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Existence, uniqueness and regularity for Kruzkov's solutions of the Burger-Carleman's system

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RÉSUMÉ. — Nous montrons l'existence et l'unicité d'une solution $(u(t), v(t))$ au sens kruzkov du système de Burger-Carleman avec condition initiale $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+ \times \mathbf{L}^1(\mathbf{R})_+$. Nous montrons que pour tout $t > 0, u(t), v(t) \in \mathbf{L}^\infty(\mathbf{R})$. Cet effet régularisant est lié à la possibilité de définir la solution au sens Kruzkov du système de Burger - Carleman.

ABSTRACT. — We prove existence and uniqueness of a Kruzkov solution $(u(t), v(t))$ of the Burger-Carleman's system with initial data $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+ \times \mathbf{L}^1(\mathbf{R})_+$. Moreover, we show that for any $t > 0, u(t), v(t) \in \mathbf{L}^\infty(\mathbf{R})$ with precise estimates. In fact, this regularizing effect is related to the possibility of defining Kruzkov's solutions for the Burger-Carleman's system.

We consider the following first order system which will be called the Burger-Carleman's system :

$$(BC) \qquad \begin{aligned} & u_t + \left(\frac{u^2}{2}\right)_x + u^2 - v^2 = 0 \qquad \text{on } [0, +\infty) \times \mathbf{R} \\ & v_t - \left(\frac{v^2}{2}\right)_x + v^2 - u^2 = 0 \qquad \text{on } [0, +\infty) \times \mathbf{R} \\ & u(0, x) = u_0(x), v(0, x) = v_0(x) \end{aligned}$$

with initial data $u_0, v_0 \in \mathbf{L}^1(\mathbf{R})_+$. We prove the existence and uniqueness of a Kruzkov's solution of (BC) (see definition 1 below) using the theory of nonlinear semigroups generated by accretive operators. We notice that

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the possibility of defining Kruzkov's solutions for (BC) when the initial data $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})^2_+$ depends on the $\mathbf{L}^1 - \mathbf{L}^\infty$ regularizing effect for homogeneous equations proved in [2]. In fact, the estimates proved in [2] imply that for any $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})^2_+$ and any $t > 0, u(t), v(t) \in \mathbf{L}^\infty(\mathbf{R})_+$ with precise estimates given below. Before stating the precise result, let us define the notion of Kruzkov's solution for (BC) :

DEFINITION 1.— *Let $T > 0$. The pair of functions $(u, v) \in \mathbf{L}^\infty([0,T], \mathbf{L}^1(\mathbf{R})_+)^2 \cap \mathbf{L}^\infty([\tau, T] \times \mathbf{R})^2$ for any $\tau > 0$ will be called a kruzkov's solution of (BC) in $[0,T] \times \mathbf{R}$ with initial data $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})^2_+$ if $(u(t), v(t)) \to (u_0, v_0) \in \mathbf{L}^1(\mathbf{R})^2$ as $t \to 0$ and*

$$\int_0^T \int_{\mathbf{R}} |u-k|\xi_t + \operatorname{sign}_0(u-k)\left[\left(\frac{u^2}{2} - \frac{k^2}{2}\right)\xi_x + (v^2 - u^2)\xi\right] dx dt \geq 0$$

$$\int_0^T \int_{\mathbf{R}} |v-k'|\eta_t + \operatorname{sign}_0(v-k')\left[\left(\frac{k'^2}{2} - \frac{v^2}{2}\right)\eta_x + (u^2 - v^2)\eta\right] dx dt \geq 0$$

holds for all $\xi, \eta \in C_0^\infty((0,T) \times \mathbf{R}), \xi, \eta \geq 0$ and all $k, k' \in \mathbf{R}$.

As it is costumary

$\operatorname{sign}_0(r) = +1$ if $r > 0, 0$ if $r < 0$

$\operatorname{sign}(r) = +1$ if $r > 0, [-1, 1]$ if $r = 0, -1$ if $r < 0$

$\operatorname{sign}^+(r) = +1$ if $r > 0, [0, 1]$ if $r = 0, 0$ if $r < 0$

Similarly one defines $\operatorname{sign}_0^+(r)$.

Then, our result says :

THEOREM 1.— *For any $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})^2_+$, there exists a unique Kruzkov's solution $(u, v) \in C([0,T], \mathbf{L}^1(\mathbf{R})^2_+)$ of (BC) in $[0,T] \times \mathbf{R}$ for any $T > 0$ with initial data (u_0, v_0) such that for any $t > 0$:*

$$(\mathbf{R}E) \quad ||u(t)||_{\mathbf{L}^\infty(\mathbf{R})} \leq \left(\frac{2}{t}||u_0 + v_0||_{\mathbf{L}^1(\mathbf{R})} + \frac{2\sqrt{2}}{\sqrt{t}}(||u_0 + v_0||_{\mathbf{L}^1(\mathbf{R})})^{3/2}\right)^{1/2}$$

The same estimate holds for $||v(t)||_{\mathbf{L}^\infty(\mathbf{R})}$. Moreover, if $(u, v), (\widehat{u}, \widehat{v})$ are two Kruzkov's solutions of (BC) in $[0,T] \times \mathbf{R}, T > 0$, corresponding to the initial data $(u_0, v_0), (\widehat{u}_0, \widehat{v}_0) \in \mathbf{L}^1(\mathbf{R})^2_+$ respectively, then for all $t \in [0,T]$.

$$\begin{aligned} &||(u(t) - \widehat{u}(t))^+||_{\mathbf{L}^1(\mathbf{R})} + ||(v(t) - \widehat{v}(t))^+||_{\mathbf{L}^1(\mathbf{R})} \\ &\leq ||(u_0 - \widehat{u}_0)^+||_{\mathbf{L}^1(\mathbf{R})} + ||(v_0 - \widehat{v}_0)^+||_{\mathbf{L}^1(\mathbf{R})} \end{aligned}$$

To begin with the proof, let us introduce the following operators A, B :

$D(A) := \{(u,v) \in \mathbf{L}^1(\mathbf{R})^2_+ : u^2, v^2 \in AC(\mathbf{R})\}$

$D(B) := \{(u,v) \in \mathbf{L}^1(\mathbf{R})^2_+ : u^2, v^2 \in \mathbf{L}^1(\mathbf{R})\}$

where $AC(\mathbf{R})$ is the set of absolutely continuous functions on $\mathbf{R}$,

$A(u,v) = \left((\frac{u^2}{2})_x, -(\frac{v^2}{2})_x \right), B(u,v) = (u^2 - v^2, v^2 - u^2)$

for $(u,v) \in D(A), (u,v) \in D(B)$ respectively. Notice that $D(A) \subset D(B)$. Thus $D(A+B) = D(A)$ and (BC) can be written in the abstract form : let $U = (u,v)$

$$(BC)_a \qquad \frac{dU}{dt} + (A+B)U = 0$$
$$U(0) = (u_0, v_0) \in \mathbf{L}^1(\mathbf{R})^2_+$$

We show that one can use the Grandall-Liggett's theorem to solve $(BC)_a$. This is the purpose of the next two lemmas. Before stating them, let us recall the definition of T-accretivity. Let E be a Banach lattice. A (in general, multivalued) operator B on E called T-accretive if

$||(x-\widehat{x})^+||_E \le ||(x - \widehat{x} + \lambda y - \lambda \widehat{y})^+||_E$ holds for all $[x,y], [\widehat{x}, \widehat{y}] \in B$ and all $\lambda > 0$.

If $E = \mathbf{L}^1(\mathbf{R}) \times \mathbf{L}^1(\mathbf{R})$ endowed with the norm

$||(u,v)||_E = \int_{\mathbf{R}} |u| + \int_{\mathbf{R}} |v|, (u,v) \in E$, then this is equivalent to say that for all $[(x_1,x_2),(y_1,y_2)], [(\widehat{x}_1\widehat{x}_2),(\widehat{y}_1\widehat{y}_2)] \in B$ there exists some $\alpha_1 \in sign^+(x_1 - \widehat{x}_1), \alpha_2 \in sign^+(x_2 - \widehat{x}_2)$ such that

$\int_{\mathbf{R}} \alpha_1(y_1 - \widehat{y}_1) + \alpha_2(y_2 - \widehat{y}_2)dx \ge 0$. Then :

LEMMA 1.— *$A + B$ is T-accretive in $\mathbf{L}^1(\mathbf{R})^2$. Moreover, for any $p \in W^{1,\infty}(\mathbf{R})$ such that $p' \ge 0$ has compact support :*

$$\int_{\mathbf{R}} p(u)w + p(v)h \; dx \ge 0 \qquad (1)$$

holds for any $(u,v) \in D(A)$ where $(w,h) = (A+B)(u,v)$.

LEMMA 2.— *For all $\lambda > 0$, Ran $(I + \lambda(A+B)) = \mathbf{L}^1(\mathbf{R})^2_+$.*

Proof of lemma 1.— Let $U = (u,v), \widehat{U} = (\widehat{u}, \widehat{v}) \in D(A)$. One easily checks that

$$\int_{\mathbf{R}} \left[\left(\frac{u^2}{2}\right)_x \left(\frac{\widehat{u}^2}{2}\right)_x \right] \mathrm{sign}_0^+(u - \widehat{u})\mathrm{d}x = \int_{\mathbf{R}} \left[\left(\frac{v^2}{2}\right)_x - \left(\frac{\widehat{v}^2}{2}\right)_x \right] \mathrm{sign}_0^+(v - \widehat{v})\mathrm{d}x$$
$$= 0$$

since $u, \widehat{u}, v, \widehat{v} \geq 0$ and sign_0^+ is an increasing function, then

$$\int_{\mathbf{R}} B(u,v)(\text{sign}_0^+(u-\widehat{u}), \text{sign}_0^+(v-\widehat{v}))\mathrm{d}x =$$

$$\int_{\mathbf{R}} [\text{sign}_0^+(u-\widehat{u}) - \text{sign}_0^+(v-\widehat{v})][(u^2-\widehat{u}^2)-(v^2-\widehat{v}^2)]\mathrm{d}x =$$

$$\int_{\mathbf{R}} [\text{sign}_0^+(u^2-\widehat{u}^2) - \text{sign}_0^+(v^2-\widehat{v}^2)][(u^2-\widehat{u}^2)-(v^2-\widehat{v}^2)]\mathrm{d}x \geq 0$$

Both remarks imply that $A+B$ is T accretive in $\mathbf{L}^1(\mathbf{R})_+^2$.

Let $\beta(r) : = r^{1/2}, r \geq 0$. Let $p \in W^{1,\infty}(\mathbf{R})$ be such that $p' \geq 0$ has compact support.

Let $j : \mathbf{R}_+ \to \mathbf{R}$ be $j(r) = \displaystyle\int_0^r (p \circ \beta)(s)\mathrm{d}s$. Then, if $z = u^2$

$$\int_{\mathbf{R}} \left(\frac{u^2}{2}\right)_x p(u)\mathrm{d}x = \int_{\mathbf{R}} \left(\frac{z}{2}\right)_x (p\circ\beta)(z)\mathrm{d}x = \frac{1}{2}\int_{\mathbf{R}} j(z)_x \,\mathrm{d}x = 0.$$

Similarly $\displaystyle\int_{\mathbf{R}} \left(\frac{v^2}{2}\right)_x p(x)\mathrm{d}x = 0$ and

$$\int_{\mathbf{R}} (u^2-v^2)p(u) + (v^2-u^2)p(v)\mathrm{d}x = \int_{\mathbf{R}} (u^2-v^2)(p(u)-p(v))\mathrm{d}x \geq 0$$

since p is increasing and $u, v \geq 0$. Putting this things together we get the inequality (1).

Proof of lemma 2.— Since the proof below is independent of the value of $\lambda > 0$ we take $\lambda = 1$. We have to solve the following equations : let $f, g \in \mathbf{L}^1(\mathbf{R})_+$.

$$u + \left(\frac{u^2}{2}\right)_x + u^2 - v^2 = f \qquad (2.1)$$

$(SP)_{f,g}$

$$v - \left(\frac{v^2}{2}\right)_x + v^2 - u^2 = g \qquad (2.2)$$

1[st] step : We work in a $\mathbf{L}^2$ – framework. Let $I_n = [-n, n]$. Let us solve the equations $(SP)_{f,g}$ for $f, g \in \mathbf{L}^2(I_n)_+$. Let β be as above. Then $(SP)_{f,g}$ is equivalent to

$$\beta(w) + \left(\frac{w}{2}\right)_x + w - h = f$$

$(SP)_{\beta,f,g}$

$$\beta(h) - \left(\frac{h}{2}\right)_x + h - w = g$$

through the change of variable $w = u^2, h = v^2$. Let $\overline{\beta}(r) = \sqrt{r}$ if $r \geq 0$, $-\sqrt{|r|}$ if $r < 0$. Let us first consider the system :

$$\overline{\beta}(w) + \left(\frac{w}{2}\right)_x + w - h = f$$

$(SP)_{\overline{\beta},f,g}$

$$\overline{\beta}(h) - \left(\frac{h}{2}\right)_x + h - w = g$$

where $f, g \in \mathbf{L}^2(I_n)$. The existence of a solution of $(SP)_{\overline{\beta},f,g}$ is a consequence of standard perturbation results for maximal monotome operators ([5]). Let $T_{\overline{\beta}} : \mathbf{L}^2(I_n)^2 \to \mathbf{L}^2(I_n)^2$ be given by $T_{\overline{\beta}}(w,h) = (\overline{\beta}(w), \overline{\beta}(h))$. Let $T : \mathbf{L}^2(I_n)^2 \to \mathbf{L}^2(I_n)^2$ with domain.

Dom$(T) = \{(w,h) \in H^1(I_n) \times H^1(I_n) : w(-n) = h(-n), w(n) = h(n)\}$ be given by $T(w,h) = \left(\frac{w_x}{2} + w - h, -\frac{h_x}{2} + h - w\right)$. Since $T_{\overline{\beta}}, T$ are maximal monotone and Dom $(T_{\overline{\beta}}) = \mathbf{L}^2(I_n)^2, T_{\overline{\beta}} + T$ is maximal monotone ([4], Corol. 2.7). Moreover, since $\overline{\beta}$ is the subgradient of a convex function, by [5], thm. 4, Int Ran$(T_{\overline{\beta}}+T) = Int($Ran $T_{\overline{\beta}}+$ Ran $T)$. But it is an exercise to see that Ran $T = \mathbf{L}^2(I_n)^2$. Therefore, Ran$(T_{\overline{\beta}}+T) = \mathbf{L}^2(I_n)^2$. Therefore, for $f, g \in \mathbf{L}^2(I_n), (SP)_{\overline{\beta},f,g}$ has a solution $(w,h) \in H^1(I_n) \times H^1(I_n)$ with $w(-n) = h(-n), w(n) = h(n)$. To go back to problem $(SP)_{\beta,f,g}$ it suffices to remark that $w, h \geq 0$ if $f, g \geq 0$. For that we multiply the first equation in $(SP)_{\overline{\beta},f,g}$by w^- and the second by h^-. Adding both equations and integrating over $\mathbf{R}$, one gets :

$$\int_{\mathbf{R}} gh^- + fw^- + (w^-)^{3/2} + (h^-)^{3/2} + (w^- - h^-)^2 + 2w^+h^- \mathrm{d}x = 0$$

Since each term in the integrand is positive, $w^- = h^- = 0, i.e., w, h \geq 0$. Thus, given $f, g \in \mathbf{L}^2(I_n)_+$, there exists $w, h \in H^1(I_n)$ with $w(-n) = h(-n), w(n) = h(n), w, h \geq 0$ which solve $(SP)_{\beta,f,g}$. Then $u = \sqrt{w}$ on I_n, 0 in $\mathbf{R} - I_n, v = \sqrt{h}$ on I_n, 0 in $\mathbf{R} - I_n$ solve $(SP)_{f,g}$.

2$^{\text{nd}}$ step : Let $f, g \in \mathbf{L}^1(\mathbf{R})_+$. Let $f_n, g_n \in \mathbf{L}^2(I_n)_+$ be such that $f_n \uparrow f, g_n \uparrow g$. Let (u_n, v_n) be the solutions of $(SP)_{fn,gn}$ found in step 1.

Notice that the accretivity of $A+B$ implies that u_n, v_n are Cauchy sequences in $\mathbf{L}^1(\mathbf{R})$. Let $u, v \in \mathbf{L}^1(\mathbf{R})_+$ be the limits of u_n, v_n in $\mathbf{L}^1(\mathbf{R})$. Now adding the corresponding equations to (2.1), (2.2) for $(SP)_{fn,gn}$ and using that $u_n, v_n \geq 0$ we get;

$$\left(\frac{u_n^2 - v_n^2}{2}\right)_x \leq f_n + g_n \tag{3}$$

Since $u_n(-n) = v_n(-n), u_n(n) = v_n(n)$, integrating from $-\infty$ to x and from x to ∞ we get $||u_n^2 - v_n^2||_\infty \leq 2||f_n + g_n||\mathbf{L}^1(\mathbf{R})$. Since, for $a, b \geq 0$, $|a-b| \leq |a^2 - b^2|^{1/2}$, the sequence $u_n - v_n$ is bounded in $\mathbf{L}^\infty(\mathbf{R})$. Then, $u_n^2 - v_n^2 = (u_n - v_n)(u_n + v_n)$ is bounded in $\mathbf{L}^1(\mathbf{R})$. From $(SP)_{fn,gn}$ it follows that $\left(\frac{u_n^2}{2}\right)_x, \left(\frac{v_n^2}{2}\right)_x$ are bounded in $\mathbf{L}^1(\mathbf{R})$. This, together with $u_n \to u, v_n \to v$ in $\mathbf{L}^1(\mathbf{R})$ implies that u_n, v_n are bounded in $\mathbf{L}^\infty(\mathbf{R})$ and $u_n^2 \to u^2, v_n^2 \to v^2$ in $\mathbf{L}^1(\mathbf{R})$. Thus

$$\left(\frac{u_n^2}{2}\right)_x \to \left(\frac{u^2}{2}\right)_x, \left(\frac{v_n^2}{2}\right)_x \to \left(\frac{v^2}{2}\right)_x \text{ in } \mathbf{L}^1(\mathbf{R}), (u,v) \in D(A)$$

and letting $n \to \infty$ in $(SP)_{fn,gn}$ we get a solution $(u,v) \in D(A)$ for $(SP)_{f,g}$.

Using the Crandall – Ligget's theorem in combination with lemmas 1 and 2 above, one gets :

PROPOSITION 1.— *For any* $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+^2$ *and any* $t > 0$, *there exists a unique mild (or semigroup) solution* $(u,v) \in C([0,T], \mathbf{L}^1(\mathbf{R})_+^2$ *of* (BC) *with initial data* $u(0) = u_0, v(0) = v_0$. *If* $(u,v), (\widehat{u}, \widehat{v})$ *are two mild solutions of* (BC) *with initial data* $(u_0, v_0), (\widehat{u}_0, \widehat{v}_0) \in \mathbf{L}^1(\mathbf{R})_+^2$ *respectively, then :*

$$||(u(t) - \widehat{u}(t))^+||_{\mathbf{L}^1(\mathbf{R})} + ||(v(t) - \widehat{v}(t))^+||_{\mathbf{L}^1(\mathbf{R})} \leq ||(u_0 \widehat{u}_0)^+||_{\mathbf{L}^1(\mathbf{R})} + ||(v_0 \widehat{v}_0)^+||_{\mathbf{L}^1(\mathbf{R})}$$

Moreover, if $u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+^2 \cap \mathbf{L}^p(\mathbf{R})_+^2, 1 \leq \infty$, *then* $(u(t), v(t)) \in \mathbf{L}^1(\mathbf{R})_+^2 \cap \mathbf{L}^p(\mathbf{R})_+^2$ *and for any* $t \geq 0$

$$||u(t)||_{\mathbf{L}^p(\mathbf{R})} + ||v(t)||_{\mathbf{L}^p(\mathbf{R})} \leq ||u_0||_{\mathbf{L}^p(\mathbf{R})} + ||v_0||_{\mathbf{L}^p(\mathbf{R})}.$$

Proof.— Just remark that the last assertion is a consequence of the inequalities (1) in Lemma 1 ([1], section 2).

Before proving the regularizing estimate (RE) let us prove that the semigroup solution (u,v) of (BC) with initial data $(u_0,v_0) \in \mathbf{L}^1(\mathbf{R})^2_+ \cap \mathbf{L}^\infty(\mathbf{R})^2_+$ obtained via the Crandal-Ligget's theorem is a Kruzkov's solution. This is a consequence of two facts : first, if $(u(t),v(t))$ is the mild solution of (BC) with initial data $(u_0,v_0) \in \mathbf{L}^1(\mathbf{R})^2_+ \cap \mathbf{L}^\infty(\mathbf{R})^2_+$ then $u(t)$ and $v(t)$ are, respectively, the mild solutions of

$$(*) \qquad u_t + \left(\frac{u^2}{2}\right)_x = \Psi(t),$$

$$(**) \qquad v_t - \left(\frac{v^2}{2}\right)_x = -\Psi(t),$$

where $\Psi(t) \equiv v^2(t) - u^2(t)$ ([10], Lemma 1.7) and second, the well known fact that mild or semigroup solutions of (*) and (**) are in fact Kruzkov's solutions of (*), (**) respectively ([1], Prop. 2.11). Writing what means that $u(t), v(t)$ are Kruzkov's solutions of (*), (**) respectively we get that $(u(t),v(t))$ is a Kruzkov solution of (BC) in the sense of definition 1. One can argue directly using only [1], Prop. 2.11. Recall that (u,v) is obtained in the following way : let $\mathcal{P}_n = \{0 = a_0^n < \ldots < a_n^n = T\}$ where $a_k^n = \dfrac{kT}{n}$. Let $u_n(t), v_n(t)$ be the step functions given by $u_n(0) = 0, v_n(0) = 0, u_n(t) = u_k^n, v_n(t) = v_k^n$ in $]a_{k-1}^n, a_k^n]$, where (u_k^n, v_k^n) are constructed as solutions of the difference scheme :

$$(DS) \qquad \begin{aligned} &\frac{u_k^n - u_{k-1}^n}{a_k^n - a_{k-1}^n} + \left(\frac{(u_k^n)^2}{2}\right)_x + (u_k^n)^2 - (v_k^n)^2 = 0 \\ &\frac{v_k^n - v_{k-1}^n}{a_k^n - a_{k-1}^n} - \left(\frac{(v_k^n)^2}{2}\right)_x + (v_k^n)^2 - (u_k^n)^2 = 0 \end{aligned}$$

with $u_0^n = u_0, v_0^n = v_0$. Then $u_n(t), v_n(t) \to u(t), v(t)$ in $\mathbf{L}^1(\mathbf{R})$ uniformly on $[0,T]$. Let $\Psi_n(t) = (v_k^n)^2 - (u_k^n)^2$ on $]a_{k-1}^n, a_k^n]$. Since $(u_0,v_0) \in \mathbf{L}^1(\mathbf{R})^2_+ \cap \mathbf{L}^\infty(\mathbf{R})^2_+$ then $\Psi_n(t) \to \Psi(t) := v(t)^2 - u(t)^2$ in $\mathbf{L}^1([0,T], \mathbf{L}^1(\mathbf{R}))$ as $n \to \infty$. Thus $u(t), v(t)$ are mild solutions of

$$\begin{cases} u_t + \left(\dfrac{u^2}{2}\right)_x = \Psi(t) \\ u(0) = u_0 \end{cases} \qquad \begin{cases} v_t - \left(\dfrac{v^2}{2}\right)_x = \Psi(t) \\ v(0) = v_0 \end{cases}$$

respectively therefore $(u(t), v(t))$ is the Kruzkov's solution of (BC) in $[0,T] \times \mathbf{R}$ with initial data (u_0, v_0) in the sense of Definition 1 ([1], Prop. 2.11).

Since $(u_0, v_0) \in \mathbf{L}^\infty(\mathbf{R})^2$, then $u, v \in \mathbf{L}^\infty([0,1] \times \mathbf{R})$.

Taking $k > ||u(.,.)||_\infty, k' > ||v(.,.)||_\infty$ and then $k < -||u(.,.)||_\infty, k' < -||v(.,.)||_\infty$ we see that u, v are distributional solutions of (BC). We can now easily show the regularizing estimate (RE) of theorem 1. Let $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+$. Let $(u_{0n}, v_{0n}) \in \mathbf{L}^1(\mathbf{R})_+^2 \cap \mathbf{L}^\infty(\mathbf{R})_+^2$ be such that $u_{0n} \uparrow u_0, v_{0n} \uparrow v_0$. Let $u_n(t), v_n(t)$ be the solutions of (BC) given by proposition 1. Using [2], Theorem 2, it follows that

$$\frac{u_n(t+h) - u_n(t)}{h} \geq -\frac{1}{t+h} u_n(t)$$

$$\frac{v_n(t+h) - v_n(t)}{h} \geq -\frac{1}{t+h} v_n(t)$$

for $t, h > 0$. This implies that for any $t > 0$ and any $t \in]0,T] u_{nt} \geq -\frac{u_n}{t}, v_{nt} \geq -\frac{v_n}{t}$ in $\mathcal{D}'((0,T) \times \mathbf{R})$. It follows that

$$\left(\frac{u_n^2 - v_n^2}{2} \right)_x \leq \frac{u_n + v_n}{t}$$

in $\mathcal{D}'((0,T) \times \mathbf{R})$. Thus, for any $\varphi \in C_0^\infty(\mathbf{R})$ with $||\varphi||_\infty \leq 1, \varphi \geq 0$:

(4)
$$\int_{\mathbf{R}} \frac{u_n^2(t,x) - v_n^2(t,x)}{2} \varphi'(x) dx \leq \frac{||u_n(t) + v_n(t)||_{\mathbf{L}^1(\mathbf{R})}}{t} \leq \frac{||u_0 + v_0||_{\mathbf{L}^1(\mathbf{R})}}{t}$$

holds *a.e.* with respect to t. Since $u_n, v_n \in C([0,T], \mathbf{L}^1(\mathbf{R})_+)$ it holds for all $t \in]0,T]$. As we remarked above, since $(u_{on}, v_{on}) \in \mathbf{L}^\infty(\mathbf{R})^2, (u_n(t), v_n(t) \in \mathbf{L}^\infty(\mathbf{R})^2$. Then, $u_n(t)^2 - v_n(t)^2 \in \mathbf{L}^1(\mathbf{R})$. Now the following argument can be justified : let x_0 be a Lebesgue point of $u_n(t)^2 - v_n(t)^2$. For each $k \in \mathbf{N}$, take $\varphi_k(x) = 0$ if $x < x_0$, $k(x - x_0)$ if $x \in]x_0, x_0 + 1/k], 1$ if $x \geq x_0 + 1/k$. Plug φ_k into (4) to get :

$$-k \int_x^{x+1/k} \frac{u_n^2(t,x) - v_n^2(t,x)}{2} dx \leq \frac{||u_n(t) + v_n(t)||_{\mathbf{L}^1(\mathbf{R})}}{t}$$

Since x_0 is a Lebesgue point of $u_n^2(t) - v_n^2(t)$, letting $k \to \infty$ we get :

$$-(u_n^2(t,x_0) - v_n^2(t,x_0)) \leq \frac{2}{t} ||u_n(t) + v_n(t)||_{\mathbf{L}^1(\mathbf{R})}$$

Taking now $\varphi_k(x) = 1$ if $x \leq x_0, 1-k(x-x_0)$ if $x \in]x_0, x_0+1/k], 0$ if $x \geq x_0$ and repeating the argument above, one gets :

$$(u_n^2(t,x_0) - v_n^2(t,x_0)) \leq \frac{2}{t}||u_n(t)+v_n(t)||_{\mathbf{L}^1(\mathbf{R})}$$

Therefore, $u_n^2(t) - v_n^2(t) \in \mathbf{L}^\infty([0,T]\times\mathbf{R})$ and

$$||u_n^2(t) - v_n^2(t)||_\infty \leq \frac{2}{t}||u_n(t)+v_n(t)||_{\mathbf{L}^1(\mathbf{R})}$$

for all $t \in]0,T]$. Since for $a, b \geq 0, |a-b| \leq |a^2-b^2|^{1/2}$, it follows that

$$||u_(t) - v_n(t)||_\infty \leq \frac{\sqrt{2}}{t^{1/2}}\left(||u_0+v_0||_{\mathbf{L}^1(\mathbf{R})}\right)^{1/2}$$

and

$$||u_n^2(t) - v_n^2(t)||_{\mathbf{L}^1(\mathbf{R})} \leq \frac{\sqrt{2}}{t^{1/2}}\left(||u_0+v_0||_{\mathbf{L}^1(\mathbf{R})}\right)^{3/2}$$

Since $u_{nt} + \left(\frac{u_n^2}{2}\right)_x + u_n^2 - v_n^2 = 0$ holds in $\mathcal{D}'((0,T)\times\mathbf{R})$ then : $\left(\frac{u_n^2}{2}\right)_x \leq v_n^2 - u_n^2 + \frac{u_n}{t}$. As before, this implies that $u_n \in \mathbf{L}^\infty([0,T]\times\mathbf{R})$ and $||u_n(t)||_\infty \leq \left\{\frac{2}{t}||u_0+v_0||_{\mathbf{L}^1(\mathbf{R})} + \frac{2\sqrt{2}}{t^{1/2}}\left(||u_0+v_0||_{\mathbf{L}^1(\mathbf{R})}\right)^{3/2}\right\}^{1/2}$

for all $n \in \mathbf{N}$ and $t > 0$, Letting $n \to \infty$ we get (RE) for $u(t)$. Similarly, (RE) holds for $v(t)$.

Now, it is easy to show that for any $(u_0, v_0) \in \mathbf{L}^1(\mathbf{R})_+^2$, the semigroup solution of (BC) given by proposition 1 is in fact a Kruzkov's solution of (BC). (RE) implies that $(u,v) \in \mathbf{L}^\infty([0,T],\mathbf{L}^1(\mathbf{R}))^2 \cap \mathbf{L}^\infty([\tau,T]\times\mathbf{R})^2$ for any $\tau > 0$. Let $u_{0n}, v_{0n} \in \mathbf{L}^1(\mathbf{R})_+ \cap \mathbf{L}^\infty(\mathbf{R})_+$ be such that $u_{0n} \uparrow u_0, v_{0n} \uparrow v_0$. As has been proved above, the semigroup solutions u_n, v_n of (BC) in $[0,T]$ with initial data u_{0n}, v_{0n} satisfy :

$$\int_0^T\int_{\mathbf{R}} |u_n - k|\zeta_t + \operatorname{sign}_0(u_n - k)\left[\left(\frac{u_n^2}{2} - \frac{k^2}{2}\right)\zeta_x + (v_n^2 - u_n^2)\zeta\right] dxdt \geq 0$$

(5)

$$\int_0^T\int_{\mathbf{R}} |v_n - k'|\eta_t + \operatorname{sign}_0(v_n - k')\left[\left(\frac{k'^2}{2} - \frac{v_n^2}{2}\right)\eta_x + (u_n^2 - v_n^2)\eta\right] dxdt \geq 0$$

for all $\zeta,\ \eta \in C_0^\infty((0,T)\times\mathbf{R}), \zeta,\eta \geq 0$, all $k, k' \in \mathbf{R}$ and all $n \in \mathbf{N}$.

Since u_n, v_n satisfy the estimate (RE), $u_n^2 - v_n^2 \to u^2 - v^2$ in $\mathbf{L}^1([\tau,T]\times\mathbf{R})$ for any $\tau \in [0,T]$ and one can let $n \to \infty$ in (5) to get

$$\int_0^T \int_{\mathbf{R}} \alpha(t,x,k)\left[(u-k)\zeta_t + \left(\frac{u^2}{2} - \frac{k^2}{2}\right)\zeta_x + (v^2 - u^2)\zeta\right] \mathrm{d}x\mathrm{d}t \geq 0$$

$$\int_0^T \int_{\mathbf{R}} \beta(t,x,k')\left[(v-k')\eta_t + \left(\frac{k'^2}{2} - \frac{v^2}{2}\right)\eta_x + (u^2 - v^2)\eta\right] \mathrm{d}x\mathrm{d}t \geq 0$$

for all $\zeta,\eta \in C_0^\infty((0,T)\times\mathbf{R}), \zeta,\eta \geq 0$ and all $k,k' \in \mathbf{R}$ where $\alpha(t,x,k) \in$ sign $(u(t,x)-k), \beta(t,x,k') \in$ sign $(v(t,x)-k')$.

Using [1], Lemme 2.2, we see that (u,v) is a Kruzkov's solution of (BC) on $[0,T]\times\mathbf{R}$ with initial data (u_0,v_0).

The uniqueness of Kruzkov's solutions of (BC) follows easily adaptating the arguments of [1], Sect. II. Firts of all we observe that if $(u,v), (\widehat{u},\widehat{v})$ are Kruzkov's solutions of (BC) on $[0,T]\times\mathbf{R}$ with respective initial data $(u_0,v_0), (\widehat{u}_0\widehat{v}_0) \in \mathbf{L}^1(\mathbf{R})^2_+$ then ([1], Prop. 2.7) there exists some $\alpha(t,x) \in$ sign $(u(t,x)-\widehat{u}(t,x))$, $\beta(t,x) \in$ sign $(v(t,x)-\widehat{v}(t,x))$ such that

$$\int_0^T \int_{\mathbf{R}} |u-\widehat{u}|\zeta_t + \alpha(t,x)\left[\left(\frac{u^2-\widehat{u}^2}{2}\right)\zeta_x + ((v^2-u^2)-(\widehat{v}^2-\widehat{u}^2))\zeta\right] \mathrm{d}x\mathrm{d}t \geq 0$$

$$\int_0^T \int_{\mathbf{R}} |v-\widehat{v}|\eta_t + \beta(t,x)\left[\left(\frac{\widehat{v}^2-v^2}{2}\right)\eta_x + ((u^2-v^2)-(\widehat{u}^2-\widehat{v}^2))\eta\right] \mathrm{d}x\mathrm{d}t \geq 0$$

holds for all $\zeta,\eta \in C_0^\infty((0,T)\times\mathbf{R}), \zeta,\eta \geq 0$. Take $\zeta,\eta \in C_0^\infty((0,T)\times\mathbf{R}), \zeta \geq 0$ in both inequalities and add them. Then, observing that

$[(u^2-\widehat{u}^2)-(v^2-\widehat{v}^2)](\beta(t,x)-\alpha(t,x))\zeta \leq 0$ a.e. one gets :

$$\int_0^T \int_{\mathbf{R}} (|u-\widehat{u}| + |v-\widehat{v}|)\zeta_x + \left[\left(\frac{u^2}{2} - \frac{\widehat{u}^2}{2}\right)\alpha + \left(\frac{v^2}{2} - \frac{\widehat{v}^2}{2}\right)\beta\right]\zeta_x \mathrm{d}x\mathrm{d}t \geq 0$$

As in [1], Lemme 2.5, one obtains : for any $\tau \in]0,T]$ fixed

$$\int_{|x|\leq R-Ct} |u(t,x)-\widehat{u}(t,x)| + |v(t,x)-\widehat{v}(t,x)|\mathrm{d}x$$
$$\leq \int_{|x|\leq R-Cs} |u(s,x)-\widehat{u}(s,x)| + |v(s,x)-\widehat{v}(s,x)|\mathrm{d}x$$

for $0 < \tau \leq s \leq t \leq T$, where C is the Lipschitz constant of the function $r \rightarrow \frac{r^2}{2}$ on $\{|r| \leq \max(|u(t,x)|, |\widehat{u}(t,x)|, |v(t,x)|, |\widehat{v}(t,x)|) : t \in [\tau, T], x \in \mathbf{R}\}$ and $R > Ct$. Thus :

$$\begin{aligned} &\int_{|x|\leq Ct} |u(t,x) - \widehat{u}(t,x)| + |v(t,x) - \widehat{v}(t,x)|\mathrm{d}x \\ &\qquad \leq \int_{\mathbf{R}} |u(s,x) - \widehat{u}(s,x)| + |v(s,x) - \widehat{v}(s,x)|\mathrm{d}x \end{aligned} \tag{6}$$

for any $0 < \tau \leq s \leq t \leq T$. Since $(u(s), v(s)) \rightarrow (u_0, v_0)$ on $\mathbf{L}^1(\mathbf{R})$ as $s \rightarrow 0$, letting $R \rightarrow \infty$ on (6) and then $\tau, s \rightarrow 0$ we get :

$$\int_{\mathbf{R}} |u(t,x) - \widehat{u}(t,x)| + |v(t,x) - \widehat{v}(t,x)|\mathrm{d}x \leq \int_{\mathbf{R}} |u_0 - \widehat{u}_0| + |v_0 - \widehat{v}_0|\mathrm{d}x$$

for any $t > 0$. From this estimate, the uniqueness of Kruzkov's solutions of (BC) follows. This finishes the proof of theorem 1.

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Références

[1] BENILAN (Ph.).— *Equations d'évolution dans un espace de Banach quelconque et applications..*— Thesis, Univ. Paris XI, Orsay, 1972.

[2] BENILAN (Ph.), CRANDALL (M, G).— *Regularizing effects for homogeneous evolution equations.* Amer. J. Math. Supplement dedicated to P. HARTMAN 1981, p. 23-29.

[3] BENILAN (Ph.), CRANDALL (M, G), PAZY (A.).— *Revolution equations governed accretive operators.*— (forthcoming book).

[4] BREZIS (H.).— *Operateurs maximaux monotones.*— North Holland. Math. Studies 5, 1973.

[5] BREZIS (H.), HARAUX (A.).— Image d'une somme d'opérateurs monotones et applications., *Israel J. Math.*, t. **23**, 1976, p. 165-186.

[6] CRANDALL (M.G).— The semigroup approach to first order quasilinear equations in several space variables, *Israel J. Math.*, t. **12**, 1972, p. 108-132.

[7] CRANDALL (M.G), LIGGETT (T.M.).— Generation of semigroups of nonlinear transformations on general Banach spaces, *Amer. J. Math.*, t. **93**, 1971, p. 265-298.

[8] KRUZKOV (S.N.).— First order quasillinear equations in several space variables., *Math. USSR - 56*, t. **10**, 1970, p. 217-243.

[9] LIU (T.P.), PIERRE (M.).— Source solutions and assymptotic behavior in conservation laws, *J. Diff. Eq.*, t. **51**, 1984, p. 419-441.

[10] PIERRE (M.).— Generation et perturbations de semigroupes de contractions non lineaires. These 3°cycle, Univ. Paris VI, 1976.

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