ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 5e série, tome 10, nº 2 (1989), p. 325-333

<http://www.numdam.org/item?id=AFST_1989_5_10_2_325_0>

Annales Faculté des Sciences de Toulouse Vol. X, n°2, 1989

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RÉSUMÉ. — On utilise la théorie des potentiels de couche et les propriétés de la solution fondamentale pour obtenir un théorème de régularité dans les espaces de Sobolev des solutions de l'équation de Poisson dans les domaines lipschitziens de $\mathbf{R}^n$. Nous obtenons comme corollaire la continuité de ces solutions pour $n = 3$.

ABSTRACT. — Using the layer potentials theory and the properties of the singular fundamental solution we obtain a regularity theorem in Sobolev spaces for solutions of Poisson's equation on Lipschitz domains in $\mathbf{R}^n$. As a consequence we have got the continuity of these solutions for $n = 3$.

I. Introduction

The regularity analysis for boundary value problems is an old field in Mathematics and a lot of works are concerned with elliptic equations. One studied the regularities of solutions in the domains, on the boundaries and at the infinite, with various conditions : regularities of domains, of second member and of boundary values etc.

For regular domains the method of differential quotient was used in the works of L.NIRENBERG [1], E. MAGENES, G. STAMPACCHIA [2], J.L.LIONS [3] etc., to obtain the expected regularities (i.e. the regularity of solution is of order $k + s$ where k is the order of the equation and s is the order of the second member). About the regularities of weak solutions and very weak solutions one can find the works of S.L.SOBOLEV [4], J.L. LIONS [3], J.L. LIONS, E. MAGENES [5], E. MAGENES [6], J. NECAS [7], P. GRISVARD [8], and the work of H. DING [9].

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In general everything goes well when domains and coefficients of the operators have sufficient smoothness. But when domains have corners there is something tricky.

Our interest in this paper is the regularity of solutions for Poisson's equation on Lipschitz domains, in particular the continuity property of these solutions with domains in $\mathbf{R}^3$, because we often encounter it when we deal with problems of concentrated loads, nonlinear problems and the estimates for Green's functions etc. For smooth domains we can obtain the continuity of solutions directly (or using the Sobolev's embedding theorem) from most of the works mentioned above. Stampacchia [10], using the maximum principle, had given the regularities of solutions for Dirichlet problems in Hölder spaces on H_0^1 domains. For a polyhedral domain, by means of splitting method and interpolation techniques, Ding (ref. [9]) had got the continuity theorems for NEUMANN and DIRICHLET-NEUMANN problems.

The works of VERCHOTA [11], KENIG [12] etc., using layer potentials to solve boundary value problems on Lipschitz domains, enable us to go further. From their works we will induce a regularity theorem related to Sobolev spaces for arbitrary $n > 2$ (for $n = 2$ it will be still valid) and a continuity theorem for the case where $n = 3$.

II. Preliminary results

To achieve the final results we need the following :

THEOREM 2.1.— *(ref. [8]) Let Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary $\partial\Omega$. For $1/p < s \leq 1$ the mapping*

$$y : u \to yu = u\,|_{\partial\Omega}$$

which is defined for $u \in C^{0,1}(\overline{\Omega})$ has a unique continuous extension as an operator from

$$W_p^1(\Omega) \text{ onto } W_p^{-\frac{1}{p}}(\partial\Omega)$$

this operator has a right continuous inverse which does not depend on p.

As a consequence of this theorem we have

THEOREM 2.2.— *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$ and $p > 1$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then the mapping*

$$y' : g \to g \otimes \delta_r$$

(where δ_r is the Dirac measure supported on $\partial\Omega$) is continuous from

$$L^p(\partial\Omega) \text{ into } W_p^{-\varepsilon-\frac{1}{q}}(\mathbf{R}^n)$$

for all $\varepsilon > 0$.

Proof. — From theorem 2.1 we have

$$y : u \to u\,|_{\partial\Omega}$$

is continuous from

$$W_q^{\varepsilon-\frac{1}{q}}(\Omega) \text{ onto } W_q^{\varepsilon}(\partial\Omega) \subset L^q(\partial\Omega)$$

take the transposition, we have the theorem.

THEOREM 2.3.— *(ref. [8]) Let Ω be a Lipschitz domain in $\mathbf{R}^n$, then the following inclusions hold :*

$$W_p^s(\Omega) \subseteq W_q^t(\Omega)$$

for $t \leq s$ and $q \geq p$ such that $s - n/p = t - n/q$

$$W_p^s(\Omega) \subset C^{k,a}(\overline{\Omega})$$

for $k < s - n/p < k+1$, where $a = s - k - n/p$, k a non-negative integer. The proof may be found in many papers concerning Sobolev spaces.

COROLLARY 2.4.—

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$ then

$$H^{1/2}(\partial\Omega) \subseteq L^p(\partial\Omega)$$

for all

$$p < \frac{2(n-1)}{n-2}$$

(for unbounded Ω, p must, at the same time, be greater than or equal to 2)

Proof. — From theorem 2.3 we have

$$H^1(\Omega) \subseteq W_p^{\varepsilon+\frac{1}{p}}(\Omega)$$

for all $\varepsilon > 0$ such that
$$2 \leq p = \frac{2(n-1)}{n-2+2\varepsilon}$$
then by theorem 2.1
$$H^{1/2}(\partial\Omega) \subseteq W_p^{\varepsilon}(\partial\Omega) \subset L^p(\partial\Omega)$$
Hence
$$H^{1/2}(\partial\Omega) \subseteq L^p(\partial\Omega)$$
for all $p < \dfrac{2(n-1)}{n-2}$ because Ω is bounded.

THEOREM 2.5.—

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$. For $g \in L'(\partial\Omega)$, we define
$$u(x) = \frac{-1}{w_n(n-2)} \int_{\partial\Omega} \frac{1}{|X-Q|^{n-2}} g(Q)dQ$$
then
$$u\,|_\Omega \in W_p^{2-\varepsilon-\frac{1}{q}}(\Omega)$$
For all $\varepsilon > 0$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. $p > 1$ and moreover
$$\|u\,|_\Omega\|_{W_p^{2-\varepsilon-\frac{1}{q}}(\Omega)} \leq C\|g\|_{L'(\partial\Omega)}$$

Proof.—Let $\phi \in D(\mathbf{R}^n)$ such that
$$\phi = 1$$
in a neighbourhood of Ω we have, in the sense of distributions
$$\Delta(\phi u) = \Delta\phi u + 2 \bigtriangledown \phi \bigtriangledown u + \phi\Delta u$$
After some calculation, using theorem 2.2 and theorem 1.1.3 of [12], we can see that :
$$\Delta u = g \otimes \delta_r \in W_p^{-\varepsilon-\frac{1}{q}}(\mathbf{R}^n)$$
Because u is harmonic out of $\partial\Omega$ and ϕ is infinitely differentiable we obtain
$$\Delta(\phi u) \in W_p^{-\varepsilon-\frac{1}{q}}(\mathbf{R}^n)$$

Noting that ϕ has compact support, from the regularity results for solutions of elliptic equation in regular domains or at interior of domains we conclude that

$$\|\phi u\|_{W_p^{2-\varepsilon-\frac{1}{q}}(supp\phi)} \leq C\|g\|_{L'(\partial\Omega)}$$

or

$$\|u\|_{W_p^{2-\varepsilon-\frac{1}{q}}(\Omega)} \leq C\|g\|_{L'(\partial\Omega)}$$

Remark 2.6.—

If Ω is unbounded it is easy to show that for any compact set $\mathbf{K}$ of $\mathbf{R}^n$ we have

$$u \in W_p^{2-\varepsilon-\frac{1}{q}}(K)$$

III. Results about layer potentials

As pointed out the results of the paper [12] remain valid when suitably interpreted for all bounded Lipschitz domains in $\mathbf{R}^n$, $n \geq 2$. With the reference [11] we arrange them as follows :

THEOREM 3.1.—

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$. Then there exists $\varepsilon > 0$ such that given $f \in L_0'(\partial\Omega)$ $1 < p < 2+\varepsilon$ we can find unique u with $N(\nabla u) \in L'(\partial\Omega)$, solution to the Neumann problem :

$$(N) \quad \begin{cases} \Delta u = 0 & in\ \Omega \\ \dfrac{\partial u}{\partial n} = f & a \cdot o \cdot on\ \partial\Omega \end{cases}$$

Moreover u has the form $(n > 2)$

$$^{(*)}u(x) = \frac{-1}{w_n(n-2)} \int_{\partial\Omega} \frac{1}{|X-Q|^{n-2}} g(Q)dQ$$

for some $g \in L_0'(\partial\Omega)$, where $L_0'(\partial\Omega)$ is defined as

$$L_0^p(\partial\Omega) = \{g \in L^p(\partial\Omega), \int_{\partial\Omega} g = 0\}$$

(For the definition of $N(\nabla u)$ see [12])

THEOREM 3.2.—

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$, then there exists $\varepsilon > 0$ such that given $f \in L_1^{\prime}(\partial\Omega)$, $1 < p < 2+\varepsilon$, there exists a unique u (modulo constants) with

$$\|N(\nabla u)\|_{L^{\prime}(\partial\Omega)} \leq C\|\nabla, f\|_{L^{\prime}(\partial\Omega)}$$

solution of the problem

$$\begin{cases} \Delta u = 0 & in\ \Omega \\ \nabla, u = \nabla, f & a.e.\ on\ \partial\Omega \end{cases}$$

Moreover u has the form

$$u(x) = \frac{-1}{w_n(n-2)} \int_{\partial\Omega} \frac{1}{|X-Q|^{n-2}} g(Q)dQ$$

for some $g \in L^{\prime}(\partial\Omega)$.

IV. Main results

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$, we set the problems :

$$(Ng) \qquad \begin{cases} \Delta u = f & in\ \Omega \\ \dfrac{\partial u}{\partial n} = h & on\ \partial\Omega \end{cases}$$

$$(Dg) \qquad \begin{cases} \Delta u = f & in\ \Omega \\ u = g & on\ \partial\Omega \end{cases}$$

THEOREM 4.1.—

Given

$$f \in L^2(\Omega)\ and\ h \in H^{1/2}(\partial\Omega)\ with\ \int_\Omega f - \int_{\partial\Omega} h = 0$$

Let $u \in H^{-1}(\Omega)/\mathbf{R}$ be the variational solution of (Ng), then there exists $\varepsilon > 0$ such that

$$u \in W_p^{2-\varepsilon-\frac{1}{q}}(\Omega)$$

for $\dfrac{1}{p} - \dfrac{1}{q} = 1.\ p \leq 2+\varepsilon < \dfrac{2(n-1)}{n-2}$ *and all* $\widehat{\varepsilon} > 0$.

Proof. — We can always find

$$v \in H^2(\mathbf{R}^n) \text{ such that } \Delta v = f \textit{ in } \Omega$$

for $f \in L^2(\Omega)$. Then $w = u - v$ verifies

$$\begin{cases} \Delta w = 0 & \textit{in } \Omega \\ \dfrac{\partial w}{\partial n} \in H^{1/2}(\partial\Omega) \end{cases}$$

from corollary 2.4

$$\frac{\partial w}{\partial n} \in L^p(\partial\Omega)$$

for $p < \dfrac{2(n-1)}{n-2}$. By theorem 2.5 and theorem 3.1, there exists $\varepsilon > 0$, for $p \leq z + \varepsilon$ and for all $\widehat{\varepsilon} > 0$ we have

$$w \in W_p^{2-\widehat{\varepsilon}-\frac{1}{q}}(\Omega)$$

(we can verify that the solution of (N) in the form $(*)$ (page 5) is also a variational solution) and therefore

$$u = w + v \in W_p^{2-\widehat{\varepsilon}-\frac{1}{q}}(\Omega)$$

because for $p \leq \dfrac{2(n-1)}{n-2}$ we have

$$H^2(\Omega) \subseteq W_p^{2-\frac{1}{q}}(\Omega)$$

COROLLARY 4.2. — *Let u be defined in theorem 4.1 and $n = 3$, we have*

$$u \in C(\overline{\Omega})$$

Proof. — From theorem 2.3 we have

$$W_p^{2-\widehat{\varepsilon}-\frac{1}{q}}(\Omega) \subset C^{0,\alpha}(\overline{\Omega})$$

with (for $p = 2 + \varepsilon$)

$$\alpha = \frac{3 - n + \varepsilon - 2\widehat{\varepsilon} - \varepsilon\widehat{\varepsilon}}{2 + \varepsilon}$$

Since we can chose $\widehat{\varepsilon}$ sufficently small to ensure $\alpha > 0$,

then

$$W_p^{2-\widehat{\varepsilon}-\frac{1}{q}}(\Omega) \subset C(\overline{\Omega})$$

Very similarly we obtain the same results for the problem (Dg) :

THEOREM 4.3.— *Given*

$$f \in L^2(\Omega) \text{ and } g \in H^{3/2}(\partial\Omega),$$

let $u \in H^1(\Omega)$ be the variational solution of (Dg) then there exists $\varepsilon > 0$ such that

$$u \in W_p^{2-\widehat{\varepsilon}-\frac{1}{q}}(\Omega)$$

for $\dfrac{1}{p} + \dfrac{1}{q} = 1.\ p \leq 2 + \varepsilon < \dfrac{2(n-1)}{n-2}$ *and all* $\widehat{\varepsilon} > 0$.

COROLLARY 4.4.— *Let u be defined in theorem 4.3 then*

$$u \in C(\overline{\Omega})$$

Acknowledgement

I would thank professor P. GRISVARD who helped and encouraged me to work in mathematics during my stay in FRANCE, and has reviewed this paper before it is published.

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(Manuscrit reçu le 21 mars 1989)