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# Volterra integral equations associated with a class of nonlinear operators in Hilbert spaces

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**RÉSUMÉ.** — Soit $H$ un espace de Hilbert réel. Nous étudions l'équation non linéaire de Volterra

$$u(t) + \int_0^t b(t-s)Au(s)\, ds \ni F(t) \ ,0 \leq t \leq T$$

où $b$ et $F$ sont respectivement une fonction scalaire et une fonction vectorielle et $A$ est un opérateur (éventuellement multivoque) qui n'est pas nécessairement monotone.

Nous démontrons, sous des hypothèses convenables, des résultats d'existence, d'unicité et de régularité pour la solution $u$.

Enfin, nous donnons des exemples qui clarifient les résultats abstraits.

**ABSTRACT.** — Let $H$ be a real Hilbert space. We study the nonlinear Volterra equation

$$u(t) + \int_0^t b(t-s)Au(s)\, ds \ni F(t) \ ,0 \leq t \leq T$$

where $b$ and $F$ are respectively a scalar and a vector valued function and $A$ is a (possibly multivalued) operator not necessarily monotone.

Under suitable hypotheses we prove various existence, uniqueness and regularity results for the solution $u$. Some examples which illustrate the abstract results are presented.

## § 0. Introduction

In this paper we discuss some existence and regularity properties of the solution of

$$u(t) + \int_0^t b(t-s)Au(s)\, ds \ni F(t) \ ,0 \leq t \leq T, \tag{0.1}$$

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where $A$ denotes a nonlinear (possibly multivalued) operator on a real Hilbert space $H$, $b : [0,T] \to \mathbf{R}$ and $F : [0,T] \to H$ are given functions. Many results concerning the existence, uniqueness and asymptotic behaviour of the solutions of (0.1), are known if $A$ is a maximal monotone operator, or more generally, an $m$-accretive operator in a Banach space $X$.

See for example, [10], [11], [12], [13], [15], [16], [21], for some results in this direction.

The aim of this paper is to give some contribution to the existence and regularity theory in the case that "$A$ is not necessarily monotone". A similar problem was discussed also in Kiffe [12], where this author has considered "non monotone" perturbations of monotone operators. Using the ideas introduced in [4] - [9], we have been able to prove some existence results for (0.1), for a large class of nonlinear operators.

We emphasize that our results enable us to treat concrete examples of operators which are not necessarily perturbations of maximal monotone operators.

This paper is organized as follows :

Section 1 contains some definitions and properties needed in the subsequent sections.

Section 2 contains some results concerning the existence and uniqueness of the "local" solution in the "nonvariational case" that is, we consider operators which are not necessarily of the form $A = \partial^- f$ (the precise meaning of the operation "$\partial^-$" is explained in section 1. In the same section we give also some sufficient conditions for the existence of the global solution.

In section 3 we analyse further properties enjoyed by the solution of (0.1) in the "variational case", that is when $A = \partial^- f$.

In particular we give some sufficient conditions which ensure the global existence of solutions, as well as, their regularity.

Finally section 4 contains some concrete examples which illustrate our abstract results.

## § 1. Preliminaries

In this paper $H$ will denote a real Hilbert space with scalar product $(\ ,\ )$ and norm $\|\ \| = (\ ,\ )^{\frac{1}{2}}$.

If $u \in H$ and $r > 0$, we set $B(u,r) = \{v \in H : \|v - u\| < r\}$.

Let $A : H \to 2^H a$ (possibly multivalued) operator defined on $D(A) = \{u \in H : Au \neq \emptyset\}$, and let $\Omega \subset H$ be an open set.

DEFINITION (1.1).— *An operator*

$$A : H \to 2^H$$

*is said $(\varphi, f)$-monotone if*

*(i) there exists a lower semicontinuous function*

$$f : \Omega \to \mathbf{R} \cup \{+\infty\}$$

*such that*

$$D(A) \subset D(f) = \{x \in \Omega : f(x) \in \mathbf{R}\}$$

*(ii) there exists a continuous function*

$$\varphi : D(f) \times \mathbf{R}^2 \to \mathbf{R}^+$$

*such that for every $u, v \in D(A)$ and $\alpha \in Au$, $\beta \in Av$ one has*

$$(\alpha - \beta, u - v) \geq (\varphi(u, f(u), \|\alpha\|) + \varphi(v, f(v), \|\beta\|))\|u - v\|^2. \tag{1.1}$$

In what it follows, if $A$ is a $(\varphi, f)$-monotone operator, we will use the standard notation

$$\|A^0 u\| = \begin{cases} \inf\{\|\alpha\| : \alpha \in Au\}, & \text{if } u \in D(A) \\ +\infty \quad , & \text{otherwise.} \end{cases}$$

A particular class of $(\varphi, f)$-monotone operators are the so called "$f$-solvable" $(\varphi, f)$-monotone operators (see [4], [9]).

DEFINITION (1.2).— *Let $A$ be a $(\varphi, f)$-monotone operator on $H$.*

*Then $A$ is said "$f$-solvable" at $u \in D(A)$ if*

*(i) for every $c > 0$, there exist $M, \lambda_0 > 0$ : for every $\lambda \in ]0, \lambda_0]$ and $v \in B(u, \lambda c)$, there exists $w \in D(A)$ :*

$$\frac{v - w}{\lambda} \in Aw, \quad \left\|\frac{v - w}{\lambda}\right\| \leq M, \; f(w) \leq M.$$

*Remark* (1.2).— If $A$ is a Lipschitz perturbation of a monotone operator, then $A$ is a $(\varphi, f)$-monotone operator with $\varphi \equiv$ suitable constant and $f \equiv 0$.

Furthermore if $A$ coincides with its maximal extension (see [20]), then $A$ is also $f$-solvable at every point $u \in D(A)$ (in this case we have $f \equiv 0$ too).

For other concrete examples of operators which satisfy the properties of Def. (1.1) and (1.2), see [4] - [5] - [6] - [7] - [8].

**$\Phi$ - convex functions**

A particularly usefull class of $(\varphi, f)$-monotone operators are obtained as it follows :

Let $\Omega$ be an open subset of $H$ and $g : \Omega \to \mathbf{R} \cup \{+\infty\}$ a given function.

As usual, we will put $D(g) = \{v \in H : g(v) \in \mathbf{R}\}$. For $u \in D(g)$ we can define

$$\partial^- g(u) = \begin{cases} \{\alpha \in H : \liminf\limits_{v \to u} \dfrac{f(v) - f(u) - (\alpha, v-u)}{\|v-u\|} \geq 0\}, & \text{if } u \in D(g); \\ \phi, & \text{otherwise.} \end{cases} \tag{1.3}$$

We will put $D(\partial^- g) = \{u \in H : \partial^- g(u) \neq \phi\}$.

It is not difficult to see that $\partial^- g(u)$ is a closed, convex subset of $H$ for every $u \in D(\partial^- g)$.

Therefore we can denote by $\operatorname{grad}^- g(u)$ the element of minimal norm of $\partial^- g(u)$.

If $\partial^- g(u)$ is not empty, we say that $g$ is "subdifferentiable" at $u$, and we will denote by $\partial^- g(u)$ the set of its subdifferentials and by $\operatorname{grad}^- g(u)$ the "subgradient" of $g(\cdot)$ at $u$.

DEFINITION (1.3).— *A lower semicontinuous function*

$$f : \Omega \to \mathbf{R} \cup \{+\infty\}$$

*is called "$\Phi$-convex" if :*

*there exists a continuous function*

$\Phi : D(f) \times \mathbf{R}^3 \to \mathbf{R}^+$ *such that* $\forall v \in D(f), \forall u \in D(\partial^- f)$ *and* $\forall \alpha \in \partial^- f(u)$ *we have*

$$f(v) \geq f(u) + (\alpha, v-u) - \Phi(u, v, f(u), f(v), \|\alpha\|) \cdot \|u-v\|^2. \tag{1.4}$$

It is known that if $f$ is a $\Phi$-convex function, then $A = \partial^- f$ is a $(\varphi, f)$-monotone operator (for a suitable $\varphi$) which is $f$-solvable at every point of $D(\partial^- f)$.

For the proof of this fact, as well as for some relevant properties of $\Phi$-convex functions, see [8].

In what it follows, if $T > 0$ and $H$ is a Hilbert space, we will denote by $AC([0,T];H)$ ($\mathrm{Lip}([0,T];H)$) the space of absolutely (Lipschitz) continuous functions, and by $BV(0,T;H)$ the space of the functions with essentially bounded variation on $]0,T[$.

If $h \in BV(0,T;H)$, we will use the convenction that

$$h(0) = \lim_{t\to 0^+} \frac{1}{t}\int_0^t h(s)ds.$$

$$h(T) = \lim_{t\to T^-} \frac{1}{T-t}\int_t^T h(s)ds.$$

If $u : [0,T] \to H$ is a continuous function we will put also

$$(u * h)(t) = \int_0^t u(t-s)\ dh(s)$$

In this paper we shall use the following definition of solution for (0.1).

DEFINITION (1.4).— *If $T > 0$, $b \in L^1(0,T;\mathbf{R})$, $F \in L^1(0,T;H)$, we say that a function $u \in L^1(0,T;H)$ is a strong solution of (0.1) on $[0,T]$ if :*

*there exists a function $W \in L^1(0,T;H)$ such that*

$$W(t) \in Au(t), \text{ a.e. on } [0,T] \tag{1.5}$$

*and*

$$u(t) + (b * W)(t) = F(t), \text{ a.e. on } [0,T] \tag{1.6}$$

## § 2. The nonvariational case

In this section we will prove the main result of this paper, namely "a local existence" result for the equation (0.1) in the general case of a $(\varphi, f)$-monotone operator $A$ which is $f$-solvable.

We start with a simple uniqueness result :

PROPOSITION (2.1).— *Let $A$ be a $(\varphi, f)$-monotone operator on $H$.*

*Assume that*

$$b \in AC([0,T];\mathbf{R}),\ b' \in BV(0,T;\mathbf{R})\ , b(0) = 1, \tag{2.1}$$

$$F \in AC([0,T];H) \tag{2.2}$$

*Let $u_i$ $(i = 1,2)$ be a strong solution of*

$$u_i(t) + (b * Au_i)(t) \ni F(t) \qquad (2.3)_i$$

*on $[0,T]$, such that for $i = 1,2$*

$$\int_0^T \varphi(u_i(s), f(u_i(s)),\ \|W_i(s)\|)\ ds < +\infty \tag{2.4}$$

*Then $u_1 = u_2$ and $W_1 = W_2$ on $[0,T]$.*

*Proof.*— Clearly $u_i \in AC([0,T];H)$. By (2.1)-(2.2) we know (see Prop. 1 of [10]) that $u_i$ satisfies

$$\begin{cases} \frac{du_i}{dt} + Au_i(t) \ni G(u_i)(t) & a.e. \text{ on } [0,T] \\ u_i(0) = F(0) \end{cases} \tag{2.5}$$

where, $\forall v \in C([0,T];H)$

$$G(v)(t) = F'(t) + (r * F')(t) - r(0)v(t) + r(t)v(0) - (v * r')(t)$$

($r$ being the unique solution of $r + b' * r = -b'$). Since

$$\|G(u_1)(t) - G(u_2)(t)\| \le \gamma(t)\ \|u_1 - u_2\|_{L^\infty(0,t;H)}\ a.e. \text{ on } [0,T] \tag{2.6}$$

where $\gamma(t) = |r(0)| + \mathrm{Var}(r;[0,t])$, (2.5) and the $(\varphi, f)$-monotonicity of $A$ give :

$$\begin{aligned} \frac{d}{dt}\|u_1(t) - u_2(t)\|^2 \le 2\ &\Big(\sum_{i=1}^{2} \varphi(u_i(t), f(u_i(t)), \|W_i(t)\|)\Big) \\ &\cdot \|u_1(t) - u_2(t)\|^2 + 2\ \|G(u_1)(t) - G(u_2)(t)\|\ \|u_1(t) - u_2(t)\| \end{aligned} \tag{2.7}$$

which, by Gronwall lemma, implies

$$\begin{aligned} \|u_1(t) - u_2(t)\| \le &\Big\{\|u_1(0) - u_2(0)\| + \int_0^t \|G(u_1)(s) - G(u_2)(s)\| ds\Big\} \\ &\cdot\ exp\Big(\int_0^t \sum_{i=1}^{2} \varphi(u_i(s), f(u_i(s)), \|W_i(s)\|) ds\Big) \end{aligned} \tag{2.8}$$

Using (2.4), (2.6) and (2.8) we conclude easily.

*Remark* (2.1).— By proposition (2.1), it follows that if $F \in Lip([0,T];H)$ and $u_i$ $(i = 1,2)$ is a strong solution of $(2.3)_i$ on $[0,T]$ such that

$$(u_i, f(u_i)) \in Lip([0,T];H) \times L^\infty(0,T;\mathbf{R}),$$

then $u_1 = u_2$, since, in this case, $W_i \in L^\infty(0,T;H)$ $(i = 1,2)$.

THEOREM (2.1).— *(Local existence)*

*Let $A$ be a $(\varphi, f)$-monotone operator which is $f$-solvable at $u_0 \in D(A)$.*

*Assume that*

$$b \in AC([0,T];\mathbf{R}) \ , \ b' \in BV(0,T;\mathbf{R}), \ b(0) = 1 \tag{2.9}$$

$$F \in AC([0,T];H) \ , \ F' \in BV(0,T;H), \ F(0) = u_0. \tag{2.10}$$

*Then there exist $\overline{T} \in ]0,T]$ and a unique strong solution $u$ of*

$$u(t) + (b * Au)(t) \ni F(t) \tag{2.11}$$

*on* $[0,\overline{T}]$.

*Furthermore*

$$u(t) \in D(A) \text{ for every } t \in [0,\overline{T}]. \tag{2.12}$$

$$(u, f(u)) \in Lip([0,\overline{T}];H) \times L^\infty(0,\overline{T};\mathbf{R}) \tag{2.13}$$

For the proof of theorem (2.1) we need the following lemma whose proof can be obtained using the same techniques of Prop. (1.4) of [9] (see also Prop. (4.1) of [9]).

LEMMA (2.1).— *Let $A$ be a $(\varphi, f)$-monotone operator which is $f$-solvable at $u_0 \in D(A)$.*

Then :

for every $C > \|A^0 u_0\|$, there exist $M \geq f(u_0)$, $\lambda_0 > 0$, $r > 0$ such that if we set

$$N = \{u \in B(u_0, r) \cap D(A) : f(u) \leq M, \ \|A^0 u\| < C\}$$

and

$$\Omega_\lambda = \{v \in H : d(v, N) < \lambda C\}, \forall \lambda \in ]0, \lambda_0]$$

the following facts hold :

$$\begin{cases} \text{for every } \lambda \in ]0,\lambda_0] \text{ there exists a map} \\ J_\lambda : \Omega_\lambda \to D(A) \\ \text{such that} \\ \text{(i)} \begin{cases} \|J_\lambda(u) - J_\lambda(v)\| \le (1-\lambda M)^{-1}\|u-v\|, \ \forall u,v \in \Omega_\lambda \\ A_\lambda\, v = \dfrac{v - J_\lambda(v)}{\lambda} \in AJ_\lambda(v), \ f(J_\lambda(v)) \le M, \|A_\lambda v\| \le M \end{cases} \\ \text{(ii)} \lim\limits_{\lambda\to 0} J_\lambda(u) = u \text{ for every } u \in N; \end{cases} \tag{2.14}$$

$$\begin{cases} \text{for every } (v_n)_n \subset D(A) \text{ and } (\alpha_n)_n \subset H \text{ with} \\ \alpha_n \in Av_n : \\ \lim\limits_n v_n = v \in B(u_0, r), \ w - \lim\limits_n \alpha_n = \alpha, \\ \|\alpha_n\| \le C, \ f(v_n) \le M \Rightarrow \alpha \in Au; \end{cases} \tag{2.15}$$

$$\begin{cases} \text{for every } u \in N \text{ there exists a unique element } A^0u \text{ sucht that} \\ \|A^0 u\| = \inf\{\|\alpha\| : \alpha \in Au\}; \end{cases} \tag{2.16}$$

$$\begin{cases} \text{for every } \lambda \in ]0,\lambda_0], \text{ the map} \\ A_\lambda : \Omega_\lambda \to H, \text{ defined by } A_\lambda = \dfrac{I-J}{\lambda} \\ \text{satisfies} \\ \text{(i)}(A_\lambda v_1 - A_\lambda v_2, v_1 - v_2) \ge -M\ \|v_1 - v_2\|^2 \\ \text{for every } v_1, v_2 \in \Omega_\lambda, \text{ and} \\ \text{(ii)} \lim\limits_{\lambda\to 0} A_\lambda u = A^0 u, \text{ for every } u \in N. \end{cases} \tag{2.17}$$

*Proof of theorem* (2.1).— We shall organize the proof as follows :

I. Solution of an approximating equation and research of a priori bounds for the approximating solutions.

II. Uniform convergence of the approximating solutions on a common interval of existence.

III. The limit of the approximating solutions is a strong solution of (2.11).

I. Set
$$C = 2\Big(1 + \|A^0 u_0\| + \|F'\|_{L^\infty(0,T;H)}\Big)$$
and let $M, \lambda_0, r, N$ and $\Omega_\lambda$ be as in the statement of the preceding lemma.

By the lower semicontinuity of $f$, and the continuity of $\varphi$, we can suppose (unless of decreasing $r$) that : for every
$$v \in B(u_0, 2r) \Longrightarrow f(v) \geq m, \ v \in \Omega$$
and
$$\omega = 2 \sup\{\varphi(u, x_1, x_2) : u \in B(u_0, 2r), \ m \leq x_1 \leq M, \ \|x_2\| \leq M + C\} < +\infty$$
and (unless of decreasing $\lambda_0$) that for every $\lambda \in ]0, \lambda_0]$ we have $J_\lambda(\Omega_\lambda) \subset B(u_0, 2r)$.

Let $\lambda \in ]0, \lambda_0]$ and
$$u_\lambda : [0, T_\lambda[ \to \Omega_\lambda, \ 0 < T_\lambda \leq T$$
be the unique strong solution of
$$u_\lambda(t) + (b * A_\lambda u_\lambda)(t) = F(t) \tag{2.18}$$
defined on its maximal interval of existence $[0, T_\lambda[$.

We know that $u_\lambda$ satisfies
$$\begin{cases} \dfrac{du_\lambda}{dt} = -A_\lambda u_\lambda(t) + G(u_\lambda)(t) \\ u_\lambda(0) = u_0 \end{cases} \tag{2.19}$$
a.e. on $[0, T_\lambda[$.

Using i) of (2.17), (2.18), we have for every $\lambda \in ]0, \lambda_0]$ and for every $T', h$ :
$$0 < T' < T' + h < T_\lambda$$
that
$$\begin{aligned} \frac{d}{dt}\|u_\lambda(t+h) - u_\lambda(t)\|^2 &\leq 2M(\|u_\lambda(t+h) - u_\lambda(t)\|^2) + \\ &+ 2\|G(u_\lambda)(t+h) - G(u_\lambda)(t)\| \ \|u_\lambda(t+h) - u_\lambda(t)\| \end{aligned} \tag{2.20}$$

a.e. on $[0,T'[$, which implies

$$\|u_\lambda(t+h)-u_\lambda(t)\| \leq \Big(\|u_\lambda(h)-u_\lambda(0)\|+ \\ + \int_0^t \|G(u_\lambda)(\tau+h)-G(u_\lambda)(\tau)\|d\tau\Big)\,.\,\exp\,(2MT'),\ \text{on } [0,T']. \tag{2.21}$$

Using (2.6) we have also

$$\sup_{0\leq s\leq t} \|u_\lambda(s+h)-u_\lambda(s)\| \leq \exp\,(2MT')\cdot \\ \cdot\Big\{\|u_\lambda(h)-u_\lambda(0)\|+\int_0^t \gamma(\tau)\sup_{0\leq\sigma\leq\tau}\|u_\lambda(\sigma+h)-u_\lambda(\sigma)\|d\tau\Big\}; \tag{2.22}$$

so

$$\sup_{0\leq s\leq t} \|u_\lambda(s+h)-u_\lambda(s)\| \leq \exp\,(2MT')\|u_\lambda(h)-u_\lambda(0)\|\cdot \\ \cdot\Big(1+\Big(\int_0^t \exp(\exp\,(2MT')\int_s^T \gamma(\tau)d\tau)ds\Big)\Big) \tag{2.23}$$

which implies, by (2.19), that

$$\|\frac{du_\lambda}{dt}(t)\| \leq \|\frac{d^+u_\lambda}{dt}(0)\|k(T') \leq \\ \leq (\|A_\lambda u_\lambda(0)\|+\|F'(0)\|)k(T'), a.e. \text{ on } [0,T'].^{\dagger} \tag{2.24}$$

Therefore, by (ii) of (2.17) and the definition of $C$, there exist $\overline{T}>0$ and $\epsilon>0$ such that (unless of decreasing $\lambda_0$), for every $\lambda\in]0,\lambda_0]$,

$$\left\|\frac{du_\lambda}{dt}(t)\right\| \leq \frac{C-\epsilon}{2} \text{ a.e. on } [0,\overline{T}]\cap[0,T_\lambda[. \tag{2.25}$$

Furthermore (unless of decreasing $\overline{T}$) by (2.25) we get,

$$\|G(u_\lambda)(t)\| = \|F'(t)+(r*F')(t)+(r*u'_\lambda)(t)\| \\ \leq \|F'\|_{L^\infty(0,T;H)}+\|r\|_{L^\infty(0,T;\mathbf{R})}\cdot\overline{T}\big(\|F'\|_{L^\infty(0,T;H)}+ \\ +\frac{C-\epsilon}{2}\big) \leq \frac{C-\epsilon}{2},\ \text{on } [0,\overline{T}]\cap[0,T_\lambda[, \tag{2.26}$$

which implies, by (2.19) that for every $\lambda\in]0,\lambda_0]$

$$\|A_\lambda u_\lambda(t)\| \leq C-\epsilon, \forall t\in[0,\overline{T}]\cap[0,T_\lambda[ \tag{2.27}$$

† Here $k(\cdot)$ is a computable function such that $\lim_{T'\to 0} k(T')=1$

and

$$\begin{aligned} &\|J_\lambda(u_\lambda(t)) - u_0\| \le \|J_\lambda(u_\lambda(t)) - u_\lambda(t)\| + \\ &+ \|u_\lambda(t) - u_0\| \le \lambda\|A_\lambda u_\lambda(t)\| + \overline{T}(\frac{C-\epsilon}{2}) \\ &\le (C-\epsilon)\big(\lambda_0 + \frac{\overline{T}}{2}\big) < r \end{aligned} \tag{2.28}$$

for every $t \in [0,\overline{T}] \cap [0,T_\lambda[$, (unless of decreasing $\lambda_0$ and $\overline{T}$).

Now, by (2.14) (i) and (2.19), we have

$$\begin{cases} \|A^0 J_\lambda(u_\lambda(t))\| \le \|A_\lambda u_\lambda(t)\| < C - \epsilon \\ f(J_\lambda(u_\lambda(t))) \le M \end{cases} \tag{2.29}$$

which implies, together with (2.28), that

$$J_\lambda(u_\lambda(t)) \in N \tag{2.30}$$

for every $t \in [0,\overline{T}] \cap [0,T_\lambda[$.

On the other hand, by (2.27) we have

$$\|u_\lambda(t) - J_\lambda(u_\lambda(t))\| = \lambda\|A_\lambda u_\lambda(t)\| \le \lambda(C-\epsilon) \tag{2.31}$$

which implies that, for every $\lambda$ in $]0,\lambda_0]$

$$0 < \inf\{d(u_\lambda(t), \partial\Omega_\lambda) : t \in [0,\overline{T}] \cap [0,T_\lambda[\}. \tag{2.32}$$

We conclude that $\overline{T} < T_\lambda$ for every $\lambda \in ]0,\lambda_0]$ since $[0,T_\lambda[$ was the maximal interval of existence of $u_\lambda$.

II. Now we want to show that $(u_\lambda)_\lambda$ converges uniformly on $[0,\overline{T}]$ to a Lipschitz function $u$ such that $f(u)$ is bounded and $u(t) \in D(A)$ for every $t \in [0,\overline{T}]$.

By (2.14), (2.19), the definition of $\omega$ and the $(\varphi, f)$ - monotonicity of $A$, we have for every $\lambda,\mu \in ]0,\lambda_0]$ and a.e. on $[0,\overline{T}]$ that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2 - \big(G(u_\lambda)(t) - G_\mu(u_\mu)(t),\ u_\lambda(t) - u_\mu(t)\big) \\ &= -(A_\lambda u_\lambda(t) - A_\mu u_\mu(t), J_\lambda(u_\lambda(t)) - J_\mu(u_\mu(t))) \\ &\quad - (A_\lambda u_\lambda(t) - A_\mu u_\mu(t), u_\lambda(t) - J_\lambda(u_\lambda(t)) - u_\mu(t) + J_\mu(u_\mu(t))) \\ &\quad \le \omega\|u_\lambda(t) - u_\mu(t)\|^2 + \omega(\lambda\|A_\lambda u_\lambda(t)\| + \mu\|A_\mu u_\mu(t)\|)^2 + \\ &\quad + 2(\lambda+\mu)(\|A_\lambda u_\lambda(t)\| + \|A_\mu u_\mu(t)\|)^2 \\ &\quad \le \omega(\|u_\lambda(t) - u_\mu(t)\|^2) + (\omega(\lambda+\mu)^2 + (\lambda+\mu)). \\ &\quad \cdot (\|A_\lambda u_\lambda(t)\| + \|A_\mu u_\mu(t)\|)^2 \le \\ &\quad \le \omega(\|u_\lambda(t) - u_\mu(t)\|^2 + 4(\lambda+\mu)(1 + (\lambda+\mu)\omega)C^2. \end{aligned} \tag{2.33}$$

Therefore, for a suitable constant $K_1 > 0$ and for every $t \in [0, \overline{T}]$, we have

$$\|u_\lambda(t) - u_\mu(t)\|^2 \leq e^{K_1 T}\Big(\int_0^t \|G(u_\lambda)(\tau) - G(u_\mu)(\tau)\|^2 d\tau +$$
$$+ (\lambda+\mu)K_1 T\Big) \leq e^{K_1 T}\Big(\int_0^t \gamma^2(\tau)\|u_\lambda - u_\mu\|^2_{L^\infty(0,\tau;H)} d\tau +$$
$$+ (\lambda+\mu)K_1 T\Big)$$

which implies, for a suitable $K_2 > 0$,

$$\|u_\lambda - u_\mu\|^2_{L^\infty(0,t;H)} \leq K_2\Big\{\int_0^t \|u_\lambda - u_\mu\|^2_{L^\infty(0,\tau;H)} d\tau + (\lambda+\mu)\Big).$$

So, for a suitable $K_3$,

$$\|u_\lambda - u_\mu\|^2_{L^\infty(0,\overline{T};H)} \leq (\lambda+\mu)K_3. \tag{2.34}$$

Therefore $(u_\lambda)_\lambda$ converges, uniformly on $[0, \overline{T}]$, to a Lipschitz map $u : [0, \overline{T}] \to \Omega$, such that

$$\Big\|\frac{du}{dt}(t)\Big\| < C \quad \text{a.e. on } [0, \overline{T}]. \tag{2.35}$$

By (2.27) we get that $\lim_{\lambda\to 0} J_\lambda(u_\lambda(t)) = u(t)$ uniformly on $[0, \overline{T}]$.

Using now the lower semicontinuity of $f$, (2.29) and (2.15) of lemma (2.1), we conclude that

$$u(t) \in N, \forall t \in [0, \overline{T}] \tag{2.36}$$

and in particular

$$m \leq f(u(t)) \leq M, \; u(t) \in D(A) \tag{2.37}$$

for every $t \in [0, \overline{T}]$.

III. Finally we want to show that $u$ is a strong solution of (2.11) on $[O, \overline{T}]$.

Since

$$\|A_\lambda u_\lambda\|_{L^\infty(0,\overline{T};H)} \leq C - \epsilon \qquad (2.38$$

for every $\lambda \in ]0, \lambda_0]$, there exists $W \in L^\infty(0, \overline{T}; H)$ such that

$$\|W\|_{L^\infty(0,\overline{T};H)} \leq C - \epsilon \tag{2.39}$$

and

$$\lim_{\lambda \to 0} A_\lambda u_\lambda = W \tag{2.40}$$

in the weak topology of $L^\infty(0, \overline{T}; H)$. Let $\lambda < \lambda_0$ be such that $(1 - \lambda\omega) > 0$. By (2.36), we know that

$$v(t) = u(t) + \lambda W(t) \text{ belongs to } \Omega_\lambda \text{ a.e. on } [0, \overline{T}].$$

Using the definition of $A_\lambda$, the $(\varphi, f)$-monotonicity of $A$, the definition of $\omega$ and (2.14) of lemma (2.1) we have

$$\begin{aligned} &\int_0^{\overline{T}} (A_\lambda v(\tau) - A_\mu u_\mu(\tau), J_\lambda(v(\tau)) - J_\mu(u_\mu(\tau)))d\tau \\ &\geq -\omega \int_0^{\overline{T}} \|J_\lambda(v(\tau)) - J_\mu(u_\mu(\tau))\|^2 d\tau \end{aligned} \tag{2.41}$$

Taking the limit as $\mu \to 0$ in (2.41), we get

$$\int_0^{\overline{T}} (1 - \lambda\omega)\|J_\lambda(v(\tau)) - u(\tau)\|^2 d\tau \leq 0 \tag{2.42}$$

which implies that $J_\lambda(v(t)) = u(t)$ and $W(t) \in Au(t)$ a.e. on $[0, \overline{T}]$.

Furthermore, passing to the limit as $\lambda \to 0$ in (2.18), we conclude that $u$ is a strong solution of (2.11).

Finally Remark (2.1) implies the uniqueness of $u$.

We conclude this section with the following :

THEOREM (2.2).— *(Global existence). Let $A$ be a $(\varphi, f)$-monotone operator on $H$ which is $f$-solvable at every point of $D(A)$.*

*Assume that :*

$$\begin{cases} \textit{for every } K > 0, \textit{ the set} \\ \{v \in D(A) : f(v) \leq K, \ \|A^0 v\| \leq K\} \\ \textit{is closed in } \Omega. \end{cases} \tag{2.43}$$

*Suppose that $u_0 \in D(A)$ and (2.9)-(2.10) hold. Let $T_0 (\leq T)$ be the supremum of $\overline{T}$ such that $u$ is a strong solution of (2.11) and (2.13) holds.*

*Then*

$$\textit{for every } t \in [0, T_0[, \ u(t) \in D(A); \tag{2.44}$$

$$\textit{if } \limsup_{t\to T_0^-}\left\{\|\frac{du}{dt}\|_{L^\infty(0,t;H)} \vee f(u(t)) \vee d(u(t),\partial\Omega)^{-1}\right\} < +\infty\dagger \tag{2.45}$$

*then* $T_0 = T$, *u is a strong solution of (2.11) on* $[0,T]$ *and*

$$(u, f(u)) \in \ Lip\ ([0,T];H) \times L^\infty(0,T;\mathbf{R}) \tag{2.46}$$

*Proof*. — We get (2.44) directly from (2.43). Suppose now that

$$\limsup_{t\to T_0^-}\left\{\|\frac{du}{dt}\|_{L^\infty(0,t;H)} \vee f(u(t)) \vee d(u(t),\partial\Omega)^{-1}\right\} < +\infty;$$

we found easily that

$$\lim_{t\to T_0^-} u(t) = \overline{u} \in \Omega \tag{2.47}$$

and by (2.43) $\overline{u} \in D(A)$, $u$ is a strong solution of (2.11) on $[0, T_0]$ and

$$(u, f(u)) \in \ \mathrm{Lip}\ ([0,T_0];H) \times L^\infty(0,T_0;\mathbf{R}) \tag{2.48}$$

Then, if $T_0 < T$, using Theor. (2.1) we can extend $u$ to a right neighborhood of $T_0$. This contradiction proves the claim.

## § 3. The variational case

The aim of this section is to examine some regularity properties of the solution obtained via Theor. (2.1), in the special case $A = \partial^- f$, $f$ being a $\Phi$-convex function, (see def. (1.3)).

THEOREM (3.1). — *Let* $A = \partial^- f$ *be the subdifferential of a* $\Phi$*-convex function* $f : \Omega \to \mathbf{R} \cup \{+\infty\}$.

*Suppose that*

$$u_0 \in D(A) \quad \textit{and } (1.5), (1.6) \quad \textit{hold} \tag{3.1}$$

*and set*

$$T_0 = \sup\{\overline{T} > 0 : u \textit{ is a strong solution of (2.11) on } [0,\overline{T}], \textit{ and (2.13) holds}\}. \tag{3.2}$$

† If $a, b \in \mathbf{R}$ we set $a \vee b = \max(a, b)$.

*Then*

$$f(u) \text{ is Lipschitz continuous on compact subsets of } [0,T_0[, \tag{3.3}$$

*and*

$$\frac{d}{dt}f(u(t)) = \left(W(t), \frac{du}{dt}(t)\right), \text{ a.e. on } [0,T_0] \tag{3.4}$$

*(where W is given by def. (1.4)).*

*Furthermore the following property holds :*

$$\left\{\begin{array}{l} \text{if there exists a continuous function } \overline{X} : D(f)^2 \times \mathbf{R}^2 \to \mathbf{R}^+ \\ \text{such that} \\ (3.6) \quad \Phi(u_1,u_2,x_1,x_2,x_3) = \overline{X}(u_1,u_2,x_1,x_2)(1+|x_3|^2) \\ \text{and} \\ (3.7) \quad \limsup\limits_{t\to T_0^-}\left(-f(u(t)) \vee d(u(t),\partial\Omega)^{-1}\right) < +\infty \\ \text{then } T_0 = T,\ u \text{ is a strong solution of (2.11) on } [0,T] \text{ and} \\ (3.8) \quad (u, f(u)) \in Lip([0,T];H) \times Lip([0,T];\mathbf{R}). \end{array}\right. \tag{3.5}$$

*Proof.* — Let $u$ and $\overline{T}$ be given by theorem (2.1).

By the properties of $\Phi$-convex functions (see (1.20) of [8]) we know that fo every $\lambda \in ]0,\lambda_0]$ the function

$$f_\lambda(v) = \frac{1}{2\lambda}\ \|v - J_\lambda(v)\|^2 + f(J_\lambda(v)) \tag{3.9}$$

is in $C^1(\Omega_\lambda)$ and

$$\text{grad}\ \ f_\lambda(v) = \frac{v - J_\lambda(v)}{\lambda} = A_\lambda(v).$$

Therefore, taking into account that a.e. on $[0,\overline{T}]$ we have

$$\frac{d}{dt}f_\lambda(u_\lambda(t)) = \left(A_\lambda u_\lambda(t), \frac{du_\lambda}{dt}(t)\right) \tag{3.10}$$

we get by (2.25)-(2.27), that $(f_\lambda(u_\lambda))_\lambda$ is a family of equilipschitz maps on $[0,\overline{T}]$, and

$$\lim_{\lambda\to 0} f_\lambda(u_\lambda(t)) = f(u(t)) \tag{3.11}$$

for every $t \in [0,\overline{T}]$, which clearly implies that $f(u)$ is Lipschitz on $[0,\overline{T}]$.

Now, let $\overline{T}_0$ be the supremum of $\overline{T}$ such that $u$ is a strong solution of (2.11) on $[0,\overline{T}]$ and

$$(u,f(u)) \in \text{ Lip }([0,\overline{T}];H) \times \text{ Lip}([0,\overline{T}]);\mathbf{R}).$$

By the preceding remark we know that $\overline{T}_0 > 0$. We want to show that $\overline{T}_0 = T_0$.

We remark that,

$$\frac{d}{dt}f(u(t)) = \left(W(t), \frac{du}{dt}(t)\right) \tag{3.12}$$

a.e. on $[0,T_0[$, $(W(t) \in \partial^- f(u(t)))$. Now, if $\overline{T}_0 < T_0$, by the definition of $T_0$, there exists $K > 0$ such that

$$\sup_{0\le t\le \overline{T}_0}\text{ess}\left\{\|\frac{du}{dt}(t)\| \vee \|W(t)\|\right\} \le K \tag{3.13}$$

which, by (3.12) implies the extendability of $f(u)$ to a Lipschitz map on $[0,\overline{T}_0]$.

Applying theorem (2.1) with initial data $u_0 = u(\overline{T}_0)$ we obtain a contradiction. To prove property (3.5), we remark, first of all, that $u$ satisfies the following problem :

$$\begin{cases} \frac{du}{dt}(t) + W(t) = G(u)(t) \text{ a.e. on } [0,T_0] \\ u(0) = u_0, \end{cases} \tag{3.14}$$

where $W$ is given by (1.5).

Using (3.12), we have for every $t \in [0,T_0[$ that

$$\begin{aligned} &\int_0^t \|\frac{du}{dt}(s)\|^2 ds \le f(u(0)) - f(u(t)) + \\ &+ \left\{\int_0^t \|G(u)(s)\|^2 ds\right\}^{\frac{1}{2}} \left\{\int_0^t \|\frac{du}{dt}(s)\|^2 ds\right\}^{\frac{1}{2}}. \end{aligned} \tag{3.15}$$

Therefore, if $0 < \epsilon < 1$, by (2.6) we get

$$\begin{aligned} &(1-\epsilon)\int_0^t \|\frac{du}{dt}(s)\|^2 ds \le f(u(0)) - f(u(t)) + \\ &+ \frac{1}{4\epsilon}\int_0^t \|G(u)(s)\|^2 ds \le f(u(0)) - f(u(t)) + \frac{1}{2\epsilon}\int_0^t \|G(u)(s) - \\ &- G(u_0)(s)\|^2 ds + \frac{1}{2\epsilon}\int_0^t \|G(u_0)(s)\|^2 ds \le \\ &\le f(u(0)) - f(u(t)) + \frac{1}{2\epsilon}\int_0^t \tau^2\gamma^2(\tau)\left(\int_0^\tau \|\frac{du}{dt}(s)\|^2 ds\right) d\tau + c_1 \end{aligned} \tag{3.16}$$

where $c_1 = \frac{1}{2\epsilon}\int_0^T \|G(u_0)(s)\|^2 ds$.

Now by (3.7), we have

$$\inf\{f(u(t)) : 0 \leq t < T_0\} \in \mathbf{R}. \tag{3.17}$$

Then, for every $t \in [0, T_0[$ and a suitable $c_2 > 0$,

$$\begin{aligned}\int_0^t \left\|\frac{du}{dt}(s)\right\|^2 ds \leq (f(u_0) - c_2)(1-\epsilon)^{-1} + \\ \int_0^t \frac{\tau^2\gamma^2(\tau)}{2\epsilon(1-\epsilon)}\left\{\int_0^\tau \left\|\frac{du}{dt}(s)\right\|^2 ds\right\} d\tau.\end{aligned} \tag{3.18}$$

Using Gronwall lemma, we obtain for every $t \in [0, T_0]$,

$$\int_0^t \left\|\frac{du}{dt}(s)\right\|^2 ds \leq (f(u_0) - c_2)\Big(1 + T \exp\Big(\int_0^T \frac{s^2\gamma^2(s)}{2\epsilon(1-\epsilon)} ds\Big)\Big)(1-\epsilon)^{-1} \tag{3.19}$$

that is

$$\int_0^{T_0} \left\|\frac{du}{dt}(s)\right\|^2 ds < +\infty \tag{3.20}$$

Therefore, $u$ is Hölder continuous, and by (3.7)

$$\lim_{t \to T_0^-} u(t) = \overline{u} \in \Omega. \tag{3.21}$$

This implies that we can extend $u$ on $[0, T_0]$ by putting $u(T_0) = \overline{u}$. In particular by (3.14) we have

$$\int_0^{T_0} \|W(s)\|^2 ds < +\infty, \tag{3.22}$$

and by (3.12), (3.20), (3.22),

$$\sup_{0 \leq t \leq T_0} |f(u(t))| < +\infty. \tag{3.23}$$

Now by (3.6), since $A = \partial^- f$, for every $u, v \in D(A)$ and $\alpha \in Au$, $\beta \in Av$ the following inequality holds

$$(\alpha - \beta, u - v) \geq -X(u, v, f(u), f(v))(1 + \|\alpha\|^2 + \|\beta\|^2) \cdot \|u - v\|^2, \tag{3.24}$$

where

$$X(u_1, u_2, x_1, x_2) = \overline{X}(u_1, u_2, x_1, x_2) + \overline{X}(u_2, u_1, x_2, x_1). \tag{3.25}$$

If now $0 < T' < T' + h < T_0$, by (3.14) we have

$$\begin{aligned}
&\frac{d}{dt}\|u(t+h) - u(t)\|^2 \leq 2X(u(t+h), u(t), f(u(t+h)),\\
&f(u(t)))(1 + \|W(t+h)\|^2 + \|W(t)\|^2)\|u(t+h) - u(t)\|^2\\
&+ 2\|G(u)(t+h) - G(u)(t)\| \ \|u(t+h) - u(t)\|\\
&\leq 2c_3(1 + \|W(t+h)\|^2 + \|W(t)\|^2)\|u(t+h) - u(t)\|^2\\
&+ 2\|G(u)(t+h) - G(u)(t)\| \ \|u(t+h) - u(t)\|. \text{ a.e. on } [0, T'],
\end{aligned}$$

for a suitable $c_3$, since by (3.23)

$$\sup\{X(u(r), u(s), f(u(r)), f(u(s)) : 0 \leq r, s \leq T_0\} < +\infty,$$

Gronwall lemma implies that for every $t \in [O, T']$ we have

$$\begin{aligned}
&\|u(t+h) - u(t)\| \leq (\|u(h) - u(0)\| + \int_0^t \|G(u)(s+h) - G(u)(s)\|)ds)\\
&\quad \exp\Big(c_3 \int_0^t (1 + W(s+h)\|^2 + \|W(s)\|^2)ds\\
&\leq c_4(\|u(h) - u(0)\| + \int_0^t \gamma(s) \sup_{0 \leq \tau \leq s} \|u(\tau + h) - u(\tau)\| ds
\end{aligned} \tag{3.26}$$

which implies that for every $t \in [0, T']$ we have,

$$\|u(t+h) - u(t)\| \leq c_4\|u(h) - u(0)\|\Big(1 + T \exp\,(c_4 \int_0^T \gamma(s)ds)\Big). \tag{3.27}$$

Therefore $\left\|\frac{du}{dt}(t)\right\| \leq c_5$ a.e. on $[0, T_0]$, which implies together to (3.7), that

$$\limsup_{t \to T_0^-}\Big(\|\frac{du}{dt}\| \vee f(u(t)) \vee d(u(t), \partial\Omega)^{-1}\Big) < +\infty. \tag{3.28}$$

Then $T_0 = T$ by theor. (2.2) and (3.8) follows since $\frac{d}{dt}f(u(t)) \in L^\infty(0, T)$.

*Remark* (3.1).— Suppose that $f$ is a $\Phi$-convex function with $\Phi$ given by (3.6), and (2.1), (2.2) hold.

Then, using the same proof of Theor. (2.1), it is easy to show that if $u_1$, $u_2$ are two strong solutions of (2.11) on $[0,T]$ such that (see (3.25))

$$\int_0^T X(u_1(s),u_2(s),f(u_1(s)),f(u_2(s)))(1+\|W_1(s)\|^2+ \\ +\|W_2(s)\|^2)ds < +\infty \tag{3.29}$$

then $u_1 = u_2$ and $W_1 = W_2$ on $[0,T]$.

In the particular situation where $\Phi$ has the form (3.6) we get the local existence of the solution also in the case that $u_0 \in D(f)$.

THEOREM (3.2).— *(Local existence)*

*Suppose that $f$ is a $\Phi$-convex function, with $\Phi$-given by (3.6) and $A = \partial^- f$.*

*Suppose that*

$$u_0 \in D(f) \text{ and (2.9), (2.10) hold.}$$

*Then, there exist $\overline{T} > 0$ and a unique strong solution $u$ of (2.11) on $[0,\overline{T}]$ such that*

$$u \in H^{1,2}(0,\overline{T};H), f(u) \in AC([0,\overline{T}];\mathbf{R}) \tag{3.30}$$

*for every $t \in ]0,\overline{T}]$ :*

$$(u,f(u)) \in \ Lip([t,\overline{T}];H) \times \ Lip([t,\overline{T}];\mathbf{R}) \ \text{ and } u(t) \in D(A). \tag{3.31}$$

*Proof.*— Let $u_0 \in D(f)$.

By [8] Prop. (1.2) we know that, there exists a sequence $(u_{0n})_n \subset D(A)$ such that

$$\lim_n u_{0n} = u_0, \ f(u_{0n}) \le f(u_0) \ (\Rightarrow \lim_n f(u_{0n}) = f(u_0)). \tag{3.32}$$

For every $n$, set

$$F_n(t) = F(t) + (u_{0n} - u_0)$$

and let $T_{0n}$ and $u_n : [0,T_{0n}[\to H$ be such that $T_{0n}$ is the supremum of $T'$ such that $u_n$ is the strong solution of

$$u_n(t) + (b * Au_n)(t) \ni F_n(t) \tag{3.33}$$

on $[0, T']$ and

$$(u_n, f(u_n)) \in \operatorname{Lip}([0, T']; H) \times L^\infty([0, T'); \mathbf{R}) \tag{3.34}$$

I - First of all we want to show that there exists $\overline{T} > 0$ such that $T_{0n} > \overline{T}$ for every $n \in \mathbf{N}$.

Without loss of generality we can suppose that there exists $r > 0$ such that

$$(u_{0n})_n \subset B(u_0, r), \overline{B(u_0, 2r)} \subset \Omega \tag{3.35}$$

and, by the lower semicontinuity of $f$, that

$$f(v) \geq m \text{ for every } v \in B(u_0, 2r). \tag{3.36}$$

As in the proof of (3.5) of Theor. (3.1), we can find a constant $c_1 > 0$, independent of $n$, such that for every $t \in [0, T_{0n}[$, we have

$$\int_0^t \left\| \frac{du_n}{dt}(s) \right\|^2 ds \leq (f(u_{0n}) - f(u_n(t)) + 1)c_1. \tag{3.37}$$

Let $0 < \overline{T} \leq T$ be sucht that

$$\sqrt{\overline{T}(f(u_{0n}) - m + 1)c_1} < r \tag{3.38}$$

Since,

$$d(u_n(t), u_0) \leq r + d(u_n(t), u_{0n}) \leq r + \sqrt{t \int_0^t \| \frac{du}{dt}(s) \|^2 ds} \tag{3.39}$$

we get, by (3.36) and (3.37) that for every $t \in [0, \overline{T}] \cap [0, T_{0n}[$, and for every $n \in \mathbf{N}$ that

$$u_n(t) \in B(u_0, 2r) \tag{3.40}$$

Using (3.37), (3.36) and Theor. (3.1), we have that $\overline{T} < T_{0n}$ for every $n \in \mathbf{N}$.

Furthermore by (3.32), (3.37), (3.36), there exists a constant $c_2 > 0$ such that for every $t \in [0, \overline{T}]$ and $n \in \mathbf{N}$, we have $m \leq f(u_n(t)) \leq c_2$.

II - Now we show that $(u_n)_n$ converges uniformly on $[0, \overline{T}]$ to a continuous function $u : [0, \overline{T}] \to \overline{B(u_0, r)}$ sucht that $u(t) \in D(A)$ a.e. on $[0, \overline{T}]$.

We recall that $u_n$ satisfies

$$\begin{cases} \frac{du_n}{dt}(t) + W_n(t) = G_n(u_n)(t) \text{ a.e. on } [0,\overline{T}] \\ u_n(0) = u_{0n} \end{cases} \tag{3.41}$$

where $G_n(v)(t) = F_n'(t) + (s * F_n')(t) - s(0)v(t) + s(t)v(0) - (v * s')(t)$, ($s$ being the unique solution of $s + b' * s = -b'$).

By decreasing $r$, we can suppose that

$$c_3 = \sup\{X(u_1,u_2,x_1,x_2) : \|u_i - u_0\| \le 2r,\ m \le x_i \le c_2,\ i = 1,2\} < +\infty$$

Using (3.24) and (3.41) we get, for every $n, m \in \mathbf{N}$ and a.e. on $[0,\overline{T}]$ that,

$$\begin{aligned} &\frac{d}{dt}\|u_n(t) - u_m(t)\|^2 \le 2\ c_3(1 + \|W_n(t)\|^2 + \|W_m(t)\|^2)\ \cdot \\ &\cdot \|u_n(t) - u_m(t)\|^2 + 2\|G_n(u_n)(t) - G_m(u_m)(t)\| \\ &\|u_n(t) - u_m(t)\|, \end{aligned} \tag{3.42}$$

which implies that for every $t \in [0,T]$,

$$\begin{aligned} &\|u_n(t) - u_m(t)\| \le (\|u_n(0) - u_m(0)\| + \\ &+ \int_0^t \|G_n(u_n)(s) - G_m(u_m)(s)\|ds)\cdot \\ &\cdot\ \exp(c_3 \int_0^t (1 + \|W_n(s)\|^2 + \|W_m(s)\|^2)ds. \end{aligned} \tag{3.43}$$

By (3.36), (3.37), (3.40) we have

$$\sup\Big\{\int_0^{\overline{T}} \|\frac{du_n}{dt}(s)\|^2 ds : n \in \mathbf{N}\Big\} < +\infty, \tag{3.44}$$

$$\{G_n(u_n)\}_{n\in\mathbf{N}} \text{ is bounded in } L^\infty(0,\overline{T};H), \tag{3.45}$$

which implies by (3.41) that

$$\sup\Big\{\int_0^{\overline{T}} \|W_n(s)\|^2 ds : n \in \mathbf{N}\Big\} < +\infty. \tag{3.46}$$

Therefore there exists a constant $c_4 > 0$, such that for every $n \in \mathbf{N}$ we have, for every $t$ in $[0,\overline{T}]$

$$
\begin{aligned}
&\|u_n(t) - u_m(t)\| \leq c_4(\|u_n(0) - u_m(0)\| + \\
&+ \int_0^t \|G_n(u_n)(s) - G_m(u_m)(s)\| ds) \leq \\
&\leq c_4(\|u_n(0) - u_m(0)\| + \int_0^t \gamma(s) \ \|u_n - u_m\|_{L^\infty(0,s,H)} ds).
\end{aligned} \tag{3.47}
$$

Using again Gronwall lemma we get, for every $n, m \in \mathbf{N}$

$$
\|u_n - u_m\|_{L^\infty(0,\overline{T};H)} \leq c_4\|u_n(0) - u_m(0)\|(1 + \overline{T} \exp\Big(\int_0^{\overline{T}} c_4\gamma(s)ds\Big) \tag{3.48}
$$

which implies that the sequence $(u_n)_n$ converges uniformly on $[0, \overline{T}]$, to a continuous function $u$ such that for every $t \in [0, \overline{T}]$, we have

$$
u(t) \subset \overline{B(u_0, r)}, \ m \leq f(u(t)) \leq c_2. \tag{3.49}
$$

Now, Fatou lemma and (3.46) imply that

$$
\lim_{n\to+\infty} \inf \|W_n(t)\| < +\infty \text{ a.e. on } [0, \overline{T}] \tag{3.50}
$$

which gives (since (2.43) holds (see (1.17) of [8]))

$$
u(t) \in D(A) \ \text{ a.e. on } [0, \overline{T}]. \tag{3.51}
$$

Using Remark (1.14) of [8] together to (3.50) we conclude that

$$
\lim_n f(u_n(t)) = f(u(t)) \text{ a.e. on } [0, \overline{T}]. \tag{3.52}
$$

III - Now for a.e. $t \in [0, \overline{T}]$ such that $u(t) \in D(A)$ and $\lim_n f(u_n(t)) = f(u(t))$ we can apply theor. (2.1) and (3.5) of theor. (3.1) to show that there exists a strong solution $\widetilde{u}$ on $[0, \overline{T} - t]$ of (2.11) such that

$$
(\widetilde{u}, f(\widetilde{u})) \in \text{ Lip}([0, \overline{T} - t]; H) \times \text{ Lip}([0, \overline{T} - t]; \mathbf{R}).
$$

Further, by a technique similar to the one used in step II, we can show that for every $\tau \in [0, \overline{T} - t], u(t + \tau) = \widetilde{u}(\tau)$.

Therefore (3.31) holds; $u \in H^{1,2}(0, \overline{T}; H)$ and $W \in L^2(0, \overline{T}; H)$.

Finally $f(u) \in AC([0,\overline{T}];H)$, since $f(u) \in \mathrm{Lip}([t,\overline{T}];\mathbf{R})$ for every $t \in ]0,\overline{T}]$ and $\frac{d}{dt}f(u) \in L^1(0,\overline{T};\mathbf{R})$ and $\lim_{t\to 0^+} f(u(t)) = f(u(0))$.

COROLLARY (3.1).— *Suppose that*

$$f : H \to \mathbf{R} \cup \{+\infty\}$$

*is a lower semicontinuous function such that there exists a continuous function* $\overline{X} : D(f)^2 \times \mathbf{R}^2 \to \mathbf{R}^+$ *such that for every* $v \in D(f), u \in D(\partial^- f)$, $\alpha \in \partial^- f(u)$ *we have*

$$f(v) \geq f(u) + (\alpha, v-u) - \overline{X}(u,v,f(u),f(v))\ (1 + \|\alpha\|^2)\ \|u-v\|^2$$

*and*

$$\inf\{f(v) : v \in H\} > -\infty.$$

*If*

$$u_0 \in D(f) \tag{3.53}$$

$$b \in AC([0,T]:\mathbf{R}), b' \in BV(0,T;\mathbf{R}), b(0) = 1, \tag{3.54}$$

$$F \in AC([0,T];H), F' \in BV(0,T;H), F(0) = u_0, \tag{3.55}$$

*then there exists a unique strong solution u of*

$$u(t) + (b * \partial^- f(u))(t) \ni F(t) \tag{3.56}$$

*on* $[0,T]$*, such that*

$$(u, f(u)) \in H^{1,2}(0,T;H) \times AC([0,T];\mathbf{R}) \tag{3.57}$$

$$\textit{for every } t \in ]0,T], (u,f(u)) \in \ \mathit{Lip}([t,T];H) \times \ \mathit{Lip}([t,T];\mathbf{R}). \tag{3.58}$$

If in addiction $u_0 \in D(\partial^- f)$, then (3.58) holds also for $t = 0$.

## § 4. Some applications of the abstract results

The aim of this section is to illustrate, by some examples the abstract results obtained in the preceding sections.

The first example is a direct application of Corollary (3.1).

*Example.*— 1 Let $\Lambda \subset \mathbf{R}^N$ be a smooth, bounded open subset and $\varphi_1, \varphi_2 \in H^{2,2}(\Lambda) \cap C(\Lambda)$ such that

$$\varphi_1 \leq \varphi_2 \qquad \text{on } \Lambda \tag{4.1}$$

$$\varphi_1 \leq 0 \leq \varphi_2 \quad \text{on } \partial\Lambda \tag{4.2}$$

Set $H = L^2(\Lambda)$ equipped with the usual norm $\|u\| = \left\{\int_\Lambda |u(x)|^2 dx\right\}^{\frac{1}{2}}$ and consider,

$$K = \{u \in H : \varphi_1 \leq u \leq \varphi_2 \quad \text{a.e. on } \Lambda\} \tag{4.3}$$

$$\Gamma = \{u \in H : \|u\| = p\},\ p > 0. \tag{4.4}$$

Furthermore suppose that :

$$\left.\begin{array}{l} \inf\{\|u\| : u \in K\} < p < \sup\{\|u\| : u \in K\} \\ \|\varphi_i\| \neq p (i = 1, 2). \qquad\qquad \dagger \\ \text{The sets } \{x : \varphi_1(x) < 0\},\ \{x : \varphi_2(x) > 0\} \text{ are connected} \end{array}\right. \tag{4.5}$$

Under the above hypotheses the following facts hold (see [18]).

Let

$$f : H \to \mathbf{R} \cup \{+\infty\}$$

be defined by

$$f(u) = \begin{cases} \frac{1}{2}\int_\Lambda |\nabla u|^2 dx & , \forall u \in H_0^{1,2}(\Lambda) \cap K \cap \Gamma \\ +\infty & , \text{otherwise;} \end{cases} \tag{4.6}$$

then,

(a) $f(\cdot)$ is lower semicontinuous on $H$

(b) $u \in D(\partial^- f) \Longleftrightarrow u \in H_0^{1,2}(\Lambda) \cap H^{2,2}(\Lambda) \cap K \cap \Gamma$

(c) for every $u \in D(\partial^- f)$ there exists $\lambda \in \mathbf{R}$ such that :

$$\operatorname{grad}^- f(u) = \begin{cases} [\Delta u + \lambda u]^- & \text{a.e. on } \{x : \varphi_1(x) < u(x) = \varphi_2(x)\} \\ -\Delta u - \lambda u & \text{a.e. on } \{x : \varphi_1(x) < u(x) < \varphi_2(x)\} \\ -[\Delta u + \lambda u]^+ & \text{a.e. on } \{x : \varphi_1(x) = u(x) < \varphi_2(x)\} \\ 0 & \text{a.e. on } \{x : \varphi_1(x) = u(x) = \varphi_2(x)\} \end{cases}$$

† For example this happens if $\Lambda$ is connected and $\varphi_1 < 0 < \varphi_2$ on $\Lambda$.

(d) there exists a continuous function $\overline{X}$ such that for every $v \in D(f)$, $u \in D(\partial^- f), \alpha \in \partial^- f(u)$ we have;

$$f(v) \geq f(u) + (\alpha, v-u) - \overline{X}(u, f(u))(1+\|\alpha\|)\ \|u-v\|^2. \tag{4.7}$$

Using (a)-(d) we get by Corollary (3.1) the following proposition.

PROPOSITION (4.1).— *Let $f$ be defined by (4.6) and let $T > 0$ be fixed. Assume that*

*(i)* $b \in AC([0,T] : \mathbf{R}), b' \in BV(0,T;\mathbf{R}),\ b(0) = 1$

*(ii)* $F(t,x) \in AC([0,T];H),\ F'(t,x) \in BV(0,T;H)$

*(iii)* $F(0,x) = u_0(x) \in H_0^{1,2}(\Lambda) \cap K \cap \Gamma$.

*Then there exists a unique strong solution of*

$$\begin{cases} u(t,x) + \displaystyle\int_0^t b(t-s)\partial^- f(u(s,x))ds \ni F(t,x) \text{ on } [0,T] \\ u(0,x) = u_0(x) \\ u(t,x) \in H_0^{1,2}(\Lambda) \cap H^{2,2}(\Lambda) \cap K \cap \Gamma,\ \forall t \in ]0,T]. \end{cases} \tag{4.8}$$

We emphasize the fact that in the above example $f$ is not a perturbation with Lipschitz gradient of a convex function (observe that the domain of $f$ is not convex).

As application of theor. (3.2) we have the following example.

*Example.*— 2 Let $\Lambda \subset \mathbf{R}^N$ be a smooth open bounded subset. Set $H = L^2(\Lambda)$. Let us consider the following problem : †

$$\begin{cases} u(t,x) + \displaystyle\int_0^t b(t-s)(-\Delta u(s,x) + g(u(s,x),x)ds = F(t,x), \\ u(t,x) \in H_0^{1,2}(\Lambda), \\ u(0,x) = u_0(x) \in H_0^{1,2}(\Lambda), \end{cases} \tag{4.9}$$

where

$$g : \mathbf{R} \times \Lambda \to \mathbf{R}$$

is a given Caratheodory function.

† In the following, we implicitely assume that $b$ and $F$ satisfy the same hypotheses as in Example 1.

In order to apply theorem (3.2) we assume that the primitive of $g(\cdot, x)$, i.e. the following function :

$$G(s,x) = \int_0^s g(\tau, x)d\tau,$$

satisfies the properties :

(i) there exists $b_0 \in \mathbf{R}$, $a_0 \in L^1(\Lambda)$ such that for every $s \in \mathbf{R}$ we have

$$G(s,x) \geq -a_0(x) - b_0|s|^{2+\frac{4}{N}}$$

(ii) there exists a Caratheodory function

$$\omega : \Lambda \times \mathbf{R}^2 \to \mathbf{R}$$

such that for $\forall r, s \in \mathbf{R}$ and a.e. on $\Lambda$ we have

$(\mathrm{ii})_N$ $|\omega(x,s,r-s)| \leq a(x) + b(|s| + |r-s|)^p$, $2 \leq p < \frac{2N}{N-2}$,†.

$$a \in L^{\frac{p}{p-2}}(\Lambda),\ a(\cdot) \geq 0,\ b \in \mathbf{R},\ \text{if } N > 2;$$

$(\mathrm{ii})_1$ if $N = 1$,

$$|\omega(x,s,r-s)| \leq a(x) + \beta(s, r-s)\ a \in L^2(\Lambda)_1\ \beta \text{ is constinuous}$$

and

$$G(r,x) \geq G(s,x) + g(s,x)(r-s) - \omega(x,s,r-s)\ |r-s|^2,$$
$$G(s,\cdot) \in L^1(\Lambda),\ \text{for a.e. } s \text{ in } \mathbf{R}.$$

Let it be

$$f(u) = \begin{cases} \frac{1}{2}\int_\Lambda\{|\Delta u|^2 + G(u,x)\}dx, & \text{if } u \in H_0^{1,2}(\Lambda) \\ +\infty, & \text{if } u \in H \setminus H_0^{1,2}(\Lambda). \end{cases}$$

It is not difficult to see †† that $f$ is a lower semicontinuous functional and that for every $u_0 \in H_0^{1,2}(\Lambda)$, there exists an $L^2(\Lambda)$-neighborhood $U$ of $u_0$ such that for every $u, v \in U \cap D(f)$ and $\alpha \in \partial^- f(u)$, we have (for some suitable $c, \epsilon > 0$) :

$$f(v) \geq f(u) + (\alpha, v-u) - c(1 + |f(u)| + |f(v)|)^\epsilon \cdot \|v-u\|^2.$$

† If $N = 2$, we may permit any $p \geq 2$

†† For a complete proof of this fact see [19].

A direct application of theor. (3.2) gives :

PROPOSITION (4.2).— *For every* $u_0 \in H_0^{1,2}(\Lambda)$ *there exist* $\overline{T} > 0$ *and a unique strong solution* $u$ *of (4.9) on* $[0, \overline{T}]$, *such that*

$$u \in H^{1,2}(0, \overline{T}; H_0^{1,2}(\Lambda)); u(t, \cdot) \in H^{2,2}(\Lambda) \cap H_0^{1,2}(\Lambda), \ \forall t \in ]0, \overline{T}],$$

$$\int_\Lambda (|\Delta u(t,x)|^2 + G(u(t,x),x))dx \in AC([0,T], \mathbf{R}).$$

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