 clearly yield $V_f = V_{C(f)} + V_{J_0(f)}$ so, for every $[a,b] \subset I$,

$$\mathrm{var}(f;a,b) = \ \mathrm{var}\ (C(f);a,b) + \ \mathrm{var}\ (J_0(f);a,b).$$

Actually, this equality will in the sequel be established as Proposition 3.2, prior to proving Theorem 3.

Theorem 3 also implies that *an element f of $lbv(I,X)$ is a jump function if and only if the same is true for V_f.* In fact, if V_f is a jump function, $C(V_f) = V_{C(f)}$ equals a constant (in fact zero), so $C(f)$ is a constant, making of f a jump function. Conversely, if f is a jump function, $C(f)$ is a constant, so $C(V_f)$ equals zero, making of V_f a jump function.

Symmetrically, f is continuous if and only if the same is true for V_f; this is elementarily established in [5] and will be used in the sequel prior to proving Theorem 3.

The most common reason one has for being interested in lbv functions (letting alone Nonsmooth Dynamics) lies in the fact that, with every $f \in lbv(I,X)$, an X-valued *measure* on the interval I is associated, usually denoted by df or Df, called the *differential measure* (or the *Stieltjes measure*) of f. A connection is naturally expected between Theorem 1 and a property which, at least in the real-valued case, is classical, namely the decomposition of a measure into the sum of two components, on of which is *diffuse* and the other *atomic*.

Actually, properties of both sorts cannot be strictly equivalent, since the correspondence between X-valued measures and elements of $lbv(I,X)$ is not one-to-one. In fact, if an lbv function f is discontinuous at some point a, the measure df is expected to exhibit at this point an atom with mass equal to the *total jump* of f, i.e. the difference between the right-limit $f^+(a)$ and the left-limit $f^-(a)$. But the very value that f takes at point a bears non relationship with df.

The differential measures of the elements of $lbv(I,X)$ are characterized by the following property, which, for infinite-dimensional X, makes of them a special class of vector measures : they have *bounded variation* (concerning this concept, see e.g. [3]) on every compact subinterval of I. In [2], such vector measures are said to have *finite variation*; the corresponding concept in Bourbaki's construction of measure theory [1] is that of a *majorable* vector measure. This property implies the existence of the nonnegative real measure $|df|$, the *modulus measure* of df (also called the *variation measure* or the *absolute value* of df). In Sec. 5, we shall prove :

THEOREM 4.— *An element f of $lbv(I,X)$ is a jump function if and only if the nonnegative real measure $|df|$ is atomic.*

Equivalently, the nonnegative real measure dV_f is atomic.

2. The variation topology

At the first stage, let us consider functions whose total variation on I is finite.

PROPOSITION 2.1.— *The vector space $bv_0(I,X)$ is a Banach space in the norm $\|f\|_{\text{var}} := var\,(f,I)$.*

Proof.— Let (f_n) be a Cauchy sequence in this norm. There exists $M \geq 0$ such that

$$\forall n \in \mathbf{N} \,:\, \operatorname{var}\,(f_n, I) \leq M. \tag{2.1}$$

In view of inequality (1.1), (f_n) is also a Cauchy sequence relative to the supremum norm $\|.\|_\infty$, so it converges in the latter norm to some function $f_\infty : I \to X$, with $f_\infty(\varrho) = 0$.

For every finite ordered set of points of I, say $S : \tau_0 < \tau_1 < \ldots < \tau_\nu$, and every $f : I \to X$, let us put the notation

$$V(f,S) = \sum_{i=1}^{\nu} |f(\tau_i) - f(\tau_{i-1})|_X.$$

Since, at every point τ_i of S, the element $f_\infty(\tau_i)$ of X equals the limit of $f_n(\tau_i)$ in the $|.|_X$ norm, one has

$$V(f_\infty, S) = \lim_{n\to\infty} \sum_{i=1}^{\nu} |f_n(\tau_i) - f_n(\tau_{i-1})|_X.$$

Due to (2.1), this is majorized by M whatever is S, hence $f_\infty \in bv_0(I,X)$.

Finally, let us prove that f_n converges to f_∞ in the $\|.\|_{\mathrm{var}}$ norm. Let $\varepsilon > 0$; in view of the assumed Cauchy property, there exists $n \in \mathbf{N}$ such that

$$p \geq n \Rightarrow \|f_n - f_p\|_{\mathrm{var}} \leq \varepsilon$$

hence, for every $p \geq n$,

$$V(f_n - f_p, S) \leq \text{ var } (f_n - f_p, I) \leq \varepsilon$$

One readily checks that, for fixed S, the mapping $f \mapsto V(f, S)$ is a semi-norm. Therefore

$$V(f_n - f_\infty, S) \leq V(f_n - f_p, S) + V(f_p - f_\infty, S) \leq \varepsilon + V(f_p - f_\infty, S).$$

By letting p tend to $+\infty$, one concludes $V(f_n - f_\infty, S) \leq \varepsilon$ for every finite sequence S, hence var $(f_n - f_\infty, I) \leq \varepsilon$, *q.e.d.* □

Let us now drop the assumption of finite total variation on I. The variation topology on $lbv_0(I, X)$ is defined by the collection of semi-norms $f \mapsto N_k(f) := \text{var } (f, K_k)$, where (K_k) denotes a nondecreasing sequence of compact subintervals whose union equals I. In addition, on may assume that all intervals K_k are large enough to contain ϱ. Therefore, the resulting topology is metrizable and Hausdorff.

PROPOSITION 2.2.— *The variation topology makes of $lbv_0(I, X)$ a complete metrizable locally convex vector space (i.e. a Fréchet space).*

Proof.— Let (f_n) be a Cauchy sequence in $lbv_0(I, X)$. By definition, for every U, a neighborhood of the origin in this space, there exists $n \in \mathbf{N}$ such that

$$p \geq n \text{ and } q \geq n \Rightarrow f_p - f_q \in U.$$

By taking as U the semi-ball

$$U_{k,\varepsilon} = \{u \in lbv_0(I, X) \ : \ N_k(u) < \varepsilon\},$$

with $k \in \mathbf{N}$ and $\varepsilon > 0$, this yields the implication

$$p \geq n \text{ and } q \geq n \Rightarrow N_k(f_p - f_q) < \varepsilon$$

Therefore the restrictions of the functions (f_n) to K_k make a Cauchy sequence in $bv_0(K_k, X)$. In view of Prop. 2.1, this sequence converges to some element f^k of the latter space. If the same construction is effected for another interval $K_{k'}$, with $k' > k$, the resulting function $f^{k'} : K_{k'} \to X$ is an extension of f^k. Inductively, a function f is constructed on the whole of I, which constitutes the limit of the sequence (f_n) in the variation topology. □

3. Construction of the jump component

Without assuming that I is compact, let us first consider the case where $f : I \to X$ has finite total variation on the whole interval. As before, a reference point ϱ is chosen in I. With every $e \in I$, let us associate the *single-step function* $s_e : I \to X$ defined by the following conditions.

The function s_e verifies

$$s_e(\varrho) = 0 \tag{3.1}$$

and admits e as its unique possible discontinuity point, with the same jumps as f at this point, i.e.

$$s_e^+(e) - s_e(e) = f^+(e) - f(e), \tag{3.2}$$

$$s_e(e) - s_e^-(e) = f(e) - f^-(e), \tag{3.3}$$

Here, the notations $^+$ and $^-$ refer to the *right-limit* and the *left-limit* of the considered functions at the considered point. By convention, $f^-(e) = f(e)$ *if the point e of I happens to be the left end of this interval and* $f^+(e) = f(e)$ *if it happens to be the right end.*

Clearly, $s_e \in bv_0(I, X)$; it equals the zero constant if and only if f is continuous at point e.

For every finite subset F of I, let us consider the step function

$$s_F = \sum_{e \in F} s_e.$$

Call $\mathcal{F}$ the totality of the finite subsets of I. The inclusion ordering makes of $\mathcal{F}$ a *directed set* [4] and one readily checks that the mapping

$$F \mapsto \text{var}\,(s_F, I) \tag{3.4}$$

of $(\mathcal{F}, \supset)$ to $\mathbf{R}$ is nondecreasing and bounded by var (f, I). Hence this mapping is a *convergent net* of real numbers.

Let us show that the net $F \mapsto s_F$ is Cauchy in the Banach norm of $bv_0(I, X)$. Let $\varepsilon > 0$; due to the convergence of the net (3.4) in $\mathbf{R}$, there exists $F \in \mathcal{F}$ such that, for every $F' \in \mathcal{F}$ containing F, one has

$$0 \leq \operatorname{var}(s_{F'}, I) - \operatorname{var}(s_F, I) \leq \frac{\varepsilon}{2}. \tag{3.5}$$

Now

$$s_{F'} = s_F + s_{F' \setminus F}; \tag{3.6}$$

this is the sum of two step functions with disjoint dicontinuity sets, therefore

$$\operatorname{var}(s_{F'}, I) = \operatorname{var}(s_F, I) + \operatorname{var}(s_{F' \setminus F}, I), \tag{3.7}$$

which, in view of (3.5) and (3.6), yields

$$\operatorname{var}(s_{F'} - s_F, I) \leq \frac{\varepsilon}{2}.$$

By the triangle inequality, there comes out that, for every F' and F'' in $\mathcal{F}$, both containing F, one has

$$\|s_{F'} - s_{F''}\|_{\text{var}} \leq \varepsilon;$$

this is the Cauchy property for the net $F \mapsto s_F$ in the Banach space $bv_0(I, X)$. It secures the existence in this space of the element

$$j_f = \lim_{F \in \mathcal{F}} s_F. \tag{3.8}$$

If, more generally than above, the given function f belongs to $lbv(I, X)$, one may cover I with a nondecreasing sequence of compact subintervals K_k, all containing ϱ. The preceding construction will be applied to the restriction of f to K_k, yielding as the limit in (3.8) some function $j_f^k \in bv_0(K_k, X)$. Observe that the single-step function s_e vanishes throughout K_k if the element e of I is not a discontinuity point of the restriction of f to K_k. When K_k is replaced by a larger subinterval $K_{\widehat{k}}$ of I, the step function s_F previously considered is replaced by some extension of it to $K_{\widehat{k}}$. For F ranging in $\mathcal{F}$, the limit found in $bv_0(K_{\widehat{k}}, X)$ is an extension of the function precedingly obtained.

By this process, an element j_f of $lbv_0(I, X)$ is inductively defined, equal to the limit of the net $F \mapsto s_F$ in this Fréchet space.

DEFINITION 3.1.— *For every $f \in lbv(I,X)$ (resp. $f \in bv(I,X)$), the function j_f constructed above is called the* jump component *of f in $lbv_0(I,X)$ (resp. in $bv_0(I,X)$).*

Clearly, when the reference point ϱ is changed, the function j_f is altered by the addition of a constant.

PROPOSITION 3.2.— *The function $c_f = f - j_f$ is continuous.*

For every $[a,b] \subset I$, one has

$$var\,(f;a,b) = var\,(j_f;a,b) + var\,(c_f;a,b). \tag{3.9}$$

Proof.— It is enough to consider the case $f \in bv(I,X)$. Let $t \in I$; the part of F consisting of the finite subsets of I which include t is cofinal in $(\mathcal{F}, \subset)$. Therefore in (3.8), one may indifferently impose on F the condition of including t. Under this condition, the function $f - s_F$ is continuous at t, since by construction $f - s_t$ is so, as well as all the functions s_e for $e \neq t$. Now, continuity at point t is preserved when taking the limit (3.8), because the norm $\|.\|_{\text{var}}$ on bv_0 majorizes the norm of uniform convergence. This proves the first assertion.

In order to establish (3.9), we shall first show that, for every $g \in bv(I,X)$ and every single-step function s_e such that $g - s_e$ is continuous at e, one has

$$\text{var}\,(g;a,b) = \text{var}\,(s_e;a,b) + \text{var}\,(g - s_e;a,b). \tag{3.10}$$

In fact, supposing $e \in [a,b]$ (otherwise the equalities that follow are trivial) one has, in view of [5], Prop. 4.3,

$$\text{var}\,(g;a,e) = \|g(e) - g^-(e)\| + \lim_{t \to e, t<e} \text{var}\,(g;a,t)$$

Now, since s_e equals a constant in [a, t], the two functions g and $g - s_e$ have the same variation on this interval, while

$$\text{var}\,(s_e;a,e) = \|s_e(e) - s_e^-(e)\| = \|g(e) - g^-(e)\|\cdot$$

This proves that (3.10) holds when $[a,b]$ is replaced by [a, e]; similar reasoning applies to $[e,b]$ and (3.10) follows by addition.

Applying (3.10) inductively, one sees that, for every finite subset F of I,

$$\text{var}\,(f;a,b) = \text{var}\,(s_F;a,b) + \text{var}\,(f - s_F;a,b).$$

Due to the definition of j_f in (3.8) and to the continuity of the mapping $g \mapsto \text{var}\,(g; a, b)$ in the variation topology, this yields equality (3.9). □

PROPOSITION 3.3.— *The mapping* $J_0 : f \mapsto j_f$ *of* $lbv(I, X)$ *to* $lbv_0(I, X)$ *is linear, idempotent and continuous in the variation topology.*

Proof.— It is enough to consider the *bv* case. From (3.1), (3.2) and (3.3) it follows that the mapping $f \mapsto s_e$ is linear. By addition, the same is true for $f \mapsto s_F$. Linearity is preserved under the process of going to the limit, used in order to construct j_f.

Equality (3.9) implies $\text{var}(j_f, I) \leq \text{var}(f, I)$, securing the continuity of J_0. That J_0 is idempotent results from Prop. 3.2; in fact, when applied to a continuous function such as c_f, the process generating the jump component clearly yields the zero constant. □

PROPOSITION 3.4.— *The range of* J_0 *is a closed subspace of* $lbv(I, X)$ *in the variation topology. It equals the closure of the set of the step functions vanishing at point* ϱ.

The mapping id $- J_0 : f \mapsto C_f$ *is a linear projector, continuous in the same topology. Its range equals the totality of the elements of* $lbv(I, X)$ *which are continuous throughout* I.

Proof.— Every element g of the range of J_0 equals the limit of a net (equivalently a sequence, since we are working in metrizable spaces) of step functions vanishing at ϱ. In fact, as J_0 is idempotent, g equals the corresponding j_g, which, by construction, equals such a limit.

That every continuous element of $lbv(I, X)$ belongs to the range of id $- J_0$ follows from the construction of the jump component : in fact this construction yields the zero constant when performed upon a continuous function. □

4. Variation functions

Theorem 2, which states the continuity of the mapping $f \mapsto V_f$ in the respective variation topologies of $lbv_0(I, X)$ and $lbv_0(I, \mathbf{R})$ is a consequence of the following. Incidentally, the vanishing of the considered functions at point ϱ is not needed here.

PROPOSITION 4.1.— *Let f and g belong to $lbv(I,X)$, with v and w as respective variation functions. Then, for every $[a,b] \subset I$, one has*

$$var\ (w - v; a, b) \leq var\ (g - f; a, b). \tag{4.1}$$

Proof.— Let $\varepsilon > 0$; there exists a finite sequence of points of $[a,b]$, say $\tau_0 < \tau_1 < \ldots < \tau_n$, such that

$$\text{var}\ (w - v; a, b) \leq \frac{1}{2}\varepsilon + \sum_{i=1}^{n} |w(\tau_i) - v(\tau_i) - w(\tau_{i-1}) + v(\tau_{i-1})|.$$

For every $i \in \{1, 2, \ldots, n\}$, there exist α_i and β_i in $[0, \varepsilon/2n]$ and a sequence

$$\tau_{i-1} = \tau_i^0 < \tau_i^1 < \ldots < \tau_i^p = \tau_i,$$

where the integer p depends on i, such that

$$v(\tau_i) - v(\tau_{i-1}) = \text{var}\ (f; \tau_{i-1}, \tau_i) = \alpha_i + \sum_{j=1}^{p} |f(\tau_i^j) - f(\tau_i^{j-1})|_X.$$

$$w(\tau_i) - w(\tau_{i-1}) = \text{var}\ (g; \tau_{i-1}, \tau_i) = \beta_i + \sum_{j=1}^{p} |g(\tau_i^j) - g(\tau_i^{j-1})|_X$$

Then

$$|w(\tau_i) - v(\tau_i) - w(\tau_{i-1}) + v(\tau_{i-1})| \tag{4.2}$$

$$= \left| \beta_i - \alpha_i + \sum_{j=1}^{p} (|g(\tau_i^j) - g(\tau_i^{j-1})|_X - |f(\tau_i^j) - f(\tau_i^{j-1})|_X) \right|.$$

Now

$$\left| |g(\tau_i^j) - g(\tau_i^{j-1})|_X - |f(\tau_i^j) - f(\tau_i^{j-1})|_X \right|$$

$$\leq |g(\tau_i^j) - g(\tau_i^{j-1}) - f(\tau_i^j) + f(\tau_i^{j-1})|_X \leq \text{var}\ (g - f; \tau_i^{j-1}, \tau_i^j).$$

Therefore, the expression in (4.2) is majorized by

$$|\beta_i - \alpha_i| + \text{var}(g - f; \tau_{i-1}, \tau_i).$$

Finally, since $|\beta_i - \alpha_i| \leq \varepsilon/2n$,

$$\text{var}\ (w - v; a, b) \leq \varepsilon + \text{var}\ (g - f; a, b).$$

As ε is arbitrary, this establishes the expected inequality. □

To prove Theorem 3, there now remains to investigate how he decomposition $f = j_f + c_f$ is reflected on a decomposition of the variation function $V_f : I \to \mathbf{R}$ (vanishing at point ϱ). Denote by V_j and V_c the respective variation functions (vanishing at point ϱ) of j_f and c_f. From equality (3.9) it follows that

$$V_f = V_j + V_c.$$

An element of $lbv(I, X)$ is continuous if and only if its variation function is continuous [5]; thus V_c is continuous. Furthermore, V_j is a real jump function. In fact, j_f equals the limit, in the variation topology, of a sequence of X – valued step functions. The corresponding variation functions are real step functions which, due to Proposition 4.1, converge to V_j in the variation topology. Using Proposition 3.3 with $X = \mathbf{R}$, one concludes that V_c and V_j respectively equal the continuous component and the jump component of V_f.

5. Differential measures

As recalled in the Introduction, with every $f \in lbv(I, X)$ there is associated its differential measure df, an X–valued measure on the interval I. Characteristically, for every compact subinterval $[a, b]$ of I, one has

$$\int_{[a,b]} df = f^+(b) - f^-(a) \tag{5.1}$$

Here again, it is agreed that, if the point b of I happens to be the right end of this interval, then by convention $f^+(b) = f(b)$; symmetrically, $f^-(a) = f(a)$ if a is the left end of I.

In particular, by taking as $[a, b]$ a singleton $\{a\}$, there comes out that

$$\int_{\{a\}} df = f^+(a) - f^-(a). \tag{5.2}$$

One thus observes that, when f is discontinuous at point a, the proper value $f(a)$ of the function bears no relationship with the measure df. Consequently, even if one agrees to treat as equivalent two functions which differ only by a constant, the correspondence between f and df cannot be one-to-one.

As standard sources concerning the theory of measures with values in a Banach space, one may refer to [1], [2], [3]. The measure df introduced above has the special property of being *majorable* in the sense of [1], i.e. in the terminology of [2] and [3], this vector measure has *finite variation* on every compact subinterval of I. This implies the existence of the nonnegative real measure on I called in [3] the *variation measure* of df or, in [2] and [5], the *modulus measure* of df. We shall denote it by $|df|$. It is naturally involved in some inequalities, with the consequence that, in the sense of the ordering of real measures, the differential measure of the variation function V_f satisfies

$$|df| \leq dV_f.$$

It is established in [5] that, when f has *aligned jumps*, i.e. for every $a \in I$ the value $f(a)$ belongs to the line segment of X with endpoints $f^-(a)$ and $f^+(a)$, then $|df| = dV_f$ (conversely, this equality implies jump alignment, provided the Banach space X is *strictly convex*, i.e the triangle inequality in this space holds as equality for flat triangles only).

The following is classical in the context of nonnegative real measures on arbitrary spaces : such a measure is said *diffuse* if it yields zero as the measure of every singleton. In contrast, the nonnegative measure is said *atomic* if it equals the supremum of a collection of nonnegative point measures. It is found that a nonnegative measure belongs to one of these two classes if and only if it is *singular* relatively to every element of the other. Furthermore, *every nonnegative real measure lets itself be uniquely decomposed into the sum of a diffuse measure and of an atomic measure.*

Since f can differ from f^- only at its discontinuity points, which form a countable subset of I, one easily obtains :

LEMMA 5.1.— *For every $f \in lbv(I, X)$ the difference $f - f^-$ (resp. the difference $f - f^+$) is a jump function.*

This lemma allows one to restrict the proof of Theorem 4 to the case where $f = f^-$, so f trivially has aligned jumps, a circumstance which makes $|df| = dV_f$. As observed in Introduction, Theorem 3 implies that f is a jump function if and only if the same is true for V_f, so we are reduced to the study of the latter, a nondecreasing real function which, in the present case, is left-continuous [5]. In particular, when the interval I is bounded from the right and contains its right end, say r, the left-continuity of V_f secures that dV_f has no atom at this point. Recall in addition that V_f has been assumed to vanish at point ϱ. Therefore, V_f may be recovered from dV_f trough the following process, a special case of a construction investigated in detail in [2] or [5].

Let us denote by $\mathcal{M}$ the linear space consisting of the real measures on I with no atom at the possible right end of this interval. Let us call L the linear operation associating with every element of $\mathcal{M}$, say dw, the function $w : I \to \mathbf{R}$ defined as

$$w(t) := \begin{cases} \int_{[\varrho,t[} dw & \text{if } \varrho \leq t \\ -\int_{[t,\varrho[} dw & \text{if } t < \varrho. \end{cases}$$

One may check (see e.g. [2] or [5]; a similar situation is also classically met in Probability) that L is a linear bijection of $\mathcal{M}$ to the subspace $\mathcal{W}$ of $lbv_0(I, \mathbf{R})$ consisting of the left-continuous elements. Trivially w has aligned jumps, so $|dw| = dV_w$ and, for every compact subinterval $[a,b]$ of I,

$$\text{var } (w; a, b) = V_w(b) - V_w(a) = \int_{[a,b[} dV_w = \int_{[a,b[} |dw|.$$

This readily yields that, when $\mathcal{W}$ is endowed with the variation topology of $lbv(I, \mathbf{R})$ and $\mathcal{M}$ with the strong topology of the space of real measures, L is bicontinuous.

By using these topologies, since $V_w = L(dV_w)$, one completes the proof of Theorem 4 through the following remarks :

1° An element of $\mathcal{W}$ is a jump function if and only if it equals the sum of a series of single-step functions.

2° An element w of $\mathcal{W}$ is a single-step function if and only if dw is a point measure.

3° An element of $\mathcal{M}$ is an atomic measure if and only if it equals the sum of a series of point measures.

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