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Families of polynomials and determining measures

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RÉSUMÉ. — Soit μ une mesure de probabilité sur une partie borélienne bornée non pluripolaire E de $\mathbf{C}^N$, on étudie l'allure de croissance des familles de polynômes ponctuellement bornées μ-presque partout sur E. On définit une fonction $\mathcal{M}(t; E, \mu)$ $(0 \leq t \leq 1)$ associée au couple (E, μ). Sous des hypothèses naturelles sur E et μ, on montre que $\mathcal{M}(1; E, \mu) = 1$ si et seulement si le couple (E, μ) satisfait à la condition polynomiale $(\mathcal{L}^*)$, généralisant la condition polynomiale de Leja dans le cas plan, si et seulement si μ est une mesure déterminante pour E par rapport à la fonction L-extrémale L_E^*.

ABSTRACT. — Given a probability measure μ on a bounded nonpluripolar Borel subset E of $\mathbf{C}^N$, we study the growth behaviour of polynomial families which are pointwise bounded μ-a.e. on E. We define a function $\mathcal{M}(t, E, \mu)$ $(0 \leq t \leq 1)$ associated to the pair (E, μ). Under natural assumptions on E and μ we prove that $\mathcal{M}(1; E, \mu) = 1$ if and only if the pair (E, μ) satisfies the polynomial condition $(\mathcal{L}^*)$ (a generalization of the Leja's condition in the plane), if and only if μ is determining for E with respect to the L-extremal function L_E^*.

0 - Introduction

Given a domain Ω in $\mathbf{C}^N$, we denote by $P(\Omega)$ the class of plurisubharmonic (plsh) functions on Ω. Let

$$\mathcal{L} := \{u \in P(\mathbf{C}^N); u(z) \leq \beta + \log(1 + |z|) \text{ in } \mathbf{C}^N\},$$

where β is a real constant depending on u. For a bounded set E in $\mathbf{C}^N$ define

$$L_E(z) := \sup\{u(z); u \in \mathcal{L},\ u \leq 0 \text{ on } E\}. \tag{0.1}$$

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The uppersemicontinuous regularization $L_E^*(z) := \limsup_{w\to z} L_E(w)$ is called the L-extremal function of E. It is known that if E is a compact subset of $\mathbf{C}$ with positive logaritmic capacity then L_E is identical with the Green function for $\mathbf{C}\backslash\widehat{E}$ with pole at infinity.

For a bounded set E in $\mathbf{C}^N$ either $L_E^* \equiv \infty$, in which case E is pluripolar (plp), or $L_E^* \in \mathcal{L}$.

DEFINITION 0.1. — *We say that a point a in $\mathbf{C}^N$ is an $\mathcal{L}$-regular point of $E \subset \mathbf{C}^N$, if $L_E^*(a) = 0$. A point $a \in \mathbf{C}^N$ such that $L_E(a) = 0$ and $L_E^*(a) > 0$ is called irregular point of E. It is clear that L_E^* is continuous at each regular point. By Bedford-Taylor theorem on negligible sets the set of irregular points of any subset E of $\mathbf{C}^N$ is plp. If E is a compact set and $L_E = L_E^*$ on E (i.e. if L_E^* is continuous at each point of E) then L_E is continuous in $\mathbf{C}^N$ and $L_E = L_E^*$. A compact set E with $L_E^* = L_E$ is called $\mathcal{L}$-regular. The set of $\mathcal{L}$-regular points of a compact $\mathcal{L}$-regular set E is identical with the polynomially convex hull $\widehat{E}$ of E.*

DEFINITION 0.2. — *A finite positive Borel measure μ on a bounded Borel set E in $\mathbf{C}^N$ is called determining for E, if for every Borel subset F of E with $\mu(F) = \mu(E)$ one has $L_F^* = L_E^*$.*

Observe that if $L_E^* = L_E$ and μ is determining for E, then for every $F \subset E$ with $\mu(F) = \mu(E)$ one has $L_F = L_E$ (because $L_F^* = L_E^* = L_E \leq L_F$).

It is known that $L_{E\cup A}^* = L_E^*$, if A is plp. Therefore $L_E^* = L_F^*$ for a subset F of E if and only if $L_F^* = 0$ quasi-almost everywhere (q.a.e.) on E. We say that a property $\mathcal{P}$ holds q.a.e. on E, if it holds for each point of E except at most of a plp subset of E.

We say that a property $\mathcal{P}$ holds quasi-star-almost-everywhere (q^*.a.e.) on E, if it holds for each point of a subset F of E with $L_F^* = L_E^*$.

It is clear that if μ is determining for E and $\mathcal{P}$ holds μ.a.e. on E then it holds q^*.a.e. on E.

DEFINITION 0.3. — *Let μ be a finite positive Borel measure on a bounded Borel set in $\mathbf{C}^N$. We say that the pair (E, μ) satisfics $(\mathcal{L}^*)$-condition at a point a of $\mathbf{C}^N$, if for every family $\mathcal{F}$ of polynomials of N-complex variables and for every number $b > 1$ the polynomial family*

$$\mathcal{F}_b := \{b^{-deg\ f} f; f \in \mathcal{F}\} \tag{0.2}$$

is uniformly bounded on a neighborhood $\mathcal{U}$ of a.

We say that the pair (E, μ) satisfies $(\mathcal{L}^*)$-condition, if for every $b > 1$ and for every polynomial family $\mathcal{F}$ bounded μ-a.e. on E the family $\mathcal{F}_b$ is uniformly bounded on a neighborhood of E.

It is clear that if E is compact then (E, μ) satisfies $(\mathcal{L}^*)$-condition, if and only if it satisfies $(\mathcal{L}^*)$ at each point of E.

All these notions are important for applications of the extremal function L_E^*. There are strict relations between them. Also are known important examples of pairs (E, μ) satisfying $(\mathcal{L}^*)$ and of determining measures (e.g. [2], [5], [6], [7], [8], [14]).

In this paper we introduce a new function $\mathcal{M}(t) \equiv \mathcal{M}(t; E, \mu)$ associated to every pair (E, μ) by the formula

$$\log \mathcal{M}(t) := \sup \big\{ \sup_E L_A ; A \subset E,\ \mu(A) \geq t\mu\ (E) \big\},\ \ 0 \leq t \leq 1.$$

It is clear that $\mathcal{M}$ is a decreasing function and $1 \leq \mathcal{M}(t) \leq +\infty$. The function $\mathcal{M}^*(t) := \lim_{\tau \uparrow t} \mathcal{M}(\tau)$ $(0 < t \leq 1)$, $\mathcal{M}^*(0) := \mathcal{M}(0)$, is decreasing and uppersemicontinuous on $[0, 1]$.

In the sequel we shall often assume (without loss of generality) that μ is a probability measure (i.e. $\mu(E) = 1$).

The function $\mathcal{M}$ appears to be a useful notion strictly related to the determining measures and the $(\mathcal{L}^*)$-condition. For example we have obtained the following results involving the function $\mathcal{M}$.

THEOREM A.— *If $E \subset \mathbf{C}^N$ is compact and μ vanishes on plp subsets of $\mathbf{C}^N$ then the following conditions are equivalent*

(i) The pair (E, μ) satisfies $(\mathcal{L}^*)$-condition;

(ii) If $u \in \mathcal{L}$ and $u \leq 0$ μ-a.e. on E, then $u \leq 0$ on E;

(iii) $\mathcal{M}^*(1) = 1$;

(iv) $\mathcal{M}(1) = 1$;

(v) μ is determining for E and E is $\mathcal{L}$-regular.

THEOREM B.— *Let $A \subset \mathbf{C}^P$, $B \subset \mathbf{C}^Q$ be two bounded Borel sets and μ, ν two probability measures on A and B, respectively. Put $\mathcal{M}_A(t) := \mathcal{M}(t; A, \mu), \mathcal{M}_B(t) := \mathcal{M}(t; B, \nu)$ and $\mathcal{M}_{A \times B}(t) := \mathcal{M}(t, A \times B, \mu \otimes \nu)$. Then*

(i) $\mathcal{M}_{A \times B}(1) \leq \mathcal{M}_A(1)\mathcal{M}_B(1)$

(ii) $\mathcal{M}^*_{A\times B}(1) \le \mathcal{M}^*_A(1)\mathcal{M}^*_B(1)$

COROLLARY .— *If $A \subset \mathbf{C}^P$, $B \subset \mathbf{C}^P$ are compact sets and μ, ν vanish on plp sets, then if the pairs (A,μ), (B,ν) satisfy one of the equivalent conditions of Theorem A then the pair $(A \times B,\ \mu \otimes \nu)$ satisfies each of the conditions.*

The equivalence of the conditions (i) and (v) was earlier obtained by LEVENBERG [6]. NGUYEN THANH VAN formulated the $(\mathcal{L}^*)$-condition in his paper [7]; his definition was inspired by LEJA's paper [5] containing as a special case so called "Polynomial Lemma", which in the present language reads as follows :

Let Γ be a rectifiable curve in the complex plane and let λ be the lenghth measure on Γ. Then the pair (Γ,μ) satisfies $(\mathcal{L}^*)$.

It is worthwhile to mention that the LEJA's paper [5] permits immediately to obtain the following estimate for the function $\mathcal{M}(t) \equiv \mathcal{M}(t;[A,b],\lambda)$:

$$\mathcal{M}(t) \le \mathcal{J}\left(\sqrt{\frac{1-t}{t-9/10}}\right),\ 9/10 < t < 1,$$

where

$$\mathcal{J}(\alpha) := \exp\int_0^1 \log\frac{x^2+\alpha^2}{x^2}\,dx \le \exp\alpha(\pi+\alpha).$$

The exact formula for the function $\mathcal{M}(t;[a,b],\lambda)$, where $[a,b]$ is a bounded interval of the real line $\mathbf{R}$ and λ is the Lebesgue measure on $\mathbf{R}$, reads as follows

$$\mathcal{M}(t;[a,t],\lambda) = 2t^{-1} - 1 + 2t^{-1}\sqrt{1-t}, \quad 0 \le t \le 1,$$

and may be easily derived from the following inequality due to DUDLEY and RANDOL [4]

$$\|f\|_{[a,b]}/\|f\|_A \le (2^{-1} - 1 + 2t^{-1}\sqrt{1-t})^{\deg f}$$

true for every polynomial f of a complex variable and for every compact set $A \subset [a,b]$ with $\lambda(A) \ge t(b-a)$, $0 \le t \le 1$.

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1 - Determining measures for arbitrary bounded Borel subsets of $\mathbf{C}^N$

Let us start with the following.

LEMMA 1.1.— *Let F be a subset of a bounded set E in $\mathbf{C}^N$. If the set*

$$G := \{z \in E \backslash F; L_F^*(z) > 0\}$$

is not plp, then there exist a nonplp subset G_o of G, a number $b > 1$ and a polynomial family $\mathcal{F}$ such that

1) $\mathcal{F}$ is bounded at each point of F.

2) $\mathcal{F}_b$ given by (0.2) is unbounded at each point $z \in G_o$.

Proof.— It is known that $F_1 := \{z \in F; L_F^*(z) > 0\}$ is plp, so there exists a function w in the class $\mathcal{L}$ with $w = -\infty$ on F_1 and $w \leq -\log 2$ on E. It is also known [10] that w can be represented in the form

$$w = \left(\limsup_{m\to\infty} \frac{1}{m} \log |P_m|\right)^*, \tag{1.1}$$

where P_m is a polynomial on $\mathbf{C}^N$ of degree $\leq m$. We shall consider two cases : either $F_1 = F$, or $F_1 \neq F$.

Case $F_1 = F$. By Bedford-Taylor theorem on negligible sets [1] the set

$$\left\{\limsup_{m\to\infty} \sqrt[m]{|P_m|} < \left(\limsup_{m\to\infty} \sqrt[m]{|P_m|}\right)^*\right\}$$

is plp. Hence there exists a non pluripolar subset G' of G such that

$$-\infty < w(z) = \limsup_{m\to\infty} \frac{1}{m} \log |P_m(z)| \text{ for } z \in G'.$$

There is a real number ϵ with $0 < \epsilon < 1$ such that the set $G_o := \{x \in G'; w(z) \geq \log \epsilon\}$ is not plp. Take any b with $1 < b < 2$. Then the family

$$\mathcal{F} := \left\{\left(\frac{2}{\epsilon}\right)^m P_m; m \geq 1\right\}$$

has the required properties. Indeed, 1) is satisfied because

$$\limsup_{m\to\infty} \sqrt[m]{\left(\frac{2}{\epsilon b}\right)^m |P_m(z)|} = 0 \text{ on } F.$$

If $z \in G_o$, we have

$$\limsup_{m\to\infty} \sqrt[m]{\left(\frac{2}{\epsilon b}\right)^m |P_m(z)|} = \frac{2}{\epsilon b} \exp w(z) > \frac{2}{b} > 1, \text{ which implies 2).}$$

Case $F_1 \neq F$. Since $F_1 \neq F$, we have $L_F^* \in \mathcal{L}$ and $u_k := \frac{1}{k}\, w + \frac{k-1}{k}\, L_F^* \in \mathcal{L}$ for every $k \geq 1$. If $z \in G$ and $w(z) > 0$ the sequence $u_k(z)$ is increasing to the limit $L_F^*(z) > 0$. Therefore there exists k such that the set $G_k := \{z \in G; u_k(z) > 0\}$ is not plp. For such k there is $\epsilon > 0$ such that

$$G' := \{z \in G; u_k(z) \geq \log(1+\epsilon)\}$$

is not plp. Write u_k in the form

$$u_k = \left(\limsup_{j\to\infty} \frac{1}{j}\, \log|P_j|\right)^* \quad (\deg P_j \leq j)$$

By the theorem on negligible sets there is a non pluripolar subset G_o of G' with

$$u_k(z) \equiv \frac{1}{k}\, w(z) + \frac{k-1}{k}\, L_F^*(z) = \limsup_{j\to\infty} \frac{1}{j}\, \log|P_j(z)|, z \in G_o.$$

The set G_o, any number b with $1 < b < 1+\epsilon$ and the polynomial family $\mathcal{F} := \{P_j; j \geq 1\}$ have the required property. Indeed $\limsup \sqrt[j]{|P_j(z)|} \leq \exp u_k(z) \leq 2^{-k} < 1$ on F, which gives 1). On the other hand, if $z \in G_o$ then

$$\limsup_{j\to\infty} \sqrt[j]{b^{-j}|P_j(z)|} = b^{-1} \exp u_k(z) \geq \frac{1+\epsilon}{b} > 1,$$

which implies 2).

LEMMA 1.2.— *If a polynomial family $\mathcal{F}$ is bounded q^*.a.e. on a subset E of $\mathbf{C}^N$, then for every $b > 1$ the family $\mathcal{F}_b$ is bounded q.a.e. on E and uniformly on a neighborhood of every $\mathcal{L}$-regular point a of E. If E is compact and $\mathcal{L}$-regular, and $\mathcal{F}$ is bounded q^*.a.e. on E then for each $b > 1$ the family $\mathcal{F}_b$ is uniformly bounded on a neighborhood of E.*

Proof.— Without loss of generality we can assume E is not plp. Let $\mathcal{F}$ be a polynomial family bounded at each point of a subset F of E with $L_F^* = L_E^*$. Put

$$E_j := \{z \in E; |f(x)| \leq j,\ \forall f \in \mathcal{F}\},\ j \geq 1 \tag{1.2}$$

Then $E_j \subset E_{j+1}$ and $F \subset E_o := \bigcup_1^\infty E_j$. Hence $L^*_{E_j} \downarrow L^*_{E_o} = L^*_F = L^*_E$. By the definition of the $\mathcal{L}$-extremal function we have

$$|f(z)| \leq j\big(\exp L^*_{E_j}(z)\big)^{\deg f}, \; z \in \mathbf{C}^N, \; j \geq 1, \; f \in \mathcal{F} \tag{1.3}$$

which implies that for each $b > 1$ the family $\mathcal{F}_b$ is bounded at every $\mathcal{L}$-regular point of E. So $\mathcal{F}_b$ is bounded q.a.e. on E. Moreover, if $L^*_E(a) = 0$, then $L^*_E(z) < b$ on a ball $|z-a| \leq r$. By Dini's argument there is j sufficiently large with $L^*_{E_j}(z) < b$ on the ball $|z-a| \leq r$, which implies by (1.3) that the family F_b is uniformly bounded on a ball $|z-a| < r$, if a is any $\mathcal{L}$-regular point of E. The proof of the remaining part of Lemma 1.2. is trivial.

THEOREM 1.3.— *Given a probability measure on a bounded Borel set E in $\mathbf{C}^N$ the following conditions are equivalent*

- *I. The measure μ is determining for E;*
- *II. If $u \in \mathcal{L}$ and $u \leq 0$ μ-a.e. on E, then $u \leq 0$ q.a.e. on E;*
- *III. If $\mathcal{F}$ is a polynomial family bounded μ-a.e. on E, then for every $b > 1$ the family $\mathcal{F}_b$ is bounded q.a.e. on E.*

Proof. $I \Rightarrow II$. Let u be a fixed function in the class $\mathcal{L}$ with $u \leq 0$ μ-a.e. on E. Put $F := \{z \in E; u(x) \leq 0\}$. Then $u(z) \leq L^*_F(z) = L^*_E(z)$. Hence $u \leq 0$ q.a.e. on E.

$I \Rightarrow III$. Let $\mathcal{F}$ be a polynomial family bounded μ.a.e. on E. Let E_j be given by (1.2). Then $E_j \subset E_{j+1}$ and $\mu(F) = \mu(E)$ for $F := \cup_1^\infty E_j$. By I $L^*_F = L^*_E$. It is known [12] that $L^*_{E_j} \downarrow L^*_F$ as $j \to \infty$. Hence by (1.3) the family $\mathcal{F}$ is bounded q*.a.e. on E, and by Lemma 1.2. the family $\mathcal{F}_b$ is bounded q.a.a. on E for every $b > 1$.

The implication $III \Rightarrow I$ follows directly from Lemma 1.1.

It remains to show that $II \Rightarrow I$. Fix $F \subset E$ with $\mu(F) = \mu(E)$ and let u be a function of the class $\mathcal{L}$ such that $u \leq 0$ on F. Then $u \leq 0$ q.a.e. on E. Hence $u \leq L^*_E$ in $\mathbf{C}^N$. By the arbitrariness of u we get $L^*_F \leq L^*_E$, which gives $L^*_F = L^*_E$, because $L_E \leq L_F$.

2 - The function $\mathcal{M}(t; E, \mu)$

Given a probability measure μ on a bounded Borel set E in $\mathbf{C}^N$ the function $\mathcal{M}$ is defined by the formula

$$\log \mathcal{M}(t) := \sup\Big\{\sup_E L_A; A \subset E, \; \mu(A) \geq t\Big\}, \; 0 \leq t \leq 1 \tag{2.1}$$

It is clear that $1 \le \mathcal{M}(t_2) \le \mathcal{M}(t_1) \le +\infty$ if $0 \le t_< t_2 \le 1$. The function $\mathcal{M}^*(t) := \limsup_{\tau\to t} \mathcal{M}(\tau)$ is also decrasing. It follows from (0.1) that

$$\log \mathcal{M}(t) = \sup\left\{\sup_E u; u \in \mathcal{L},\ u \le 0 \text{ on } A,\ A \subset E,\ \mu(A) \ge t\right\}, \qquad (2.2)$$

which implies

$$\sup_E u - \sup_A u \le \log \mathcal{M}\big(\mu(A)\big), \text{ if } u \in \mathcal{L},\ A \subset E, \qquad (2.3)$$

$$L_A(z) \le \log \mathcal{M}\big(\mu(A)\big) + L_E(z),\ z \in \mathbf{C}^N,\ A \subset E, \qquad (2.4)$$

where A is any Borel subset of E.

Remark 2.1. — If μ vanishes on plp sets then

$$\mathcal{M}(t) \equiv \mathcal{M}_1(t) := \sup\left\{\sup_E(\exp L_A^*); A \subset E,\ \mu(A) \ge t\right\}.$$

Indeed, it is clear that $\mathcal{M} \le \mathcal{M}_1$. In order to prove the opposite inequality observe that given t with $0 \le t \le 1$ and $m \in \mathbf{R}$ with $m < \mathcal{M}_1(t)$ these exists $A \subset E$ sucht that $\mu(A) \ge t$ and $\sup_E L_A^* > \log m$. Put $A_o := \{z \in A; L_A(z) = L_A^*(z)\}$. Then $\mu(A_o) = \mu(A) \ge t$ and $L_A^* \ge L_{A_o}$. Hence $\log m < \sup_E L_A^* \le \sup_E L_{A_o} \le \mathcal{M}(t)$. By the arbitrariness of m we get $\mathcal{M}_1(t) \le \mathcal{M}(t)$.

Remark 2.2. — If $\mathcal{M}^*(1) = 1$, then $\mathcal{M}$ is continuous at $t = 1$ and $\mathcal{M}(1) = 1$. Hence, if $\mathcal{M}^*(1) = 1$ then $L_{A_n} \to L_E$ for every sequence A_n of Borel subsets of E such that $\mu(A_n) \to \mu(E)$.

Remark 2.3. — If μ vanishes on plp sets and $\mathcal{M}(1) = 1$ then $L_E^* = 0$ on E. In particular, if E is compact and $\mathcal{M}(1) = 1$ then E is $\mathcal{L}$-regular. Indeed, put $E_o := \{z \in E; L_E^*(z) = 0\}$. Then $\mu(E_o) = 1$ and $L_E^* \le L_{E_o} \le \log \mathcal{M}(1) = 0$ on E, i.e. $L_E^* = 0$ on E. It is clear that the pairs (E, μ) for which $\mathcal{M}(1) = 1$ or $\mathcal{M}^*(1) = 1$ are of great importance for applications of the $\mathcal{L}$-extremal function.

PROPOSITION 2.4. — *Define*

$$\mathcal{M}_{\mathcal{P}}(t) := \sup\left\{(\|f\|_E/\|f\|_A)^{1/deg\ f};\ deg\ f \ge 1,\ A \subset\subset E,\ \mu(A) \ge t\right\}$$

where the sup is taken over all polynomials f of degree ≥ 1 and over all compact sets $A \subset E$. If E is compact, then

$$\mathcal{M}_{\mathcal{P}}(t) \le \mathcal{M}(t) \le \mathcal{M}_{\mathcal{P}}^*(t),\ 0 \le t \le 1$$

Proof. — Fix t with $0 \leq t \leq 1$. Given any number m with $m < \mathcal{M}(t)$, take $A \subset E$ with $\mu(A) \geq t$ and $\sup_E L_A > \log m$. Next choose u in the class $\mathcal{L}$ with $u \leq 0$ on A and $\sup_E u > \log m$. By the Approximation Lemma [13] there is a sequence $u_\nu := \max_{1 \leq j \leq s_\nu} \frac{1}{n_j} \log |f_j|$, where f_j is a polynomial of degree $\leq n_j$, such that $u_\nu \downarrow u$ as $\nu \to \infty$. Given $\epsilon > 0$ there exists a compact set $K \subset A$ with $\mu(K) \geq t - \epsilon$. Take ν so large that $\sup_K u_\nu < \epsilon$ and choose j with $\sup_E \frac{1}{n_j} \log |f_j| > \log m$. Then $m\, e^{-\epsilon} \leq \big(\|f_j\|_E / \|f_j\|_K\big)^{1/n_j} \leq \mathcal{M}_{\mathcal{P}}\big(\mu(K)\big) \leq \mathcal{M}_{\mathcal{P}}(t - \epsilon)$. Hence $m \leq \mathcal{M}^*_{\mathcal{P}}(t)$. By the arbitrariness of m we get $\mathcal{M}(t) \leq \mathcal{M}^*_{\mathcal{P}}(t)$. The inequality $\mathcal{M}_{\mathcal{P}}(t) \leq \mathcal{M}(t)$ is obvious.

PROPOSITION 2.5. — *If E is nonplp compact set in $\mathbf{C}^N$ then $\mathcal{M}_{\mathcal{P}}(t) = \lim_{n\to\infty} B_n^{1/n}(t) = \sup_{n \geq 1} B_n^{1/n}(t)$, where*

$$B_n(t) := \sup\{\|f\|_E;\, deg\ f \leq n, \|f\|_A = 1,\ A \subset\subset E,\ \mu(A) \geq t\}.$$

Proof. — Given $m \geq 1$ and c with $0 < c^m < B_m(t)$, let A be a compact subset of E with $\mu(A) \geq t$ and let f_m be a polynomial of degree $\leq m$ such that $\|f_m\|_A = 1$ and $\|f_m\|_E > c^m$. Every natural number $n \geq m$ can be written in the form $n = km + r$ with $0 \leq r < m$. Observe that

$$c^{km} < \|f_m\|_E^k \leq B_n(t).$$

Hence $\liminf_{n\to\infty} B_n^{1/n}(t) \geq c$, which implies that $B_m^{1/m}(t) \leq \liminf_{n\to\infty} B_n^{1/n}(t)$, and consequently we get the required result.

THEOREM 2.6. — *Let $A \subset \mathbf{C}^p$, $B \subset \mathbf{C}^q$ be bounded Borrel sets and let μ, ν be probability measures on A and B, respectively. Put $\mathcal{M}_A(t) := \mathcal{M}(t; A, \mu)$, $\mathcal{M}_B(t) := \mathcal{M}(t; B, \nu)$ and $\mathcal{M}_{A\times B}(t) := \mathcal{M}(t; A \times B, \lambda)$ with $\lambda := \mu \otimes \nu$.*

Then

(i) $\mathcal{M}_{A\times B}(1) \leq \mathcal{M}_A(1)\ \mathcal{M}_B(1)$

(ii) $\mathcal{M}^*_{A\times B}(1) \leq \mathcal{M}^*_A(1)\ \mathcal{M}^*_B(1)$

Proof. — (i) Let $E \subset A \times B$ with $\lambda(E) = 1$ and put $B^z := \{w \in B; (z,w) \in E\}$. Then $\nu(B^z) = 1$ μ-a.e. on A. Let $u \in \mathcal{L}(\mathbf{C}^p \times \mathbf{C}^q)$, $u \leq 0$ on E. Then $u(z,w) \leq \log \mathcal{M}_B(1) + L_B(w)$ for all $z \in A_o$ and $w \in \mathbf{C}^q$, where $A_o \subset A$ and $\mu(A_o) = 1$. Hence

$$u(z,w) \leq \log \mathcal{M}_B(1) + L_B(w) + \log \mathcal{M}_A(1) + L_A(z),\ (z,w) \in \mathbf{C}^p \times \mathbf{C}^q.$$

Hence by (2.2) one gets (i).

(ii) Let m be a fixed number with $m < \log \mathcal{M}^*_{A\times B}(1)$. There is a sequence E_n of Borel subsets of $A \times B$ such that $\lambda(E_n) \geq 1 - 2^{-n}$ and $\sup_{A\times B} L_{E_n} > m$. Define

$$B_n^z := \{w \in B; (z,w) \in E_n\},\ z \in A,\ n \geq 1$$
$$A_{n\epsilon} := \{z \in A; \nu(B_n^z) \geq 1 - \epsilon\},\ n \geq 1,\ 0 < \epsilon < 1$$

We claim that $\mu(A_{n\epsilon}) \to 1$ as $n \to \infty$. Indeed,

$$\lambda(E_n) = \int_A \nu(B_n^z) d\mu(z) = \int_{A_{n\epsilon}} + \int_{A\setminus A_{n\epsilon}} \leq \mu(A_{n\epsilon}) + (1 - \mu(A_{n\epsilon}))\ (1-\epsilon)$$
$$= 1 - \epsilon + \epsilon\mu(A_{n\epsilon}).$$

Hence $\liminf_{n\to\infty} \mu(A_{n\epsilon}) \geq 1$, which implies the claim. Fix $n \geq 1$ and let u be a function of the class $\mathcal{L}(\mathbf{C}^p \times \mathbf{C}^q)$ with $u \leq 0$ on E_n. Then for every fixed z in A we have $u(z,w) \leq 0$ on B_n^z. Therefore

$$u(z,w) \leq \log \mathcal{M}_B(\nu(B_n^z)) + L_B(w)$$

which implies

$$u(z,w) \leq \log \mathcal{M}_B(1-\epsilon) + L_B(w) \text{ for } z \in A_{n\epsilon},\ w \in \mathbf{C}^q.$$

Hence

$$u(z,w) \leq \log \mathcal{M}_B(1-\epsilon) + L_B(w) + \log \mathcal{M}_A(\mu(A_{n\epsilon})) + L_A(z)$$

for all $(z,w) \in \mathbf{C}^p \times \mathbf{C}^q$. By the arbitrariness of u we can replace u by $L_{E_n}(z,w)$. Then we get

$$m < \sup_{A\times B} L_{E_n} \leq \log \mathcal{M}_B(1-\epsilon) + \log \mathcal{M}_A(\mu(A_{n\epsilon})),\ n \geq 1,\ 0 < \epsilon < 1.$$

After passing to the limits, first with n to ∞ and next with ϵ to 0, we get $m \leq \log \mathcal{M}^*_B(1) + \log \mathcal{M}^*_A(1)$. By the arbitrariness of m we get (ii).

The following corollary is important for applications of the function $\mathcal{M}$.

COROLLARY 2.7.— *If $\mathcal{M}_A(1) = 1$, $\mathcal{M}_B(1) = 1$, (resp. $\mathcal{M}^*_A(1) = 1$, $\mathcal{M}^*_B(1) = 1$) then $\mathcal{M}_{A\times B}(1) = 1$ (resp. $\mathcal{M}^*_{A\times B}(1) = 1$).*

Exemple 2.8 .— Let $I = \{a, b\}$ be an interval of the real line $\mathbf{R}$ with end points a, b such that $-\infty < a < b < +\infty$. Then

$$\mathcal{M}(t) \equiv \mathcal{M}(t; I, \lambda_1) = 2t^{-1} - 1 + 2\, t^{-1}\sqrt{1-t},\ 0 \le t \le 1,$$

λ_1 denoting the Lebesgue measure on $\mathbf{R}$.

Proof.— Without loss of generality we may assume that $I = [a, b]$ is closed. By [4] for every polynomial f of degree $\le n$, $\|f\|_A / \|f\|_I \le B_n(t)$, if $A \subset\subset I$ and $\lambda_1(A) \ge t(b-a)$, where

$$B_n(t) := \frac{1}{2}\left[(2t^{-1} - 1 + 2t^{-1}\sqrt{1-t})^n + (2t^{-1} - 1 - 2t^{-1}\sqrt{1-t})^n\right].$$

Moreover, if A is a subinterval of I with a common end point, this bound is best possible. Therefore by Proposition 2.5 we have

$$\mathcal{M}_{\mathcal{P}}(t) = \sup_{n\ge 1} B_n^{1/n}(t) = 2t^{-1} - 1 + 2t^{-1}\sqrt{1-t},\ 0 \le t \le 1.$$

By Proposition 2.4. $\mathcal{M}_{\mathcal{P}} = \mathcal{M}$.

Remark 2.9.— If Ω is a bounded open set in $\mathbf{R}^N$ (resp. in $\mathbf{C}^N$) then for every determining measure μ for Ω one has $\mathcal{M}(1; \Omega, \mu) = 1$. Indeed, it is known [12] that $L_\Omega^* = L_\Omega$. So if $F \subset \Omega$ and $\mu(F) = 1$, then $L_F = L_\Omega$ which implies that $\mathcal{M}(1; \Omega, \mu) = 1$. As an example of such μ one can take the Lebesgue measure λ_N in $\mathbf{R}^N$ (resp. λ_{2N} in $\mathbf{C}^N$).

If μ is a probability measure on Ω such that $\mathcal{M}^*(1; \Omega, \mu) = 1$, then the closure E of Ω, $E = \overline{\Omega}$, is an $\mathcal{L}$-regular compact. Indeed, let K_n be an increasing sequence of $\mathcal{L}$-regular compact subsets of Ω such that $\mu(\Omega) = \lim \mu(K_n)$ and $\Omega = \cup_1^\infty K_n$. Then

$$\log \mathcal{M}\big(\mu(K_n)\big) \ge \sup_\Omega L_{K_n} = \sup_E L_{K_n} \ge \sup_E L_E^*,\ n \ge 1,$$

which implies that $L_E^* = 0$ on E, i.e. E is $\mathcal{L}$-regular.

Example 2.10 .— We shall now construct a bounded open subset Ω of $\mathbf{C}$ with the following progerties.

1) $E := \overline{\Omega}$ is $\mathcal{L}$-regular.
2) For every probability measure μ on Ω, $\mathcal{M}^*(1; \Omega, \mu) > 1$.
3) There exists no finite positive Borel mesure μ on Ω such that the pair (Ω, μ) satisfies the $(\mathcal{L}^*)$-condition.

Indeed, let $\{a_n\}$ be a discrete sequence in the upper half plan $\{Im\ z > 0\}$ such that each point of $I = [0,1]$ is a limit of a subsequence of $\{a_n\}$ and the sequence $\{a_n\}$ has no other limit points. There exists a sequence of positive real numbers $\{r_n\}$ such that

$$L_{\Omega_n}(z) \geq 5 - (2^{-1} + \ldots + 2^{-n}),\ z \in I,\ n \geq 1, \tag{$*$}$$

with $\Omega_n := \cup_{j=1}^{n} \{|z - a_j| < r_j\}$. Namely, it is clear that $L_{\Omega_1}(z) > 5 - 2^{-1}$ on I, if $r_i > 0$ is sufficiently small. Suppose $r_1, \ldots, r_n$ are already chosen so that (*) is satisfied. Put $\Omega(r) := \Omega_n \cup \{|z - a_{n+1}| < r\}$. Then $L_{\Omega(r)} \uparrow L_{\Omega_n}$ in $\mathbf{C}\backslash\{a_{n+1}\}$ as $r \uparrow 0$. By Dini's argument the convergence is uniform on I. Hence (*) is satisfied for $n+1$ with $r = r_{n+1}$ sufficiently small. The open set $\Omega := \cup_1^\infty \Omega_n$ has the required properties. It is clear that $E := \overline{\Omega}$ is $\mathcal{L}$-regular. If μ is a finite positive Borel measure on Ω, then $\log \mathcal{M}(\mu(\Omega_n)) \geq \sup_\Omega L_{\Omega_n} = \sup_E L_{\Omega_n} \geq 4$ $(n \geq 1)$. Hence $\mathcal{M}^*(1; \Omega, \mu) \geq 4$. The set $G := \{z \in E : L_\Omega^*(z) > 0\}$ contains the interval I, so G is not plp. By Lemma 1.1 the pair (Ω, μ) does not satisfy $(\mathcal{L}^*)$ (see also theorem 3.1).

PROPOSITION 2.11.— *If μ is determining for a nonpluripolar bounded Borel set in $\mathbf{C}^N$, μ vanishes on plp sets and $\mathcal{M}^*(1; E, \mu) = 1$ then (E, μ) satisfies $(\mathcal{L}^*)$.*

Proof.— Given a polynomial family $\mathcal{F}$ bounded μ-a.e. on E, let E_j be the sequence of subsets of E defined by (1.2). Then $E_j \uparrow F$ with $\mu(F) = 1$. Therefore $L_{E_j}^* \downarrow L_F^* = L_E^* = L_E$ (see Remark 2.3). Given $b > 1$ the set $\Omega_b := \{L_F^* < \sqrt{b}\}$ is an open neighborhood of E. By (1.2) and (2.4)

$$|f(z)| \leq j\ \left(\mathcal{M}(\mu(E_j)) \exp L_E^*(z)\right)^{\deg f},\ f \in \mathcal{F},\ j \geq 1.$$

If j is sufficiently large the family $\mathcal{F}_b$ is bounded by j uniformly on Ω_b.

Problem 2.12 .— Let $\Delta = \{|z| < 1\}$ be the unit disk on the complex plane $\mathbf{C}$. Let θ denote the lenghth measure on the boundary $\partial\Delta$ of Δ and let λ_2 be the Lebesgue measure on $\mathbf{C} \equiv \mathbf{R}^2$. Compute the functions $\mathcal{M}(t; \partial\Delta, \theta)$ and $\mathcal{M}(t; \Delta, \lambda_2)$, $0 \leq t \leq 1$.

3 - Determining measures for bounded Borel sets with $\mathcal{L}$-regular closure

The main result of this section is given by the following.

THEOREM 3.1. — *Let μ be a probability measure on a bounded Borel set E in $\mathbf{C}^N$ such that $\overline{E}$ is $\mathcal{L}$-regular. Then the following conditions are equivalent.*

(1) The pair (E,μ) satisfies $(\mathcal{L}^*)$-condition;

(2) If $u \in \mathcal{L}$ and $u \le 0$ μ-a.e. on E then $u \le 0$ on $\overline{E}$;

(3) $\mathcal{M}^*(1) \equiv \mathcal{M}^*(1;E,\mu) = 1$ and $L_E = L_{\overline{E}}$;

(4) $\mathcal{M}(1) \equiv \mathcal{M}(1;E,\mu) = 1$ and $L_E = L_{\overline{E}}$;

(5) If $A \subset E$ and $\mu(A) = 1$, then $L_A = L_{\overline{E}}$;

(6) For every $b > 1$ there exists a neighforhood Ω of $\overline{E}$ such that for every polynomial family $\mathcal{F}$ bounded μ-a.e. on E the family $\mathcal{F}_b$ (given by (0.2)) is uniformly bounded on Ω;

(7) If $\mathcal{F}$ is a polynomial family bounded μ-a.e. on E then for every number $b > 1$ the family $\mathcal{F}_b$ is bounded q.a.e. on $\overline{E}$.

Proof. — (1) $\Rightarrow$ (2). Let u be a function of the class $\mathcal{F}$ with $u \le 0$ μ-a.e. on E. The function u can be written in the form

$$u = \left(\limsup_{j\to\infty} \frac{1}{j} \log|f_j|\right)^*$$

where f_j is a polynomial of degree $\le j$. Given any fixed number $b > 1$ the polynomial family $\mathcal{F} := \{b^{-j} f_j; j \ge 1\}$ is bounded μ-a.e. on E. By (1) there are a constant $M > 0$ and a neighborhood Ω of E such that

$$\|f_j\|_\Omega \le M\ b^{2j},\ j \ge 1,$$

which implies $\|f_j\|_{\overline{E}} \le M\ b^{2j} (j \ge 1)$. Hence by the definition of $L_{\overline{E}}$ we obtain $\frac{1}{j}\log|f_j(z)| \le \frac{1}{j}\log M + 2\log b + L_E(z)$ in $\mathbf{C}^N$ $(j \ge 1)$. Therefore $u(z) \le 2\log b$ on $\overline{E}$. By the arbitrariness of $b > 1$ we get $u \le 0$ on $\overline{E}$.

(2) $\Rightarrow$ (3). If (2) is satisfied, then $L_E \le L_{\overline{E}} \le L_E$, so that $L_E = L_{\overline{E}}$. It remains to show that $\lim_{t\uparrow 1} \mathcal{M}(t) = 1$. Suppose there exists $b > 1$ with $\mathcal{M}(t) > b$ for all t with $0 < t < 1$. Let A_n be Borel subsets of E such that

$$\mu(A_n) \ge 1 - 2^{-n} \text{ and } \sup_E\left(\exp L_{A_n}\right) > b\ (n \ge 1) \qquad (*)$$

Put $E_n := A_n \cap A_{n+1} \cap \ldots$ and observe that $E_{n+1} \supset E_n$, $E_n \subset A_n$ and

$$\begin{aligned}\mu(E_n) &= \mu(A_n) - \mu(A_n \backslash E_n) \ge \mu(A_n) - \mu(E\backslash E_n)\\ &\ge \mu(A_n) - \sum_{j=0}^{\infty} \mu(E\backslash A_{n+j}) \ge 1 - 2^{-n} - \sum_{j=0}^{\infty} 2^{-n-j} = 1 - 3.2^{-n},\end{aligned}$$

which implies that $\mu(E_n) \to 1$. Put $F := \cup E_n$. Then $L^*_{E_n} \downarrow L^*_F$ and $\mu(F) = 1$. By (2) $L_F \leq L_{\overline{E}}$ and since $L_{\overline{E}} \leq L_F$, we get $L_F = L_{\overline{E}}$. By Dini's argument $L_{E_n} \leq L^*_{E_n} < \log b$ on $\overline{E}$, if $n > n_o = n_o(b)$. This however contradicts the second inequality of (*). Therefore $\mathcal{M}^*(1) = \lim_{t\uparrow 1} \mathcal{M}(t) = 1$.

(3) $\Rightarrow$ (4) obvious.

(5) $\Rightarrow$ (6) If $b > 1$, then the set $\Omega_b := \{z \in \mathbf{C}^N; L_E(z) < \sqrt{b}\}$ is by (5) an open neighborhood of $\overline{E}$. Let $\mathcal{F}$ be a polynomial family bounded μ-a.e. on E. Put $E_k := \{z \in E; |f(z)| \leq k, \ \forall_{f\in\mathcal{F}}\}$. Then $E_k \subset E_{k+1}$ and $\mu(E_k) \uparrow 1$. Hence by (5) $L^*_{E_k} \downarrow L^*_A = L_{\overline{E}}$ with $A := \cup_1^\infty E_k$, the convergence being uniform on $\overline{E}$. Hence $L_{E_k} \leq \frac{1}{2}\log b$ on $\overline{E}$ if $k > k_o$. It is clear that

$$\begin{aligned} |f(z)| &\leq k(\exp L_{E_k}(z))^{\deg f} \\ &\leq k\left(\exp\left[\frac{1}{2}\log b + L_E(z)\right]\right)^{\deg f} \\ &\leq k\ b^{\deg f}, r \text{ if } z \in \Omega_b,\ f \in \mathcal{F},\ k > k_o. \end{aligned}$$

(6) $\Rightarrow$ (7) is obvious.

(7) $\Rightarrow$ (1) follows from lemma 1.2.

4 - Determining measures for compact sets in $\mathbf{C}^N$

THEOREM 4.1.— *If μ is a probability measure on a compact set E in $\mathbf{C}^N$ vanishing on plp subsets of E, then the following conditions are equivalent.*

(i) The pair (E,μ) satisfies $(\mathcal{L}^*)$;

(ii) If $u \in \mathcal{L}$ and $u \leq 0$ μ-a.e. on E, then $u \leq 0$ on E;

(iii) $\mathcal{M}^*(1, E, \mu) = 1$;

(iv) $\mathcal{M}(1; E, \mu) = 1$;

(v) μ si determining for E and E is $\mathcal{L}$-regular.

Proof.— First observe that each of the conditions (i), (ii), (iii), (iv) implies $\mathcal{L}$-regularity of E, and next apply Theorem 3.1.

Example 4.2 .— (most likely well known to the reader). Let E be a compact subset in the complex plane. Assume E has a positive logarithmic capacity $c(E)$. By the classical potential theory there exists a unique probability measure λ with support on E such that

$$\log c(E) = \int_E \int_E \log|z-\zeta| d\lambda(z) d\lambda(\zeta) = \sup_\mu \int_E \int_E \log|z-\zeta| d\mu(z) d\mu(\zeta)$$

the supremum being taken over all probability measures μ on E. The measure λ is called the *equilibrium measure of* E. We shall show that λ is determining for E. Indeed, if F is a Borel subset of E with $\lambda(F) = 1$ there is (by Choquet capacitability theorem) a sequence F_n of compact subsets of F with $c(F_n) \nearrow c(F)$. Without loss of generality we may assume E is contained in the disk $|z| < 1/2$. Then

$$\begin{aligned}\log c(E) \geq \log c(F_n) &\geq \frac{1}{\lambda^2(F_n)} \int_{F_n} \int_{F_n} \log|z-\zeta|\ d\lambda(z) d\lambda(\zeta) \\ &\geq \frac{1}{\lambda^2(F_n)} \log c(E).\end{aligned}$$

Therefore $c(F_n) \uparrow c(E) = c(F)$. For all suficiently large n the function $u_n(z) := L^*_{F_n}(z) - L^*_E(z)$, $u_n(\infty) := \log[c(E)/c(F_n)]$, is harmonic in $\overline{\mathbf{C}} \backslash \widehat{E}$, $u_{n+1} \leq u_n$ and $u_n(\infty) \downarrow 0$. By Harnack's theorem $u_n \downarrow 0$ locally uniformly in $\overline{\mathbf{C}} \backslash \widehat{E}$. The function $u := \lim L^*_{F_n}$ is subharmonic on $\mathbf{C}$, $u \geq L^*_E$ on $\mathbf{C}$, and $u = L^*_E$ in $\mathbf{C} \backslash \widehat{E}$ as well as at each regular point of $\partial \widehat{E}$. By the generalized maximum principle for subharmonic function, $u \leq 0$ on $\widehat{E}$ except at most the polar set of irregular points of $\partial \widehat{E}$. On the other hand $u \geq 0$ on $\mathbf{C}$. Therefore $u = L^*_E$. Observe that $L_E \leq L_F \leq L_{F_n}$ $(n \geq 1)$. Hence $L^*_E = L^*_F$. It follows that μ is determining for E. Hence by theorem 3.1, if E is an $\mathcal{L}$-regular subset of $\mathbf{C}$ and λ is the equilibrium measure of E, then the pair (E, λ) satisfies each of the equivalent conditions of theorem 4.1.

Remark 4.3. — Given a norm $\mathcal{N}$ on $\mathbf{C}^N$ the logarithmic capacity $c(E) \equiv c(E, \mathcal{N})$ of a bounded subset E of $\mathbf{C}^N$ is defined by the formula

$$-\log c(E) := \limsup_{\mathcal{N}(z) \to \infty} \left[L_E(z) - \log \mathcal{N}(z) \right].$$

If E is a probability measure on E with $\mathcal{M}(1; E, \mu) = 1$, then for every $F \subset E$ with $\mu(F) = \mu(E)$ one has $c(F) = c(E)$.

On the plane, if E is bounded and $F \subset E$, then : $c(F) = c(E) \Leftrightarrow L^*_F = L^*_E$, which implies that μ is determining for E iff $F \subset E$, $\mu(F) = 1 \Rightarrow c(F) = c(E)$ (i.e.; iff μ is determining in the sense of ULLMAN [14]).

If $N \geq 2$ and $F \subset E$, it is clear that $L_F^* = L_E^* \Rightarrow \forall \mathcal{N} c(F, \mathcal{N}) = c(E, \mathcal{N})$. But we do not know whether the inverse implication is true.

The aim of the following example is to illustrate an application of theorem 2.6.

Example 4.4 . — Let Ω be a bounded open set in $\mathbf{R}^N$ (resp. in $\mathbf{C}^N$). Then it is know that λ_N (resp. λ_{2N}) is determining for Ω. We can propose the following proof of this result.

It is sufficient to consider the case of R^N (because by (1.1) for every $u \in \mathcal{L}(\mathbf{C}^N)$ there is $\widetilde{u} \in \mathcal{L}(\mathbf{C}^{2N})$ such that $\widetilde{u}(x_1, y_1, \ldots, x_N, y_N) = u(x_1 + iy_1, \ldots, x_N + iy_N)$ for $(x_1 + iy_1, \ldots, x_N + iy_N) \in \mathbf{C}^N \equiv \mathbf{R}^{2N}$. Hence, if $u \in \mathcal{L}(\mathbf{C}^N)$ and $u \leq 0$ $\lambda_{2N} - a.e.$ on $\Omega \subset \mathbf{C}^N$ then $u \leq 0$ on Ω.). Let $u \in \mathcal{L}(\mathbf{C}^N)$ and let $u \leq 0$ $\lambda_N - a.e.$ on Ω. Given a point $a = (a_1, \ldots, a_n)$ in Ω, let $Q := \{|x_j - a_j| \leq r \ (j = 1, \ldots, N)\}$ be a closed cube with center a contained in Ω. Since by Theorem 4.1 (via example 2.8) λ_1 is determining for $[a_j - r, \ a_j + r]$, so by theorem 2.6 the measure λ_N is determining for the cube Q. Therefore $u \leq 0$ on Q. By the arbitrariness of Q we get $u \leq 0$ on Ω. Hence $L_\Omega = L_F^*$ for every $F \subset \Omega$ with $\lambda_N(F) = \lambda_N(\Omega)$.

Let $I^N = [0,1]^N$ be the unit cube in $\mathbf{R}^N$. If A is a nonsingular affine mapping of $\mathbf{R}^N$ onto itself, then the set $P := A(I^N)$ is called a parallelepiped.

Let Ω be a bounded open subset of $\mathbf{R}^N$ such that for each point $b \in \overline{\Omega}$ there exists a parallelepiped P such that $P \subset \Omega \cup \{b\}$ and $b \in P$. Then $\overline{\Omega}$ is $\mathcal{L}$-regular and the pair (Ω, λ_N) satisfies each of the equivalent conditions of theorem 3.1.

Indeed, it is easy to see that each parallelepiped P is $\mathcal{L}$-regular. Therefore $\overline{\Omega}$ is $\mathcal{L}$-regular, because $L_{\overline{\Omega}} \leq L_P$. We already know that the pair (I^N, λ_N) satisfies $(\mathcal{L}^*)$. Hence for every parallelepiped P the pair (P, λ_N) satisfies $(\mathcal{L}^*)$. Therefore the pair (Ω, λ_N) satisfies $(\mathcal{L}^*)$ at each point of $\overline{\Omega}$, which implies that (Ω, λ_N) satisfies (6) of Theorem 3.1.

5 - Polynomial inequality of Bernstein-Markov type and pairs (E, μ) atisfying the $(\mathcal{L}^*)$-condition

DEFINITION 5.1. — *Let p be a positive number, E a bounded Borel set in $\mathbf{C}^N$ and μ a probability measure on E. We say that the triple (p, E, μ) has Bernstein-Markov Property, if for every $b > 1$ there exist a positive constant M and a neighborhood G of E such that for every polynomial f of*

N complex variables one has

$$\|f\|_G \leq M\ b^{deg\ f}\|f\|_{\mu p} \tag{BM}$$

with $\|f\|_{\mu p} := \left(\int_E |f(x)|^p\ d\mu(z)\right)^{1/p}$.

It was shown in [11] that if (E,μ) satisfies $(\mathcal{L}^*)$ and μ satisfies some density condition, then the triple (p, E, μ) has BMP for every $p > 0$. Due to a remark by A. ZERIAHI the density condition may be dropped and one gets the following.

THEOREM 5.1. — *Let E be a Borel subset of $\mathbf{C}^N$ and let μ be a positive measure on E such that (E,μ) satisfies $(\mathcal{L}^*)$. Then for every $p > 0$ the triple (p, E, μ) has the Bernstein-Markov Property (BMP).*

Proof. — Let $s(f)$ denote the degree of f. It is sufficient to prove that for every $p > 0$ and for every $b > 1$ there exists a constant $M > 0$ such that for every polynomial f

$$\|f\| := \|f\|_E \leq M\ b^{s(f)}\|f\|_{\mu p}.$$

Suppose the statement is not true. Then we can find $p > 0$, $b > 1$ and a sequence of polynomials f_k such that

$$\|f_k\| > k^k\ b^{s(f_k)}\|f\|_{\mu p} \text{ for } k \geq 1. \tag{5.1}$$

It follows that $\|f_k\| > 0$ and $0 < \|f_k\|_{\mu p} < +\infty$ $(k \geq 1)$. We claim that for every $q > 1$ and every $\eta > 1$ the sequence of polynomials $g_k := \eta^{-k}\ q^{-s(f_k)} f_k/\|f_k\|_{\mu p}$ is bounded μ-a.e. on E. Indeed, following NGUYEN THANH VAN [8], put $E_{nk} := \{z \in E; |g_k(z)| \geq n\}$, $E_n := \cup_{k=1}^{\infty} E_{nk}$ and observe that

$$\mu(E_n) \leq \sum_{k=1}^{\infty} n^{-p}\eta^{-kp}q^{-qs(f_k)} \leq n^{-p}/\eta^{p-1},\ n \geq 1,$$

whence it follow that $\{g_k\}$ is bounded μ-a.e. on E. Now by the assuption (E,μ) satisfies $(\mathcal{L}^*)$, so that we can find $G \supset E$ and $M > 0$ such that $\|g_k|_G \leq M\ q^{s(t_k)}$, $k \geq 1$. Hence

$$\|f_k\|_G \leq M\ \eta^k\ q^{2s(f_k)}\|f\|_{\mu p},\ k \geq 1 \tag{5.2}$$

Put $q = b^{1/2}$. Then (5.1) and (5.2) imply

$$k^k < M\ \eta^k,\ k \geq 1,$$

which is an absurd.

THEOREM 5.2. — *If $\mathcal{M}^*(1; E, \mu) = 1$ and there is $p > 0$ such that the triple (p, E, μ) has the BMP, then (E, μ) satisfies $(\mathcal{L}^*)$.*

Proof. — Take $b > 1$ and let $\mathcal{F}$ be a polynomial family bounded μ-a.e. on E. Define E_j by formula (1.2). Then $\mu(E_j) \uparrow 1$ and

$$|f(z)| \leq j\ \mathcal{M}(\mu(E_j))^{\deg f} \text{for all } z \in E,\ f \in \mathcal{F},\ j \geq 1.$$

Hence by BMP

$$\|f\|_G \leq j\ M\left[b\ \mathcal{M}(\mu(E_j))\right]^{\deg f},\ f \in \mathcal{F},\ j \geq 1,$$

which implies the required result.

COROLLARY. — *If $\mathcal{M}^*(1; E, \mu) = 1$, then the pair (E, μ) satisfies $(\mathcal{L}^*)$ if and only if for every $p > 0$ (for some $p > 0$) the triple (p, E, μ) has the BMP.*

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