ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 5e série, tome 10, nº 1 (1989), p. 37-64

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Annales Faculté des Sciences de Toulouse Vol. X, n°1, 1989

Asymptotic behaviour of a viscoplastic bingham fluid in porous media with periodic structure

ALAIN BRILLARD[1]

RÉSUMÉ. — Un fluide de BINGHAM viscoplastique et incompressible circule lentement dans un milieux poreux Ω_ε contenant une répartition ε–périodique d'inclusions identiques. On étudie le comportement asymptotique de la vitesse $\vec{u}_\varepsilon$ du fluide, solution de

$$\underset{(H_0^1(\Omega))^N}{\mathrm{Min}}\Big(\frac{1}{2}\int_{\Omega_\varepsilon}\sum_{ij}(e_{ij}(\vec{u}))^2dx+g\int_{\Omega_\varepsilon}\Big(\sum_{ij}(e_{ij}(\vec{u}))^2\Big)^{1/2}dx$$
$$+\delta_{V(\Omega_\varepsilon)}(\vec{u})-\int_{\Omega_\varepsilon}\vec{f}\cdot\vec{u}dx\Big)$$

lorsque le paramètre ε tend vers 0. Les lois limites de BRINKMAN et de DARCY sont obtenues par des méthodes d'épi-convergence.

ABSTRACT. — A viscoplastic Bingham fluid slowly flows in a porous medium Ω_ε containing an ε–periodic distribution of identical inclusions. We study the behaviour of the velocity $\vec{u}_\varepsilon$, solution of the minimization problem

$$\underset{(H_0^1(\Omega))^N}{\mathrm{Min}}\Big(\frac{1}{2}\int_{\Omega_\varepsilon}\sum_{ij}(e_{ij}(\vec{u}))^2dx+g\int_{\Omega_\varepsilon}\Big(\sum_{ij}(e_{ij}(\vec{u}))^2\Big)^{1/2}dx$$
$$+\delta_{V(\Omega_\varepsilon)}(\vec{u})-\int_{\Omega_\varepsilon}\vec{f}\cdot\vec{u}dx$$

when the parameter ε goes to 0. Epi-convergence methods are systematically used in order to derive the nonlinear BRINKMAN and DARCY limit laws, according to the size of the inclusions.

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1. Introduction

Let Ω be a bounded open and smooth subset of R^N ($N = 2$ or 3) and T be a smooth open subset of the unit ball $B(1)$ of R^N. The porous medium Ω_ε is equal to

$$\Omega_\varepsilon = \Omega \backslash U_{i=1}^{I(\varepsilon)} \overline{T_{\varepsilon i}},$$

where $T_{\varepsilon i}$ is a r_ε–homothetic of T disposed at the center $x_{\varepsilon i}$ of the i–th ε–cell $Y_{\varepsilon i}$ of a rectangular ε–mesh covering Ω :

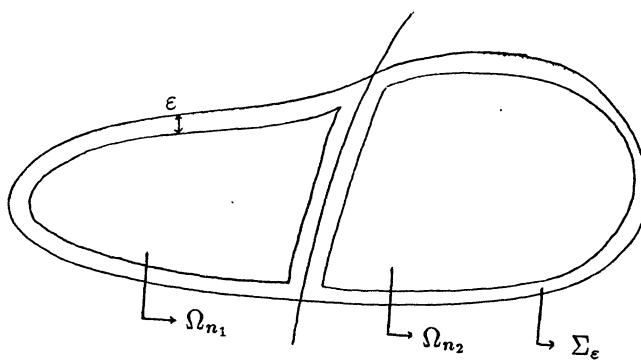

Figure 1

(notice that the number $I(\varepsilon)$ of solid inclusions $T_{\varepsilon i}$ is equivalent to Vol $(\Omega)/\varepsilon^N$). The problem we are interested in is the behaviour of the solution $\vec{u}_\varepsilon$ of the minimization problem [7]

$$\underset{\vec{u} \in (H_0^1(\Omega))^N}{\text{Min}} (F^\varepsilon(\vec{u}) - \int_\Omega \vec{f} \cdot \vec{u}\, dx), \tag{1}$$

where $\vec{f}$ belongs to $(L^2(\Omega))^N$ ($\vec{f}$ represents the exterior forces) and F^ε is the functional defined on $(H_0^1(\Omega))^N$ by

$$F^\varepsilon(\vec{u}) = \frac{1}{2} \int_\Omega \sum_{ij} (e_{ij}(\vec{u}))^2 dx + g \int_\Omega (\sum_{ij} (e_{ij}(\vec{u}))^2)^{1/2}\, dx + \delta_{V(\Omega_\varepsilon)}(\vec{u}), \tag{2}$$

$e_{ij}(\vec{u})$ being the deformation (symmetric) tensor : $\frac{1}{2}(D_j\, u_i + D_i\, u_j)$, g being a nonnegative real, $\delta_{V(\Omega_\varepsilon)}$ denoting the indicator function of the closed subspace $V(\Omega_\varepsilon)$ of $(H_0^1(\Omega_\varepsilon))^N$

$$V(\Omega_\varepsilon) = \{\vec{u} \in (H_0^1(\Omega_\varepsilon))^N / \operatorname{div} \vec{u} = \sum_{i=1}^N D_i\, u_i = 0 \text{ in } \Omega\}.$$

The coefficient g is equal to 0 for newtonian fluids and is strictly positive for Bingham fluids. The case $g = 0$, has been studied in [2], [5], [9], [11], [12]. Let $P^{\varepsilon}\vec{u}_{\varepsilon}$ be the canonical extension of $\vec{u}_{\varepsilon}$ taking the value $\vec{0}$ on the inclusions (thanks to the no-slip condition on the boundary of the solid inclusions $T_{\varepsilon i}$), then the asymptotic behaviour of the fluid is described in the following Theorem :

THEOREM 1.1.— *(Brinkman's law : case $g = 0$) [5], [11]*

Suppose that : $\lim_{\varepsilon \to 0}(r_{\varepsilon}/\varepsilon) = 0$. *Then the sequence* $(P_{\varepsilon}\vec{u}_{\varepsilon})_{\varepsilon}$ *converges in the weak topology of* $(H_0^1(\Omega))^N$ *to the solution* $\vec{u}_0^*$ *of*

$$\left|\begin{array}{ll} -\Delta\vec{u}_0^* + M\vec{u}_0^* = -\operatorname{grad} p_0^* + \vec{f} & \text{in } \Omega, \\ \operatorname{div}\vec{u}_0^* = 0 & \text{in } \Omega, \\ \vec{u}_0^* = \vec{0} & \text{on } \partial\Omega, \end{array}\right.$$

where M is the symmetric matrix given by :

for every k, l in $\{1, \ldots, N\}$:

$$M_{kl} = \lim_{\varepsilon \to 0}\left(\frac{1}{\varepsilon^N}\int_{B(\varepsilon/4)\setminus\overline{T}_{\varepsilon}} \sum_{ij}(e_{ij}(\vec{w}_{\varepsilon}^k)e_{ij}(\vec{w}_{\varepsilon}^l))\,dx\right), \tag{3}$$

$\vec{w}_{\varepsilon}^k$ *being the solution of the local minimization problem :*

$$Min \int_{B(\varepsilon/4)\setminus\overline{T}_{\varepsilon}} \sum_{ij}(e_{ij}(\vec{w}))^2\,dx, \tag{4}$$

$$\left|\begin{array}{l} \vec{w} = \vec{e}_k \text{ on } \partial B(\varepsilon/4) \\ \vec{w} = \vec{0} \text{ on } \partial T_{\varepsilon}, \\ \operatorname{div}\vec{w} = 0 \text{ in } B(\varepsilon/4)\setminus\overline{T_{\varepsilon}}, \\ \vec{w} \in (H^1(B(\varepsilon/4)))^N \end{array}\right.$$

($\vec{e}_k$ is the kth canonical vector of R^N).

Remark 1.2.—

1 – From the definition (3) of the matrix M, we infer the existence of a critical size r_{ε}^c of the inclusions :

$$r_{\varepsilon}^c = \left|\begin{array}{ll} C\,\varepsilon^3 & \text{if } N = 3\,,\ (C \in R^{+*}) \\ \exp(-C\varepsilon^{-2}) & \text{if } N = 2, \end{array}\right.$$

such that

a) if $\lim_{\varepsilon\to 0}(r_\varepsilon/r_\varepsilon^c) = 0$, then the matrix M is null and $\vec{u}_0^*$ is the solution of the Stokes problem in Ω.

b) if $\lim_{\varepsilon\to 0}(r_\varepsilon/r_\varepsilon^c) = +\infty$, then $\vec{u}_0^*$ is equal to $\vec{0}$ (the fluid is at rest in Ω).

c) if $\lim_{\varepsilon\to 0}(r_\varepsilon/r_\varepsilon^c)$ belongs to R^{+*}, then Brinkman's law contains a "strange term" $M\vec{u}_0^*$, which depends on the size and shape of the inclusions, through the capacity problem (4).

2 – The solution $\vec{u}_0^*$ of Brinkman's law is the solution of the following minimization problem :

$$\operatorname*{Min}_{(H_0^1(\Omega))^N}(F^*(\vec{u}) - \int_\Omega \vec{f}\cdot\vec{u}\,dx),$$

where F^* is the functional defined on $(H_0^1(\Omega))^N$ by :

$$F^*(\vec{u}) = \frac{1}{2}\int_\Omega \sum_{ij}(e_{ij}(\vec{u}))^2\,dx + \frac{1}{2}\sum_{kl}\int_\Omega M_{kl}\,u_k\,u_l\,dx + \delta_{V(\Omega)}(\vec{u}). \qquad (5)$$

In [5], thanks to the use of epi-convergence methods, have been given : the behaviour of the dissipation energy of the fluid, the convergence of the internal pressure (in some duality sense), the convergence of the solution of the Stokes evolution problem, the convergence of the spectra, some indications about first-order correctors.

In the next paragraph of the present work, the asymptotic behaviour of a Bingham fluid ($g > 0$), slowly flowing in the above-described porous medium, will be studied in the situation : $\lim_{\varepsilon\to 0}(r_\varepsilon/\varepsilon) = 0$.

We will prove that the asymptotic velocity $\vec{u}_0$ of the Bingham fluid is the solution of the minimization problem : $\operatorname*{Min}_{\vec{u}\in(H_0^1(\Omega))^N}(F(\vec{u}) - \int_\Omega \vec{f}\cdot\vec{u}\,dx)$, where F is the functional defined on $(H_0^1(\Omega))^N$ by

$$\begin{aligned} F(\vec{u}) = {} & \frac{1}{2}\int_\Omega \sum_{ij}(e_{ij}(\vec{u}))^2\,dx + \frac{1}{2}\sum_{kl}\int_\Omega M_{kl}\,u_k\,u_l\,dx \\ & + g\int_\Omega (\sum_{ij}(e_{ij}(\vec{u}))^2)^{1/2}\,dx + \delta_{V(\Omega)}(\vec{u}). \end{aligned}$$

In this non-newtonian case, the limit functional F still contains the "strange term" $\frac{1}{2}\sum_{kl}\int_\Omega M_{kl}\,u_k\,u_l\,dx$, where M is the matrix given by (3). The non-newtonian term $g\int_\Omega (\sum_{ij}(e_{ij}(\vec{u}))^2)^{1/2}\,dx$, appearing in the functional $F^\varepsilon(2)$

and also in the limit functional F, does not alter this strange term. The proof of this result, presented in Theorem 2.1 is based on the two following arguments :

– the explicit computation of the solution $(\vec{w}_\varepsilon^k,\ q_\varepsilon^k)$ of (4) when T is equal to the unit ball $B(1)$ (see [5]), or the existence of pointwise estimates on this solution when T is a smooth subset of $B(1)$ (see [11]). These computations or estimates imply that for every smooth divergence-free function $\vec{v}$ in $(H_0^1(\Omega))^N$, there exists a sequence $(P^\varepsilon \vec{v}_\varepsilon^0)_\varepsilon$ converging to $\vec{v}$ in the weak topology of $(H_0^1(\Omega))^N$ and such that :

$$\lim_{\varepsilon\to 0} \frac{1}{2} \int_{\Omega_\varepsilon} \sum_{ij} (e_{ij}(\vec{v}_\varepsilon^0))^2\, dx = F^*(\vec{v}),$$

$$\lim_{\varepsilon\to 0} g \int_{\Omega_\varepsilon} (\sum_{ij} (e_{ij}(\vec{v}_\varepsilon^0))^2)^{1/2}\, dx = g \int_{\Omega} (\sum_{ij} (e_{ij}(\vec{v}))^2)^{1/2}\, dx$$

(see Proposition 2.3),

– the lower semicontinuity, with respect to the weak topology of $(H_0^1(\Omega))^N$ of this non-newtonian term.

Then, we study the behaviour of the solution of the evolution problem associated to (1) (see Proposition 2.4).

In the last part of this paragraph, we consider the case of a coefficient g depending on ε. We prove that, either the Bingham fluid has a linear asymptotic behaviour (when $\lim_{\varepsilon\to 0} g(\varepsilon) = 0$), either the sequence $(P^\varepsilon \vec{u}_\varepsilon)_\varepsilon$ converges to $\vec{0}$ very fastly (when $\lim_{\varepsilon\to 0} g(\varepsilon) = +\infty$) (see Remark 2.5).

The particular case $r_\varepsilon = a\varepsilon (0 < a < 1/2)$ has been studied, for newtonian fluids, through the method of asymptotic expansions in [2, 9, 12] and by means of epi-convergence methods in [5].

THEOREM 1.3.— *(Darcy's law, case $g = 0$)*

Suppose $r_\varepsilon = a\varepsilon$ $(0 < a < 1/2)$. Then, the sequence $\left(\frac{1}{\varepsilon^2} P^\varepsilon \vec{u}_\varepsilon\right)_\varepsilon$ converges in the weak topology of $(L^2(\Omega))^N$ to the function $\vec{u}_1^*$

$$\begin{cases} \vec{u}_1^* = K(-\,grad\, p_1^* + \vec{f}), \\ \vec{u}_1^* \in H(\Omega) = \{u \in (L^2(\Omega))^N \quad / \ \operatorname{div} \vec{u} = 0 \text{ in } \Omega,\ \vec{u}\cdot\vec{n} = 0 \text{ on } \partial\Omega\}, \\ \qquad\qquad\qquad\qquad (\vec{n} \text{ is the outer normal to } \partial\Omega), \end{cases}$$

where K is the symmetric positive definite matrix given by :

for every k, l in $\{1, \ldots, N\}$:

$$K_{kl} = \int_{Y \setminus \overline{aT}} \sum_{ij} e_{ij}(\vec{z}^k)\, e_{ij}(\vec{z}^l)\, dy \;=\; \int_{Y \setminus \overline{aT}} \vec{z}^k \cdot \vec{e}_l\, dy,$$

$\vec{z}^k$ being the solution of the minimization problem :

$$\begin{array}{ll} Min & \left(\dfrac{1}{2} \displaystyle\int_{Y \setminus \overline{aT}} \sum_{ij} (e_{ij}(\vec{z}))^2\, dy - \int_{Y \setminus \overline{aT}} \vec{z} \cdot \vec{e}_k\, dy \right). \\ \vec{z}\, Y - \textit{periodic} & \\ \vec{z} = \vec{0} \textit{ on } aT & \\ \operatorname{div} \vec{z} = 0 \textit{ in } Y \setminus \overline{daT} & \end{array}$$

Remark 1.4.— $\vec{u}_1^*$ is the solution of the following minimization problem :

$$\operatorname*{Min}_{\vec{u} \in H(\Omega)} \left(\int_\Omega j(\vec{u}(x))\, dx - \int_\Omega \vec{f} \cdot \vec{u}\, dx \right),$$

where j is the function defined from R^N into R by :

$$\forall \eta \in R^N : j(\eta) = \frac{1}{2} \sum_{kl} (K^{-1})_{kl}\, \eta_k\, \eta_\ell.$$

An elementary computation shows that the function j may be written in the following form :

$$\forall \eta \in R^N : \; j(\eta) = \operatorname*{Min}_{\xi \in R^N} \left(\frac{1}{2} \int_{Y \setminus \overline{aT}} \sum_{ij} (e_{ij}(\vec{z}_\xi^*))^2\, dy + \right.$$
$$\left. + K^{-1}\eta \cdot (\eta - \int_{Y \setminus \overline{aT}} \vec{z}_\xi^*\, dy) \right),$$

where $\vec{z}_\xi^*$ is the Y–periodic function equal to $\sum_k \xi_k\, \vec{z}^k$. Notice that $\vec{z}_\xi^*$ is the solution of :

$$\begin{array}{ll} \text{Min} & \left(\dfrac{1}{2} \displaystyle\int_{Y \setminus \overline{aT}} \sum_{ij} (e_{ij}(\vec{z}))^2\, dy - \int_{Y \setminus \overline{aT}} \vec{z} \cdot \xi\, dy \right). \\ \vec{z} Y - \text{periodic} & \\ \vec{z} = 0 \text{ on } aT & \\ \operatorname{div} \vec{z} = 0 \text{ in } Y \setminus \overline{aT} & \end{array} \qquad (6)$$

This case : $r_\varepsilon = a\varepsilon (0 < a < 1/2)$ leads, for a Bingham fluid (that is when g is strictly positive and changed into $\varepsilon\, g$, through a dimension analysis of the coefficients [10]) to a nonlinear Darcy's law :

$$\begin{cases} \vec{u}_1 & = A_0\,(\vec{f} - \operatorname{grad} p_1), \\ \vec{u}_1 & \in \ H(\Omega), \end{cases}$$

where A_0 is the function from R^N into R^N equal to :

$$A_0 : \begin{pmatrix} R^N & \to R^N \\ \xi & \to \displaystyle\int_{Y\setminus\overline{aT}} \vec{z}_\xi(y)\, dy, \end{pmatrix}$$

$\vec{z}_\xi$ being the solution of the minimization problem :

$$\begin{array}{ll} \text{Min} & \left(\dfrac{1}{2} \displaystyle\int_{Y\setminus\overline{aT}} \sum_{ij} (e_{ij}(\vec{z}))^2 dy + g \int_{Y\setminus\overline{aT}} (\sum_{ij} (e_{ij}(\vec{z}))^2)^{1/2}\, dy - \right. \\ \vec{z}\, Y-\text{periodic} & \qquad\qquad \left. - \displaystyle\int_{Y\setminus\overline{aT}} \vec{z}\cdot\xi\, dy \right) \\ \vec{z} = \vec{0} \text{ on } aT & \\ \operatorname{div} \vec{z} = 0 \text{ in } Y & \end{array}$$

(compare to (6)).

This result has been partially proved in [10], by means of asymptotic expansions of the solution $\vec{u}_\varepsilon$ of (1). The purpose of the third part of the present work is to furnish a different proof of this nonlinear Darcy's law, by means of epi-convergence methods.

We shall first prove that $(\varepsilon^{-2} P^\varepsilon \vec{u}_\varepsilon)_\varepsilon$ is a bounded sequence in $(L^2(\Omega))^N$. Noticing that $\varepsilon^{-2} P^\varepsilon \vec{u}_\varepsilon$ is the solution of a minimization problem :

$$\underset{(H_0^1(\Omega))^N}{\text{Min}} \left(G^\varepsilon(\vec{u}) - \int_\Omega \vec{f}\cdot\vec{u}\, dx \right),$$

where

$$G^\varepsilon(\vec{u}) = \frac{\varepsilon^2}{2} \int_\Omega \sum_{ij} (e_{ij}(\vec{u}))^2\, dx + \varepsilon g \int_\Omega (\sum_{ij} (e_{ij}(\vec{u}))^2)^{1/2} dx + \delta_{V(\Omega_\varepsilon)}(\vec{u}),$$

the nonlinear Darcy's law will be derived from the computation of the epi-limit G of the sequence $(G^\varepsilon)_\varepsilon$ in the weak topology of $(L^2(\Omega))^N$:

$$G(\vec{u}) = \int_\Omega j_0(\vec{u}(x))\,dx - \int_\Omega \vec{f}\cdot\vec{u}\,dx, \qquad \text{with}$$

$$\forall \eta \in R^N : j_0(\eta) = \operatorname*{Min}_{\xi\in R^N}\left(\frac{1}{2}\int_{Y\setminus\overline{aT}}\sum_{ij}(e_{ij}(\vec{z}))^2\,dy + \right.$$
$$+g\int_{Y\setminus\overline{aT}}(\sum_{ij}(e_{ij}(\vec{z}_\xi))^2)^{1/2}\,dy +$$
$$\left. +A_0^{-1}\eta\cdot(\eta - \int_{Y\setminus\overline{aT}}\vec{z}_\xi\,dy)\right).$$

The last part of this study deals with the convergence of the internal pressure of the Bingham fluid. Thanks to Tartar's idea [13], we shall prove the existence of an extension $\overline{p}_\varepsilon$ of the internal pressure p_ε of the fluid, such that $(\overline{p}_\varepsilon)_\varepsilon$ converges in the strong topology of $L^2(\Omega)/R$, to a function p_2. We conjecture that this function p_2 is in fact equal to the limit pressure p_1, appearing in the nonlinear Darcy's limit law.

Let us now recall the main properties of epi-convergence, which will be used throughout this study.

DEFINITION 1.5 [1], [6].—

Let (X,τ) be a metric vector space and $(F^\varepsilon)_\varepsilon$ a sequence of functionals from X into $\overline{R}$. Then, $(F^\varepsilon)_\varepsilon$ epi-converges to F, in the topology τ, if

$$\forall x \in X : (\tau - \liminf_{\varepsilon\to 0} F^\varepsilon)(x) = (\tau - \limsup_{\varepsilon\to 0} F^\varepsilon)(x) = F(x),$$

where : $(\tau - \liminf_{\varepsilon\to 0} F^\varepsilon)(x) = \operatorname*{Min}_{(x_\varepsilon)_\varepsilon} \liminf_{\varepsilon\to 0} F^\varepsilon(x_\varepsilon),$

$(\tau - \limsup_{\varepsilon\to 0} F^\varepsilon)(x) = \operatorname*{Min}_{(x_\varepsilon)_\varepsilon} \limsup_{\varepsilon\to 0} F^\varepsilon(x_\varepsilon),$

$(x_\varepsilon)_\varepsilon$ being any sequence converging to x, in the topology τ.

Equivalently, this equality is satisfied if the two following assertions are fulfilled

$$\text{for every } x \text{ in } X, \text{ there exists a sequence } (x_\varepsilon^0)_\varepsilon \quad (7)$$
$$\tau\text{—converging to } x, \text{ such that } \limsup_{\varepsilon\to 0} F^\varepsilon(x_\varepsilon^0) \le F(x),$$
$$\text{for every } x \text{ in } X \text{ and for every sequence } (x_\varepsilon)_\varepsilon \quad (8)$$
$$\tau\text{—converging to } x : \liminf_{\varepsilon\to 0} F^\varepsilon(x_\varepsilon) \ge F(x).$$

This variational convergence is well-fitted to the asymptotic analysis of minimization problems :

THEOREM 1.6 ([1] Theorem 1.10).—

Suppose that $(F^\varepsilon)_\varepsilon$ epi-converges to F, in the topology τ and let $\overline{x}_\varepsilon$ be an o_ε–minimizer of F^ε, that is : $F^\varepsilon(\overline{x}_\varepsilon) \leq \operatorname*{Inf}_{x\in X} F^\varepsilon(x) + o_\varepsilon$, with $\lim_{\varepsilon\to 0} o_\varepsilon = 0$. Suppose, moreover, that the sequence $(\overline{x}_\varepsilon)_\varepsilon$ is τ–relatively compact.

Then, for every τ–converging subsequence $(\overline{x_{\varepsilon'}})_{\varepsilon'}$ of $(\overline{x_\varepsilon})_\varepsilon$, with $\overline{x} = \lim_{\varepsilon'\to 0} \overline{x_{\varepsilon'}}$:

$$F(\overline{x}) = \operatorname*{Min}_X F(x).$$

and moreover $\lim_{\varepsilon'\to 0} F^{\varepsilon'}(\overline{x}_{\varepsilon'}) = F(\overline{x})$.

The next result shows the stability of epi-convergence, with respect to continuous perturbations.

PROPOSITION 1.7 ([1] Theorem 2.15.).—

Suppose that $(F^\varepsilon)_\varepsilon$ epi-converges to F, in the topology τ and G is continuous for this topology τ. Then $(F^\varepsilon + G)_\varepsilon$ epi-converges to $F + G$ in the topology τ.

NOTATIONS :

$L^2(\Omega)$, $H_0^1(\Omega)$ are the classical function spaces, while $(L^2(\Omega))^N$ and $(H_0^1(\Omega))^N$ consist of vector-valued functions whose components belong to $L^2(\Omega)$ or $H_0^1(\Omega)$.

$\mathcal{V}(\Omega)$ is the space of smooth divergence-free functions :

$$\mathcal{V}(\Omega) = \{\vec{u} \in C_0^\infty(\Omega) / \operatorname{div} \vec{u} = \sum_{i=1}^N D_i\, u_i = 0 \text{ in } \Omega\},$$

$\mathcal{V}(\Omega)$ is dense in $V(\Omega)$ for the topology of $(H_0^1(\Omega))^N$ and in $H(\Omega)$ for the topology of $(L^2(\Omega))^N$, where

$V(\Omega) = \{\vec{u} \in (H_0^1(\Omega))^N / \operatorname{div} \vec{u} = 0 \text{ in } \Omega\}$,
$H(\Omega) = \{\vec{u} \in (L^2(\Omega))^N / \operatorname{div} \vec{u} = 0 \text{ in } \Omega \text{ and } \vec{u}\cdot\vec{n} = 0 \text{ on } \partial\Omega$, where $\vec{n}$ is the outer normal to the (smooth) boundary $\partial\Omega\}$.

δ_A is the indicator function of the set A : $\delta_A(x) = \left| \begin{array}{ll} 0 & \text{if } x \text{ belongs to } A, \\ +\infty & \text{elsewhere.} \end{array} \right.$

χ_A is the characteristic function of the set A : $\chi_A(x) = \left| \begin{array}{l} 1 \text{ if } x \text{ belongs to } A, \\ 0 \text{ elsewhere.} \end{array} \right.$

For every z in $L^1(Y)$ (Y is the unit cell $\left[-\frac{1}{2}, \frac{1}{2}\right]^N$ of R^N), $\tilde{z}$ denotes the mean value of z on Y

$$\tilde{z} = \int_Y z(y)\, dy.$$

A function z defined on Y is called $Y-$periodic if it takes the same values on the opposite sides of Y.

II. Asymptotic behaviour of the solution $\vec{u}_\varepsilon$ of (1) when $g > 0$ and $\lim_{\varepsilon \to 0}(r_\varepsilon/\varepsilon) = 0$

P^ε denotes the canonical extension operator from $(H_0^1(\Omega_\varepsilon))^N$ into $(H_0^1(\Omega))^N$:

$$P^\varepsilon \vec{u} = \left| \begin{array}{l} \vec{u} \text{ on } \Omega_\varepsilon, \\ \vec{0} \text{ on } \bigcup_i T_{\varepsilon i}. \end{array} \right.$$

From the strict positivity of g, Poincare and Korn's inequalities in Ω [7], one first deduces from (1), that $(P^\varepsilon \vec{u}_\varepsilon)_\varepsilon$ is bounded in $(H_0^1(\Omega))^N$.

THEOREM 2.1.

The sequence $(P^\varepsilon \vec{u}_\varepsilon)_\varepsilon$ (canonical extension of the solution of (1)) converges in the weak topology of $(H_0^1(\Omega))^N$ to the solution $\vec{u}_0$ of :

$$\operatorname*{Min}_{\vec{u} \in (H_0^1(\Omega))^N} (F(\vec{u}) - \int_\Omega \vec{f} \cdot \vec{u}\, dx),$$

where F is the functional defined by

$$\begin{aligned} F(\vec{u}) = \frac{1}{2} \int_\Omega \sum_{ij} (e_{ij}(\vec{u}))^2 dx + \frac{1}{2} \sum_{kl} \int_\Omega M_{kl}\, u_k\, u_l\, dx + \\ + g \int_\Omega (\sum_{ij} (e_{ij}(\vec{u})^2))^{1/2} dx + \delta_{V(\Omega)}(\vec{u}), \end{aligned} \tag{9}$$

M being the symmetric matrix given by (3).

Moreover, the dissipation energy of the fluid :

$$\frac{1}{2} \int_{\Omega_\varepsilon} \sum_{ij} (e_{ij}(\vec{u}_\varepsilon))^2\, dx + g \int_{\Omega_\varepsilon} (\sum_{ij} (e_{ij}(\vec{u}_\varepsilon))^2)^{1/2}\, dx$$

converges to the energy of the asymptotic fluid :

$$\frac{1}{2}\int_\Omega \sum_{ij}(e_{ij}(\vec{u}_0))^2\,dx+\frac{1}{2}\int_\Omega \sum_{kl} M_{kl}(\vec{u}_0)_k\,(\vec{u}_0)_l\,dx+g\int_\Omega (\sum_{ij}(e_{ij}(\vec{u}_0))^2)^{1/2}\,dx$$

This Theorem 2.1 is a straightforward consequence of the following result,

THEOREM 2.2.

The sequence $(F^\varepsilon)_\varepsilon$ of functionals defined by (2) epi-converges to the functional F defined by (9), in the weak topology of $(H_0^1(\Omega))^N$,

through theorem 1.6, since

– the functional F is convex and strictly coercive on $(H_0^1(\Omega))^N$ (hence the minimizer of F is unique and the whole sequence $(P^\varepsilon\vec{u}_\varepsilon)_\varepsilon$ converges to $\vec{u}_0$),

– the functional : $\vec{u} \longrightarrow \int_\Omega \vec{f}\cdot\vec{u}\,dx$, is continuous for the weak topology of $(H_0^1(\Omega))^N$.

Proof of Theorem 2.2 :

Noticing that $V(\Omega)$ is a closed subspace of $(H_0^1(\Omega))^N$, for the weak topology of this space, one proves that for every $\vec{v}$ in $(H_0^1(\Omega))^N$, which is not divergence-free in Ω :

$$((w-(H_0^1(\Omega))^N)-\limsup_{\varepsilon\to 0} F^\varepsilon)(\vec{v}) = ((w-(H_0^1(\Omega))^N)-\liminf_{\varepsilon\to 0} F^\varepsilon)(\vec{v}) = +\infty.$$

In order to prove Theorem 2.2, one has to verify the assertions (7) and (8) which take the special form :

For every $\vec{v}$ in $V(\Omega)$, there exists a sequence $(P^\varepsilon\vec{v}_\varepsilon^0)_\varepsilon$, converging in the weak topology of $(H_0^1(\Omega))^N$ to $\vec{v}$, with $\vec{v}_\varepsilon^0$ in $V(\Omega_\varepsilon)$, such that

$$\limsup_{\varepsilon\to 0} F^\varepsilon(\vec{v}_\varepsilon^0) \le F(\vec{v}), \tag{10}$$

for every $\vec{v}$ in $V(\Omega)$ and for every sequence $(P^\varepsilon\vec{v}_\varepsilon)_\varepsilon$ converging to $\vec{v}$ in the weak topology of $(H_0^1(\Omega))^N$, with $\vec{v}_\varepsilon$ in $V(\Omega_\varepsilon)$:

$$\liminf_{\varepsilon\to 0} F^\varepsilon(\vec{v}_\varepsilon) \ge F(\vec{v}). \tag{11}$$

Suppose first that α_N is finite, with

$$\alpha_N = \left| \begin{array}{ll} \lim\limits_{\varepsilon\to 0}(r_\varepsilon\varepsilon^{-3}) & \text{if } N=3, \\ \lim\limits_{\varepsilon\to 0}\dfrac{\varepsilon^{-2}}{|\log r_\varepsilon|} & \text{if } N=2. \end{array}\right. \tag{12}$$

The verification of (10) and (11) is a consequence of the following Proposition, the technical proof of which being postponed in Appendix 1 :

PROPOSITION 2.3.— *Suppose that $\vec{v}$ belongs to $\mathcal{V}(\Omega)$ and α_N is finite. Then, there exists a function $\vec{v}_\varepsilon^0$ in $V(\Omega_\varepsilon)$ satisfying*

a) $(P^\varepsilon \vec{v}_\varepsilon^0)_\varepsilon$ converges to $\vec{v}$ in the weak topology of $(H_0^1(\Omega))^N$.

b) $\left(\int_{\Omega_\varepsilon} \sum_{ij} (e_{ij}(\vec{v}_\varepsilon^0))^2\, dx\right)_{\varepsilon_2}$ *converges to*

$$\int_\Omega \sum_{ij} (e_{ij}(\vec{v}))^2\, dx + \sum_{kl} \int_\Omega M_{kl}\, v_k\, v_l\, dx.$$

c) there exists a constant C independant of $\vec{v}$ such that for every $\vec{w}$ in $V(\Omega)$ and for every sequence $(P^\varepsilon \vec{w}_\varepsilon)_\varepsilon$ converging to $\vec{w}$, in the weak topology of $(H_0^1(\Omega))^N$, with $\vec{w}_\varepsilon$ in $V(\Omega_\varepsilon)$:

$$\liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \sum_{ij} e_{ij}(\vec{v}_\varepsilon^0) e_{ij}(\vec{w}_\varepsilon)\, dx \geq \int_\Omega \sum_{ij} e_{ij}(\vec{v}) e_{ij}(\vec{w})\, dx - C \|\vec{w}\|_{(L^2(\Omega))^N} \|\vec{v}\|_{(L^2(\Omega))^N}.$$

d) $\left(\int_{\Omega_\varepsilon} (\sum_{ij} (e_{ij}(\vec{v}_\varepsilon^0))^2)^{1/2}\, dx\right)_\varepsilon$ *converges to* $\int_\Omega (\sum_{ij} (e_{ij}(\vec{v}))^2)^{1/2} dx.$

Let us prove (10) for a smooth divergence-free function $\vec{v}$ in $\mathcal{V}(\Omega)$. Proposition 2.3 a), b) and d) imply the existence of $(P^\varepsilon \vec{v}_\varepsilon^0)_\varepsilon$ converging to $\vec{v}$ in the weak topology of $(H_0^1(\Omega))^N$ and such that :

$$\lim_{\varepsilon\to 0} F^\varepsilon(\vec{v}_\varepsilon^0) = F(\vec{v}).$$

Therefore, (10) is proved for a smooth divergence-free function $\vec{v}$ in $\mathcal{V}(\Omega)$.

If $\vec{v}$ belongs to $V(\Omega)$, there exists a sequence $(\vec{v}_n)_n$ of smooth divergence-free functions in $\mathcal{V}(\Omega)$, converging to $\vec{v}$ in the strong topology of $(H_0^1(\Omega))^N$. For every n, Proposition 2.3 guarantees the existence of $(P^\varepsilon(\vec{v}_n)_\varepsilon^0)_\varepsilon$ converging to $\vec{v}_n$, in the weak topology of $(H_0^1(\Omega))^N$, such that :

$$\lim_{\varepsilon\to 0} F^\varepsilon((\vec{v}_n)_\varepsilon^0) = F(\vec{v}_n).$$

Hence

$$\limsup_{n\to+\infty} \quad \limsup_{\varepsilon\to 0} F^\varepsilon((\vec{v}_n)_\varepsilon^0) \leq F(\vec{v}),$$

thanks to the continuity of F, with respect to the strong topology of $(H_0^1(\Omega))^N$. Using the diagonalization argument of Corollary 1.16. [1], one derives the existence of a subsequence $(n(\varepsilon))_\varepsilon$ growing to $+\infty$ such that :

$$\limsup_{\varepsilon\to 0} F^\varepsilon((\vec{v}_{n(\varepsilon)})_\varepsilon^0) \leq F(\vec{v}),$$

and $((P^\varepsilon(\vec{v}_{n(\varepsilon)})_\varepsilon^0)_\varepsilon$ converges to $\vec{v}$ in the weak topology of $(H_0^1(\Omega))^N$. Take $\vec{v}_\varepsilon^0 = (\vec{v}_{n(\varepsilon)})_\varepsilon^0$: (10) is proved for $\vec{v}$ in $V(\Omega)$ (and α_N finite).

Let us now prove (11) when α_N is finite : take $\vec{v}$ in $V(\Omega)$, $(P^\varepsilon\vec{v}_\varepsilon)_\varepsilon$ (resp. $(\vec{v}_n)_n$) converging to $\vec{v}$ in the weak (resp. strong) topology of $(H_0^1(\Omega))^N$, $\vec{v}_\varepsilon$ in $V(\Omega_\varepsilon)$, $\vec{v}_n$ in $\mathcal{V}(\Omega)$.

$$\begin{aligned}
\liminf_{\varepsilon\to 0} F^\varepsilon(\vec{v}_\varepsilon) &\geq \frac{1}{2}\liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\sum_{ij}(e_{ij}(\vec{v}_\varepsilon))^2\,dx+\\
&\quad + g\liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon}(\sum_{ij}(e_{ij}(\vec{v}_\varepsilon))^2)^{1/2}\,dx\\
&\geq \frac{1}{2}\liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\sum_{ij}(e_{ij}((\vec{v}_n)_\varepsilon^0))^2\,dx+\\
&\quad + \liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\sum_{ij} e_{ij}((\vec{v}_n)_\varepsilon^0)\, e_{ij}(\vec{v}_\varepsilon - (\vec{v}_n)_\varepsilon^0)\,dx+\\
&\quad + g\liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon}(\sum_{ij}(e_{ij}(\vec{v}_\varepsilon))^2)^{1/2}\,dx.
\end{aligned}$$

Thanks to Proposition 2. 3. b) and c), one obtains

$$\begin{aligned}
\liminf_{\varepsilon\to 0} F^\varepsilon(\vec{v}_\varepsilon) &\geq \frac{1}{2}\int_\Omega\sum_{ij}(e_{ij}(\vec{v}_n))^2\,dx + \frac{1}{2}\sum_{kl}\int_\Omega M_{kl}\,(\vec{v}_n)_k\,(\vec{v}_n)_l\,dx+\\
&\quad + \int_\Omega\sum_{ij} e_{ij}(\vec{v}_n)e_{ij}(\vec{v}-\vec{v}_n)\,dx - C\|\vec{v}_n\|_{(L^2(\Omega))^N}\|\vec{v}-\vec{v}_n\|_{(L^2(\Omega))^N}+\\
&\quad + g\liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon}(\sum_{ij}(e_{ij}(\vec{v}_\varepsilon))^2)^{1/2}\,dx.
\end{aligned}$$

Moreover the functional

$$\vec{v}\to\int_\Omega(\sum_{ij}(e_{ij}(\vec{v}))^2)^{1/2}\,dx$$

is convex and continuous for the strong topology of $(H_0^1(\Omega))^N$, hence lower semicontinuous for the weak topology of this space.

Let n go to $+\infty$, and use the property of $(\vec{v}_n)_n$: (11) is proved.

Suppose finally that α_N is equal to $+\infty$.

Observe first that for every $\vec{v}$ in $V(\Omega)$ and for every α in R^{+*} :

$$F^\varepsilon(\vec{v}) \geq F^\varepsilon_\alpha(v),$$

where F^ε_α is the functional F^ε corresponding to the case $\alpha_N = \alpha$ and $T = B(1)$. From the properties of the matrix M, one immediately deduces

$$((w - (H^1_0(\Omega))^N) - \limsup_{\varepsilon\to 0} F^\varepsilon)(\vec{v}) = ((w - (H^1_0(\Omega))^N) - \liminf_{\varepsilon\to 0} F^\varepsilon)(\vec{v}) =$$
$$= \left| \begin{array}{l} 0 \text{ if } \vec{v} = \vec{0}\,(a.e.) \text{ in } \Omega, \\ +\infty \text{ if } \vec{v} \text{ is not equal to } \vec{0} \text{ in } \Omega. \end{array} \right.$$

This ends the proof of Theorem 2.2.

Let us conclude this paragraph, giving the convergence of the solution of the evolution problems.

PROPOSITION 2.4.— *Take T_0 in R^{+*}, $(\vec{f}_\varepsilon)_\varepsilon$ (resp. $(\vec{x}_\varepsilon)_\varepsilon$) a sequence converging to $\vec{f}$ in $L^2(0,T_0;(L^2(\Omega))^N)$ (resp. to $\vec{x}$ in $(L^2(\Omega))^N$). Then the sequence $(P^\varepsilon\vec{u}_\varepsilon(.,.))_\varepsilon$ of extensions of the solution $\vec{u}_\varepsilon$ of*

$$\left| \begin{array}{l} \dfrac{\partial \vec{u}_\varepsilon}{\partial t} + \partial F^\varepsilon(\vec{u}_\varepsilon) \ni \vec{f}_\varepsilon \text{ in }]0,T_0[\times\Omega_\varepsilon, \\ \vec{u}_\varepsilon(.,t) \in V(\Omega_\varepsilon) \text{ for } t > 0, \\ \vec{u}_\varepsilon(x,0) = \vec{x}_\varepsilon, \end{array} \right.$$

converges in the strong topology of $L^2(0,T_0;(L^2(\Omega))^N)$ to the solution $\vec{u}_0$ of

$$\left| \begin{array}{l} \dfrac{\partial \vec{u}_0}{\partial t} + \partial F(\vec{u}_0) \ni \vec{f}_0, \\ \vec{u}_0(.,t) \in V(\Omega) \text{ for } t > 0, \\ \vec{u}_0(x,0) = \vec{x}. \end{array} \right.$$

Moreover

$$\int_0^{T_0} t \left| \frac{\partial \vec{u}_\varepsilon}{\partial t} - \frac{\partial \vec{u}_0}{\partial t} \right|^2 dt \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Proof of Proposition 2.4.

See [1] Theorem 3.74.

Remark 2.5.— In this section, we have supposed g to be a constant. Let us now consider the case of a coefficient g equal to some function $g(\varepsilon)$, that is

$$F^\varepsilon(\vec{u}) = \frac{1}{2}\int_\Omega \sum_{ij} (e_{ij}(\vec{u}))^2\, dx + g(\varepsilon)\int_\Omega \ (\sum_{ij}(e_{ij}(\vec{u}))^2)^{1/2}\, dx + \delta_{V(\Omega_\varepsilon)}(\vec{u}).$$

1 – If $\lim_{\varepsilon\to 0} g(\varepsilon) = 0$, then one immediately proves, with the arguments developped in the proof of Theorem 2.2, that $(F^\varepsilon)_\varepsilon$ epi-converges in the weak topology of $(H_0^1(\Omega))^N$, to the functional F^* defined by (5). In this case, the (weakly) non-newtonian fluid has a linear asymptotic behaviour.

2 – If $\lim_{\varepsilon\to 0} g(\varepsilon) = +\infty$ and if $\vec{f}$ belongs to $(C^0(\overline{\Omega}))^N$, for example, then $((g(\varepsilon))^p P^\varepsilon \vec{u}_\varepsilon)_\varepsilon$ converges to $\vec{0}$, in the strong topology of $(H_0^1(\Omega))^N$ for every integer p. Indeed, observe first that :

$$\int_\Omega \sum_{ij} (e_{ij}(P^\varepsilon \vec{u}_\varepsilon))^2\, dx + g(\varepsilon) \int_\Omega (\sum_{ij} (e_{ij}(P^\varepsilon \vec{u}_\varepsilon))^2)^{1/2}\, dx =$$
$$= \int_\Omega \vec{f} \cdot P^\varepsilon \vec{u}_\varepsilon\, dx.$$

Since $(P^\varepsilon \vec{u}_\varepsilon)_\varepsilon$ is bounded in $(H_0^1(\Omega))^N$, one proves that $\left(\int_\Omega (\sum_{ij} (e_{ij}(P^\varepsilon \vec{u}_\varepsilon))^2)^{1/2}\, dx\right)_\varepsilon$ converges to 0. Therefore (see [15]) $(P^\varepsilon \vec{u}_\varepsilon)_\varepsilon$ converges to $\vec{0}$ in the appropriate space $LD(\Omega)$, hence in $(L^1(\Omega))^N$ and finally in $(H_0^1(\Omega))^N$. The preceding equality implies that $(g(\varepsilon)\ P^\varepsilon \vec{u}_\varepsilon)_\varepsilon$ converges to $\vec{0}$ in $LD(\Omega)$ and in $(L^1(\Omega))^N$.

Multiply the preceding equality by $g(\varepsilon)$. One now proves that $((g(\varepsilon))^2\, P^\varepsilon \vec{u}_\varepsilon)_\varepsilon$ converges to $\vec{0}$ in $(L^1(\Omega))^N$ and in $(H_0^1(\Omega))^N$. The assertion is proved in a recurrent way.

III. The case $g > 0$ and $r_\varepsilon = a\varepsilon (0 < a < 1/2)$: The nonlinear Darcy's law.

Notice first that in the present case, the coefficient g has to be changed into $g\varepsilon$ [10]. Let us denote $\vec{u}_\varepsilon$ the solution of the minimization problem

$$\operatorname*{Min}_{(H_0^1(\Omega))^N} \left(\frac{1}{2} \int_\Omega \sum_{ij} (e_{ij}(\vec{u})^2\, dx + g\varepsilon \int_\Omega (\sum_{ij} (e_{ij}(\vec{u}))^2)^{1/2}\, dx + \delta_{V(\Omega_\varepsilon)}(\vec{u}) \right.$$
$$\left. - \int_\Omega \vec{f} \cdot \vec{u}\, dx \right) \tag{13}$$

According to the results obtained in Theorem 2.2, $(P^\varepsilon \vec{u}_\varepsilon)_\varepsilon$ converges to $\vec{0}$ in the weak topology (and in fact the strong topology) of $(H_0^1(\Omega))^N$, since

α_N is equal to $+\infty$, in this case. The purpose of this paragraph is to study the rate of convergence.

LEMMA 3.1.— *The sequence $(\varepsilon^{-2}P^\varepsilon\vec{u}_\varepsilon)_\varepsilon$ is bounded in $(L^2(\Omega))^N$. $\varepsilon^{-2}P^\varepsilon\vec{u}_\varepsilon$ is the solution of the minimization problem*

$$\underset{\vec{v}\in(L^2(\Omega))^N}{Min}(G^\varepsilon(\vec{v})-\int_\Omega \vec{f}\vec{v}\,dx),$$

where G^ε is the functional defined on $(L^2(\Omega))^N$ by

$$G^\varepsilon(\vec{v})=\frac{\varepsilon^2}{2}\int_\Omega\sum_{ij}(e_{ij}(\vec{v}))^2\,dx+\varepsilon g\int_\Omega(\sum_{ij}(e_{ij}(\vec{v}))^2)^{1/2}\,dx+\delta_{V(\Omega_\varepsilon)}(\vec{v}). \quad (14)$$

Proof of Lemma 3.1

The first assertion follows from (13), thanks to Korn's inequality in Ω, the positivity of g and the following estimate of Poincaré's inequality in Ω_ε, the proof of which is trivial :

LEMMA 3.2.— *There exists a strictly positive constant C such that :*

$$\forall u\in H^1(Y),\ u=0 \text{ on } aT:\ \int_{Y\setminus\overline{aT}}u^2\,dy\le C\int_{Y\setminus\overline{aT}}|\operatorname{grad}u|^2\,dy,$$

$$\forall u\in H^1(Y_\varepsilon), u=0 \text{ on } T_\varepsilon:\ \int_{Y_\varepsilon\setminus\overline{T}_\varepsilon}u^2\,dx\le C\varepsilon^2\int_{Y_\varepsilon\setminus\overline{T}_\varepsilon}|\operatorname{grad}\ u|^2\,dx,$$

$$\forall u\in H_0^1(\Omega_\varepsilon):\ \int_{\Omega_\varepsilon}u^2\,dx\le C\varepsilon^2\int_{\Omega_\varepsilon}|\operatorname{grad}\ u|^2\,dx.$$

In order to describe the limit of the sequence $(\varepsilon^{-2}P^\varepsilon\vec{u}_\varepsilon)_\varepsilon$, in the weak topology of $(L^2(\Omega))^N$, one has to compute the epi-limit G of the sequence $(G^\varepsilon)_\varepsilon$ in this topology. Notice that the functional : $\vec{v}\longrightarrow\int_\Omega\vec{f}\cdot\vec{v}\,dx$ is continuous for the weak topology of $(L^2(\Omega))^N$ (see Theorem 1.6 and Proposition 1.7).

From the divergence-free condition contained in G^ε, one first deduces that the domain of G in contained in $H(\Omega)$, the closure of $V(\Omega)$, with respect to the (weak) topology of $(L^2(\Omega))^N$. Hence, one has to prove the following assertions, deduced from (7) and (8) :

for every $\vec{v}$ in $H(\Omega)$, there exists a sequence $(P^\varepsilon\vec{v}_{\varepsilon,0})_\varepsilon$ converging to $\vec{v}$ in the weak topology of $(L^2(\Omega))^N$, with $\vec{v}_{\varepsilon,0}$ in $V(\Omega_\varepsilon)$, such that

$$\limsup_{\varepsilon\to0}\quad G^\varepsilon(\vec{v}_{\varepsilon,0})\le G(\vec{v}), \quad (15)$$

for every $\vec{v}$ in $H(\Omega)$ and for every sequence $(P^\varepsilon \vec{v}_\varepsilon)_\varepsilon$ converging to $\vec{v}$ in the weak topology of $(L^2(\Omega))^N$, with $\vec{v}_\varepsilon$ in $V(\Omega_\varepsilon)$:

$$\liminf_{\varepsilon \to 0} \quad G^\varepsilon(\vec{v}_\varepsilon) \geq G(\vec{v}). \tag{16}$$

The verification of (15) and (16) requires the local character of G^ε and the density of the set of piecewise constant functions in $(L^2(\Omega))^N$:

DEFINITION 3.3.— *A function $\vec{u}$ in $(L^2(\Omega))^N$ is called piecewise constant if there exists a finite family $(\Omega_p)_{1 \leq p \leq p(\vec{u})}$ of smooth open subsets of Ω, with $\Omega \backslash \overline{\bigcup_p \Omega_p}$ negligible, such that $\vec{u}$ is a constant $\vec{u}_p$ on Ω_p.*

The verification of (15) and (16) also requires the study of the local minimization problem :

For every ξ in R^N, let $\vec{z}_\xi$ be the solution of

$$\text{Min} \quad \left(\frac{1}{2} \int_{Y \backslash \overline{aT}} \sum_{ij} (e_{ij}(\vec{z})^2 \, dy + \right. \tag{17}$$

$$\left. + g \int_{Y \backslash \overline{aT}} (\sum_{ij} (e_{ij}(\vec{z}))^2)^{1/2} dy - \xi \cdot \widetilde{\vec{z}} \right).$$

$\vec{z}$ Y–periodic
div $\vec{z} = 0$ on Y
$\vec{z} = \vec{0}$ on aT

Let A_0 be the function : $\begin{pmatrix} R^N & \longrightarrow & R^N \\ \xi & \longrightarrow & \widetilde{\vec{z}}_\xi \end{pmatrix}$.

The properties of A_0 are summarized in the following Lemma :

LEMMA 3.4.— *(see [10] and Appendix 2)*

a) For every λ in R^N, there exists $\xi(\lambda)$ in R^N such that : $A_0(\xi(\lambda)) = \lambda$.

b) this $\xi(\lambda)$ is not unique, but if ξ_1 and ξ_2 belong to $A_0^{-1}(\lambda)$:

$$\frac{1}{2} \int_{Y \backslash \overline{aT}} \sum_{ij} (e_{ij}(\vec{z}_{\xi_1}))^2 \, dy + g \int_{Y \backslash \overline{aT}} (\sum_{ij} (e_{ij}(\vec{z}_{\xi_1}))^2)^{1/2} \, dy =$$

$$= \frac{1}{2} \int_{Y \backslash \overline{aT}} \sum_{ij} (e_{ij}(\vec{z}_{\xi_2}))^2 \, dy + g \int_{Y \backslash \overline{aT}} \left(\sum_{ij} (e_{ij}(\vec{z}_{\xi_2}))^2 \right)^{1/2} dy,$$

c) there exist two constants c, C independant of ξ such that :

$$c|\xi|^2 \leq A_0 \xi \cdot \xi \leq C|\xi|^2, \qquad (0 < c \leq C)$$

d) there exist a symmetric matrix valued function m_ξ and a function q_ξ verifying

$$\sum_i (m_\xi)_{ii} = 0.$$

$$\sum_{ij} (m_\xi)_{ij}(m_\xi)_{ij} \leq 1 \, a.e. \; in \; Y,$$

$$\sum_{ij} (m_\xi)_{ij} e_{ij}(\vec{z}_\xi) = \left(\sum_{ij} (e_{ij}(\vec{z}_\xi))^2 \right)^{1/2} \quad a.e. \; in \; Y,$$

$$-\sum_j D_j e_{ij}(\vec{z}_\xi) - g \sum_j D_j (m_\xi)_{ij} = -D_i q_\xi + \xi_i \; in \; Y.$$

Let us denote by j_0 the function from R^N into R defined by

$$\begin{aligned} j_0(\eta) = \frac{1}{2} \int_{Y \setminus \overline{aT}} \sum_{ij} (e_{ij}(\vec{z} A_0^{-1} \eta))^2 dy + \\ + g \int_{Y \setminus \overline{aT}} \Big(\sum_{ij} (e_{ij}(\vec{z} A_0^{-1} \eta))^2 \Big)^{1/2} dy. \end{aligned} \tag{18}$$

(this is justified by Lemma 3.4 b)).

From (17) and Lemma 3.4 c), one immediately infers that j_0 has a quadratic growth, namely : $j_0(\eta) \leq C + C'|\eta|^2 (0 \leq C, C')$. Moreover, an elementary computation (deduced from (17)) proves that j_0 is convex.

Consequently, the functional $\int_\Omega j_0(\vec{v}(x))\, dx$ is well-defined on $(L^2(\Omega))^N$. Moreover, this functional G is convex and continuous, with respect to the strong topology of $(L^2(\Omega))^N$.

Verification of (15) with $G(\vec{v}) = \int_\Omega j_0(\vec{v}(x))\, dx$.

Take any $\vec{v}$ in $H(\Omega)$. Let $(\vec{v}_n)_n$ be any sequence of piecewise constant functions converging to $\vec{v}$ in the strong topology of $(L^2(\Omega))^N$:
$\vec{v}_n = \vec{v}_{np}$ on $\Omega_{np} (1 \leq p \leq p(n))$, with $\Omega \setminus \overline{\bigcup_p \Omega_{np}}$ negligible. For every p Lemma 3.4 implies the existence of an Y–periodic function $\vec{z}_{\xi(\vec{v}_{np})}$ such that the mean value $\widetilde{\vec{z}}_{\xi(\vec{v}_{np})}$ is equal to $\vec{v}_{np}$. Then we define $\vec{z}^\varepsilon_{\xi(\vec{v}_{np})}$ as the Y_ε–periodic function equal to $\vec{z}_{\xi(\vec{v}_{np})} \left(\frac{\cdot}{\varepsilon}\right)$ in Y_ε and ε–periodically extented in Ω_{np}.

Lemma 4.1 of [12] implies that $(\vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})})_\varepsilon$ converges in the weak topology of $(L^2(\Omega_{np}))^N$ to $\widetilde{\vec{z}}_{\xi(\vec{v}_{np})} = \vec{v}_{np}$.

The main difficulty of the problem is to gather all these different Y_ε–periodic functions $\vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})}$. This may be done in the following way.

First, define $\sum_\varepsilon$ as the set $\{x \in \Omega / d(x, \partial\Omega \cup \bigcup_p \partial\Omega_{np}) < \varepsilon\}$.

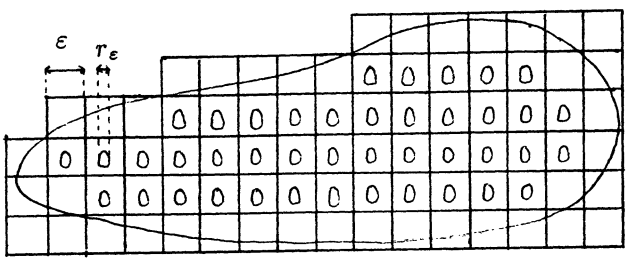

Figure2

Let $\vec{z}_{n\varepsilon}$ be the solution of the Stokes problem in $\sum_\varepsilon$, with boundary conditions equal either to $\vec{0}$ (on $\partial\Omega$ or on the boundary of the pieces of inclusions included in $\sum_\varepsilon$) or $\vec{z}^{\varepsilon}_{\xi(v_{np})}$ (on $\partial\Omega_{np} \cap \partial\sum_\varepsilon$).

One verifies that the sequence $(\varepsilon\|\vec{z}_{n\varepsilon}\|_{H^1(\Sigma_\varepsilon))^N})_\varepsilon$ converges to 0, when ε converges to 0. Then, one deduces, as in Lemma 3.2, that the sequence $(\|\vec{z}_{n\varepsilon}\|_{(L^2(\Sigma_\varepsilon))^N})_\varepsilon$ converges to 0.

Finally, we define : $\vec{v}_{n,\varepsilon,0} = \left| \begin{array}{ll} \vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})} & \text{in } \Omega_{np}\backslash\overline{\Sigma_\varepsilon}, \\ \vec{z}_{n\varepsilon} & \text{in } \Sigma_\varepsilon. \end{array} \right.$

Trivially $\vec{v}_{n,\varepsilon,0}$ belongs to $V(\Omega_\varepsilon)$ and $(P^\varepsilon\vec{v}_{n,\varepsilon,0})_\varepsilon$ converges to $\vec{v}_n$ in the weak topology of $(L^2(\Omega))^N$. The computation of $G^\varepsilon(\vec{v}_{n,\varepsilon,0})$ is quite trivial, since

$$\begin{aligned} G^\varepsilon(\vec{v}_{n,\varepsilon,0}) \leq & \sum_p \frac{\varepsilon^2}{2} \int_{\Omega_{np}} \sum_{ij} (e_{ij}(\vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})}))^2 \, dx \\ & + \varepsilon g \int_{\Omega_{np}} \left(\sum_{ij} (e_{ij}(\vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})}))^2 \right)^{1/2} dx \\ & + \frac{\varepsilon^2}{2} \int_{\Sigma_\varepsilon} \sum_{ij} (e_{ij}(\vec{z}_{n\varepsilon}))^2 \, dx + \varepsilon g \int_{\Sigma_\varepsilon} \left(\sum_{ij} (e_{ij}(\vec{z}_{n\varepsilon}))^2 \right)^{1/2} dx \end{aligned}$$

Hence, from the definition of $\vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})}$ and (18) :

$$\limsup_{\varepsilon\to 0} G^{\varepsilon}(\vec{v}_{n,\varepsilon,0}) \leq \sum_p \int_{\Omega_{np}} j_0(\vec{v}_{np})\,dx = \int_{\Omega} j_0(\vec{v}_n(x))\,dx = G(\vec{v}_n).$$

Therefore :

$$\limsup_{n\to+\infty} \quad \limsup_{\varepsilon\to 0} G^{\varepsilon}(\vec{v}_{n,\varepsilon,0}) \leq G(\vec{v}).$$

The diagonalization argument of Corollary 1.16 [1] implies the existence of a subsequence $(n(\varepsilon))_\varepsilon$ growing to $+\infty$ such that $(P^{\varepsilon}\vec{v}_{n(\varepsilon),\varepsilon,0})_\varepsilon$ converges to $\vec{v}$, in the weak topology of $(L^2(\Omega))^N$, and

$$\limsup_{\varepsilon\to 0} G^{\varepsilon}(\vec{v}_{n(\varepsilon),\varepsilon,0}) \leq G(\vec{v}).$$

(15) is proved, taking $\vec{v}_{\varepsilon,0} = \vec{v}_{n(\varepsilon),\varepsilon,0}$.

Verification of (16) with : $G(\vec{v}) = \displaystyle\int_{\Omega} j_0(\vec{v}(x))\,dx$.

Choose any $\vec{v}$ in $H(\Omega)$. Let $(\vec{v}_n)_n$ be any sequence of piecewise constant functions in Ω, converging to $\vec{v}$ in the strong topology of $(L^2(\Omega))^N$: $\vec{v}_n = \vec{v}_{np}$ in Ω_{np} with $\Omega\backslash\overline{\bigcup_p \Omega_{np}}$ negligible.

Suppose that $(P^{\varepsilon}\vec{v}_\varepsilon)_\varepsilon$ converges to $\vec{v}$ in the weak topology of $(L^2(\Omega))^N$, with $\vec{v}_\varepsilon$ in $V(\Omega_\varepsilon)$. We may suppose that :

$$\limsup_{\varepsilon\to 0} \quad G^{\varepsilon}(\vec{v}_\varepsilon) < +\infty,$$

otherwise, (16) is trivially true.

For every p, we choose a smooth function $\emptyset_{np}$ in C_0^∞ (Ω_{np}) satisfying $0 \leq \emptyset_{np} \leq 1$, and we write :

$$\begin{aligned} G^{\varepsilon}(\vec{v}_\varepsilon) \geq \sum_p \Bigg\{ & \frac{\varepsilon^2}{2} \int_{\Omega_{np}} \sum_{ij} (e_{ij}(\vec{v}_\varepsilon))^2\, \emptyset_{np}\,dx \\ & + \varepsilon g \int_{\Omega_{np}} (\sum_{ij} (e_{ij}(\vec{v}_\varepsilon))^2)^{1/2}\, \emptyset_{np}\,dx \Bigg\} \\ \geq \sum_p \Bigg\{ & \frac{\varepsilon^2}{2} \int_{\Omega_{np}} \sum_{ij} (e_{ij}(\vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})}))^2\, \emptyset_{np}\,dx \end{aligned}$$

$$
\begin{aligned}
&+ \varepsilon g \int_{\Omega_{np}} (\sum_{ij} (e_{ij}(\vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})}))^2)^{1/2} \, \emptyset_{np} \, dx \\
&+ \varepsilon^2 \int_{\Omega_{np}} \sum_{ij} e_{ij}(\vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})}) e_{ij}(\vec{v}_\varepsilon - \vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})}) \, \emptyset_{np} \, dx \\
&+ \varepsilon g \int_{\Omega_{np}} (\sum_{ij} (e_{ij}(\vec{v}_\varepsilon))^2)^{1/2} \, \emptyset_{np} \, dx \\
&- \varepsilon g \int_{\Omega_{np}} (\sum_{ij} (e_{ij}(\vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})}))^2)^{1/2} \, \emptyset_{np} \, dx \Bigg\} .
\end{aligned}
$$

From the definition of $\vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})}$, the smoothness of $\emptyset_{np}$ and the definition of j_0 (18), we derive :

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \Bigg(\sum_p \frac{\varepsilon^2}{2} \int_{\Omega_{np}} \sum_{ij} (e_{ij}(\vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})}))^2 \, \emptyset_{np} \, dx \\
+ \varepsilon g \int_{\Omega_{np}} (\sum_{ij} (e_{ij}(\vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})}))^2)^{1/2} \, \emptyset_{np} \, dx \Bigg) \\
= \sum_p \int_{\Omega_{np}} j_0(\vec{v}_{np}) \, \emptyset_{np} \, dx .
\end{aligned}
$$

From Lemma 3.4. d) and the smoothness of $\emptyset_{np}$, we derive :

$$
\begin{aligned}
\liminf_{\varepsilon \to 0} \sum_p \Bigg\{ \varepsilon^2 \int_{\Omega_{np}} \sum_{ij} e_{ij}(\vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})}) e_{ij}(\vec{v}_\varepsilon - \vec{z}^{\varepsilon}_{\xi(\vec{v}_{np})}) \, \emptyset_{np} \, dx \\
+ \varepsilon g \int_{\Omega_{np}} (\sum_{ij} (e_{ij}(\vec{v}_\varepsilon))^2)^{1/2} \, \emptyset_{np} \, dx \\
- \varepsilon g \int_{\Omega_{np}} (\sum_{ij} (e_{ij}(\vec{z}^{\varepsilon}_{\xi(v_{np})}))^2)^{1/2} \, \emptyset_{np} \, dx \Bigg\} \\
\geq \sum_p \int_{\Omega_{np}} \xi(\vec{v}_{np}) \, (\vec{v} - \vec{v}_{np}) \, \emptyset_{np} \, dx .
\end{aligned}
$$

Let $\emptyset_{np}$ increase to the characteristic function $\mathcal{X}_{\Omega_{np}}$ of Ω_{np} :

$$
\begin{aligned}
\liminf_{\varepsilon \to 0} G^{\varepsilon}(\vec{v}_\varepsilon) &\geq \sum_p \left\{ \int_{\Omega_{np}} j_0(\vec{v}_{np}) \, dx + \int_{\Omega_{np}} \xi(\vec{v}_{np}) \, (\vec{v} - \vec{v}_{np}) \, dx \right\} . \\
&\geq \int_{\Omega} j_0(\vec{v}_n) \, dx + \int_{\Omega} \xi(\vec{v}_n) \, (\vec{v} - \vec{v}_n) \, dx .
\end{aligned}
$$

Let n increase to $+\infty$: (16) is proved, thanks to the continuity of G, with respect to the strong topology of $(L^2(\Omega))^N$.

Finally, we have proved :

THEOREM 3.5.— *The sequence $(G^\varepsilon)_\varepsilon$ of functionals defined by (14) epi-converges in the weak topology of $(L^2(\Omega))^N$ to the functional G defined on $(L^2(\Omega))^N$ by :*

$$G(\vec{v}) = \int_\Omega j_0(\vec{v}(x))\,dx + \delta_{H(\Omega)}(\vec{v}),$$

where j_0 is defined by (18).

Through Theorem 1.6, we deduce of Theorem 3.5 :

COROLLARY 3.6.— *There exists a function p_1 in $L^2(\Omega)/R$ such that the sequence $\left(\frac{1}{\varepsilon^2} P^\varepsilon \vec{u}_\varepsilon\right)_\varepsilon$ converges in the weak topology of $(L^2(\Omega))^N$ to $A_0(\vec{f}$ -grad $p_1)$*

Moreover,

$$\left(\frac{1}{\varepsilon^2}\int_\Omega \sum_{ij}(e_{ij}(P^\varepsilon\vec{u}_\varepsilon))^2\,dx + \frac{g}{\varepsilon}\int_\Omega (\sum_{ij}(e_{ij}(P^\varepsilon\vec{u}_\varepsilon))^2)^{1/2}\,dx\right)_\varepsilon$$

converges to

$$\int_\Omega \left(\frac{1}{2}\int_{Y\setminus\overline{aT}} \sum_{ij}(e_{ij}(\vec{z}_{\vec{f}-\text{grad } p_1}))^2\,dy + \right.$$
$$\left. + g\int_{Y\setminus\overline{aT}} (\sum_{ij}(e_{ij}(\vec{z}_{\vec{f}-\text{grad } p_1}))^2)^{1/2} dy\right)\,dx.$$

Remark 3.7.— J.L. LIONS and E. SANCHEZ-FALENCIA proved in [10] that p_1 is not unique in $L^2(\Omega)/R$. In fact, if ξ is sufficiently small, $A_0\,\xi$ is equal to 0. (see [10] 4.3). However, two determinations of the limit pressure p_1 provide the same mean velocity (see Appendix 2).

Remark 3.8.— In the case $N = 2$, the construction of the test function $\vec{v}_{\varepsilon,0}$ satisfying (15) may be done in the following way. First, one deduces from Lemma 2.5 of [14], through a trivial change of scale, the following property of Y_ε–periodic, divergence-free functions.

LEMMA 3.9.— *For every Y_ε–periodic, divergence-free function $\widehat{z}$ in $(L^2(Y_\varepsilon^2))$, there exists an Y_ε–periodic function $\widehat{z}$ in $H^1(Y_\varepsilon)$ verifying :*

$$curl\,\widehat{z} = (D_2\,\widehat{z}, -D_1\,\widehat{z}) = \vec{z} \text{ in } Y_\varepsilon,$$

$$\|\widehat{z}\|_{L^2(Y_\varepsilon)} \leq C\varepsilon\|\vec{z}\|_{(L^2(Y_\varepsilon))^2}\,;\ \|\operatorname{grad}\widehat{z}\|_{(L^2(Y_\varepsilon))^2} \leq C\|\vec{z}\|_{(L^2(Y_\varepsilon))^2},$$

where C is a constant independant of ε and $\vec{z}$.

We also need the following restriction operator R^ε :

PROPOSITION 3.10 [13].— *There exists a linear restriction operator R^ε from $(H_0^1(\Omega))^N$ into $(H_0^1(\Omega_\varepsilon))^N$ satisfying :*

a) for every $\vec{u}$ in $(H_0^1(\Omega_\varepsilon))^N$: $R^\varepsilon P^\varepsilon \vec{u} = \vec{u}$ (P^ε is the canonical extension operator),

b) $R^\varepsilon(V(\Omega)) \subset V(\Omega_\varepsilon)$,

b) there exists a constant C independant of ε and $\vec{u}$ in $(H_0^1(\Omega))^N$ such that

$$\|R^\varepsilon\vec{u}\|_{(L^2(\Omega_\varepsilon))^N} \leq C\|\vec{u}\|_{(L^2(\Omega))})^N + C\varepsilon\|\operatorname{grad}\vec{u}\|_{(L^2(\Omega))^{N^2}}$$

$$\|\operatorname{grad}\ R^\varepsilon\vec{u}\|_{(L^2(\Omega_\varepsilon))^{N^2}} \leq \frac{C}{\varepsilon}\|\vec{u}\|_{(L^2(\Omega))^N} + C\|\operatorname{grad}\vec{u}\|_{(L^2(\Omega))^{N^2}}$$

And then we define :

$$\vec{v}_{\varepsilon,0} = R^\varepsilon\left(\operatorname{curl}(\sum_p \widehat{z}^\varepsilon_{\xi(\vec{v}_{np})}\chi\Omega_{np}\ \emptyset_{n\varepsilon})\right),$$

where $\widehat{z}^\varepsilon_{\xi(\vec{v}_{np})}$ is the function associated to the Y_ε–periodic, divergence-free function $\widehat{z}^\varepsilon_{\xi(v_{np})}$, by Lemma 3.9 and $\emptyset_{n\varepsilon}$ is a smooth function in $C_0^1(\overline{\Omega})$ and identically equal to 1 in $\Omega\backslash\overline{\Sigma}_\varepsilon$.

IV – Convergence of the internal pressure when $r_\varepsilon = a\,\varepsilon(0 < a < 1/2)$

THEOREM 4.1.— *There exists an extension $\overline{p}_\varepsilon$ of the internal pressure of the Bingham fluid, such that $(\overline{p}_\varepsilon)_\varepsilon$ converges, in the strong topology of $L^2(\Omega)/R$.*

Proof of Theorem 4.1.

Through Proposition 3.10b), define (as in [13]) grad $\overline{p}_\varepsilon$ the element of $(H^{-1}(\Omega))^N$ satisfying :

$$< \operatorname{grad} \overline{p}_\varepsilon, \vec{u} >=< \operatorname{grad} p_\varepsilon,\ R^\varepsilon \vec{u} >$$

for every $\vec{u}$ in $(H_0^1(\Omega))^N$.

From (13), one deduces the following estimate :

$$\| \operatorname{grad}\ p_\varepsilon \|_{H^{-1}(\Omega_\varepsilon))^N} \leq C\varepsilon.$$

Then Proposition 3.10c) implies the boundedness of (grad $\overline{p}_\varepsilon)_\varepsilon$ in $(H^{-1}(\Omega))^N$.

As in [13], we then prove the convergence of $(\overline{p}_\varepsilon)_\varepsilon$, in the strong topology of the quotient space $L^2(\Omega)/R$, to some limit point p_2. We conjecture that this limit point p_2 is, in fact, one "limit pressure" appearing in the limit law established in Corollary 3.6 (see Remark 3.7).

APPENDIX 1 : *The proof of Proposition 2.3.*

Let us first mention the following property of the divergence-free functions (which is an immediate consequence of Theorem 4.9 of [11], see also [14]) :

PROPOSITION A.1.— *a) Suppose $N = 3$: for every divergence-free function $\vec{w}$ in $(L^2(B(r)))^3$, there exists a divergence-free function $\widehat{\vec{w}}$ in $(H^1(B(r)))^3$ satisfying*

$\widehat{\vec{w}} \cdot \vec{n} = 0$ *on $\partial B(r)$ ($\vec{n}$ is the outer normal to $\partial B(r)$),*

curl $\widehat{\vec{w}} = \vec{w}$ in $B(r)$,

$\|\widehat{\vec{w}}\|_{(L^2(B(r)))^3} \leq C\, r \|\vec{w}\|_{(L^2(B(r)))^3}$,

$\| \operatorname{grad} \widehat{\vec{w}}\|_{(L^2(B(r)))^9} \leq C\|\vec{w}\|_{(L^2(B(r)))^3}$,

b) Suppose $N = 2$: for every divergence-free function $\vec{w}$ in $(L^2(B(r)))^2$, there exists a function $\widehat{w}$ in $H^1(B(r))$ satisfying :

$\widehat{w} = 0$ *on $\partial B(r)$,*

curl $\widehat{w} = w$ in $B(r)$,

$\|\widehat{w}\|_{L^2(B(r))} \leq C\, r \|\vec{w}\|_{(L^2(B(r)))^2}$,

$\| \operatorname{grad} \widehat{w}\|_{(L^2(B(r)))^2} \leq C\|\vec{w}\|_{(L^2(B(r)))^2}$,

where C is a constant independant of $r(r > 0)$ and $\vec{w}$.

Let us now prove the Proposition 2.3. Suppose that T is equal to $B(1)$ and define

$$\vec{v}_\varepsilon^0(x) = \begin{cases} \vec{v}(x) - \sum_{k=1}^N (\vec{e}_k - \vec{w}_\varepsilon^k(x)) v_k(x_{\varepsilon i}) - rot(\widehat{\vec{v}}_{\varepsilon i}\emptyset_{\varepsilon i})(x) \text{in } B^i(\varepsilon/4), \\ \vec{v}(x) \qquad\qquad\qquad\qquad\qquad\qquad \text{in } Y_{\varepsilon i}\backslash B^i(\varepsilon/4) \end{cases} \quad (19)$$

where $x_{\varepsilon i}$ is the center of $Y_{\varepsilon i}$ and $B^i(\varepsilon/4)$ (the i–thε cell and the i–th ball), $\widehat{\vec{v}}_{\varepsilon i}$ is the divergence-free function associated to $\vec{v}(\cdot) - \vec{v}(x_{\varepsilon i})$ in $B^i(r_\varepsilon^{2/3})$, by Proposition A.1, $\emptyset_{\varepsilon i}$ is a cut-off function in $C^\infty(\Omega; [0,1])$ whose support is contained in $B^i\left(r_\varepsilon^{2/3}\right)$ and identically equal to 1 in $B^i(r_\varepsilon)$.

In the present case, the solution $(\vec{w}_\varepsilon^k, q_\varepsilon^k)$ of the minimization problem (4) is computable in terms of radial functions (see [5]). The assertions of Proposition 2.3 are immediate consequences of these computations.

When T is a smooth open subset of the unit ball $B(1)$ of R^N, one has to modify the test functions $\vec{v}_\varepsilon^0$, given in (19), in order to use the following pointwise estimates on the solution $(\vec{w}_\varepsilon^k, q_\varepsilon^k)$ of (4) :

PROPOSITION A.2 ([11] Lemma 2.2).— *for every $s_\varepsilon(s_\varepsilon < \varepsilon)$ and c $(c > 0)$, there exists a constant C such that*

if $N = 3$: for every x in $B(s_\varepsilon)$ verifying $d(x, T_\varepsilon) > c\, r_\varepsilon$

$$|\vec{e}_k - \vec{w}_\varepsilon^k(x)| \le C\frac{r_\varepsilon}{d(x,T_\varepsilon)} \ ; \ |D_i\vec{w}_\varepsilon^k(x)| \le C\frac{r_\varepsilon}{d(x,T_\varepsilon)^2};$$
$$|D_iD_j\vec{w}_\varepsilon^k(x)| \le C\frac{r_\varepsilon}{d(x,T_\varepsilon)^3} \ ; \ |q_\varepsilon^k(x)| \le C\frac{r_\varepsilon}{d(x,T_\varepsilon)^2}$$

if $N = 2$: for every x in $B(s_\varepsilon)$ verifying $d(x, T_\varepsilon) > \ exp\,(-c|\log r_\varepsilon|^{1/2})$

$$|\vec{e}_k - \vec{w}_\varepsilon^k(x)| \le C\frac{|\log d(x,T_\varepsilon)| + 1}{|\log r_\varepsilon|} \ ; \ |D_i\vec{w}_\varepsilon^k(x)| \le C\frac{1}{|\log r_\varepsilon| d(x,T_\varepsilon)};$$
$$|D_iD_j\vec{w}_\varepsilon^k(x)| \le C\frac{1}{|\log r_\varepsilon| d(x,T_\varepsilon)^2} \ ; \ |q_\varepsilon^k(x)| \le C\frac{1}{|\log r_\varepsilon| d(x,T_\varepsilon)}.$$

Define :

$$\vec{v}_\varepsilon^0(x) = \begin{cases} \vec{v}(x) - \sum_k \operatorname{rot}(\widehat{\vec{w}}_{\varepsilon,i}^k \psi_{\varepsilon i})(x) v_k(x_{\varepsilon i}) - \operatorname{rot}(\widehat{\vec{v}}_{\varepsilon i}\emptyset_{\varepsilon i})(x) \text{ in } B^i(\varepsilon/4), \\ \vec{v}(x) \text{ in } Y_{\varepsilon i}\backslash B^i(\varepsilon/4), \end{cases}$$

where $\psi_{\varepsilon i}$ is a cut-off function whose support is contained in $B^i\left(\frac{\varepsilon}{4} - \frac{r_\varepsilon}{2}\right)$ and identically equal to 1 in $B^i\left(\frac{\varepsilon}{8} - \frac{r_\varepsilon}{2}\right)$,

$\widehat{\vec{w}}^k_{\varepsilon,i}$ is associated to the divergence-free function $\vec{e}_k - \vec{w}^k_\varepsilon$ in $B^i\left(\frac{\varepsilon}{4} - \frac{r_\varepsilon}{2}\right)$ by Proposition A.1.

The assertions of Proposition 2.3 are quite immediate consequences of these pointwise estimates.

APPENDIX 2 : *Proof of Lemma 3.4*

Let us first prove some properties of the function A_0 :

1) A_0 is monotone in the sense that for every ξ_1 and ξ_2 in R^N :

$$(A_0\xi_1 - A_0\xi_2)\cdot(\xi_1 - \xi_2) \geq 0.$$

From (17) we derive :

$$\int_{Y\setminus\overline{aT}} \sum_{ij} (e_{ij}(\vec{z}_{\xi_1}))e_{ij}(\vec{z}_{\xi_2} - \vec{z}_{\xi_1})\,dy + g\int_{Y\setminus\overline{aT}} (\sum_{ij}(e_{ij}(\vec{z}_{\xi_2}))^2)^{1/2}\,dy$$
$$-g\int_{Y\setminus\overline{aT}} (\sum_{ij}(e_{ij}(\vec{z}_{\xi_1}))^2)^{1/2}\,dy \geq \xi_1\cdot\left(\widetilde{\vec{z}}_{\xi_1} - \widetilde{\vec{z}}_{\xi_2}\right) \tag{20}$$

and a similar inequality where ξ_1 and ξ_2 are exchanged. Add these two inequalities :

$$(A_0\xi_1 - A_0\xi_2)\cdot(\xi_1 - \xi_2) \geq \int_{Y\setminus\overline{aT}} \sum_{ij}(e_{ij}(\vec{z}_{\xi_1} - \vec{z}_{\xi_2}))^2\,dy. \tag{21}$$

and the monotonicity of A_0 is proved.

2) A_0 is continuous. This property of A_0 may be proved through an epi-convergence argument : let H_ξ be the functional associated to (17).

Let $(\xi_n)_n$ be any sequence converging to ξ in R^N. One trivially proves that $(H_{\xi_n})_n$ epi-converges to H_ξ in the strong topology of $(H^1(Y))^N$.

Theorem 1.6 then implies that $(\vec{z}_{\xi_n})_n$ converges to $\vec{z}_\xi$ in this topology.

Hence, $(A_0(\xi_n))_n$ converges to $A_0\xi$.

3) $\lim\limits_{|\xi|\to+\infty} A_0\xi\cdot\xi/|\xi| = +\infty.$

We introduce $\vec{z}^*_\xi$ the solution of the following minimization problem :

$$\text{Min} \qquad \left(\frac{1}{2}\int_{Y\setminus\overline{aT}}\sum_{ij}(e_{ij}(\vec{z}))^2\,dy - \xi\cdot\widetilde{\vec{z}}\right). \tag{22}$$

$\vec{z}$ Y–periodic
$\vec{z} = \vec{0}$ on aT
div $\vec{z} = 0$ in Y

Notice that $\vec{z}^*_\xi$ depends linearly on ξ, that is : $\vec{z}^*_\xi = \sum_k \xi_k\,\vec{z}^*_k$, where $\vec{z}^*_k$ is the solution of (22) with $\xi = \vec{e}_k$.

From (17), one derives

$$\frac{1}{2}\int_{Y\setminus\overline{aT}}\sum_{ij}(e_{ij}(\vec{z}_\xi))^2dy + g\int_{Y\setminus\overline{aT}}(\sum_{ij}(e_{ij}(\vec{z}_\xi))^2)^{1/2}dy - \xi\cdot\widetilde{\vec{z}}_\xi \leq$$
$$\leq \frac{1}{2}\int_{Y\setminus\overline{aT}}\sum_{ij}(e_{ij}(\vec{z}^*_\xi))^2dy + g\int_{Y\setminus\overline{aT}}(\sum_{ij}(e_{ij}(\vec{z}^*_\xi))^2)^{1/2}dy - \xi\cdot\widetilde{\vec{z}}^*_\xi.$$

Hence, from Lemma 3.4 d), which has been proved by Lions and Sanchez-Palencia in [10], and from (22), one derives :

$$\frac{1}{2}\int_{Y\setminus\overline{aT}}\sum_{ij}(e_{ij}(\vec{z}_\xi))^2dy \leq -\frac{1}{2}\int_{Y\setminus\overline{aT}}\sum_{ij}(e_{ij}(\vec{z}^*_\xi))^2dy +$$
$$+g\int_{Y\setminus\overline{aT}}(\sum_{ij}(e_{ij}(\vec{z}^*_\xi))^2)^{1/2}dy$$

And finally, using (21)

$$A_0\xi\cdot\xi \geq \int_{Y\setminus\overline{aT}}\sum_{ij}(e_{ij}(\vec{z}^*_\xi))^2dy - 2g\int_{Y\setminus\overline{aT}}(\sum_{ij}(e_{ij}(\vec{z}^*_\xi))^2)^{1/2}dy. \tag{23}$$

These three properties of A_0 prove that A_0 is maximal monotone and onto (see [3]), and Lemma 3.4 a) is proved.

Lemma 3.4 b) is a consequence of (17), since $\widetilde{\vec{z}}_{\xi_1} = \widetilde{\vec{z}}_{\xi_2}$.

Lemma 3.4 c) is a consequence of (23) and following inequality which is deduced from (22) :

$$\frac{1}{2}\int_{Y\setminus\overline{aT}}\sum_{ij}(e_{ij}(\vec{z}^*_\xi))^2dy - \xi\cdot\widetilde{\vec{z}}^*_\xi \leq \frac{1}{2}\int_{Y\setminus\overline{aT}}\sum_{ij}(e_{ij}(\vec{z}_\xi))^2dy - \xi\cdot\widetilde{\vec{z}}_\xi$$
$$\leq \frac{1}{2}A_0\xi\cdot\xi - A_0\xi\cdot\xi \quad \text{(see (21))}.$$

ACKNOWLEDGMENT

Let me express my thanks to the referee who pointed to my attention the necessity of the discussion presented in the Remark 2.5.

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(Manuscrit reçu le 5 janvier 1988)