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CONDITIONS ON THE PROJECTIVE CURVATURE TENSOR OF HYPERSURFACES IN EUCLIDIAN SPACE

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Résumé : Nous étudions les hypersurfaces d'un espace Euclidien satisfaisant à certaines conditions sur le tenseur de courbure projective et nous obtenons des charactérisations locales de certains types des immersions. Nous donnons une charactérisation de certaines hypersurfaces de révolution qui sont, dans le cas 3-dimensionel, des généralisations de la caténoide.

Summary : Hypersurfaces of a Euclidean space satisfying certain conditions on the projective curvature tensor are studied and local characterizations of certain types of immersions are obtained. A characterization is given of certain hypersurfaces of revolution that are, in the 3-dimensional case, generalizations of the catenoid.

I. - INTRODUCTION

In this paper we study hypersurfaces of a Euclidean space satisfying one of the conditions R·P = 0, P·C = 0, C P = 0, P·P = 0, P·R = 0, P·Q = 0 or Q·P = 0, where R denotes the Riemann-Christoffel curvature tensor, Q the Ricci endomorphism, C the Weyl conformal curvature

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tensor and P the Weyl projective curvature tensor of the hypersurface and where the first tensor acts on the second as a derivation.

Riemannian manifolds and submanifolds satisfying similar conditions have been studied by various authors. For references one can consult [3] and [4].

We will prove the following theorems.

THEOREM 1. *Let* $f : (M^n,g) \to \mathbb{E}^{n+1}$ *be an isometric immersion of an n-dimensional Riemannian manifold in* $\mathbb{E}^{n+1}$ $(n > 2)$. *Then the following assertions are equivalent :*

(i) (M^n,g) *satisfies* $R \cdot P = 0$,

(ii) (M^n,g) *satisfies* $R \cdot R = 0$,

(iii) f *is*

(a) *congruent to the inclusion of an open part of a hypersphere* S^n *of* $\mathbb{E}^{n+1}$, *or*

(b) *congruent to the inclusion of an open part of an elliptic hypercone* C^n *of* $\mathbb{E}^{n+1}$, *or*

(c) *an immersion with type-number at most* 2 *in every point, or*

(d) *a locally extrinsic product of the inclusion of an* n_1*-sphere* S^{n_1} *in* $\mathbb{E}^{n_1+1}$ *and the inclusion of an* $(n-n_1)$*-plane* $\mathbb{E}^{n-n_1}$ $(n_1 \in \{3,\dots,n-1\})$, *i.e.* $f(M)$ *is an open part of a spherical hypercylinder, or*

(e) *a locally extrinsic product of the inclusion of an elliptic hypercone in* $\mathbb{E}^{n_1+1}$ *and the inclusion of an* $(n-n_1)$*-plane* $\mathbb{E}^{n-n_1}$ $(n_1 \in \{3,\dots,n-1\})$.

For the equivalence (ii) ⇔ (iii) and elliptic hypercones, see [3].

THEOREM 2. *Let* $f : (M^n,g) \to \mathbb{E}^{n+1}$ *be an isometric immersion of an n-dimensional Riemannian manifold in* $\mathbb{E}^{n+1}$ $(n > 3)$. *Then the following assertions are equivalent:*

(i) (M^n,g) *satisfies* $P \cdot C = 0$,

(ii) (M^n,g) *satisfies* $C \cdot P = 0$,

(iii) (M^n,g) *satisfies* $C \cdot R = 0$,

(iv) (M^n,g) *is conformally flat.*

The equivalence (iii) ⇔ (iv) was shown in [4].

THEOREM 3. *Let* $f : (M^n,g) \to \mathbb{E}^{n+1}$ *be an isometric immersion of an n-dimensional Riemannian manifold in* $\mathbb{E}^{n+1}$ $(n > 2)$. *Then the following assertions are equivalent :*

(i) (M^n,g) *satisfies* $P\cdot R = 0$,

(ii) (M^n,g) *satisfies* $P\cdot P = 0$,

(iii) (M^n,g) *satisfies* $P\cdot Q = 0$,

(iv) (M^n,g) *satisfies* $P = 0$,

(v) f *is congruent to the inclusion of an open part of a hypersphere* S^n *of* $\mathbb{E}^{n+1}$ *or* f *is a cylindrical immersion.*

THEOREM 4. *Let* $f : (M^n,g) \to \mathbb{E}^{n+1}$ *be an isometric immersion* $(n > 2)$. *Then* (M^n,g) *satisfies* $Q\cdot P = 0$ *if and only if*

(i) f *is congruent to the inclusion of an open part of a hypersphere* S^n *of* $\mathbb{E}^{n+1}$, *or*

(ii) *there exists an open dense subset* U *of* M *such that each restriction* f_α *of* f *to a connected component* U_α *of* U *is*

(a) *a cylindrical immersion, or*

(b) *an immersion which is locally congruent around each point in* U_α *to the inclusion of a hypersurface of revolution* K_c^n $(c \in \mathbb{R}_0^+)$.

For a description of the hypersurfaces K_c^n, see section 6. In particular, for n = 3, the hypersurfaces K_c^3 of $\mathbb{E}^4$ are hypercatenoids (in this respect, see also [1]).

2. - BASIC FORMULAS

Let (M^n,g) be a (connected) n-dimensional Riemannian manifold $(n \geqslant 2)$. In the following X,Y,Z denote vector fields which are tangent to M^n. ∇ is the Levi-Civita connection of (M^n,g) and R is the Riemann-Christoffel curvature tensor of (M^n,g). Q is the (1,1)-tensor related to the Ricci tensor S of (M,g) by $g(QX,Y) = S(X,Y)$ for all X and Y. $\tau = \operatorname{tr} Q$ is the scalar curvature of (M,g). $X \wedge Y$ is the (1,1)-tensor field defined by $(X \wedge Y)(Z) := g(Z,Y)X - g(Z,X)Y$. The *Weyl conformal curvature tensor* and the *Weyl projective curvature tensor* are defined by

$$C(X,Y) := R(X,Y) - \frac{1}{n-2}(QX \wedge Y + X \wedge QY) + \frac{\tau}{(n-1)(n-2)} X \wedge Y, \tag{2.1}$$

$$P(X,Y) := R(X,Y) - \frac{1}{n-1}(X \wedge Y) \circ Q. \tag{2.2}$$

Let $f : (M^n,g) \to \mathbb{E}^{n+1}$ be an immersion of (M^n,g) in an (n+1)-dimensional Euclidean space. Let ξ be a local normal section on f. Then the *second fundamental form* h and the *second* *fundamental tensor* A of f are defined by the formulas of Gauss and Weingarten : $\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi$ and $\tilde{\nabla}_X \xi = -AX$ ($\tilde{\nabla}$ is the standard connection of $\mathbb{E}^{n+1}$). A is related to h by $h(X,Y) = g(AX,Y)$. We will not distinguish between A_p and its matrix ($p \in M$). The *type-number* of f *in* $p \in M$ is the rank of A_p. The equation of Codazzi is given by $(\nabla_X A)Y = (\nabla_Y A)X$ and the equation of Gauss is given by

$$R(X,Y) = AX \wedge AY. \tag{2.3}$$

Let $p \in M$. In the following x,y,z denote vectors in T_pM. Let $x \wedge y$ denote the endomorphism $T_pM \to T_pM : z \mapsto g(z,y)x - g(z,x)y$. Since A_p is symmetric, there exists an orthonormal basis $\{e_1,\dots,e_n\}$ of (T_pM,g_p) consisting of eigenvectors of A_p, i.e. such that

$$Ae_i = \lambda_i e_i , \tag{2.4}$$

where $\lambda_i \in \mathbb{R}$ for each $i \in \{1,\dots,n\}$. $\lambda_1,\dots,\lambda_n$ are called the *principal curvatures* of f *in* p. (2.1), (2.2), (2.3) and (2.4) imply that

$$\begin{aligned}
R(e_i,e_j) &= c_{ij} e_i \wedge e_j ,\\
Q e_i &= \mu_i e_i ,\\
C(e_i,e_j) &= a_{ij} e_i \wedge e_j ,\\
P(e_i,e_j)e_k &= \left(c_{ij} - \frac{\mu_k}{n-1}\right)(\delta_{kj}e_i - \delta_{ki}e_j) ,
\end{aligned}$$

(2.5) where

$$\begin{aligned}
c_{ij} &= \lambda_i \lambda_j ,\\
\mu_i &= \lambda_i(\mathrm{tr}A - \lambda_i) ,\\
a_{ij} &= c_{ij} - \frac{1}{n-2}(\mu_i + \mu_j) + \frac{(\mathrm{tr}A)^2 - \mathrm{tr}A^2}{(n-1)(n-2)}
\end{aligned}$$

for all i,j and k in $\{1,\dots,n\}$.

Let $\bar{\lambda}_1,\dots,\bar{\lambda}_r$ denote the mutually distinct eigenvalues of A_p with multiplicities $s_1,\dots,s_r$ respectively. Denote by V_α the space of eigenvectors with eigenvalue $\bar{\lambda}_\alpha$ ($\alpha \in \{1,\dots,r\}$). If $e_i,e_k \in V_\alpha$ and $e_j,e_\ell \in V_\beta$, then $c_{ij} = c_{k\ell}$, $\mu_i = \mu_k$ and $a_{ij} = a_{k\ell}$ ($i,j,k,\ell \in \{1,\dots,n\}$ and $\alpha,\beta \in \{1,\dots,r\}$). We define numbers $\bar{c}_{\alpha\beta} := c_{ij}$, $\bar{\mu}_\alpha := \mu_i$ and $\bar{a}_{\alpha\beta} = a_{ij}$ where $e_i \in V_\alpha$ and $e_j \in V_\beta$ ($i,j \in \{1,\dots,n\}$ and $\alpha,\beta \in \{1,\dots,r\}$).

According to Lemma 2.1 in [8] there exist n continuous functions $\lambda_1 \leqslant \dots \leqslant \lambda_n$ on the domain of ξ on M such that for each p in M the eigenvalues of A_p are given by $\lambda_1(p),\dots,\lambda_n(p)$.

It easily follows that the subsets $M_r = \{ p \in M \mid$ the number of distinct eigenvalues of A_p is at least $r \}$ of M are open ($i \in \{1,...,n\}$). $U := M_n \cup \mathrm{int}(M_{n-1} \setminus M_n) \cup ... \cup \mathrm{int}(M_1 \setminus M_2)$ is an open dense subset of M such that on each connected component of U the number of distinct eigenvalues is constant, the multiplicities of the eigenvalues are constant and the eigenvalue functions are differentiable (see [9]).

(M^n,g) is called *(locally) conformally flat* if (M,g) is (locally) conformally equivalent to $\mathbb{E}^n$. It is well known that (M^n,g) is conformally flat if and only if $C = 0$ for $n \geq 4$. We recall that every surface is conformally flat and that $C = 0$ for every 3-dimensional Riemannian manifold. It is well known that (M^n,g) is *locally projectively equivalent to* $\mathbb{E}^n$ (i.e. around each point of M^n there exists a mapping to $\mathbb{E}^n$ preserving geodesics) if and only if $P = 0$ for $n \geq 3$. Every surface satisfies $P = 0$.

f is called *totally umbilical* if its second fundamental tensor is proportional to the identity map everywhere. It is well known that f is totally umbilical if and only if f is congruent to the inclusion of an open part of a hypersphere or a hyperplane [2].

f is called *quasi-umbilical* if for each point p in M A_p has an eigenvalue with multiplicity at least n−1. For $n \geq 4$, E. Cartan proved that f is quasi-umbilical if and only if (M^n,g) is conformally flat. We remark that $C = 0$ in p if and only if A_p has an eigenvalue with multiplicity at least n−1 if $n \geq 4$ (i.e. also the «pointwise» version of Cartan's result holds).

f is called *cylindrical* if rank $A_p \leq 1$ for each p in M. f is cylindrical if and only if (M^n,g) is locally flat. A complete cylindrical immersion is a cylinder over a plane curve [5].

Concerning the notations $P{\cdot}C = 0$, $C{\cdot}P = 0$, $P{\cdot}Q = 0$,... we say for example that (M^n,g) satisfies $P{\cdot}C = 0$ if and only if $P(X,Y).C = 0$ for all vector fields X and Y tangent to M, where P(X,Y) acts as a derivation on the algebra of tensor fields on M, i.e.

$$(P(X,Y){\cdot}C)(Z,W)V = P(X,Y)C(Z,W)V - C(P(X,Y)Z,W)V - C(Z,P(X,Y)W)V - C(Z,W)P(X,Y)V$$

for X,Y,Z,V,W tangent to M^n. The derivation R(X,Y). is the derivation $\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$. $R(X,Y){\cdot}g = 0$ and $C(X,Y){\cdot}g = 0$ for all vector fields X and Y while in general $P(X,Y){\cdot}g \neq 0$ and $Q{\cdot}g \neq 0$. We remark that $P{\cdot}g = 0$ if and only if (M^n,g) is Einstein.

For any (1,3)-tensor field T on M we define $(C_{1,4}T)(Y,Z) = \sum_{i=1}^{n} g(T(E_i,Y)Z,E_i)$ and $(C_{1,3}T)(Y,Z) = \sum_{i=1}^{n} g(T(E_i,Y)E_i,Z)$ for all vector fields Y,Z and any local orthonormal frame field $\{E_1,...,E_n\}$. The following lemma shows that certain derivations commute with certain contractions.

LEMMA 2.1. *Let* B *be a* (1,1)*-tensor field and* T *a* (1,3)*-tensor field on* M. *Then*

(i) $C_{1,4}(B\cdot T) = B\cdot(C_{1,4}T)$,

(ii) $C_{1,3}(B\cdot T) = B\cdot(C_{1,3}T)$ *if* B *is antisymmetric,*

(iii) $(B\cdot P)(X,Y)Z = (B\cdot R)(X,Y)Z - \frac{1}{n-1}\left\{(B\cdot S)(Z,Y)X - (B\cdot S)(Z,X)Y\right\}$ *for all vector fields* X,Y,Z.

Proof. (i) We have

$$\begin{aligned}(C_{1,4}(B\cdot T))(Y,Z) &= \sum_{i=1}^{n} g((B\cdot T)(E_i,Y)Z,E_i) = \\
&= \sum_{i=1}^{n} \{g(BT(E_i,Y)Z,E_i) - g(T(BE_i,Y)Z,E_i) \\
&\quad - g(T(E_i,BY)Z,E_i) - g(T(E_i,Y)BZ,E_i)\} \\
&= \sum_{i,j=1}^{n} g(BE_j,E_i)\,g(T(E_i,Y)Z,E_j) - \sum_{i,j=1}^{n} g(T(E_j,Y)Z,E_i)g(BE_i,E_j) \\
&\quad - (C_{1,4}T)(BY,Z) - (C_{1,4}T)(Y,BZ) \\
&= \sum_{i,j=1}^{n} g(BE_i,E_j)g(T(E_j,Y)Z,E_i) - \sum_{i,j=1}^{n} g(T(E_j,Y)Z,E_i)g(BE_i,E_j) \\
&\quad + (B\cdot(C_{1,4}T))(Y,Z) \\
&= (B\cdot(C_{1,4}T))(Y,Z)\end{aligned}$$

for all vector fields Y,Z.

(ii) We have

$$\begin{aligned}(C_{1,3}(B\cdot T))(Y,Z) &= \sum_{i=1}^{n} g((B\cdot T)(E_i,Y)E_i,Z) = \\
&= \sum_{i=1}^{n} \{g(BT(E_i,Y)E_i,Z) - g(T(BE_i,Y)E_i,Z) \\
&\quad - g(T(E_i,BY)E_i,Z) - g(T(E_i,Y)BE_i,Z)\} \\
&= - \sum_{i=1}^{n} g(T(E_i,Y)E_i,BZ) - \sum_{i,j=1}^{n} g(T(E_j,Y)E_i,Z)g(BE_i,E_j) \\
&\quad - \sum_{i=1}^{n} g(T(E_i,BY)E_i,Z) - \sum_{i,j=1}^{n} g(T(E_i,Y)E_j,Z)g(BE_i,E_j)\end{aligned}$$

$$= - (C_{1,3}T)(Y,BZ) - \sum_{i,j=1}^{n} g(T(E_i,Y)E_j,Z)g(BE_j,E_i)$$
$$- (C_{1,3}T)(BY,Z) - \sum_{i,j=1}^{n} g(T(E_i,Y)E_j,Z)g(BE_i,E_j)$$
$$= (B\cdot(C_{1,3}T))(Y,Z)$$

for all vector fields Y,Z if B is antisymmetric.

(iii) This is proved by a straightforward computation. ∎

3. - THE CONDITION $R\cdot P = 0$

The proof of the equivalence (ii) ⇔ (iii) in Theorem 1 was given in [3]. We show that each Riemannian manifold satisfying $R\cdot R = 0$ also satisfies $R\cdot P = 0$ and conversely (*).

Suppose that a Riemannian manifold (M^n,g) satisfies $R\cdot R = 0$. By Lemma 2.1 (i) and (iii) (M^n,g) also satisfies $R\cdot P = 0$ since $C_{1,4}R = S$. Conversely, assume that (M^n,g) is a Riemannian manifold with $R\cdot P = 0$. It is easily seen that $C_{1,3}P = -\frac{n}{n-1} S + \frac{\tau}{n-1} g$. By Lemma 2.1 (ii) (M^n,g) satisfies $R\cdot(C_{1,3}P) = 0$. Moreover, since $R\cdot g = 0$, this shows that $R\cdot S = 0$. Lemma 2.1 (iii) then implies that (M^n,g) satisfies $R\cdot R = 0$. This finishes the proof of Theorem 1. ∎

4. - THE CONDITIONS $P\cdot C = 0$ AND $C\cdot P = 0$

The equivalence of (iii) and (iv) in Theorem 2 was shown in [4] and the implications (iv) ⇒ (i) and (iv) ⇒ (ii) are evident.

Proof of (i) ⇒ (iv). Let $f : (M^n,g) \to \mathbb{E}^{n+1}$ be an isometric immersion of a Riemannian manifold satisfying $P\cdot C = 0$. We shall show that $C = 0$.

Let $p \in M^n$ and choose a basis $\{ e_1,\dots,e_n \}$ of T_pM^n satisfying (2.4). Using the formulas (2.5), we find that

(*) *We thank R. Deszcz for pointing this out to us.*

$$
\begin{aligned}
(P(e_i,e_j)\cdot C)(e_k,e_\ell)e_m = \Big\{ & \delta_{jk}\delta_{\ell m}\,(c_{ij} - \frac{\mu_k}{n-1})(a_{k\ell} - a_{i\ell}) \\
& - \delta_{j\ell}\delta_{km}\,(c_{ij} - \frac{\mu_\ell}{n-1})(a_{k\ell} - a_{ik})\Big\}\, e_i \\
+ \Big\{ & - \delta_{ik}\delta_{\ell m}\,(c_{ij} - \frac{\mu_k}{n-1})(a_{k\ell} - a_{ik}) \\
& + \delta_{i\ell}\delta_{km}\,(c_{ij} - \frac{\mu_\ell}{n-1})(a_{k\ell} - a_{jk})\Big\}\, e_j \\
+ \Big\{ & - \delta_{i\ell}\delta_{jm}\Big[(c_{ij} - \frac{\mu_m}{n-1})a_{k\ell} - (c_{ij} - \frac{\mu_\ell}{n-1})a_{jk}\Big] \\
& + \delta_{im}\delta_{j\ell}\Big[(c_{ij} - \frac{\mu_m}{n-1})a_{k\ell} - (c_{ij} - \frac{\mu_\ell}{n-1})a_{ik}\Big]\Big\}\, e_k \\
+ \Big\{ & \delta_{ik}\delta_{jm}\Big[(c_{ij} - \frac{\mu_m}{n-1})a_{k\ell} - (c_{ij} - \frac{\mu_k}{n-1})a_{j\ell}\Big] \\
& - \delta_{im}\delta_{jk}\Big[(c_{ij} - \frac{\mu_m}{n-1})a_{k\ell} - (c_{ij} - \frac{\mu_k}{n-1})a_{i\ell}\Big]\Big\}\, e_\ell
\end{aligned}
$$

for all i,j,k and ℓ in $\{1,\dots,n\}$. For mutually distinct i,j and k in $\{1,\dots,n\}$, we obtain from $(P(e_i,e_j)\cdot C)(e_k,e_i)e_j = 0$, $(P(e_i,e_j)\cdot C)(e_k,e_i)e_k = 0$ and $(P(e_i,e_k)\cdot C)(e_i,e_k)e_i = 0$ that

$$(4.1)\qquad (\mu_i - \mu_j)a_{ik} = 0,$$

$$(4.2)\qquad ((n-1)c_{ij} - \mu_i)(a_{ik} - a_{jk}) = 0,$$

and

$$(4.3)\qquad (\mu_i - \mu_k)a_{ik} = 0.$$

Now suppose $C \neq 0$ in p. We shall then show that a contradiction follows. We may assume that $a_{12} \neq 0$. Taking $i = 1$, $k = 2$ and $j \in \{3,\dots,n\}$ in (4.1) and (4.3), we obtain that $\mu_1 = \mu_2 = \dots = \mu_n$. This gives that

$$(4.4)\qquad (\lambda_i - \lambda_j)(\mathrm{tr}A - \lambda_i - \lambda_j) = 0$$

for all mutually distinct i and j in $\{1,\dots,n\}$.

Let $\bar\lambda_1,\dots,\bar\lambda_r$ denote the mutually distinct eigenvalues of A in p and let $s_1,\dots,s_r$ be their respective multiplicities.

Suppose $r \geqslant 3$. Take mutually distinct α,β,γ in $\{1,\dots,r\}$. Then (4.4) implies that $\operatorname{tr}A - \bar{\lambda}_\alpha - \bar{\lambda}_\beta = 0$ and that $\operatorname{tr}A - \bar{\lambda}_\alpha - \bar{\lambda}_\gamma = 0$. This gives a contradiction. Assume $r = 2$. (4.2) is equivalent to

$$\lambda_i(\lambda_i - \lambda_j)(\operatorname{tr}A - \lambda_i - (n-1)\lambda_j)(\operatorname{tr}A - \lambda_i - \lambda_j - (n-2)\lambda_k) = 0 \tag{4.5}$$

for all mutually distinct i,j and k in $\{1,\dots,n\}$. We may suppose that $s_2 > 1$. Choosing mutually distinct i,j and k in $\{1,\dots,n\}$ in (4.5) such that $\lambda_i = \bar{\lambda}_2$ and $\lambda_j = \lambda_k = \bar{\lambda}_1$, we find that $\bar{\lambda}_2 = 0$. (4.4) now implies that $s_1 = 1$. From (2.5) it is easily seen that $C = 0$ in p (A_p has an eigenvalue with multiplicity $n-1$). This gives a contradiction.

If $r = 1$, (2.5) shows that $C = 0$ in p, which again contradicts our initial assumption $C \neq 0$ in p. This proves the implication.

Proof of (ii) ⇔ *(iii).* In the same way as in section 3 we can prove that (M^n,g) satisfies $C\cdot R = 0$ if and only if it satisfies $C\cdot P = 0$. This finishes the proof of Theorem 2. ■

5. - THE CONDITIONS P·P = 0, P·R = 0 AND P·Q = 0

First we will prove the following lemmas.

LEMMA 5.1. *Let* $f : (M^n,g) \to \mathbb{E}^{n+1}$ *be an isometric immersion of an* n*-dimensional Riemannian manifold* ($n > 2$). *Then the following statements are equivalent :*

(i) (M^n,g) *satisfies* $P\cdot R = 0$,

(ii) (M^n,g) *satisfies* $P\cdot P = 0$,

(iii) *for each* $p \in M^n$ A_p *is one of the following types :*

(a) λI_n *with* $\lambda \in \mathbb{R}_0$,

(b) $\begin{pmatrix} \lambda & 0 \\ 0 & 0_{n-1} \end{pmatrix}$ *with* $\lambda \in \mathbb{R}$.

Proof. It is easy to check that the implication (iii) ⇒ (i) holds : in fact $P = 0$ if (iii) is true. Next we show that (i) implies (ii). Suppose that (M^n,g) satisfies $P\cdot R = 0$. By Lemma 2.1 (i) and (iii), (M^n,g) then also satisfies $P\cdot P = 0$. Finally, we prove that (ii) implies (iii). Suppose that (M^n,g) satisfies $P\cdot P = 0$.

Let $p \in M^n$ and choose a basis $\{e_1,...,e_n\}$ for T_pM satisfying (2.4). Using the formulas (2.5), we find that

$$(P(e_i,e_j)\cdot P)(e_k,e_i)e_\ell = \delta_{k\ell}(c_{ij} - \frac{\mu_i}{n-1})(c_{ik} - c_{jk})e_j$$

$$+ \delta_{j\ell}\{(c_{ij} - \frac{\mu_i}{n-1})(c_{jk} - \frac{\mu_\ell}{n-1}) - (c_{ij} - \frac{\mu_\ell}{n-1})(c_{ik} - \frac{\mu_i}{n-1})\} e_k$$

for all mutually distinct i, j and k in $\{1,...,n\}$ and all ℓ in $\{1,...,n\}$. We obtain from $(P(e_i,e_j)\cdot P)(e_k,e_i)e_j = 0$ and $(P(e_i,e_j).P)(e_k,e_i)e_k = 0$ that

$$(5.1) \qquad \lambda_i\lambda_j(\lambda_i - \lambda_j)(\mathrm{tr}A - \lambda_i - \lambda_j - (n-2)\lambda_k) = 0$$

and

$$\lambda_i\lambda_k(\lambda_i - \lambda_j)(\mathrm{tr}A - \lambda_i - (n-1)\lambda_j) = 0$$

for all mutually distinct i, j and k in $\{1,...,n\}$. λ_k (5.1) $- \lambda_j$ (5.2) gives that

$$(5.3) \qquad \lambda_i\lambda_j\lambda_k(\lambda_i - \lambda_j)(\lambda_k - \lambda_j) = 0$$

for all mutually distinct i, j and k in $\{1,...,n\}$.

Let $\bar\lambda_1,...,\bar\lambda_r$ denote the mutually distinct eigenvalues of A in p and let $s_1,...,s_r$ be their respective multiplicities.

Suppose $r \geq 4$. Then (5.3) yields $\bar\lambda_\alpha\bar\lambda_\beta\bar\lambda_\gamma(\bar\lambda_\alpha - \bar\lambda_\beta)(\bar\lambda_\gamma - \bar\lambda_\beta) = 0$ for mutually distinct α, β, γ and δ in $\{1,...,r\}$. We may therefore assume that $\bar\lambda_\alpha = 0$. (5.3) now gives that $\bar\lambda_\beta\bar\lambda_\gamma\bar\lambda_\delta(\bar\lambda_\beta - \bar\lambda_\gamma)(\bar\lambda_\delta - \bar\lambda_\gamma) = 0$, which is impossible. We conclude that $r \leq 3$.

Assume $r = 3$. It then follows from (5.3) that $\bar\lambda_1\bar\lambda_2\bar\lambda_3\ (\bar\lambda_1 - \bar\lambda_2)(\bar\lambda_3 - \bar\lambda_2) = 0$ which implies that, for instance, $\bar\lambda_1 = 0$. Choosing i, j and k in $\{1,...,n\}$ such that $\lambda_i = \bar\lambda_2$, $\lambda_j = \bar\lambda_3$ and $\lambda_k = \bar\lambda_1$, (5.1) gives that

$$(5.4) \qquad (s_2 - 1)\bar\lambda_2 + (s_3 - 1)\bar\lambda_3 = 0.$$

Furthermore, for i, j and k in $\{1,...,n\}$ such that $\lambda_i = \bar\lambda_2$, $\lambda_j = \bar\lambda_1$ and $\lambda_k = \bar\lambda_3$, (5.2) yields

$$(5.5) \qquad (s_2 - 1)\bar\lambda_2 + s_3\bar\lambda_3 = 0.$$

(5.4) contradicts (5.5). So $r \neq 3$.

Suppose $r = 2$. Then we may assume that $s_2 \geq 2$. Taking mutually distinct i, j and k in $\{1,...,n\}$ such that $\lambda_i = \lambda_k = \bar{\lambda}_2$ and $\lambda_j = \bar{\lambda}_1$, (5.3) gives that $\bar{\lambda}_1 \bar{\lambda}_2^2 (\bar{\lambda}_2 - \bar{\lambda}_1)^2 = 0$. We conclude that $\bar{\lambda}_1 = 0$ or $\bar{\lambda}_2 = 0$. First we show that $\bar{\lambda}_2 = 0$. Suppose that $\bar{\lambda}_2 \neq 0$. Then $\bar{\lambda}_1 = 0$. It follows from (5.2), taking the same choice for the indices i, j and k as above, that $\mathrm{tr}A = \bar{\lambda}_2$. This would mean that $s_2 = 1$. This contradicts one of our initial assumptions. Secondly, we show that $s_1 = 1$. Suppose $s_1 \geqslant 2$. Then we can choose mutually distinct i,j and k in $\{1,...,n\}$ such that $\lambda_i = \lambda_k = \bar{\lambda}_1$ and $\lambda_j = \bar{\lambda}_2 = 0$. Formula (5.2) now gives that $\bar{\lambda}_1 = \mathrm{tr}A$, from which we conclude that $s_1 = 1$. This is in contradiction with the assumption $s_1 \geqslant 2$. This shows that the matrix of A_p in the basis $\{e_1,...,e_n\}$ has one of the desired forms.

Finally, the case r = 1 is trivial. This finishes the proof of Lemma 5.1. ■

LEMMA 5.2. *Let* $f : (M^n,g) \to \mathbb{E}^{n+1}$ *be an isometric immersion of an* n*-dimensional Riemannian manifold* (n > 2). *Then* (M^n,g) *satisfies* $P\cdot Q = 0$ *if and only if for each point* p *in* M^n A_p *is one of the following types :*

(a) λI_n *with* $\lambda \in \mathbb{R}_0$,

(b) $$\left(\begin{array}{c|c} (s_2-1)\lambda I_{s_1} & 0 \\ \hline 0 & -(s_1-1)\lambda I_{s_2} \end{array}\right)$$ *with* $\lambda \in \mathbb{R}_0$, $s_1, s_2 \in \mathbb{N} \setminus \{0,1\}$ *and* $s_1 + s_2 = n$.

(c) $$\left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & 0_{n-1} \end{array}\right)$$ *with* $\lambda \in \mathbb{R}$.

Proof. Let $i : (M^n,g) \to \mathbb{R}^{n+1}$ be an isometric immersion of an n-dimensional Riemannian manifold. Let p be a point in M and choose a basis $\{e_1,...,e_n\}$ for T_pM satisfying (2.4). Using the formulas (2.5), we find that $(P(e_i,e_j)\cdot Q)e_k = (c_{ij} - \frac{\mu_k}{n-1})\{(\mu_k - \mu_i)\delta_{jk}e_i - (\mu_k - \mu_j)\delta_{ik}e_j\}$ for all i,j and k in $\{1,...,n\}$. From this we learn that $P\cdot Q = 0$ in p if and only if $(P(e_i,e_j)\cdot Q)e_i = 0$ for all mutually distinct i and j in $\{1,...,n\}$. This implies that $P\cdot Q = 0$ if and only if

$$(5.6) \qquad \lambda_i(\lambda_i - \lambda_j)(\mathrm{tr}A - \lambda_i - \lambda_j)(\mathrm{tr}A - \lambda_i - (n-1)\lambda_j) = 0$$

for all mutually distinct i and j in $\{1,...,n\}$.

Let $\bar{\lambda}_1,...,\bar{\lambda}_r$ denote the mutually distinct eigenvalues of A in p and let $s_1,...,s_r$ be their respective multiplicities. We will show that $P\cdot Q = 0$ in p if and only if

$$(5.7) \qquad \mathrm{tr}A - \bar{\lambda}_\alpha - \bar{\lambda}_\beta = 0$$

for all distinct α and β in $\{1,...,r\}$. For different α and β in $\{1,...,r\}$, (5.6) gives that

$$\bar{\lambda}_\alpha(\mathrm{tr}A - \bar{\lambda}_\alpha - \bar{\lambda}_\beta)(\mathrm{tr}A - \bar{\lambda}_\alpha - (n-1)\bar{\lambda}_\beta) = 0. \tag{5.8}$$

and

$$\bar{\lambda}_\beta(\mathrm{tr}A - \bar{\lambda}_\beta - \bar{\lambda}_\alpha)(\mathrm{tr}A - \bar{\lambda}_\beta - (n-1)\bar{\lambda}_\alpha) = 0. \tag{5.9}$$

Substracting (5.9) from (5.8) gives $(\bar{\lambda}_\alpha - \bar{\lambda}_\beta)(\mathrm{tr}A - \bar{\lambda}_\alpha - \bar{\lambda}_\beta)^2 = 0$. So $P{\cdot}Q = 0$ in p implies (5.7). The other implication is trivial.

Now it is easy to see that immersions for which all second fundamental tensors have the form described in the lemma are immersions of Riemannian manifolds satisfying $P{\cdot}Q = 0$. Next we show the converse.

Assume that $r \geqslant 3$. Choose mutually distinct α, β and γ in $\{1,...,r\}$. Then, by (5.7) we have $\mathrm{tr}A - \bar{\lambda}_\alpha - \bar{\lambda}_\beta = 0$ and $\mathrm{tr}A - \bar{\lambda}_\alpha - \bar{\lambda}_\gamma = 0$. This gives $\bar{\lambda}_\beta = \bar{\lambda}_\gamma$, which is impossible.

Suppose that $r = 2$. First we assume that $s_1 \geqslant 2$ and $s_2 \geqslant 2$. (5.7) learns that $P{\cdot}Q = 0$ if and only if A in p has the form (b) in the lemma. If, say, $s_1 = 1$, then $\bar{\lambda}_2 = 0$ by (5.7). So A in p has the form (c) in the lemma.

If $r = 1$, then A in p has one of the desired forms. This proves the lemma. ∎

Now we prove Theorem 3. Using Lemma 5.1 and Lemma 5.2 it is easy to see that (v) ⇒ (i), (i) ⇒ (ii) and (ii) ⇒ (iii) hold. The equivalence (iv) ⇔ (v) is well known. Thus, we only must show that (iii) implies (v).

Call

$$M_1 := \{p \in M \mid A_p = \lambda(p) I_{T_pM} \text{ for some } \lambda(p) \in \mathbb{R}_0\}$$

and

$$M_2 := \left\{ p \in M \;\middle|\; \begin{array}{l} A_p = \left(\begin{array}{c|c} (s_2(p)-1)\lambda(p) I_{s_1(p)} & \\ \hline & -(s_1(p)-1)\lambda(p) I_{s_2(p)} \end{array}\right) \\ \text{for some } s_1(p), s_2(p) \in \mathbb{N} \setminus \{0,1\} \text{ and some } \lambda(p) \in \mathbb{R}_0 . \end{array} \right\}$$

M_1 and M_2 are open.

First, we show that $M_2 = \phi$. Suppose that $M_2 \neq \phi$ and let W_2 be a connected component of M_2. By Proposition 2.3 in [8], the distributions $T_1 := \{X \in TW_2 \mid A_X = (s_2-1)\lambda X\}$ and

$T_2 := \{ X \in TW_2 \mid AX = -(s_1-1)\lambda X \}$ are differentiable and involutive and λ is a constant function on W_2. We show that T_1 and T_2 are parallel. Let X_1 and Y_1 (resp. X_2 and Y_2) be vector fields with values in T_1 (resp. T_2). The equation of Codazzi $(\nabla_{X_1}A)X_2 = (\nabla_{X_2}A)X_1$ then gives that $(A + (s_1-1)\lambda)\nabla_{X_1}X_2 = (A - (s_2-1)\lambda)\nabla_{X_2}X_1$. From this we obtain that $(A + (s_1-1)\lambda)\nabla_{X_1}X_2 = 0$ and $(A - (s_2-1)\lambda)\nabla_{X_2}X_1 = 0$. Therefore, $\nabla_{X_1}X_2$ has only values in T_2 and $\nabla_{X_2}X_1$ has only values in T_1.

Furthermore, $0 = X_1 < Y_1,Z_2 > = < \nabla_{X_1}Y_1,Z_2 > + < Y_1,\nabla_{X_1}Z_2 > = < \nabla_{X_1}Y_1,Z_2 >$ for each vector field Z_2 with values in T_2. This shows that $\nabla_{X_1}Y_1$ always has only values in T_1. Similarly, $\nabla_{X_2}Y_2$ always has only values in T_2. The equation of Gauss gives that

$$R(X_1,X_2) = -(s_1-1)(s_2-1)\lambda^2 X_1 \wedge X_2. \tag{5.10}$$

On the other hand, $g(R(X_1,X_2)X,Y) = g(R(X,Y)X_1,X_2) = g(\nabla_X\nabla_Y X_1 - \nabla_Y\nabla_X X_1 - \nabla_{[X,Y]}X_1,X_2) = 0$ for all vector fields X and Y tangent to W_2 since T_1 is parallel. This gives a contradiction with (5.10). This proves that $M_2 = \phi$.

Suppose $M_1 \neq \phi$. Let W_1 be a connected component of M_1. W_1 is open. $f_{|W_1}$ is totally umbilical. In particular, λ is a constant function on W_1. W_1 is closed as well : since the eigenvalue functions of A can be chosen to be continuous functions (see [8]), $A_q = \lambda I_{T_qM}$ (with $\lambda \in \mathrm{IR}_0$) for each q in $\overline{W}_1$, i.e. $\overline{W}_1 \subset W_1$. Since M^n is connected, $W_1 = M^n$ and f is a totally umbilical immersion.

If $M_1 = \phi$, f is a cylindrical immersion. This finishes the proof of Theorem 3. ∎

6. - THE CONDITION Q·P = 0

A. LEMMA 6.1. *Let* $f : (M^n,g) \to \mathrm{IE}^{n+1}$ *be an isometric immersion of an* n*-dimensional Riemannian manifold.* (M^n,g) *satisfies* $Q\cdot P = 0$ *if and only if for each* p *in* M^n A_p *is one of the following types :*

(a) λI_n *with* $\lambda \in \mathrm{IR}_0$,

(b) $\left(\begin{array}{c|c} \lambda I_{n-1} & 0 \\ \hline 0 & (2-n)\lambda \end{array}\right)$ *with* $\lambda \in \mathrm{IR}_0$,

(c) $\left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & 0_{n-1} \end{array}\right)$ *with* $\lambda \in \mathrm{IR}$.

Proof. Let $f : (M^n,g) \to \mathbb{E}^{n+1}$ be an isometric immersion of an n-dimensional Riemannian manifold. Let p be a point in M^n and choose a basis $\{e_1,...,e_n\}$ for T_pM^n satisfying (2.4). Using the formulas (2.5) we find that $(Q\cdot P)(e_i,e_j)e_k = (\frac{\mu_k}{n-1} - c_{ij})\{\delta_{jk}(\mu_j+\mu_k)e_i - \delta_{ik}(\mu_i+\mu_k)e_j\}$ for all i,j and k in $\{1,...,n\}$. From this we learn that $Q\cdot P = 0$ in p if and only if $(Q\cdot P)(e_i,e_j)e_i = 0$ for all distinct i and j in $\{1,...,n\}$. This implies that $Q\cdot P = 0$ if and only if

$$\lambda_i(trA - \lambda_i)(trA - \lambda_i - (n-1)\lambda_j) = 0 \tag{6.1}$$

for all different i and j in $\{1,...,n\}$. Let i,j and k be mutually distinct indices in $\{1,...,n\}$. Then $\lambda_i(trA - \lambda_i)(trA - \lambda_i - (n-1)\lambda_j) = 0$ and $\lambda_i(trA - \lambda_i)(trA - \lambda_i - (n-1)\lambda_k) = 0$. Substraction yields that

$$\lambda_i(trA - \lambda_i)(\lambda_j - \lambda_k) = 0. \tag{6.2}$$

Conversely, (6.2) implies (6.1). Therefore, (M^n,g) satisfies $Q\cdot P = 0$ if and only if (6.2) is fulfilled for all mutually distinct i,j and k in $\{1,...,n\}$. It is easy to see now that $Q\cdot P = 0$ if all A_p have one of the forms described in the lemma. Next, we show the converse.

Let $\bar{\lambda}_1,...,\bar{\lambda}_r$ denote the mutually distinct eigenvalues of A in p and let $s_1,...,s_r$ be their respective multiplicities. First, suppose $r \geqslant 3$. Now, (6.2) implies that $\bar{\lambda}_\alpha(trA - \bar{\lambda}_\alpha) = 0$ for each $\alpha \in \{1,...,r\}$. This shows that A has at most two distinct eigenvalues. This contradicts our initial assumption.

Suppose that $r = 2$. If $s_1 \geqslant 2$, then (6.2) gives that $\bar{\lambda}_1(trA - \bar{\lambda}_1) = 0$ (take i and j with $\lambda_i = \lambda_j = \bar{\lambda}_1$ and k with $\lambda_k = \bar{\lambda}_2$). In the same way, if $s_2 \geqslant 2$, then $\bar{\lambda}_2(trA - \bar{\lambda}_2) = 0$. If $s_1 \geqslant 2$ and $s_2 \geqslant 2$, the only possibility is that, say, $\bar{\lambda}_1 = 0$ and $\bar{\lambda}_2 = trA \neq 0$. This is impossible as $trA = s_2\bar{\lambda}_2 \neq \bar{\lambda}_2$. Therefore, we may assume that for instance $s_2 = 1$. If $\bar{\lambda}_1 = 0$, A_p has one of the forms described in the lemma. If $\bar{\lambda}_1 \neq 0$, then $\bar{\lambda}_2 = (2-n)\bar{\lambda}_1$.

The case $r = 1$ is trivial. This proves the lemma. ■

B. EXAMPLES

It is clear that (M^n,g) satisfies $Q\cdot P = 0$ if $f : (M^n,g) \to \mathbb{E}^{n+1}$ is a cylindrical immersion or a totally umbilical immersion, since in these cases (M^n,g) satisfies $P = 0$. Now we will give a non-trivial example of a hypersurface satisfying $Q\cdot P = 0$.

Let $\gamma : I \to \mathbb{E}^{n+1} : u \mapsto (u,\varphi(u),0,...,0)$ be a plane curve in $\mathbb{E}^{n+1}$ lying in the x_1x_2-plane and suppose $\varphi(u) > 0$ for all u. Let (M^n,g) be the hypersurface of revolution in $\mathbb{E}^{n+1}$ obtained by rotation of γ around the x_1-axis, i.e.

$$M = \{ (u,\varphi(u)\cos\theta_2,\varphi(u)\sin\theta_2\cos\theta_3,\ldots,\varphi(u)\sin\theta_2\sin\theta_3\ldots\cos\theta_n,\varphi(u)\sin\theta_2\sin\theta_3\ldots\sin\theta_n) \mid$$

$$u \in I, \theta_2,\ldots,\theta_n \in \mathbb{R} \}$$

with the induced differentiable and geometric structure. Let F be the obvious parametrization of M and call $p = F(\bar{u},\bar{\theta}_2,\ldots,\bar{\theta}_n)$,($\bar{u} \in I$ and $\bar{\theta}_2,\ldots,\bar{\theta}_n \in [0,2\pi]$). Then T_pM is spanned by the vectors

$$(\frac{\partial F}{\partial u}(\bar{u},\bar{\theta}_2,\ldots,\bar{\theta}_n))_p = (1,\varphi'(\bar{u})\cos\bar{\theta}_2,\varphi'(\bar{u})\sin\bar{\theta}_2\cos\bar{\theta}_3,\ldots,$$

$$\varphi'(\bar{u})\sin\bar{\theta}_2\sin\bar{\theta}_3\ldots\cos\bar{\theta}_n,$$

$$\varphi'(\bar{u})\sin\bar{\theta}_2\sin\bar{\theta}_3\ldots\sin\bar{\theta}_n)_p,$$

$$(\frac{\partial F}{\partial \theta_2}(\bar{u},\bar{\theta}_2,\ldots,\bar{\theta}_n))_p = (0,-\varphi(\bar{u})\sin\bar{\theta}_2,\varphi(\bar{u})\cos\bar{\theta}_2\cos\bar{\theta}_3,\ldots,$$

$$\varphi(\bar{u})\cos\bar{\theta}_2\sin\bar{\theta}_3\ldots\cos\bar{\theta}_n,$$

$$\varphi(\bar{u})\cos\bar{\theta}_2\sin\bar{\theta}_3\ldots\sin\bar{\theta}_n)_p,$$

$$(\frac{\partial F}{\partial \theta_n}(\bar{u},\bar{\theta}_2,\ldots,\bar{\theta}_n))_p = (0,0,0,\ldots,-\varphi(\bar{u})\sin\bar{\theta}_2\sin\bar{\theta}_3\ldots\sin\bar{\theta}_n,$$

$$\varphi(\bar{u})\sin\bar{\theta}_2\sin\bar{\theta}_3\ldots\cos\bar{\theta}_n)_p$$

and

$$\xi_p := (-\varphi'(\bar{u}),\cos\bar{\theta}_2,\sin\bar{\theta}_2\cos\bar{\theta}_3,\ldots,\sin\bar{\theta}_2\sin\bar{\theta}_3\ldots\sin\bar{\theta}_{n-1}\cos\bar{\theta}_{n-1},$$

$$\sin\bar{\theta}_2\sin\bar{\theta}_3\ldots\sin\bar{\theta}_{n-1}\sin\bar{\theta}_n)_p$$

is a normal vector in p. Then, if $W(\bar{u}) > 0$ is defined by $W^2(\bar{u}) := \| \xi_p \|^2 = 1 + \varphi'(\bar{u})^2$, $U(p) = \frac{\xi_p}{W(\bar{u})}$ is a unit normal vector in p. We find that $(\frac{\partial F}{\partial u}(\bar{u},\bar{\theta}_2,\ldots,\bar{\theta}_n))_p$ is an eigenvector of A_p with eigenvalue $\frac{\varphi''(\bar{u})}{W^3(\bar{u})}$ and that $(\frac{\partial F}{\partial \theta_2}(\bar{u},\bar{\theta}_2,\ldots,\bar{\theta}_n))_p,\ldots,(\frac{\partial F}{\partial \theta_n}(\bar{u},\bar{\theta}_2,\ldots,\bar{\theta}_n))_p$ are eigenvectors all with the same eigenvalue $\frac{-1}{W(\bar{u})\varphi(\bar{u})}$. Consequently, A_p has the form described in (b) of Lemma 6.1 if and only if φ satisfies the following differential equation :

(*) $$\varphi''\varphi = (n-2)(1+\varphi'^2).$$

Next, we describe the solutions of this differential equation.

Take $c \in \mathbb{R}_0^+$ and let $c' := c^{-1/n-2}$. Consider the function $h_{n,c}$ given by

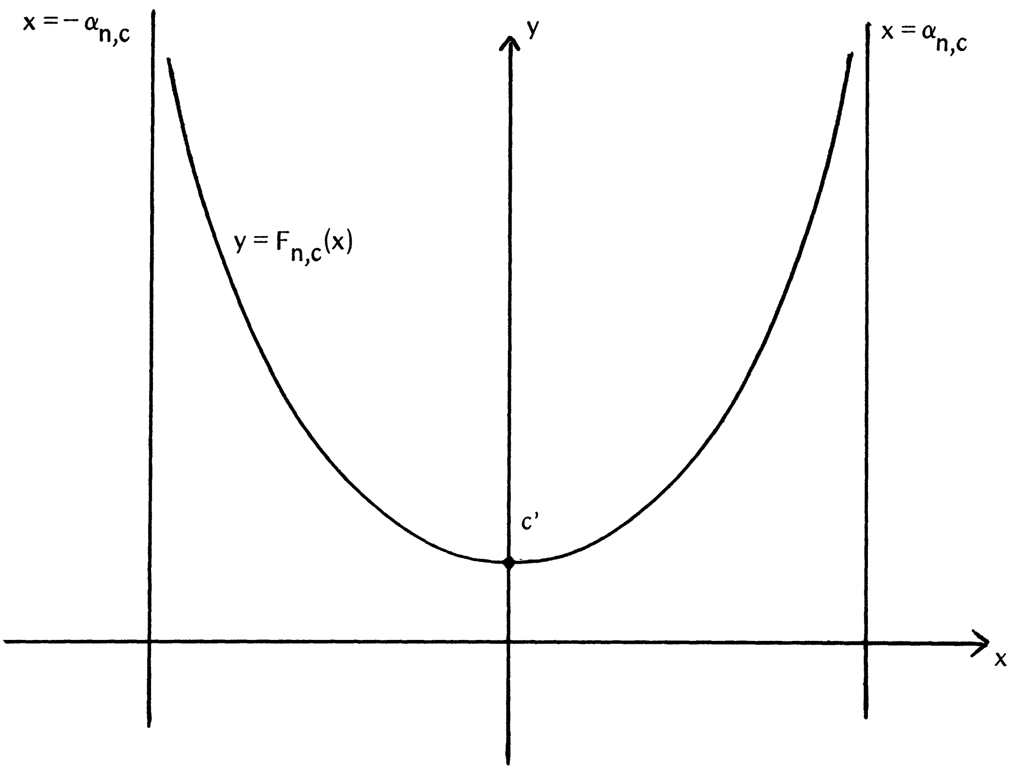
$$h_{n,c} :]c',+\infty[\to \mathbb{R} : x \mapsto \int_{c'}^{x} \frac{d\rho}{\sqrt{c^2\rho^{2(n-2)}-1}}.$$

Since $h'_{n,c} \neq 0$ everywhere, we can define the inverse function $g_{n,c} := h_{n,c}^{-1}$. It can be shown that $g_{n,c}$ is defined on $]0,\alpha_{n,c}[$ with $\alpha_{n,c} \in \mathbb{R}_0^+$ for all $n > 3$ and $\alpha_{3,c} = +\infty$. Next we consider the function $F_{n,c}$ given by

$$F_{n,c} :]-\alpha_{n,c} ; \alpha_{n,c}[\to \mathbb{R}$$

$$x \longrightarrow \begin{cases} g_{n,c}(-x) & \text{if } x < 0 \\ c' & \text{if } x = 0 \\ g_{n,c}(x) & \text{if } x > 0 \end{cases}$$

For $n > 3$:



For each solution $\varphi : I \to \mathbb{R}$ of the equation (*) there exist numbers $c \in \mathbb{R}_0^+$ and $b \in \mathbb{R}$ such that $\varphi(x) = F_{n,c}(x+b)$ for all x in I. We remark that $F_{3,c}(x) = \frac{1}{c}\cosh cx$ for all x in $\mathbb{R}$, i.e. γ is a catenary.

Call K_c^n the hypersurface of revolution obtained by rotation of the curve $\gamma_{n,c} :]-\alpha_{n,c} ; \alpha_{n,c}[\to \mathbb{E}^{n+1} : u \mapsto (u, F_{n,c}(u), 0, \ldots, 0)$ around the x_1-axis. All hypersurfaces of revolution on $\mathbb{E}^{n+1}$ such that all second fundamental tensors have the form described in Lemma 6.1 (b) are open parts of a K_c^n.

C. PROOF OF THEOREM 4.

It is clear from A and B that one of the implications holds. We now prove the other one. Suppose that (M^n, g) satisfies $Q \cdot P = 0$. The lemma determines the possible forms for the second fundamental tensors.

First, suppose that there is a point p in M with A_p a multiple of I_{T_pM}. In the same way as in the previous section, this implies that f is a totally umbilical immersion.

Next, we assume that M has no umbilical points. Call $W = \{ p \in M \mid \text{rank } A_p = n \}$. Then W is open. Call $U = W \cup \text{int}(M \setminus W)$. Then U is an open dense subset of M. Take a connected component U_α of U. If $U_\alpha \subset \text{int}(M \setminus W)$ then $f_\alpha := f|_{U_\alpha}$ is a cylindrical immersion. We next consider the case $U_\alpha \subset W$. We will need some more lemmas.

Define $T_1 := \{ x \in TU_\alpha \mid Ax = \lambda x \}$ and $T_2 := \{ x \in TU_\alpha \mid Ax = (2-n)\lambda x \}$. By Proposition 2.3 in [8], T_1 and T_2 are differentiable involutive distributions and λ is constant along integral manifolds of T_1. Furthermore, for X_1 a vector field with values in T_1 and X_2 a vector field with values in T_2, the equation $(\nabla_{X_1} A)X_2 - (\nabla_{X_2} A)X_1 = 0$ of Codazzi implies that

$$\nabla_{X_2} X_1 \text{ takes its values in } T_1 \tag{6.3}$$

and that $(A - (2-n)\lambda)\nabla_{X_1} X_2 + (X_2 \,.\, \lambda)X_1 = 0$. If $(\nabla_{X_1} X_2)_1$ denotes the component of $\nabla_{X_1} X_2$ in T_1,

$$(\nabla_{X_1} X_2)_1 = \frac{X_2 \,.\, \ln\lambda}{1-n} X_1 . \tag{6.4}$$

(6.3) implies that

$$\nabla_{X_2} Y_2 \text{ takes its values in } T_2 \tag{6.5}$$

for each vector field Y_2 with values in T_2.

For each p in U_α we write $M_1(p)$ for an integral manifold of T_1 through p and $\gamma_p : I \to M$ for an integral curve of T_2 through p. We assume that $\gamma_p(0) = p$ and that γ_p is parametrized by arclength. Around any p in U_α we can choose a local orthonormal frame field $\{E_1,\dots,E_n,E_{n+1}\}$ for $\mathbb{E}^{n+1}$ which is adapted to f_α and such that furthermore $E_1,\dots,E_{n-1}$ span T_1 and E_n spans T_2. (6.4) and (6.5) imply that

$$\nabla_{E_n} E_n = 0 \tag{6.6}$$

and that

$$\nabla_{E_i} E_n = \frac{E_n \,.\, \ln\lambda}{1-n} E_i . \tag{6.7}$$

In the following lemma we study the shape of the immersions $(f_1)_p := f\,|_{M_1(p)}$.

LEMMA 6.2. *For each* p *in* U_α, $f(M_1(p))$ *is an open part of an* $(n-1)$*-dimensional sphere in* $\mathbb{E}^{n+1}$ *with radius* $\dfrac{1}{\sqrt{\left(\frac{E_n \,.\, \ln\lambda}{n-1}\right)^2 + \lambda^2}}$. *Consequently,* $(f_1)_p$ *is local injective.*

Proof. Let $q \in M_1(p)$. If $\{E_1,\dots,E_n,E_{n+1}\}$ is a frame field around q as above, the normal bundle of $(f_1)_p$ is spanned by E_n and E_{n+1}. Let A'_{E_n} and $A'_{E_{n+1}}$ be the second fundamental tensors of f_1 and denote by D' the normal connection of f_1. Then $\tilde{\nabla}_{E_i} E_{n+1} = -AE_i = -\lambda E_i$, $(i \in \{1,\dots,n-1\})$. This yields that

$$A'_{E_{n+1}} = \lambda I_{T_q(M_1(p))} . \tag{6.8}$$

We also have that $\tilde{\nabla}_{E_i} E_n = \nabla_{E_i} E_n = \dfrac{E_n \,.\, \ln\lambda}{1-n} E_i$, $(i \in \{1,\dots,n-1\})$, by (6.7). So

$$A'_{E_n} = \frac{E_n \,.\, \ln\lambda}{n-1} I_{T_q}(M_1(p)) . \tag{6.9}$$

This proves the lemma. ∎

Let $\mathbb{E}^n(p)$ be the unique hyperplane of $\mathbb{E}^{n+1}$ containing $f(M_1(p))$, call ν_p the normal in this hyperplane on $f(M_1(p))$ in p and let m(p) be the center of the sphere. Then

$$\nu_p = \frac{\lambda(p)E_{n+1}(p) + \frac{E_n(p) \,.\, \ln\lambda}{n-1} E_n(p)}{\sqrt{\left(\frac{E_n(p) \,.\, \ln\lambda}{n-1}\right)^2 + \lambda(p)^2}} \tag{6.10}$$

and

$$(6.11) \qquad m(p) = f(p) + \frac{\vec{\nu}_p}{\sqrt{\left(\frac{E_n(p) \cdot \ln\lambda}{n-1}\right)^2 + \lambda(p)^2}}.$$

($\vec{\nu}_p$ is the vectorpart of ν_p).

Next, we study the shape of the image $f \circ \gamma_p$ of the integral curves.

LEMMA 6.3. *For each* p *in* U_α, $f \circ \gamma_p$ *is a plane curve with nowhere zero curvature.*

Proof. Let $q \in \mathrm{im}\gamma_p$. If $\{E_1,\dots,E_n,E_{n+1}\}$ is a frame field around q as above, then

$$(f \circ \gamma_p)' = E_n,$$

$$(6.12) \qquad (f \circ \gamma_p)'' = \tilde{\nabla}_{E_n} E_n = (2-n)\lambda E_{n+1},$$

$$(f \circ \gamma_p)''' = (2-n)(E_n.\lambda)E_{n+1} - (2-n)^2\lambda^2 E_n.$$

Since $(f \circ \gamma_p)' \wedge (f \circ \gamma_p)'' \wedge (f \circ \gamma_p)''' = 0$, $f \circ \gamma_p$ is a plane curve. From (6.12) it is clear that the curvature of $f \circ \gamma_p$ is nowhere zero. ■

Call $\mathbb{E}^2(p)$ the unique plane in $\mathbb{E}^{n+1}$ containing $\mathrm{im}(f \circ \gamma_p)$. $\mathbb{E}^2(p)$ is the plane through f(p) spanned by $E_n(p)$ and $E_{n+1}(p)$. It is clear from (6.10) and (6.11) that $m(p) \in \mathbb{E}^2(p)$. We prove the following lemma concerning the position of the planes $\mathbb{E}^2(p)$.

LEMMA 6.4. *Let* $p \in M$. *Then there is a line* $\ell(p)$ *in* $\mathbb{E}^{n+1}$ *such that* $\ell(p) = \mathbb{E}^2(p) \cap \mathbb{E}^2(q)$ *for each* q *in* $M_1(p)$ *which is distinct from* p *and for which* f(q) *is not the antipodal point of* f(p). *Moreover,* $m(p) \in \ell(p)$ *and* $\ell(p) \perp \mathbb{E}^n(p)$.

Proof. Let $q \in M_1(p)$ with $q \neq p$ and f(q) not the antipodal point of f(p). We prove that $\mathbb{E}^2(p) \neq \mathbb{E}^2(q)$. $\mathbb{E}^2(p) \cap \mathbb{E}^n(p)$ contains f(p) and m(p). $\mathbb{E}^2(p) \not\subset \mathbb{E}^n(p)$ since the normal η_p on $\mathbb{E}^n(p)$ lies in $\mathbb{E}^2(p)$. So $\mathbb{E}^2(p) \cap \mathbb{E}^n(p)$ is the line f(p)m(p). This line f(p)m(p) intersects $f(M_1(p))$ in at most 2 points : f(p) and possibly the antipodal point of f(p). Since $f(q) \in f(M_1(p))$ and f(q) is neither of these points, $f(q) \notin f(p)m(p)$. As $f(q) \in \mathbb{E}^n(p)$, this shows that $f(q) \notin \mathbb{E}^2(p)$. In any case $m(p) = m(q) \in \mathbb{E}^2(p) \cap \mathbb{E}^2(q)$ and $\vec{\eta}_p = \vec{\eta}_q$ is a common direction of $\mathbb{E}^2(p)$ and $\mathbb{E}^2(q)$. Therefore $\mathbb{E}^2(p) \cap \mathbb{E}^2(q)$ is the line $\ell(p)$ through m(p) in the direction $\vec{\eta}_p$. This line does not depend on q.

It is clear from the construction of $\ell(p)$ that $m(p) \in \ell(p)$ and that $\ell(p) \perp \mathbb{E}^n(p)$. ■

For $p \in U_\alpha$ choose a coordinate system $\mu : U \to]-\epsilon,\epsilon[^n : q \mapsto (x^1(q),...,x^n(q))$ around $p = \mu^{-1}(0,...,0)$ such that for each choice of numbers $a_1,...,a_n \in]-\epsilon,\epsilon[$ the sets $q \in U \mid x^n(q) = a^n$ are integral manifolds of T_1 and the curves $]-\epsilon,\epsilon[\to U : t \to \mu^{-1}(a_1,...,a_{n-1},t)$ are integral curves of T_2 (see [6] p. 182). We prove the following lemma concerning the position of the centers m(q) and the lines $\ell(q)$.

LEMMA 6.5. *Let* $p \in M$ *and suppose that* $\mu : U \to]-\epsilon,\epsilon[^n$ *is a coordinate system around* p *as above. Then, for each* $q \in U$, $\ell(q) = \ell(p)$, $m(q) \in \ell(p)$ *and* $\ell(p) \perp \mathbb{E}^n(q)$.

Proof. Suppose that $\mu(q) = (c_1,...,c_n)$. Call $q' := \mu^{-1}(0,...,0,c_n)$, $q'' = \mu^{-1}(c_1,...,c_{n-1},0)$. Then $\mathbb{E}^2(q) = \mathbb{E}^2(q'')$ and $\mathbb{E}^2(p) = \mathbb{E}^2(q')$, which implies that $\ell(p) = \mathbb{E}^2(p) \cap \mathbb{E}^2(q'') = \mathbb{E}^2(q') \cap \mathbb{E}^2(q) = \ell(q)$. The other statements in Lemma 6.5 now easily follow from Lemma 6.4. ∎

Now, we can finish the proof of Theorem 4. Suppose $p \in U_\alpha$ and let $\mu : U \to]-\epsilon,\epsilon[^n$ be a coordinate system around p as before. Call γ_p the curve $\gamma_p :]-\epsilon,\epsilon[\to U_\alpha : t \mapsto \mu^{-1}(0,...,0,t)$. Determine the line $\ell(p)$ in the way shown by Lemma 6.4. Call M' the hypersurface of $\mathbb{E}^{n+1}$ obtained by rotation of $f \circ \gamma_p$ around $\ell(p)$. We will show that $f(U) \subset M'$. Take $q = \mu^{-1}(c_1,...,c_{n-1},c_n) \in U$ and let $q' = \mu^{-1}(0,...,0,c_n)$. Then $f(M_1(q)) = f(M_1(q'))$ is an open part of a sphere in $\mathbb{E}^n(q) \perp \ell(p)$ with center $m(q) \in \ell(p)$ having the point f(q') in common with $f \circ \gamma_p$. This shows that $f(q) \in M'$. From the discussion in B it is clear that $f|_{U_\alpha}$ is congruent to the inclusion of an open part of a K^n_c.

REFERENCES

[1] D.E. BLAIR. *«On a generalization of the catenoid»*. Can. J. Math., Vol. XXVII, n° 2, 1975, pp. 231-236.

[2] B.Y. CHEN. *«Geometry of submanifolds»*. Marcel Dekker, New York, 1973.

[3] J. DEPREZ. *«Semi-parallel hypersurfaces»*. To appear in Rend. Sem. Mat. Univ. Politecnico Torino.

[4] J. DEPREZ, P. VERHEYEN and L. VERSTRAELEN. *«Characterizations of conformally flat hypersurfaces»*. Czechoslovak Math. J. 35 (110) 1985.

[5] P. HARTMAN and L. NIRENBERG. *«On spherical image maps whose Jacobians do not change sign»*. Amer. J. Math. 8 (1959), 901-920.

[6] S. KOBAYASHI and K. NOMIZU. *«Foundations of differential geometry I»*. Interscience Publ. 1963, London.

[7] K. NOMIZU. *« On hypersurfaces satisfying a certain condition on the curvature tensor»*. Tôhoku Math. J. 20 (1968), 46-59.

[8] P.J. RYAN. *«Homogeneity and some curvature conditions for hypersurfaces»*. Tôhoku Math. J. 21 (1969), 363-388.

[9] Z.I. SZABO. *«Structure theorems on Riemannian spaces satisfying* $R(X,Y) \cdot R = 0$. *I. The local version»*. J. Differential Geometry 17 (1982), 531-582.

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