Winter Braids Lecture Notes

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Abstract

These notes are the outcome of a mini-course on TQFTs held at the edition of Winter Braids in Pau in February 2015. We define the notion of TQFT and provide the first basic examples obtained via the universal construction and via Frobenius algebras. After recalling some basic notions on the mapping class groups of surfaces, we concentrate on the Reshetikhin-Turaev construction via the skein theoretical approach: we first define the skein module of a 3-manifold and the RT invariants; then we apply the universal construction to get the RT SU(2)-TQFTs. We conclude with an overview of the main results on these TQFTs and on some recent developments. An appendix summarizes the basic notions and facts in category theory used here.

1. Introduction

These notes are the outcome of a mini course held at the edition of Winter Braids in Pau in February 2015. The goal of the course was to give an introduction to the notion of TQFT and a taste of how the famous SU(2)-Reshetikhin-Turaev TQFTs can be constructed using skein theory, as explained by Blanchet, Habegger, Masbaum and Vogel [7]; then to provide a rapid overview of the main results on these TQFTs and on some new developments.

As it often happens in mathematics, TQFTs were discovered gradually before their formal definition was provided; they made a first appearance in A. Schwarz's paper [45] and their first example was introduced by E. Witten in his fundamental paper [49] who also conjectured the existence of a family of TQFTs relating Chern-Simons theory and the Jones polynomials of knots in [48]. Witten's approach was based on path integrals in infinite dimensions and it has not yet been formalized; still his papers stimulated the development of the domain now known as quantum topology. In [43] Reshetikhin and Turaev constructed a family of invariants of three manifolds having exactly the same properties as those discussed in Witten's papers: even if their approach is totally different (and based on the representation theory of quantum groups) it is now commonly accepted that these invariants are the mathematical formulation of Witten's. In this paper we will refer to these invariants as Reshetikhin-Turaev invariants (or RT-invariants for short), because Witten's approach based on Chern-Simons theory will not be discussed here. In [5] Atiyah formalized the notion of Topological Quantum Field Theory and later Blanchet, Habegger, Masbaum and Vogel [7] constructed a family of TQFTs based on Reshetikhin-Turaev invariants which complete Witten's programme; in [46] Turaev generalized the construction of TQFTs using modular categories. The study of TQFTs is now a wide field also due to the more recent ideas of extended TQFTs, categorification (which I will not discuss in these introductory notes) and non semi-simple TQFTs (to which I will dedicate a subsection in the final part of this paper).

1.1. Structure of these notes

In the first section, after defining TQFTs via a categorical language (of which I synthesize in the Appendix the necessary notions) I recall the so called “universal construction" [7] and some of its properties. The second section is the devoted to provide the very first examples (in dimension $1+1$) and to answer some natural questions. The third section details some basic facts on mapping class groups whose representations issued from TQFTs are of special interest. In particular I detail the construction of a central extension of these groups which is key to the proper construction of the Reshetikhin-Turaev TQFTs in dimension $2+1$. The fourth section details the notion of skein module of a manifold and introduces the reader to the art of computing in skein modules (the “skein theory"). At the end of the section I define the Reshetikhin-Turaev invariants and compute them for the manifolds of the form $\Sigma \times \mathbb{S}^1$. In the last section we start by detailing why a suitable modification of the category of surfaces is needed in order to get finite dimensionality of the vector spaces associated to surfaces. Then we apply the universal construction to the Reshetikhin-Turaev invariants in order to get TQFTs. We provide a sketchy proof of the fact that the so-obtained structures are indeed TQFTs.

The last subsection is devoted to discuss some of the properties of the so obtained quantum representations of the mapping class group, without providing proofs. We also cite some properties of the recent “non semi-simple TQFTs" [8] and compare it with those of the standard Reshetikhin-Turaev TQFTs studied here.

1.2. Acknowledgements.

I wish to thank the organizers of Winter Braids for proposing to give a course on TQFTs and to write these notes and the referee for his/her careful reading and helpful comments.

2. The category Cob_n

In this section we define the starting point of the notion of TQFT, namely the category of cobordisms which will be “represented" by a TQFT functor later on. We single out some key properties (e.g. monoidality, existence of duals, the fact that the object associated to a sphere is a Frobenius algebra) of the category which will be automatically reflected by a TQFT.

All manifolds will be smooth compact and oriented and all the maps will be smooth unless explicitly stated the contrary.

Definition 2.1. Two diffeomorphisms between manifolds $f, g : M \to N$ are :

- *homotopic* : if there exists a map $h : M \times [0, 1] \to N$ such that $h|_{M\times\{0\}} = f$ and $h|_{M\times\{1\}} = g$.
- *pseudo-isotopic*: if there exists an embedding $h : M \times [0, 1] \to N \times [0, 1]$ such that $h|_{M\times\{0\}} = f \times \{0\}$, $h|_{M\times\{1\}} = g \times \{1\}$.
- *isotopic*: if there exists an embedding $h : M \times [0, 1] \to N \times [0, 1]$ such that $h|_{M\times\{0\}} = f \times \{0\}$, $h|_{M\times\{1\}} = g \times \{1\}$ and for each t, $h_t := h|_{M\times\{t\}} \subset N \times \{t\}$.

Remark 2.2. Clearly isotopy $\implies$ pseudo-isotopy $\implies$ homotopy. The reverse implications are false in general in dimensions ≥ 3 (see for instance [30] for an example in dimension 3 of maps which are pseudo-isotopic but non isotopic). On contrast, in dimension 2 they are all true : this is the content of Baer’s theorem (see [16], Theorem 2.1).

Definition 2.3. The category Cob_n is the category whose objects are the $n-1$-dimensional manifolds (which typically we will denote with the letters Σ) and whose morphisms are 5-tuples $\mathrm{Mor}(\Sigma_-, \Sigma_+) = \{(W, \partial_+ W, f_+, \partial_- W, f_-)\}/\sim$ where

1. W is a n-manifold,
2. $\partial W = \partial_- W \sqcup \partial_+ W$ (oriented with the outward vector first convention),
3. $f_- : \Sigma_- \to \partial W_-$ (resp. $f_+ : \Sigma_+ \to \partial W_+$) are diffeomorphisms which reverse (resp. preserve) the orientation,

and we say that two 5-tuples $(W, \partial_+ W, f_+, \partial_- W, f_-)$ and $(W', \partial_+ W', f_+, \partial_- W', f_-)$ are equivalent ($\sim$) if there exists an orientation preserving diffeomorphism $\psi : W \to W'$ such that:

$$\psi(\partial_+ W) = \partial_+ W', \qquad f'_+ = \psi \circ f_+, \qquad \psi(\partial_- W) = \partial_- W', \qquad f'_- = \psi \circ f_-.$$

The composition of cobordisms :

$$\mathcal{W}_1 = (W_1, \partial_+ W_1, f_+, \partial_- W_1, f_-) \in \mathrm{Mor}(\Sigma_-, \Sigma) \text{ and}$$
$$\mathcal{W}_2 = (W_2, \partial_+ W_2, g_+, \partial_- W_2, g_-) \in \mathrm{Mor}(\Sigma, \Sigma_+) \text{ is defined as}$$
$$\mathcal{W}_2 \circ \mathcal{W}_1 = (W_2 \sqcup_{g_- \circ f_+^{-1}} W_1, \partial_+ W_2, g_+, \partial_- W_1, f_-) \in \mathrm{Mor}(\Sigma_-, \Sigma_+), \text{ where}$$
$$W_2 \sqcup_{g_- \circ f_+^{-1}} W_1 := (W_1 \sqcup W_2)/\{x \sim y \iff x \in \partial_- W_2, y \in \partial_+ W_1 \text{ and } x = g_- \circ f_+^{-1}(y)\}.$$

Remark 2.4. As we defined morphisms as diffeomorphisms classes of cobordisms, a little thinking is worth concerning the definition of the composition of two morphisms we gave (which used explicit representatives). Remark indeed that if $\mathcal{W}_1$ is equivalent to $\mathcal{W}'_1$ via a diffeomorphism $\psi : \mathcal{W}_1 \to \mathcal{W}'_1$ then $\mathcal{W}_2 \circ \mathcal{W}_1$ is equivalent to $\mathcal{W}_2 \circ \mathcal{W}'_1$ via the diffeomorphism defined as $Id \sqcup \psi : \mathcal{W}_2 \sqcup \mathcal{W}_1 \to \mathcal{W}_2 \sqcup \mathcal{W}'_1$ and which passes to the quotients as if $x \in \partial_- W_2$, $y \in \partial_+ W_1$ and $x = g_- \circ f_+^{-1}(y)$ then it also holds $x = g_- \circ (f'_+)^{-1}(\psi(y))$.

Furthermore we should also point out that to be fully rigorous, since we are glueing smooth manifolds, we should take the care of picking collars of the boundary components and glue them using the collars so to endow the resulting manifold with a smooth atlas. We leave this technical detail to the reader, and we limit ourselves to remarking that the fact that the result is well defined is a consequence of the uniqueness up to isotopy of the collar of the boundary.

Observe that the identity morphism Id_Σ is $(\Sigma \times [-1, 1], \Sigma \times \{-1\}, Id, \Sigma \times \{1\}, Id)$. More in general if $f \in Diff_+(\Sigma)$ then we define the cobordism $C_f := (\Sigma \times [-1, 1], \Sigma \times \{-1\}, f, \Sigma \times \{1\}, \mathrm{Id})$: the following holds :

Lemma 2.5. 1. *The semigroup* $\mathrm{Mor}(\emptyset, \emptyset)$ *is the abelian semigroup freely generated by oriented diffeomorphism classes of connected* $n+1$*-manifolds. Its only invertible element is the class of the empty manifold.*

2. *For each* Σ *the map* $Diff_+(\Sigma) \ni f \to C_f \in \mathrm{Mor}(\Sigma, \Sigma)$ *is a homomorphism whose kernel is* $\{f | f$ *is pseudo-isotopic to the identity*$\}$.

Proof. **1)**.The fact that $\mathrm{Mor}(\emptyset, \emptyset)$ is a semigroup is true in general, furthermore, by definition of the composition of two cobordisms, if those cobordisms have empty boundary, their composition is the diffeomorphism class of their disjoint union. The identity cobordism is $\emptyset \times [-1, 1] = \emptyset$ and it is invertible.

2). We need to prove that $C_f \circ C_g = C_{f \circ g}$. By definition the cobordism C_g can be also represented as $(\Sigma \times [-1, 1], \Sigma \times \{-1\}, f \circ g, \Sigma \times \{1\}, f)$ (indeed the diffeomorphism f can be extended to the whole C_g via $f \times Id$). Now it becomes evident that the composition of the two cobordisms the composition $C_f \circ C_g$ is the cobordism $(\Sigma \times [-1, 3], \Sigma \times \{3\}, Id, \Sigma \times \{-1\}, f \circ g) = C_{f \circ g}$. The cobordism C_f is equivalent to the cobordism $C_{Id} = Id_\Sigma$ iff there exists a diffeomorphism $\phi : \Sigma \times [-1, 1] \to \Sigma \times [-1, 1]$ such that

$$\phi(x, 1) = (x, 1) \text{ and } \phi(f(x), -1) = (x, -1)\ \forall x \in \Sigma.$$

Up to a re-parametrization of the $[-1, 1]$ factor this is precisely saying that f is pseudo-isotopic to Id (see Definition 2.1). 2.5

The category Cob_n has naturally much more structure than what was given above. Observe first that a monoidal structure in Cob_n is given by the disjoint union : $\Sigma_1 \otimes \Sigma_2 := \Sigma_1 \sqcup \Sigma_2$, and the unit object $\mathbb{1}$ is the empty manifold $\emptyset$. Furthermore, the natural diffeomorphisms $\Sigma_1 \sqcup \Sigma_2 \to \Sigma_2 \sqcup \Sigma_1$ induce a symmetry on the monoidal structure: Cob_n is then a symmetric monoidal category (see Definition A.9).

Observe furthermore Cob_n is a pivotal category: each object Σ has a left and right dual object $\overline{\Sigma}$ which is the same manifold with the opposite orientation and there are morphisms $\eta : \mathbb{1} \to \Sigma \otimes \overline{\Sigma}$ (defined as $\eta := (\Sigma \times [-1,1], \Sigma \times \{\pm 1\}, Id \sqcup Id, \emptyset, \emptyset)$) and $\epsilon : \overline{\Sigma} \otimes \Sigma \to \mathbb{1}$ (defined as $\epsilon := (\Sigma \times [-1,1], \emptyset, \emptyset, \Sigma \times \{\pm 1\}, Id \sqcup Id)$) which satisfy the triangle identities (see the Appendix A.3 for the general definitions on pivotal categories).

From now on we will consider Cob_n as a symmetric pivotal category.

Definition 2.6 (Frobenius algebra in $\mathscr{C}$)**.** A Frobenius algebra A in a monoidal category $\mathscr{C}$ is a 5-tuple $(A, \mu, 1, \Delta, \epsilon)$ where :

1. $\mu : A \otimes A \to A$ is associative (i.e. $\mu \circ (\mu \otimes Id) = \mu \circ (Id \otimes \mu)$)

2. $1 \in \mathrm{Mor}(\mathbb{1}, A)$ is such that $\mu \circ (1 \otimes Id) = Id = \mu \circ (Id \otimes 1)$;

3. $\Delta : A \to A \otimes A$ is co-associative (i.e. $\Delta \otimes Id \circ \Delta = Id \otimes \Delta \circ \Delta$);

4. $\epsilon : A \to \mathbb{1}$ is a co-unit i.e. it is such that $\epsilon \otimes Id \circ \Delta = Id = Id \otimes \epsilon \circ \Delta$.

5. The Frobenius Law holds : $\Delta \circ \mu = (Id \otimes \mu) \circ (\Delta \otimes Id) = (\mu \otimes Id) \circ (Id \otimes \Delta)$.

Furthermore, if $\mathscr{C}$ is symmetric with symmetry s we say that A is commutative if it holds $\mu \circ s = \mu$, cocommutative if $s \circ \Delta = \Delta$. A Frobenius algebra is a Frobenius algebra in the category $\mathcal{V}ec$ of $\mathbb{C}$-vector spaces.

Remark 2.7. If $\mathscr{C}$ is a pivotal symmetric category and $(A, \mu, 1, \Delta, \epsilon)$ is a Frobenius algebra in $\mathcal{C}$ then:

1. also $(A^*, \Delta^*, \epsilon^*, \mu^*, 1^*)$ is a Frobenius algebra in $\mathscr{C}$. If A is commutative then A^* is cocommutative and if A is cocommutative A^* is commutative.

2. if $Z : \mathscr{C} \to Vect$ is a braided monoidal functor (see Definition A.11) and A is commutative, then $Z(A)$ is a commutative Frobenius algebra in $Vect$, that is a Frobenius algebra.

Let $\mathbb{S}_n$ be the n-dimensional sphere seen as the round unit sphere in $\mathbb{R}^{n+1}$ and oriented as the outside of the round unit radius ball $\mathcal{B}_n$ of center the origin. Let $1 \in \mathrm{Mor}(\emptyset, \mathbb{S}_n)$ be the cobordism represented by $\mathcal{B}_n$ and let μ be the $n+1$ cobordism from $\mathbb{S}_n \otimes \mathbb{S}_n \to \mathbb{S}_n$ formed by the “pant" i.e. the complement of two disjoint copies of the round ball of radius 1 whose centers are in coordinates $(\pm 2, 0, \cdots, 0) \in \mathbb{R}^{n+1}$ inside the round ball of radius 4 and center the origin (the boundary components of μ are to identified with $\mathbb{S}_n$ by means of the obvious compositions of translations and positive homogeneous dilatations). Similarly let Δ, ϵ be the $n+1$-cobordisms obtained by reversing the orientations of μ and 1 respectively.

Lemma 2.8. *$(\mathbb{S}_n, \mu, 1, \Delta, \epsilon)$ is a commutative Frobenius algebra in Cob_n. As a consequence also its dual $(\overline{\mathbb{S}}_n, \Delta^*, \epsilon^*, \mu^*, 1^*)$ is a commutative Frobenius algebra in Cob_n.*

Proof. The proof is left to the reader. 2.8

Remark that in a pivotal category the dual of an object is unique up to isomorphism and a Frobenius algebra object is self dual (the pairing being $\epsilon \circ \mu : A \otimes A \to \mathbb{1}$). In particular this implies that there is an isomorphism between $\mathbb{S}_n$ and $\overline{\mathbb{S}}_n$: it can be checked that it is given by the cobordism $(\mathbb{S} \times [0,1], Id|_{\mathbb{S} \times \{0\}}, \overline{Id}|_{\mathbb{S} \times \{1\}})$ where $\overline{Id} : \mathbb{S} \to \mathbb{S}$ is the map $\overline{Id}(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, -x_{n+1})$.

3. Quantization functors, TQFTs and the universal construction

In this section we “represent” the category Cob_n defined precedently. We define the notion of TQFT and spell out some of the consequences of the intrinsic properties of Cob_n. We also recall the universal construction and reprove a result of Turaev which states that two non degenerate TQFTs having the same invariants are isomorphic.

Definition 3.1 (Various notions of n-dimensional TQFTs)**.** Let $\mathcal{V}ec$ be the symmetric monoidal category of vector spaces over $\mathbb{C}$ (not necessarily of finite dimension).

- A quantization functor is a functor $Z : \mathrm{Cob}_n \to \mathcal{V}ec$ such that $Z(\emptyset) = \mathbb{C}$.
- A finite quantization functor is a quantization functor $Z : \mathrm{Cob}_n \to \mathcal{V}ec$ such that $Z(\Sigma)$ is finite dimensional for all Σ.
- A TQFT (sometimes also called $(n-1)+1$-TQFT) is a symmetric monoidal functor $Z : \mathrm{Cob}_n \to \mathcal{V}ec$.

(We warn the reader that the first two notions are used but usually do not have a specific name in the literature). A quantization functor is *non-degenerate* (or *cobordism generated*) if for each Σ it holds

$$Z(\Sigma) = \mathrm{Span}_{\mathbb{C}}\{Z(\mathrm{Mor}(\emptyset, \Sigma)\}.$$

Lemma 3.2. *A TQFT Z is also a finite quantization functor. Furthermore* $\dim(Z(\Sigma)) = Z(\Sigma \times \mathbb{S}^1)$.

Proof. The hypothesis of $Z(\emptyset) = \mathbb{C}$ is included of that of symmetric monoidal functor. The finite dimensionality comes from the triangle identities satisfied for each object Σ :

$$Z(\Sigma) \longrightarrow Z(\Sigma) \otimes \mathbb{1} \xrightarrow{\mathrm{Id}\otimes\eta} Z(\Sigma) \otimes \overline{Z(\Sigma)} \otimes Z(\Sigma) \xrightarrow{\epsilon\otimes\mathrm{Id}} Z(\Sigma) = Id_{Z(\Sigma)}$$

Indeed if $\eta(1) = \sum_{i=1}^d e_i \otimes f_i$ then the span of $Id_{Z(\Sigma)}$ must be contained in the span of $f_i, i = 1, \ldots d$. The last equality comes from the fact that the composition of the evaluation and co-evaluation in $\mathcal{V}ec$ is the trace of the identity. 3.2

Remark 3.3. One may replace the monoidal category $\mathcal{V}ec$ with the category of finitely generated projective modules over a commutative ring A. The notion of finite dimensionality is then to be replaced with finite generation and $Z(\emptyset) = A$.

The following is a direct consequence of Lemma 2.5:

Lemma 3.4. *Let Z be a quantization functor:*

1. $Z : \mathrm{Mor}(\emptyset, \emptyset) \to \mathbb{C}$ *is a diffeomorphism invariant of* $n+1$*-manifolds which is multiplicative under disjoint union.*
2. *For each* Σ *and each* $f \in Diff^+(\Sigma)$ *let* $M_f : (\Sigma \times [-1, 1], \Sigma \times \{-1\}, f, \Sigma \times \{1\}, \mathrm{Id}) \in \mathrm{Mor}(\Sigma, \Sigma)$*. Then* $Z(M_f) \in \mathrm{End}(Z(\Sigma))$ *is a representation of* $Diff^+(\Sigma)$ *whose kernel includes the diffeomorphisms pseudo-isotopic to the identity.*

Proposition 3.5 (Universal construction, [7] Proposition 1.1)**.** *Let* $Z : \mathrm{Mor}(\emptyset, \emptyset) \to \mathbb{C}$ *be a diffeomorphism invariant of* $n+1$*-manifolds which is multiplicative under disjoint union. There exists a unique non-degenerate quantization functor, which we will denote also by Z, whose restriction to* $\mathrm{Mor}(\emptyset, \emptyset)$ *is Z. Furthermore Z is a lax monoidal functor.*

Proof. Define $V(\Sigma) := \mathrm{Span}\{Mor(\emptyset,\Sigma)\}$ and $V'(\Sigma) := \mathrm{Span}\{Mor(\Sigma,\emptyset)\}$. Define a pairing $\langle\cdot,\cdot\rangle : V'(\Sigma)\otimes V(\Sigma)\to\mathbb{C}$ by extending linearly the bracket defined on the bases as $\langle M_2, M_1\rangle = Z(M_2\circ M_1)$. Let then $Z(\Sigma) := V(\Sigma)/Ann(V'(\Sigma))$ where $Ann(V'(\Sigma)) := \{v\in V(\Sigma)|\langle w,v\rangle = 0\ \forall w\in V'(\Sigma)\}$ and similarly let $Z'(\Sigma) := V'(\Sigma)/Ann(V(\Sigma))$ where $Ann(V(\Sigma)) := \{w\in V'(\Sigma)|\langle w,v\rangle = 0\ \forall v\in V(\Sigma)\}$. It is straightforward to check that this defines a functor into $\mathcal{V}ec$ which by construction is non-degenerate. By construction, for each Σ (possibly non connected) there is a non degenerate pairing $\langle\cdot,\cdot\rangle_\Sigma : V'(\Sigma)\otimes V(\Sigma)\to\mathbb{C}$.

The last statement is proved as follows : let Σ_1,Σ_2 be two $(n-1)$-manifolds, then there is a natural map $i_{\Sigma_1,\Sigma_2} : Z(\Sigma_1)\otimes Z(\Sigma_2)\to Z(\Sigma_1\sqcup\Sigma_2)$ defined by extending linearly the map sending a pair M_1, M_2 of manifolds bounded by Σ_1 and Σ_2 respectively to $M_1\sqcup M_2$. This map is well defined as if $[M_1] = 0\in Z(\Sigma_1)$ then $[M_1\sqcup M_2]$ will also be null in $Z(\Sigma_1\sqcup\Sigma_2)$ as every closed manifold obtained by capping $M_1\sqcup M_2$ can also be seen as a closed manifold obtained by capping M_1 alone. Similarly there is a map $d'_{\Sigma_1,\Sigma_2} : V'(\Sigma_1)\otimes V'(\Sigma_2)\to V'(\Sigma_1\sqcup\Sigma_2)$. Furthermore d_{Σ_1,Σ_2} and d'_{Σ_1,Σ_2} are injective: indeed the restriction of the pairing $\langle\cdot,\cdot\rangle_{\Sigma_1\sqcup\Sigma_2}$ to their images is by construction equal to $\langle\cdot,\cdot\rangle_{\Sigma_1}\langle\cdot,\cdot\rangle_{\Sigma_2}$ and thus non-degenerate. An element in the kernel of d_{Σ_1,Σ_2} is then in the kernel of $\langle\cdot,\cdot\rangle_{\Sigma_1}\langle\cdot,\cdot\rangle_{\Sigma_2}$ and hence is zero. 3.5

The following is straightforward:

Proposition 3.6. *Let Z be a non-degenerate n-TQFT and suppose that for each $M\in\mathrm{Mor}(\emptyset,\emptyset)$ it holds $\overline{Z(M)} = Z(\overline{M})$. Then for each Σ there is a $\mathbb{C}$-antilinear isomorphism $i : Z(\Sigma)\to Z'(\Sigma)$ defined by extending $\mathbb{C}$ anti-linearly the map $\bar{\cdot} : Mor(\emptyset,\Sigma)\to Mor(\overline{\Sigma},\emptyset)$ defined by $[M]\to[\overline{M}]$. This equips $Z(\Sigma)$ with a $Mod(\Sigma)$-invariant hermitian form $\langle\cdot,\cdot\rangle$.*

Definition 3.7 (Operations with TQFTs)**.** If Z_1, Z_2 are TQFTs then :

- $Z_1\otimes Z_2$ is the TQFT associating to each Σ the vector space $Z_1(\Sigma)\otimes Z_2(\Sigma)$ and to each cobordism the tensor product of the associated maps.
- A morphism $f : Z_1\to Z_2$ is a natural transformation between Z_1 and Z_2.
- Z_1 and Z_2 are isomorphic if there are morphisms $f : Z_1\to Z_2$ and $g : Z_2\to Z_1$ such that $g\circ f = Id_{Z_1}$ and $f\circ g = Id_{Z_2}$.

Theorem 3.8 (Turaev, [46] Theorem 3.7)**.** *If Z_1, Z_2 are n-TQFTs which coincide on $Mor(\emptyset,\emptyset)$ and such that Z_1 is non-degenerate, then Z_1 and Z_2 are isomorphic.*

Proof. Observe first that $\dim(Z_1(\Sigma)) = Z_1(\Sigma\times\mathbb{S}^1) = \dim(Z_2(\Sigma))$, $\forall\Sigma$. Since Z_i are TQFTs there are natural pairings $\langle,\rangle_i : Z_i(\overline{\Sigma})\otimes Z_i(\Sigma)\to\mathbb{C}$ induced by the duality in Cob_n. Now let $\beta_i(\Sigma) = Z_i(\Sigma)/Ann(Z_i(\overline{\Sigma}))$. Since Z_1 is non-degenerate then $\beta_1(\Sigma) = Z_1(\Sigma)$. Furthermore there is a well defined and injective functorial map $i : \beta_1(\Sigma)\to\beta_2(\Sigma)$ defined on manifolds M bounded by Σ by $i(M) = [Z_2(M)]$. The map is well defined as if $Z_1(M') = Z_1(M)$ then for all $N\in\mathcal{V}'(\Sigma)$ it holds :

$$0 = \langle Z_1(N), Z_1(M) - Z_1(M')\rangle_1 = Z_2(N\circ M) - Z_2(N\circ M') = \langle Z_2(N), Z_2(M) - Z_2(M')\rangle_2$$

so $Z_2(M) - Z_2(M')\in Ann(Z_2(\overline{\Sigma}))$. The same argument shows that the map is injective. But since $\beta_1(\Sigma) = Z_1(\Sigma)$ and $\dim(Z_1(\Sigma)) = \dim(Z_2(\Sigma))$ the map is an isomorphism. 3.8

4. Some examples

The preceding section left open some very natural questions on TQFTs: we now spell these out and provide examples to support the answer.

Question 4.1. *Do there exist different TQFTs having the same associated invariants of closed manifolds? If one applies the universal construction to the invariant of closed manifolds associated to a TQFT, does he get a TQFT? Will it be identical to the starting one?*

In this section we will answer the above questions (respectively by “yes", “not in general", “not in general") by looking at examples of TQFTs in dimension 2. Let’s observe first that if $n=2$ then each object of Cob_n is a tensor product of copies of $\mathbb{S}^1$ and so to know a TQFT it is sufficient to know $Z(\mathbb{S}^1)$ which by Remark 2.7 is a commutative Frobenius algebra. This was observed and studied by various authors, see for instance [15],[1] or [28]:

Theorem 4.2. *A $1+1$-TQFT is uniquely determined by the Frobenius algebra structure of $Z(\mathbb{S}^1)$. Reciprocally, given a commutative Frobenius algebra A there exists a unique TQFT Z such that $Z(\mathbb{S}^1)=A$.*

One implication of the theorem is easy: $Z(\mathbb{S}^1)$ must be a Frobenius algebra because of the topological properties of the surfaces obtained by glueing pants and discs. The harder part of the theorem is to check that the assignment of a commutative Frobenius algebra to a circle does indeed provide a TQFT : this boils down to check that in the category Cob_2 there are no new relations among the pants associated to the product and coproduct.

Exercise 4.3. Let A be a commutative Frobenius algebra. Prove that then the bilinear form $\langle x,y\rangle := \epsilon(xy)$ is non-degenerate and satisfies $\langle xy,z\rangle = \langle x,yz\rangle,\ \forall x,y,z\in A$. Reciprocally prove that if A is a commutative, unital algebra equipped with a non-degenerate form having these properties then A is a Frobenius algebra.

Solution 4.4. The identity $\langle xy,z\rangle = \langle x,yz\rangle,\ \forall x,y,z\in A$ is a direct consequence of the associativity of the product in A. The non-degeneracy of $\epsilon(xy)$ is a direct consequence of the general fact that a “Frobenius algebra in a monoidal category is dual to itself”. More explicitly, if y is an element of the annihilator of $\langle\cdot,\cdot\rangle$ then it holds:

$$y = Id(y) = (Id\otimes\epsilon\circ\mu)\circ(\Delta(1)\otimes Id)y = 0.$$

Reciprocally, given a bilinear non degenerate form $\langle\cdot,\cdot\rangle$ such that $\langle xy,z\rangle = \langle x,yz\rangle,\ \forall x,y,z\in A$, then we can define $\epsilon : A\to\mathbb{C}$ as $\epsilon(x)=\langle 1,x\rangle,\ \forall x\in A$; observe that A is a finite dimensional algebra (as it admits a non degenerate bilinear pairing with itself). Let $x_i, i\in I$ be a (finite) basis of A and let $M_{i,j} := \epsilon(x_i x_j), i,j\in I$; clearly $\det(M)\neq 0$ and we may define $\Delta(1)\in A\otimes A$ as $\Delta(1) := \sum_{i,j}(M^{-1})_{i,j}x_i\otimes x_j$. By construction it holds: $(Id\otimes\epsilon\circ\mu)(\Delta(1)\otimes Id)(x)=x,\ \forall x\in A$. Indeed we have, letting $x=\sum_{k\in I} a_k x_k$ (for some coordinates $a_k\in\mathbb{C}$):

$$(Id\otimes\epsilon\circ\mu)(\Delta(1)\otimes x) = (Id\otimes\epsilon\circ\mu)\sum_{i,j,k\in I} a_k M^{-1}_{i,j}x_i\otimes x_j\otimes x_k = \sum_{i,j,k\in I} a_k M^{-1}_{i,j}M_{j,k}x_i = x.$$

And similarly it holds $(\epsilon\circ\mu\otimes Id)(Id\otimes\Delta(1))(x)=x,\ \forall x\in A$. Then one may define $\Delta_L : A\to A\otimes A$ by $\Delta_L(x) = (Id\otimes\mu)\circ(\Delta(1)\otimes x) = \sum_{i,k\in I} M^{-1}_{i,k}x_i\otimes(x_k\cdot x)$. Let also: $\Delta_R(x) = (\mu\otimes Id)\circ(x\otimes\Delta(1)) = \sum_{i,k\in I} M^{-1}_{i,k}(x\cdot x_i)\otimes x_k$. We claim that $\Delta_L=\Delta_R$ and so we may just drop the index L or R in the notation. Indeed by the non-degeneracy of $\langle,\rangle$ it is sufficient to check the following :

$$(\epsilon\circ\mu\otimes\epsilon\circ\mu)\circ(Id\otimes\Delta_L\otimes Id) = (\epsilon\circ\mu\otimes\epsilon\circ\mu)\circ(Id\otimes\Delta_R\otimes Id).$$

Now, using $(Id\otimes\epsilon\circ\mu)(\Delta(1)\otimes Id)(x)=x$, the left hand side equals $\epsilon(\mu(\mu\otimes Id))$. Similarly the right hand side becomes $\epsilon(\mu(Id\otimes\mu))$ but these are equal by the hypothesis $\langle xy,z\rangle = \langle x,yz\rangle,\ \forall x,y,z\in A$. The fact that $(\epsilon\otimes Id)\circ\Delta = Id = (Id\otimes\epsilon)\circ\Delta$ is now straightforward as for instance using $\Delta=\Delta_L$ and the fact that $\epsilon=\epsilon\circ\mu\circ(1\otimes Id)$ we have:

$$(\epsilon\otimes Id)\circ\Delta_L(x) = (\epsilon\circ\mu\otimes Id)\circ(Id\otimes Id\otimes\mu)\circ(1\otimes\Delta(1)\otimes x) =$$
$$(\epsilon\circ\mu\otimes\mu)\circ(1\otimes\Delta(1)\otimes x) = \mu\circ(\epsilon\circ\mu\otimes Id\otimes Id)\circ(1\otimes\Delta(1)\otimes x) = \mu(1\cdot x) = x$$

where again we used the identity $(\epsilon\circ\mu\otimes Id)(Id\otimes\Delta(1))(x)=x,\ \forall x\in A$. (We advise the reader to draw a picture translating the above identities.) We leave to the reader to prove the coassociativity of Δ. Finally, for what concerns $\Delta\circ\mu = (Id\otimes\mu)\circ(\Delta\otimes Id) = (\mu\otimes Id)\circ(Id\otimes\Delta)$, let us prove the first equality using the expression $\Delta=\Delta_L$:

$$\Delta_L\circ\mu = (Id\otimes\mu)\circ(\Delta(1)\otimes\mu) = (Id\otimes\mu)\circ(Id\otimes Id\otimes\mu)\circ(\Delta(1)\otimes Id\otimes Id) =$$
$$= (Id\otimes\mu)\circ(Id\otimes\mu\otimes Id)\circ(\Delta(1)\otimes Id\otimes Id) = (Id\otimes\mu)\circ(\Delta\otimes Id)$$

where in the second equality we used the associativity of μ and in the third the definition of Δ_L.

We will use extensively the following exercise :

Exercise 4.5. Let A be a commutative Frobenius algebra and fix a basis x_i of A as a $\mathbb{C}$-vector space; let $x_i^* \in A$ be the element defined so that $\epsilon(x_i^* x_j) = \delta_{i,j}$ and finally let $\theta = \sum_i x_i x_i^*$. If Z is a $1+1$-TQFT such that $Z(\mathbb{S}^1) = A$ then the value of Z on a closed surface of genus $g \geq 0$ is $\epsilon(\theta^g)$. In particular its value on $\mathbb{S}^1 \times \mathbb{S}^1$ is $\dim_{\mathbb{C}}(A)$.

Example 4.6. Let A be the de Rham cohomology of your favorite compact complex manifold M. It is a commutative Frobenius algebra by endowing it with the pairing given by $\epsilon(\omega_1 \cdot \omega_2) := \int_{[M]} \omega_1 \wedge \omega_2$, where $[M]$ is the fundamental class of M. In particular for $\mathbb{CP}^1$ one gets the algebra $\mathbb{C}[X]/X^2$ which is at the base of the construction of Khovanov homology. Notice that $\Delta(1) = 1 \otimes x + x \otimes 1$ and $\Delta(x) = x \otimes x$ and that these values can be computed starting from the ϵ form (evaluation on the fundamental cycle of $\mathbb{CP}^1$). The associated TQFT evaluates each sphere to 0 each torus to 2 and each other connected surface to 0. Let $\Sigma_g \in \mathrm{Mor}(\emptyset, \mathbb{S}^1)$ be the complement of a disc in a genus g oriented surface. If we apply the universal construction we immediately see that $Z(\mathbb{S}^1) = \mathrm{Span}_{\mathbb{C}}\{\Sigma_0, \Sigma_1\}$ and letting $\Sigma_{g,h} := \Sigma_g \sqcup \Sigma_h$ and $Y_k = \Sigma_k \setminus D^2$ then it is not difficult to realize that the vectors $\Sigma_{i,j}, Y_k, i, j, k \in \{0,1\}$ generate $Z(\mathbb{S}^1 \sqcup \mathbb{S}^1)$ but they are not independent as the coupling matrix (i.e. expressing $\epsilon \circ m$) written in the basis $\Sigma_{0,0}, \Sigma_{1,0}, \Sigma_{0,1}, \Sigma_{1,1}, Y_0, Y_1$ is :

$$\begin{pmatrix} 0 & 0 & 0 & 4 & 0 & 2 \\ 0 & 0 & 4 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 & 2 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

whose rank is 4. Actually as the rank of the first 4×4 minor is 4 the vectors $\Sigma_{i,j}$ form a basis of $Z(\mathbb{S}^1 \sqcup \mathbb{S}^1)$. More in general it is not difficult to check that $Z(\mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1)$ is $Z(\mathbb{S}^1) \otimes \cdots \otimes Z(\mathbb{S}^1)$ and thus Z is a TQFT. Indeed, denoting Σ^g the cobordism from $\mathbb{S}^1$ to $\emptyset$ represented by a genus g surface with one boundary component, then one can verify that $Id_{Z(\mathbb{S}^1)} = \frac{1}{2}\big(Z(\Sigma_0) \circ Z(\Sigma^1) + Z(\Sigma_1) \circ Z(\Sigma^0)\big)$. Topologically this identity tells us that the image of the cobordism given by an annulus is the same as a linear combination of that of the cobordisms formed by a disc and a once punctured torus. This allows to split the image via Z of any cobordism Σ from $\emptyset$ to a $\mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1$ into a linear combination of morphisms associated via Z to a disjoint union of surfaces with only one boundary component and so to show that $Z(\Sigma) \in Z(\mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1)$ belongs also to $Z(\mathbb{S}^1) \otimes \cdots \otimes Z(\mathbb{S}^1)$. Indeed, for each surface Σ, one can use the above identity to express the morphism $Z(\Sigma)$ as a liner combination of morphisms associated to the surfaces obtained by compressing Σ along an essential curve; iterating this, and choosing essential curves which separate the different boundary components of the initial surface, one can then reduce to disjoint union of surfaces with only one boundary component. (For instance, if $\Sigma : \emptyset \to \mathbb{S}^1 \sqcup \mathbb{S}^1$ is an annulus with two boundary components then, compressing along the core of the annulus we get $Z(\Sigma) = \frac{1}{2}\big(Z(\Sigma_0) \otimes Z(\Sigma_1) + Z(\Sigma_1) \otimes Z(\Sigma_0)\big)$.

In this case if we apply the universal construction to invariants of the TQFT associated to the Frobenius algebra $H^*(\mathbb{CP}^1)$ we recover the initial TQFT. But this is not always the case as the following examples show.

Exercise 4.7. If $A = H^*(\mathbb{CP}^n)$ what is the value of $Z(X_g)$ where X_g is the connected surface of genus g?

Solution 4.8. In the Frobenius algebra $\mathbb{C}[x]/x^{n+1}$ we have $\epsilon(x^a) = 0$ unless $a = n$, so that $\theta = \sum_{i=0}^n m(x^i \otimes x^{n-i}) = (n+1)x^n$. Hence $Z(X_g) = 0$ unless $g = 1$ in which case we have $Z(\mathbb{S}^1 \times \mathbb{S}^1) = n+1$.

Example 4.9. Let $A' = H^*(\mathbb{CP}^1 \times \mathbb{CP}^1; \mathbb{C})$ i.e. $A = \mathbb{C}[x, y]/\{x^2, y^2\}$. Then $\theta_{A'} = 4xy$ and $\theta^g_{A'} = 0\ \forall g > 1$ so that $Z_{A'}(\mathbb{S}^2) = 0$, $Z_{A'}(\mathbb{S}^1 \times \mathbb{S}^1) = 4$ and $Z_{A'}(\Sigma_g) = 0\ \forall g > 1$. These values coincide with those of the case $A = H^*(\mathbb{CP}^3)$. This shows that two TQFTs may have the same invariants without being isomorphic (indeed A and A' are not isomorphic : check it!).

Example 4.10. Let Σ_g be the complement of a disc in a genus g oriented surface and $\Sigma_{g,h} := \Sigma_g \sqcup \Sigma_h$, $Y_k := \Sigma_k \setminus D^2$. If we apply the universal construction to the functor Z of the preceding example then we have $Z(\mathbb{S}^1) = \mathrm{Span}_{\mathbb{C}}\{\Sigma_0, \Sigma_1\}$, and it is not difficult to check that $Z(\mathbb{S}^1 \sqcup \mathbb{S}^1)$ is generated by the images through Z of $\Sigma_{0,0}, \Sigma_{1,0}, \Sigma_{0,1}, \Sigma_{1,1}, Y_0, Y_1$ and writing the pairing matrix in the basis $\Sigma_{0,0}, \Sigma_{1,0}, \Sigma_{0,1}, \Sigma_{1,1}, Y_0, Y_1$ we get :

$$\begin{pmatrix} 0 & 0 & 0 & 16 & 0 & 4 \\ 0 & 0 & 16 & 0 & 4 & 0 \\ 0 & 16 & 0 & 0 & 4 & 0 \\ 16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 4 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

whose rank is 5. Then $\dim_{\mathbb{C}}(Z(\mathbb{S}^1 \sqcup \mathbb{S}^1)) = 5$ and so Z is not a TQFT but just a finite quantization functor. (Prove finiteness as an exercice !) Remark furthermore that the so-obtained functor is different from both functors Z_A and $Z_{A'}$ associated to the Frobenius algebras A and A' in the preceding example : indeed those functors were by definition TQFTs (i.e. monoidal) while Z is not; moreover $\dim_{\mathbb{C}}(Z(\mathbb{S}^1)) = 2, \dim_{\mathbb{C}}(Z_A(\mathbb{S}^1)) = 4 = \dim_{\mathbb{C}}(Z_{A'}(\mathbb{S}^1))$.

Example 4.11. Let Z be the multiplicative invariant of n-manifold to be defined on connected ones as $Z(M) = \exp(\chi(M))$ (the Euler characteristic). Then the universal construction gives for every $\Sigma \in \mathrm{Cob}_n$ that $Z(\Sigma) = \mathbb{C}$ if Σ is cobordant to $\emptyset$ and $Z(\Sigma) = 0$ else, and $Z(W) = \exp(\chi(W) - \chi(\partial W_+)) \in \mathbb{C} = \mathrm{Hom}(\mathbb{C}, \mathbb{C})$ for each cobordism W.

Example 4.12. Let $n = 2$ and for each connected manifold M let $Z(M) = k^{b_1(M)}$ for some $k \in \mathbb{R} \setminus \{\pm 1, 0\}$ (the exponential of the first Betti number). Applying the universal construction one sees that, with the notation of the preceding example, $\Sigma_g = k^{2g}\Sigma_0$ in $Z(\mathbb{S}^1)$ and that thus $Z(\mathbb{S}^1)$ is one dimensional. Similarly in $Z(\mathbb{S}^1 \sqcup \mathbb{S}^1)$ it holds $Y_h = k^{2h} Y_0$ and $Y_0 \neq \Sigma_0 \sqcup \Sigma_0$ so that $Z(\mathbb{S}^1 \sqcup \mathbb{S}^1) = \mathrm{Span}_{\mathbb{C}}\{Y_0, \Sigma_0 \sqcup \Sigma_0\}$ and so Z is just a finite quantization functor but not a TQFT.

Let us then conclude by remarking that the following corollary of Theorem 3.8:

Corollary 4.13. *If Z is a degenerate TQFT the result of the universal construction on Z is a quantization functor but not a TQFT.*

Proof. By definition of universal construction, if the universal construction applied to Z gives a TQFT, let us call it $U : \mathrm{Cob}_n \to \mathcal{V}ec$, then it is a non-degenerate TQFT. But by Theorem 3.8 if it coincides with Z on closed cobordisms then also Z must be non-degenerate and this is excluded by hypothesis. 4.13

5. Generalities on mapping class groups

In this section we recall the definition of mapping class group of a surface, of Dehn twist, we recall the statement of the Baer's theorem and of the Nielsen-Thurston classification of mapping classes. We conclude by recalling the notion of central extension of a group and defining a central extension of the mapping class group of a closed surface which will be needed later on.

5.1. Basic definitions.

Let $\Sigma^b_{g,p}$ be the complement of b open disjoint discs and p points $\{q_1,\dots,q_p\}$ in a closed oriented surface Σ_g of genus g. Let

$$Homeo^+(\Sigma^b_{g,p},\partial\Sigma^b_{g,p}) = \{f:\Sigma^b_{g,p}\to\Sigma^b_{g,p}|f \textit{ orientation preserving homeomorphism}$$
$$\textit{such that } f|_{\partial\Sigma^b_{g,p}} = Id \textit{ and } f\big(\{q_1,\dots,q_p\}\big)=\{q_1,\dots,q_p\}\}$$

endowed with the compact open topology.

Definition 5.1 (Mapping class group)**.** The mapping class group of $\Sigma^b_{g,p}$ is

$$Mod(\Sigma^b_{g,p}) := \pi_0\Big(Homeo^+(\Sigma^b_{g,p},\partial\Sigma^b_{g,p})\Big).$$

Its elements are called mapping classes. If $b=0$ we may also consider $Mod^{\pm}(\Sigma^b_{g,p}) = \pi_0\Big(Homeo^{\pm}(\Sigma^b_{g,p},\partial\Sigma^b_{g,p})\Big)$, where $Homeo^{\pm}$ is the set of diffeomorphism preserving $\{q_1,\dots,$ $\dots,q_p\}$ but possibly reversing the orientation.

Remark 5.2. By definition a mapping class must be the identity on $\partial\Sigma^b_{g,p}$ but may permute the punctures q_i.

Exercise 5.3. Prove that $Mod(\Sigma^1_{0,0}) = Mod(\Sigma^0_{0,1}) = Mod(\Sigma^0_{0,0}) = \{Id\}$.

Example 5.4 (Dehn twist in the annulus)**.** Let us parametrize the oriented annulus $\Sigma^2_{0,0}$ as $([-1,1]\times[0,2\pi])/\sim$ where $(x,\theta)=(y,\theta') \iff x=y$ and $\theta-\theta'\in 2\pi\mathbb{Z}$. The right handed Dehn-twist is the class in $Mod(\Sigma^2_{0,0})$ of the diffeomorphism $T(x,\theta)=(x,\theta-\pi(x+1))$.

Exercise 5.5. Prove that $Mod(\Sigma^2_{0,0})=\mathbb{Z}$ and that a generator is the right handed Dehn-twist.

Lemma 5.6. *Let $\Sigma^c_{g',p+e}$ be an oriented surface containing $p+e$ marked points $\{p_1,\dots p_p,$ $q_1,\dots q_e\}$ and let $i:\Sigma^b_{g,p}\to\Sigma^c_{g',p+e}$ be an embedding sending the marked points of $\Sigma^b_{g,p}$ to the points $\{p_1,\dots p_p\}$ and such that $\{q_1\dots,q_e\}\cap i(\Sigma^b_{g,p})=\emptyset$. Then there is an induced morphism $i_*:Mod(\Sigma^b_{g,p})\to Mod(\Sigma^c_{g',p+e})$.*

Proof. Each diffeomorphism and isotopy relative to $\partial\Sigma^b_{g,p}\cup\{p_1,\dots,p_p\}$ can be extended via the identity on $\Sigma^c_{g',p+e}\setminus i(\Sigma^b_{g,p})$. 5.6

Remark 5.7. Remark that we make no requirement on the image through i of the boundary components of $\Sigma^b_{g,p}$.

Definition 5.8 (Dehn twist)**.** Let $c\subset\Sigma^b_{g,p}$ be a simple closed curve in the complement of the marked points of $\Sigma^b_{g,p}$. and let $i:A\to\Sigma^n_{g,p}$ be an embedding of an oriented annulus such that $i(\{0\}\times\mathbb{S}^1)=c$. The right handed Dehn-twist along c is $i_*(T)$ where T was defined in Example 5.4.

Remark 5.9. By unicity up to isotopy of the regular neighborhood of c the definition does not depend on the choice of i.

Recall that $H_1(\Sigma^b_{g,p};\mathbb{Z})$ is equipped with a $\mathbb{Z}$-valued antisymmetric bilinear form $i(\cdot,\cdot)$ given by the algebraic intersection number of closed oriented curves.

Exercise 5.10. Prove that $i(\cdot,\cdot)$ is degenerate iff $p+b>1$.

By an abuse of notation we shall denote by $Sp(H_1(\Sigma^b_{g,p};\mathbb{Z}))$ the groups of automorphisms of the abelian group $H_1(\Sigma^b_{g,p};\mathbb{Z})$ preserving the bilinear form $i(\cdot,\cdot)$. Clearly, the natural action of $Mod(\Sigma^b_{g,p})$ on $H_1(\Sigma^b_{g,p};\mathbb{Z})$ induces a morphism $h_*:Mod(\Sigma^b_{g,p})\to Sp(H_1(\Sigma^b_{g,p};\mathbb{Z}))$.

Exercise 5.11. Show that h_* is not surjective if $b>0$.

When $b > 0$ plenty of exceptional cases occur and should be taken care of. So from here on we will often suppose that $b = 0$, unless the proofs and statements do not require special adaptation to the case $b > 0$. Hence we will write simply $\Sigma_{g,p}$ for $\Sigma^0_{g,p}$.

Proposition 5.12. *For $\Sigma_{1,0}$ and $\Sigma_{1,1}$ h_* is an isomorphism.*

Proof. We give a very sketchy proof. Let us parametrize Σ as $[0,1]\times[0,1]/\sim$ where $(x,y)\sim(x',y') \iff x-x'\in\mathbb{Z}$ and $y-y'\in\mathbb{Z}$, and if $p=1$ set $p_1=(\frac{1}{2},\frac{1}{2})$. Observe that $Sp(H_1) = SL_2(\mathbb{Z})$ which is known to be generated by the following two matrices :

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Furthermore these matrices can be easily realized by two diffeorphisms of Σ : namely respectively $f(x,y)=(x+y,y)$ and $g(x,y)=(-y,x)$ (in the case $p=0$, while for $p=1$ one should be a little more careful when writing f in order to avoid moving p_1). Thus we are left to prove injectivity. Observe that in $\Sigma^0_{0,p}$ each primitive homology class is represented by exactly one connected, oriented simple closed curve in $\Sigma^0_{0,p}$. This implies that if $\phi\in Mod(\Sigma)$ is such that $i_*(\phi)=Id$ then up to isotopy we can suppose that $\phi(x,0)=(x,0)$ and also $\phi(0,y)=(0,y)$. But then ϕ is isotopic to the identity because it is induced by a mapping class in the disc (if $p=0$) or in the punctured disc (if $p=1$). 5.12

More in general the following holds (for a proof see Theorem 6.4 in [18]) :

Theorem 5.13. *$\forall g,p$ the homomorphism $h_* : Mod(\Sigma_{g,p}) \to Sp(H_1(\Sigma_{g,p};\mathbb{Z}))$ is surjective.*

Definition 5.14 (Torelli group)**.** The Torelli group is $Tor(\Sigma^b_{g,p}) := ker(h_*)$.

5.2. Nielsen Thurston classification of diffeomorphisms.

Recall that $Mod(\mathcal{T}^2)=SL(2;\mathbb{Z})$, and let $M\neq Id\in SL(2;\mathbb{Z})$. Clearly $\det(M)=1$ and the following three cases are possible :

- $|\operatorname{tr}(M)|\leq 2$: in this case M represents an elliptic isometry of the hyperbolic plane $\mathbb{H}^2$. Furthermore the order of M can be only $2,3,4$ or 6 (exercise!). So M is periodic.
- $|\operatorname{tr}(M)|=2$: in this case M represents a parabolic isometry of $\mathbb{H}^2$ and there is a rational eigenvector of M with eigenvalue ±1 : representing it by coprime integers, we get a simple closed curve in $\mathcal{T}^2$ preserved by M. Thus M is said to be *reducible*.
- $|\operatorname{tr}(M)|>2$: in this case M represents a hyperbolic isometry of $\mathbb{H}^2$. In this case there are two distinct eigenvectors one with eigenvalue λ *s.t.* $|\lambda|>1$ (“dilatating") and one with eigenvalue λ^{-1} (“contracting"). This gives two transverse foliations in $\mathcal{T}^2$ which are kept invariant by M. In this case we say that M is *Anosov*.

The above classification actually has been generalized by Thurston to all punctured surfaces. In order to do so let us fix the following :

Definition 5.15 (Singular foliation of $\Sigma_{g,p}$)**.** 1. A singular foliation of $\Sigma_{g,p}$ is a smooth foliation of the complement of finitely many “singular points" $\{x_1,\dots,x_k\}\subset\Sigma_{g,p}$ such that for each point x_i or p_j there exists a local smooth chart of $\Sigma_{g,p}$ around the point in which the foliation is the pre-image of the horizontal foliation of $\mathbb{R}^2=\mathbb{C}$ (i.e. the foliation by the lines $y=h$) by the map $z\to\sqrt{z}^r$ for $r\geq 3$ (or, around the punctures also $r=1$ is allowed) : see Figure 5.1. (Here define $\sqrt{z}$ by cutting along the negative real axis : the preimage of the horizontal foliations is easily seen to be a smooth out of the origin.)

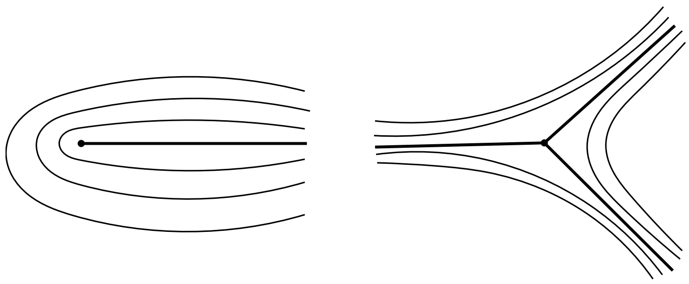


Figure 5.1: The local structure of a singular foliation around a puncture (on the left hand side) and around a singular point (in the right hand side, corresponding to the case $r = 3$ in Definition 5.15).

2. Given a singular foliation F on $\Sigma_{g,p}$, a transverse arc is a smooth path $c : [0, 1] \to \Sigma_{g,p}$ which is everywhere transverse to F (and in particular avoids the singular points); an isotopy of transverse arcs is an isotopy among transverse arcs such that the endpoints of the arcs are moved along the leafs of F they are initially contained in. Let A be the set of transverse isotopy classes of arcs.

3. Given a singular foliation F on $\Sigma_{g,p}$, a transverse measure is a map $\mu : A \to \mathbb{R}_+$ which is additive by concatenation of smooth transverse arcs and which is locally absolutely continuous with respect to the measure $|dy|$. (More explicitly if $\alpha : [0, 1] \to \Sigma$ is a smooth arc transverse to a singular foliation F and whose support is contained in a local chart with values in $\mathbb{R}^2$ with coordinates (x, y) in which the leaves of F are of the form $y = constant$ then $\mu(\alpha) = |\int_0^1 (y(\alpha(t)))'\, dt| = |y(\alpha(1)) - y(\alpha(0))|$. More in general, if the image of α is not contained in a local chart as above, one first cuts α into small pieces having this property and then sums their contributions up to compute $\mu(\alpha)$.

4. A homeomorphism $\tilde{f}$ acts on a measured singular foliation (F, μ) by
$$\tilde{f}(F, \mu) := (\tilde{f}(F), f_*(\mu))$$
where $f_*(\mu)(c) = \mu(f^{-1}(c))$.)

Definition 5.16. A class $f \in Mod(\Sigma_{g,p})$ is periodic if $\exists k > 0$ such that $f^k = Id \in Mod(\Sigma_{g,p})$. It is *reducible* if there exists a family $c_1, \dots c_k$ of pairwise disjoint oriented simple closed curves (each not bounding discs or once punctured discs) such that $f(c_i) = c_i$ (up to isotopy). We say that f is *pseudo-anosov* if there exist a representative $\tilde{f}$ of the class f and two transversally measured singular foliations $(F_\pm, \mu_\pm)$ such that $\tilde{f}(F_\pm) = F_\pm$ and a constant $\lambda > 1$ such that $\tilde{f}(F_\pm) = F_\pm$ and $\tilde{f}_*(\mu_\pm) = \lambda^{\pm 1}\mu_\pm$.

There are plenty of good references for the following fundamental result among which we mention [26] Theorem 0.1, [18] Theorem 13.1, or [17]:

Theorem 5.17 (Nielsen-Thurston classification of self-diffeomorphisms of surfaces)**.** *Let $f \in Mod(\Sigma_{g,p})$ then there exists a finite family of disjoint simple closed curves $c_1, \dots c_n$ such that for each component S_i of $\Sigma_{g,p} \setminus (c_1 \cup \dots \cup c_n)$, letting k_i be the least positive integer such that $f^{k_i}(S_i) = S_i$ then $(f|_{S_i})^{k_i}$ is either a periodic or a pseudo-Anosov self-diffeomorphism of S_i.*

5.3. Generalities on group cohomology and central extensions

In this subsection we rapidly recall some basic facts about group cohomology and central extensions we will use in the next subsection. The expert reader may just skip it. For full details on group cohomology and its relation to group extensions, the interested reader may consult [10].

Suppose that we have a morphism ρ from a group G into a quotient of a group S by its center Z. We would like to lift it to a morphism $\rho' : G \to S$. To do so we could fix a system of generators of G and choose arbitrarily lifts $\rho'(g_i)$ of $\rho(g_i)$. For this to provide a morphism the relations of G should be satisfied; this is in general not possible. In particular let's fix the whole G as the set of generators and as set of relations consider those of the form $R = \{(g_1 g_2) g_2^{-1} g_1^{-1}, g_1, g_2 \in G\}$. In order to find a lift ρ' we must be able to find $\rho'(g_i)$ so that $\rho'(g_1 g_2)\rho'(g_2)^{-1}\rho'(g_1)^{-1} = 1 \in Z$. So observe that if we pick an arbitrary lift then the maps $C(g_1, g_2) := \rho'(g_1 g_2)\rho'(g_2)^{-1}\rho'(g_1)^{-1}$ give a map $C : R \to Z$. Furthermore observe that for each three-tuple $(g_1, g_2, g_3) \in G^3$ it will automatically hold that the product of the values of C on the boundary of the tetrahedron whose faces are formed by the triangles associated to the relations $c(g_2, g_3), c(g_1 g_2, g_3)^{-1}, c(g_1, g_2 g_3), c(g_1, g_2)$ is $1 \in Z$. More explicitly, the reader may prove as an exercice that the following 2-cocycle condition holds : $c(g_2, g_3)c(g_1 g_2, g_3)^{-1}c(g_1, g_2 g_3)c(g_1, g_2)^{-1} = 1$. This says that the map C is a "two cycle" for G with coefficients in Z (seen as a trivial G module):

Definition 5.18 (Group cohomology)**.** Let G be a group, Z be an abelian group which is a G module and for each $n \geq 1$ let $C^n(G; Z) = \{c : G^n \to Z\}$. Let $\delta_n : C^n \to C^{n+1}$ be defined as follows :

$$\delta(c)(g_1, \ldots, g_{n+1}) = g_1 \cdot c(g_2, \ldots g_{n+1}) + \sum_{i=1}^{n} (-1)^i c(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) + \\ + (-1)^{n+1} c(g_1, \ldots, g_n)$$

where we use additive notation. It turns out that $\delta_{n+1} \circ \delta_n = 0\ \forall n$, so one defines $Z^n(G; Z) = \ker(\delta_n), B^n(G; Z) = Im(\delta_{n-1})$ and $H^n(G; Z) = Z^n(G; Z)/B^n(G; Z)$.

Observe furthermore that if we replace $\rho'(g_i)$ with $z_i \rho'(g_i)$ and $\rho'(g_1 g_2)$ by $z_{12}\rho'(g_1 g_2)$ then $C(g_1, g_2)$ gets multiplied by $z_{12} z_2^{-1} z_1^{-1}$ and this is precisely a one coboundary in the above cohomology (where we are using multiplicative notation). So the question we would like to ask is whether up to changing simultaneously the map C in all its components by a one-coboundary as above we can reduce it to the map $c(g_i, g_j) = 1,\ \forall g_i, g_j \in G$, which cohomologically translates to whether the 2-cohomology class represented by $[C]$ is trivial or not.

This shows that the obstruction to lift ρ to a representation into S is a cohomology class $[C] \in H^2(G; Z)$.

Stated differently, given a 2-cocycle with values in Z we can define a central extension $\tilde{G}$ of G by setting $(g_1, 1) \cdot (g_2, 1) := (g_1 g_2, c(g_1, g_2))$. The associativity of the product is assured by the above 2-cocycle condition :

$$\big((g_1, 1)(g_2, 1)\big)(g_3, 1) = (g_1 g_2 g_3, c(g_1, g_2)c(g_1 g_2, g_3)) = \tag{5.1}$$

$$(g_1 g_2 g_3, c(g_2, g_3)c(g_1, g_2 g_3)) = (g_1, 1)\big((g_2, 1)(g_3, 1)\big). \tag{5.2}$$

The projection on the first factor $\pi : \tilde{G} \to G$ has kernel given by the elements of the form $(1, z), z \in Z$ and is thus central. We finally have the following exact sequence $1 \to Z \to \tilde{G} \to G \to 1$ and it turns out that two sequences are isomorphic iff they are associated to cohomologous cocycles. In particular the sequence splits iff we can lift ρ.

5.4. The Maslov index and the Meyer cycle

In our case we shall associate a cocycle to $G = Mod(\Sigma_g)$ with coefficients in $\mathbb{Z}$ known as the Meyer cocycle (see [46] Chapter 3 or [22] for more details). We remark (but we will not use this in what follows) that Harer proved that $H^2(Mod(\Sigma_g); \mathbb{Z}) = \mathbb{Z}$ for all $g \geq 3$. To define the cocycle let us first define what the Maslov index is :

Definition 5.19. The Maslov index of three lagrangian subspaces $\mathcal{L}_i, i = 1, 2, 3$ of $H_1(\Sigma_g; \mathbb{Q})$ is the integer $\mu(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ defined as the signature of the bilinear symmetric form $\odot$ on

$(\mathcal{L}_1 + \mathcal{L}_2) \cap \mathcal{L}_3$ defined by $(a_1 + a_2) \odot (b_1 + b_2) = a_2 \cdot b_1$. (Here $a_i, b_i \in \mathcal{L}_i$ for $i = 1, 2$, and $a_1 + a_2, b_1 + b_2 \in \mathcal{L}_3$, and $\cdot$ denotes the symplectic intersection form.)

Exercise 5.20. Prove that the above defined form is well defined and symmetric. (Hint: use the fact that $\mathcal{L}_i$ are lagrangian.)

The following is a key property of the Maslov index :

Lemma 5.21 ([46], Lemma 3.6)**.** *The Maslov index changes sign under an odd permutation of the three lagrangians. Furthermore if $\mathcal{L}_i, i = 1, \ldots 4$ are lagrangian subspaces of $H_1(\Sigma_g; \mathbb{Q})$ and $f \in Mod(\Sigma_g)$ is any mapping class then it holds :*

$$\mu(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) - \mu(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_4) + \mu(\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_4) - \mu(\mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4) = 0 \tag{5.3}$$

$$\mu(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) = \mu(f_*(\mathcal{L}_1), f_*(\mathcal{L}_2), f_*(\mathcal{L}_3)). \tag{5.4}$$

Definition 5.22.

- An extended surface is a pair $(\Sigma_g, \mathcal{L})$ where $\mathcal{L} \subset H_1(\Sigma_g; \mathbb{Q})$ is a lagrangian subspace with respect to the intersection form in homology.
- An extended mapping class is a pair $(f, n) \in Mod(\Sigma_g) \times \mathbb{Z}$.

Fix an extended surface $\overline{\Sigma} := (\Sigma_g, \mathcal{L}_1)$ and let $\widetilde{Mod}(\Sigma_g)$ be the set $\{(f, n) | f \in Mod(\Sigma_g) \times \mathbb{Z}\}$ with the following operation :

$$(g, m) \cdot (f, n) = (g \cdot f, n + m - \mu(f_*(\mathcal{L}), \mathcal{L}, g_*^{-1}(\mathcal{L})) \tag{5.5}$$

where by f_*, g_* we mean the morphisms induced on homology by f and g.

Lemma 5.23. *The above defined operation endows $\widetilde{Mod}(\Sigma_g)$ of a group structure which is a $\mathbb{Z}$-central extension of $Mod(\Sigma_g)$.*

Proof. By the preceding general discussion it is sufficient to prove that $c(g, f) := -\mu(f_*(\mathcal{L}), \mathcal{L}, g_*^{-1}(\mathcal{L}))$ is a $\mathbb{Z}$-valued 2-cocycle. This (in additive notation) is the 2-cocycle condition on three classes $f, g, h \in Mod(\Sigma_g)$:

$$-\mu(f_*(\mathcal{L}), \mathcal{L}, g_*^{-1}(\mathcal{L})) + \mu((f \cdot g)_*(\mathcal{L}), \mathcal{L}, h_*^{-1}(\mathcal{L})) + \tag{5.6}$$

$$-\mu(f_*(\mathcal{L}), \mathcal{L}, (g \cdot h)_*^{-1}(\mathcal{L})) + \mu(g_*(\mathcal{L}), \mathcal{L}, h_*^{-1}(\mathcal{L})) = \tag{5.7}$$

$$-\mu(g_* f_*(\mathcal{L}), g_*(\mathcal{L}), \mathcal{L}) + \mu(h_* g_* f_*(\mathcal{L}), h_*(\mathcal{L}), \mathcal{L}) + \tag{5.8}$$

$$-\mu(h_* g_* f_*(\mathcal{L}), h_* g_*(\mathcal{L}), \mathcal{L}) + \mu(h_* g_*(\mathcal{L}), h_*(\mathcal{L}), \mathcal{L}) = \tag{5.9}$$

$$-\mu(h_* g_* f_*(\mathcal{L}), h_* g_*(\mathcal{L}), h_*(\mathcal{L})) + \mu(h_* g_* f_*(\mathcal{L}), h_*(\mathcal{L}), \mathcal{L}) + \tag{5.10}$$

$$-\mu(h_* g_* f_*(\mathcal{L}), h_* g_*(\mathcal{L}), \mathcal{L}) + \mu(h_* g_*(\mathcal{L}), h_*(\mathcal{L}), \mathcal{L}) = 0 \tag{5.11}$$

where we used equivariance of the Maslov index and in the last equality we applied Lemma 5.21.

5.23

6. The skein module and Reshetikhin-Turaev invariants

In this section we defined the Kauffman skein module $\mathcal{S}(M)$ of a 3-manifold M and its “rational versions" $\mathcal{S}_{\mathbb{Q}}(M)$ and $\mathcal{S}_{A_0}(M)$ needed to properly use the Jones-Wenzl idempotents. We then define the reduced skein module $\mathcal{S}^{red}_{A_0}(M)$ and prove a result allowing to “do skein calculus" directly in $\mathcal{S}^{red}_{A_0}(M)$. We then define the Reshetikhin-Turaev invariants of a three-manifold and prove that they are indeed invariants. We conclude by proving the Verlinde formula.

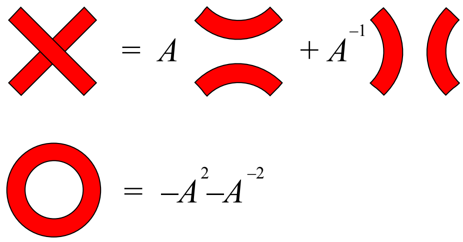


Figure 6.1: The Kauffman bracket relations.

6.1. The Kauffman module

Recall that a framing for a link L in a 3-manifold M is a non-zero vector field defined along L which is always transverse to L, seen up to isotopy. A link is framed if it is endowed with the choice of a framing. The *Kauffman skein module* of an oriented 3-manifold M (introduced independently by Przytycki [38] and Turaev [47], see also [31]) is the $\mathbb{Z}[A^{\pm 1}]$-module $\mathcal{S}(M)$ generated by all isotopy classes of framed links in M, modulo the Kauffman bracket relations shown in Fig. 6.1. An element of $\mathcal{S}(M)$ is called a *skein*.

Proposition 6.1 ([38], Theorems 2.3 and 3.1, or [6] Proposition 1.1)**.**

1. *Let* $M = \Sigma_g \times [-1, 1]$ *then* $\mathcal{S}(M)$ *is free* $\mathbb{Z}[A^{\pm 1}]$*-module generated by the multicurves in* $\Sigma_g \times \{0\}$ *(i.e. possibly empty disjoint unions of simple closed curves none of which bounds a disc in* Σ_g*).*

2. *One can define a non-commutative, associative product on* $\mathcal{S}(\Sigma_g \times [-1, 1])$ *via* $\alpha \cdot b := [\alpha \cup b]$ *where in* $\alpha \cup b$ *one first pushes* α *by isotopy near* $\Sigma_g \times \{1\}$ *and* b *near* $\Sigma_g \times \{-1\}$*.*

3. *If* $i : M \hookrightarrow N$ *is an embedding then there is an induced map* $i_* : \mathcal{S}(M) \to \mathcal{S}(N)$*. Furthermore if* $N \setminus i(M)$ *is a union of* 3*-balls then* i_* *is an isomorphism.*

Proof. 1). The idea of the proof is to use the fact that each framed link L can be represented by a diagram with crossings (as above) in Σ_g and that any two such diagrams are related by a finite sequence of “Reidemeister moves". Then to check that if one applies first all the desingularizations to a diagram of L and then replaces all the trivial components by factors $-A^2 - A^{-2}$ then the result *does not depend* on the initial diagram of L. This provides a normal form for every equivalence class in $\mathcal{S}(\Sigma_g \times [-1, 1])$. 2). The associativity of the product can be easily verified by observing that

$$\Sigma_g \times [-1, 1] \simeq \Sigma_g \times [0, 3] \simeq (\Sigma_g \times [0, 1]) \sqcup_{\Sigma_g \times \{1\}} (\Sigma_g \times [1, 2]) \sqcup_{\Sigma_g \times \{2\}} (\Sigma_g \times [2, 3]).$$

3). The first statement is obvious. For what concerns the second statement : surjectivity is due to the fact that every framed link in N is isotopic to one into $i(M)$; injectivity comes from the fact that any isotopy between two links in $i(M)$ can be supposed to avoid the balls of $N \setminus i(M)$. 6.1

Remark 6.2. When $A = -1$ the algebra structure one gets on $\mathcal{S}(\Sigma_g \times [-1, 1])$ is commutative: it turns out that this algebra is isomorphic to the algebra of regular functions on the space of representations of $\pi_1(\Sigma_g)$ into $SU(2)$ up to conjugation (see [11], [12]).

Corollary 6.3 (Kauffman [27])**.** *The spaces* $\mathcal{S}(\mathbb{S}^3)$ *and* $\mathcal{S}(D^2 \times [0, 1])$ *are spanned by the class of the empty link.*

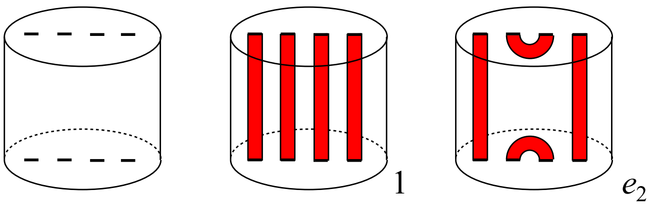


Figure 6.2: The Kauffman bracket skein module $\mathcal{S}(M)$ of the cylinder with $2n$ marked points: here $n = 4$ (left). The space $\mathcal{S}(M)$ is an algebra generated by the elements $1, e_1, \ldots, e_{n-1}$: here we draw e_2 (right).

Proof. Embed $\mathbb{S}^2 \times [-1,1]$ into $\mathbb{S}^3$, then observe that two framed links in $\mathbb{S}^2 \times [-1,1]$ are isotopic iff they also are isotopic in $\mathbb{S}^3$. Then $\mathcal{S}(\mathbb{S}^2 \times [-1,1]) = \mathcal{S}(\mathbb{S}^3)$ and we can apply the preceding proposition and conclude by observing that the only multicurve in $\mathbb{S}^2$ is the empty one. The proof for $\mathcal{S}(D^2 \times [0,1])$ is similar. 6.3

Corollary 6.4 ([38] Theorem 3.13)**.** *Let* $A = \mathbb{S}^1 \times [-1,1] \times [-1,1]$*. Then* $\mathcal{S}(A)$ *is the free commutative* $\mathbb{Z}[A^{\pm 1}]$ *algebra generated by the framed knot* $z = \mathbb{S}^1 \times \{0\} \times \{0\}$ *framed by a vector field tangent to* $\mathbb{S}^1 \times [-1,1] \times \{0\}$*: so* $\mathcal{S}(A) = \mathbb{Z}[A^{\pm 1}, z]$.

In other words every skein in $\mathcal{S}(\mathbb{S}^3)$ is equivalent to $k \cdot \varnothing$ for a well-defined complex number k, which is the *evaluation* of the skein. In order to compute the scalar k associated to each skein s in $\mathcal{S}(\mathbb{S}^3)$ or $\mathcal{S}(D^2)$ a full set of computational rules has been set up, now known as “skein theory" or “recoupling theory". The following section is devoted to recalling the basic objects of this theory.

6.2. The Jones-Wenzl projectors, $\mathcal{S}_{\mathbb{Q}}$ and $\mathcal{S}_{A_0}$

We define the *quantum integers*

$$[n] = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}} = A^{-2n+2} + A^{-2n+6} + \ldots + A^{2n-6} + A^{2n-2}$$

and note that $[n]$ is a Laurent polynomial in A whose zeroes are contained in the set S of roots of unity different from $\pm 1, \pm i$. Therefore these polynomials have non-zero evaluations at all the complex numbers which are non-zero and do not belong to S. In what follows, given a 3-manifold M we will need to be able to divide by some set of quantum integers $[n]$ the elements of $\mathcal{S}(M)$; this can be done in two possible ways :

1. we may set $\mathcal{S}_{\mathbb{Q}}(M) := \mathbb{Q}(A) \otimes_{\mathbb{Z}[A^{\pm 1}]} \mathcal{S}(M)$; then $\mathcal{S}_{\mathbb{Q}}(M)$ is a $\mathbb{Q}(A)$-vector space and we can divide by any Laurent polynomial in A;

2. or we may fix a value A_0 of A which is not a zero of any $[n]$ in our set and then consider the $\mathbb{C}$-vector space $\mathcal{S}_{A_0}(M) := \mathbb{C} \otimes_{\mathbb{Z}[A^{\pm 1}]} \mathcal{S}(M)$ (where $\mathbb{C}$ is seen as $\mathbb{Z}[A^{\pm 1}]$-module via the evaluation at A_0).

Remark 6.5. When considering $\mathcal{S}_{\mathbb{Q}}(M)$ we may also see it as a $\mathbb{Z}[A^{\pm 1}]$-module which contains $\mathcal{S}(M)$ as the submodule of the elements of the form $1 \otimes s, s \in \mathcal{S}(M)$. We will call these elements the integral elements of $\mathcal{S}_{\mathbb{Q}}(M)$.

The reason why we will need to divide by $[n]$ is given by the definition of the *Jones-Wenzl projectors* which we now recall.

There is a natural boundary version of the skein module. Let M be an oriented manifold with boundary and ∂M contain some disjoint oriented segments as in Fig. 6.2-(left). The skein

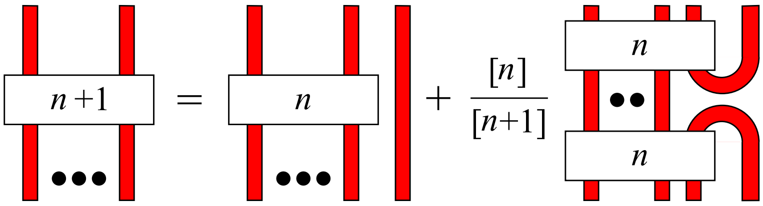


Figure 6.3: The $(n+1)^{th}$ Jones-Wenzl projector is defined recursively with this formula.

module $\mathcal{S}(M)$ is then defined as above by taking framed links and rectangles intersecting ∂M in those segments.

For instance, we may take M to be a cylinder $[0,1]\times[0,1]\times[-1,1]$ with $2n$ segments as in Fig. 6.2-(left) (so that the endpoints of the strands have coordinates of the form $(*,0,\pm 1)$). Cylinders can be stacked over each other, and hence $\mathcal{S}(M)$ and $\mathcal{S}_{\mathbb{Q}}(M)$ have natural algebra structures (called the *Temperly-Lieb algebra* and often denoted $\mathcal{T}_n$) whose multiplicative identity element is the skein 1 shown in Fig. 6.2-(centre). We define the elements $e_1,\dots,e_{n-1}$ as suggested in Fig. 6.2-(right): it is easy to prove that $\mathcal{S}(M)$ (resp. $\mathcal{S}_{\mathbb{Q}}(M)$) is generated as a $\mathbb{Z}[A^{\pm 1}]$-algebra (resp. $\mathbb{Q}(A)$-algebra) by the elements $1,e_1,\dots,e_{n-1}$.

The n-th *Jones-Wenzl projector* $f_n\in\mathcal{S}_{\mathbb{Q}}([0,1]\times[0,1]\times[-1,1],2n)$ defined inductively as in Fig. 6.3. It satisfies the following remarkable properties [32, Lemma 2]:

$$f_n\circ f_n=f_n,\quad f_n\circ e_i=e_i\circ f_n=0\quad\forall i. \tag{6.1}$$

So f_n is a projector which “kills” the skeins with short returns like the e_i's. Let I_n be the ideal generated by $e_1,\dots,e_{n-1}$: it follows from the recursive definition that

$$f_n=1+i_n\quad\text{for some } i_n\in I_n.$$

Let now $\hat{}:\mathcal{S}_{\mathbb{Q}}([0,1]\times[0,1]\times[-1,1],2n)\to\mathcal{S}_{\mathbb{Q}}([0,1]\times[0,1]\times\mathbb{S}^1)$ be the map which associates to each skein in $\mathcal{S}_{\mathbb{Q}}([0,1]\times[0,1]\times[-1,1],2n)$ its trivial closure (i.e. the skein in the annulus obtained by identifying $(x,y,1)\sim(x,y,-1),\ \forall x,y\in[0,1]$).

Exercise 6.6. Let $T_n:=\hat{f}_n$; using Corollary 6.4 observe that there is a $\mathbb{Q}(A)$-algebra structure on $\mathcal{S}_{\mathbb{Q}}([0,1]\times[0,1]\times\mathbb{S}^1)$. Prove that it holds $T_n\cdot T_1=T_{n+1}+T_{n-1},\ \forall n\geq 1$. Conclude that $T_n\in\mathcal{S}([0,1]\times[0,1]\times S^1)$ i.e. it is an integral skein (see Remark 6.5).

Definition 6.7 (Colored Jones polynomials)**.** The n^{th} colored Jones polynomial of a framed link $L\subset\mathbb{S}^3$ is the element of $\mathcal{S}(\mathbb{S}^3)=\mathbb{Z}[A^{\pm 1}]$ represented by cabling the link L with the element $T_n\in\mathcal{S}([0,1]\times[0,1]\times S^1)$ defined in Exercise 6.6.

Exercise 6.8. Prove that if L is a framed link in $\mathbb{S}^3$ then for each n the n^{th} colored Jones polynomial of L is indeed a Laurent polynomial.

For the following exercice, recall that if $k\subset\mathbb{S}^3$ is an oriented knot, then there exists a oriented surface whose boundary is k, called the Seifert surface. The intersection of a Seifert surface with the boundary of a regular neighborhood of k (which is a torus T^2) is a simple closed curve λ, parallel to k and providing the so called “Seifert framing" for k. It turns out that the homology class $[\lambda]\in H_1(T^2;\mathbb{Z})$ does not depend on the choice of the initial Seifert surface.

Exercise 6.9. Prove by recurrence that if u is the unknot in $\mathbb{S}^3$ framed by its Seifert framing then $J_n(u)=(-1)^n[n+1]$ where $[k]:=\frac{A^{2k}-A^{-2k}}{A^2-A^{-2}}$.

6.3. Ribbon graphs

The Jones-Wenzl projectors can be used to define skeins associated not only to links but also to graphs in a simple combinatorial way. A *ribbon graph* $Y \subset M$ is a 3-valent graph with a two-dimensional oriented thickening considered up to isotopy (it is the natural generalization of a framed link). Given $A_0 \in \mathbb{C}^*$ let $r(A_0) := min\{r > 0 | [r]_{A_0} = 0\}$ and let M be a compact oriented three manifold.

Definition 6.10 (Coloring, A_0-definable and A_0-admissible coloring)**.**

- A coloring on a ribbon graph $Y \in M$ is the assignment of an integer (color) to each edge of Y so that the three numbers a, b, c coloring the edges adjacent to any vertex satisfy the following conditions: $a+b+c \in 2\mathbb{N}$, and $a+b-c \geq 0, b+c-a \geq 0, c+a-b \geq 0$.

- Given $A_0 \in \mathbb{C}^*$ we say that the coloring is A_0-*definable* if the color of each edge is $\leq r(A_0)-1$.

- Given $A_0 \in \mathbb{C}^*$ we say that the coloring is A_0-*admissible* if the color of each edge is $< r(A_0)-1$ and $a+b+c \leq 2r(A_0)-4$ (where a, b, c are as above).

Remark 6.11. The terminology "A_0-definable" and "A_0-admissible" coloring appear in this text for the first time : let's explain their meaning and origin. As already stated, if A_0 is not a parameter but a complex number, then in order to divide by $[n]!$ one needs to make sure that this coefficient is non-zero. This can happen only if A_0 is a root of unity and in this case iff $n \geq r(A_0)$. So the definition of the Jones-Wenzl idempotents containing at least one color greater than or equal to $r(A_0)+1$ makes no sense in this case. The set of "A_0-definable" colorings is exactly the set of colorings in which only correctly defined Jones-Wenzl idempotents are used (i.e. colors $\leq r(A_0)$). Still, in the litterature a strictly smaller set of colorings is commonly used as a set of "colorings" when A is a root of unity: this is the set of what we call "A_0-admissible" ones. This is related to Lemma 6.25 : some of the "A_0-definable" colorings encode elements of the skein modules which are null in the reduced skein modules. We thus distinguish these two sets by our terminology.

The inequalities imposed on the colors around vertices allow to associate uniquely to a coloring c on Y a skein Y_c in $\mathcal{S}_{\mathbb{Q}}(M)$ and in $\mathcal{S}_{A_0}(M)$ (if the coloring is A_0-definable) as suggested in Fig. 6.4. Indeed observe that the value at A_0 of the quantum integer $[n]$ is non-zero for all $n < r(A_0)$ but $[r]_{A_0} = 0$, and hence the evaluations at A_0 of the Jones-Wenzl projectors $f_1, \ldots, f_{r-1}$ are defined whereas that of f_r is not, see Fig. 6.3. Therefore the values at A_0 of the ribbon graphs are defined only when all colorings are smaller or equal than $r-1$ (i.e. the coloring is A_0-definable); otherwise, working in $\mathcal{S}_{\mathbb{Q}}(M)$ all the colored ribbon graphs are defined. A framed link can be viewed as a colored ribbon graph without vertices whose components are colored with 1.

Remark 6.12. An A_0-admissible coloring is also A_0-definable but the converse is false. Furthermore in order to be able to associate an element of $\mathcal{S}_{A_0}(M)$ to a colored ribbon graph we only need to know that the coloring is A_0-definable: for the moment we are not yet using the definition of A_0-admissible coloring.

Three basic planar ribbon graphs in $\mathbb{S}^3$ are shown in Fig. 6.5. Since $\mathcal{S}_{\mathbb{Q}}(\mathbb{S}^3) = \mathbb{Q}(A)$, every such ribbon graph provides a complex number which can be expressed as a rational function in the variable A; these functions are typically expressed in terms of the quantum integers $[n]$.

We take from [31] and [37] (Theorem 1 and 2) the evaluations of the graphs , , and . We recall the usual factorial notation

$$[n]! = [1] \cdots [n]$$

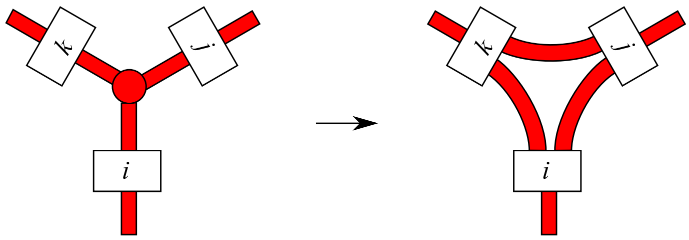


Figure 6.4: A coloring c on a ribbon graph Y determines a skein $Y_c \in \mathcal{S}_{\mathbb{Q}}(M)$: replace every edge with a projector, and connect them at every vertex via non intersecting strands contained in the depicted bands. For instance there are exactly $i+j-k$ bands connecting the projectors i and k.

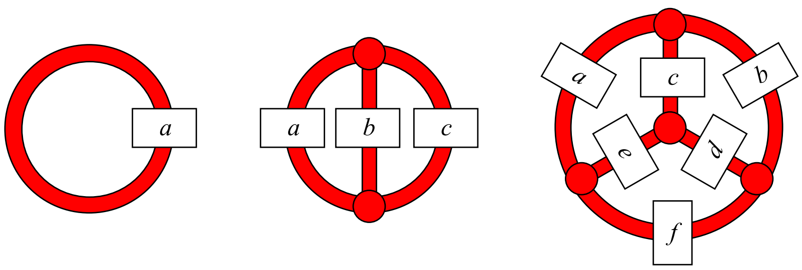


Figure 6.5: Three important planar ribbon graphs in $\mathbb{S}^3$.

with the convention $[0]! = 1$. Similarly one defines multinomial coefficients replacing standard factorials with quantum factorials:

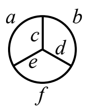
$$\begin{bmatrix} n \\ n_1, \dots n_k \end{bmatrix} = \frac{[n]!}{[n_1]! \cdots [n_k]!}.$$

When using multinomial coefficients we always suppose that $n = n_1 + \dots + n_k$. The evaluations of $\bigcirc$, $\ominus$ and $\circledast$ are:

(6.2) $$\bigcirc_a = (-1)^a [a+1],$$

(6.3) $$\ominus_{a,b,c} = (-1)^{\frac{a+b+c}{2}} \frac{\left[\frac{a+b+c}{2}+1\right]! \left[\frac{a+b-c}{2}\right]! \left[\frac{b+c-a}{2}\right]! \left[\frac{c+a-b}{2}\right]!}{[a]![b]![c]!},$$

(6.4)



$$= \frac{\prod_{i=1}^{3} \prod_{j=1}^{4} [\square_i - \Delta_j]}{[a]![b]![c]![d]![e]![f]!} \times$$
$$\sum_{z=\max \Delta_j}^{\min \square_i} (-1)^z \begin{bmatrix} z+1 \\ z-\Delta_1, z-\Delta_2, z-\Delta_3, z-\Delta_4, \square_1 - z, \square_2 - z, \square_3 - z, 1 \end{bmatrix}.$$

In the latter equality, triangles and squares are defined as follows:

$$\Delta_1 = \frac{a+b+c}{2}, \ \Delta_2 = \frac{a+e+f}{2}, \ \Delta_3 = \frac{d+b+f}{2}, \ \Delta_4 = \frac{d+e+c}{2},$$
$$\square_1 = \frac{a+b+d+e}{2}, \ \square_2 = \frac{a+c+d+f}{2}, \ \square_3 = \frac{b+c+e+f}{2}.$$

The formula (6.4) for the planar tetrahedron was first proved by Masbaum and Vogel [37]. We note that the evaluations are rational functions with poles in $S \cup \{0, \infty\}$. It is actually easy to check from the definitions that the evaluation of any ribbon graph in $\mathbb{S}^3$ is a rational function with poles contained in $S \cup \{0, \infty\}$. The following remark will be used often in what follows :

Remark 6.13. Let $A_0 \in \mathbb{C}^*$ and $a, b, c \leq r(A_0) - 1$ such that $a + b + c \in 2\mathbb{N}$, $a + b \geq c$, $b + c \geq a$, $c + a \geq b$. If $a + b + c < 2r(A_0) - 2$ then $\ominus_{a,b,c}$ is a rational function which has no pole at A_0 and its value at A_0 is non-zero. Otherwise it has a simple zero at A_0.

6.4. Computing in skein modules

A colored ribbon graph gives an element of $\mathcal{S}_{\mathbb{Q}}(M)$ by cabling its edges by the Jones-Wenzl projectors as explained in the preceding section and connecting the strands around the vertices in the unique planar way without self-retours. The following two theorems allow to compute easily the value of the so obtained skein for any colored ribbon graph in $\mathcal{S}_{\mathbb{Q}}(\mathbb{S}^3)$ and to simplify skeins in $\mathcal{S}_{\mathbb{Q}}(M)$ for any compact oriented three manifold.

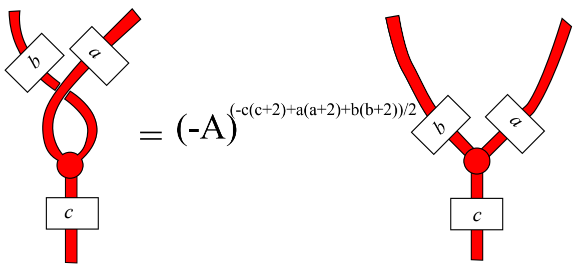
Theorem 6.14 ([31] Chapter 7 Theorem 2 and Remark 10)**.** *Let M be a compact oriented three manifold and $s \in \mathcal{S}_{\mathbb{Q}}(M)$. If s contains a portion as that in the left part of Figure 6.6 then s is also equal to the linear combination of skeins in $\mathcal{S}_{\mathbb{Q}}(M)$ which differ from s only in the ball, as depicted ball in the right part of the same figure. This equivalence is also known as* Whitehead move *and the coefficient of the f^{th}-summand in the figure is the* quantum 6j-symbol, *denoted:*

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}.$$

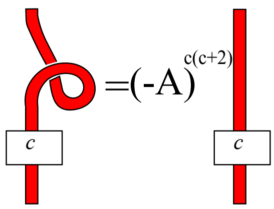
In particular, when $c = 0$ then $a = b$ and $d = e$ and applying the Whitehead move (after rotating the picture by 90° degrees) one recovers the fusion rule *depicted in Figure 6.7.*

Remark 6.15. In the Whitehead move (and hence in the fusion move) the sum ranges over all the finitely many values providing a coloring of the right-most graph (see Definition 6.10).

Theorem 6.16 ([37] Theorem 3)**.** *Let M be a compact oriented three manifold. The following local equalities hold in $\mathcal{S}_{\mathbb{Q}}(M)$ for any admissible coloring:*



In particular if $c = 0$ then $a = b$ and one has the following :



Exercise 6.17. Draw your favorite framed knot in $\mathbb{S}^3$ and compute its n^{th} colored Jones polynomial by using the above two theorems and Formula (6.4).

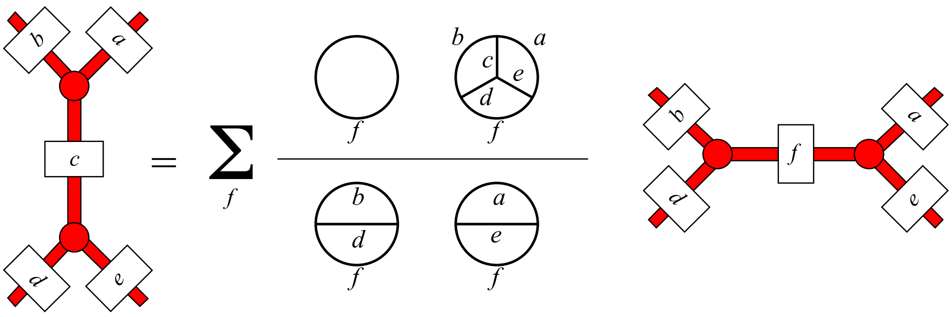


Figure 6.6: The *Whitehead move*: the summation is over all the admissible colors (and is hence finite).

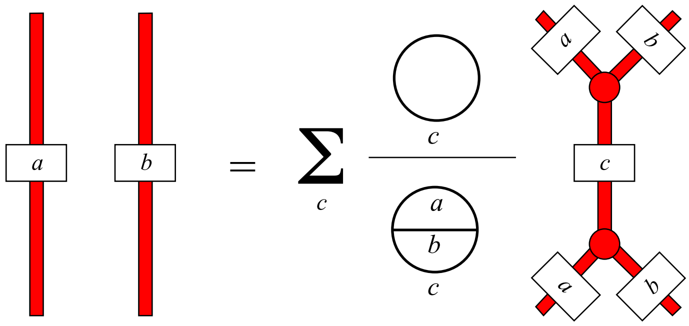
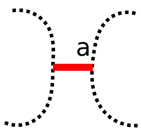


Figure 6.7: The *fusion rule*: it is a special case of the Whitehead move.

Exercise 6.18. Prove that if a colored ribbon graph $Y_c \subset \mathbb{S}^3$ contains an edge colored by a such that the complement of the arc is the disjoint union of two colored graphs $Y'_{c'}$ and $Y''_{c''}$, then the following holds in $\mathcal{S}'_{\mathbb{Q}}(\mathbb{S}^3)$:



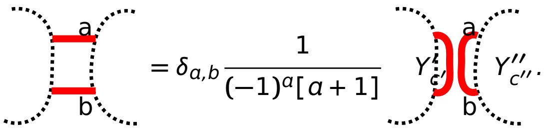
$$= \delta_{a,0} Y'_{c'} \sqcup Y''_{c''}.$$

Furthermore prove the same statement for $A_0 \in \mathbb{C}^*$ if all the colors are less than $r(A_0)$ and considering $Y_c, Y'_{c'}, Y''_{c''}$ as skeins in $\mathcal{S}_{A_0}(\mathbb{S}^3)$.

Solution 6.19. Apply iteratively Kauffman's rule to the diagram of $Y'_{c'}$ until it is reduced to a linear combination of planar graphs. If $a > 0$ then each such graph must contain an arc whose endpoints are on the same side of the a^{th} Jones-Wenzl projector coloring the disconnecting edge, thus by Equation (6.1) it is 0. If $a = 0$ then $Y_c = Y'_{c'} \sqcup Y''_{c''}$ and the claim is evident.

For what concerns the last statement, remark that when working in $\mathcal{S}_{A_0}(\mathbb{S}^3)$, the restriction on the colors being less than $r(A_0) - 1$ is needed in order for the colored ribbon graph to provide a well defined element of $\mathcal{S}_{A_0}(\mathbb{S}^3)$ (the Jones Wenzl idempotents have zero denominators for colors bigger than $r(A_0) - 1$), but the argument is the same as above.

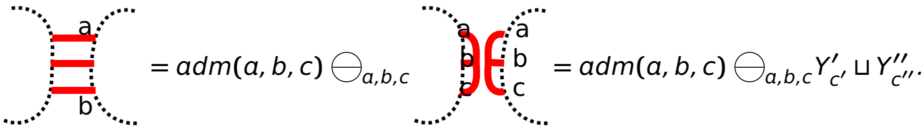
Exercise 6.20. Prove that if a ribbon graph $Y \subset \mathbb{S}^3$ is the connected sum of two ribbon graphs the following holds in $\mathcal{S}_{\mathbb{Q}}(\mathbb{S}^3)$:



Furthermore prove the same statement for $A_0 \in \mathbb{C}^*$ if all the colors are less than $r(A_0)-1$ and considering Y_c as a skein in $\mathcal{S}_{A_0}(\mathbb{S}^3)$.

Solution 6.21. Operate a fusion along the two parallel edges and apply Exercice 6.18 to conclude. To prove the statement in $\mathcal{S}_{A_0}(\mathbb{S}^3)$, observe first that the restriction on the colors being less than $r(A_0)-1$ is just in order for the colored ribbon graph to provide a well defined element of $\mathcal{S}_{A_0}(\mathbb{S}^3)$. Now remark that multiplying by $(-1)^a[a+1]$ the equality one gets an identity in $\mathcal{S}(\mathbb{S}^3)$. Then passing it to $\mathcal{S}_{A_0}(\mathbb{S}^3)$ one gets the thesis because $[a+1] \neq 0$ at A_0 as $a < r(A_0)-1$.

Exercise 6.22. Let $Y_c \subset \mathbb{S}^3$ be a colored ribbon graph containing three edges colored respectively by a, b, c such that the complement of their midpoints in a diagram of Y_c has exactly two connected components; let $Y'_{c'}, Y''_{c''}$ be the colored ribbon graphs obtained by cutting Y along these midpoints and glueing back two trivalent vertices (see the figure here below). Let also $adm(a,b,c)$ be 1 if $a+b+c \in 2\mathbb{N}$ and $a+b \geq c, a+c \geq b, b+c \geq a$ and 0 else. Prove that the following holds in $\mathcal{S}_{\mathbb{Q}}(\mathbb{S}^3)$:



Prove furthermore that, given $A_0 \in \mathbb{C}^*$, if all the colors are less than $r(A_0)-1$ and $a+b+c < 2r-2$ then the same equality holds in $\mathcal{S}_{A_0}(\mathbb{S}^3)$.

Solution 6.23. Apply one fusion to the a and b-colored edges and then apply the the result of Exercise 6.4. To prove the statement in $\mathcal{S}_{A_0}(\mathbb{S}^3)$ observe first that the restriction on the colors being less than $r(A_0)-1$ is in order for the colored ribbon graph to provide a well defined element of $\mathcal{S}_{A_0}(\mathbb{S}^3)$. Then up to multiplying by the denominator of the fraction in the equation in $\mathcal{S}_{\mathbb{Q}}(\mathbb{S}^3)$ one can reduce to an equation in $\mathcal{S}(\mathbb{S}^3)$ and in order to conclude it is sufficient to check that the coefficients in the equation have non-zero evaluation at A_0. This is the case as a direct inspection to the formula providing $\ominus$, shows that under the condition $a+b+c < 2r-2$ the evaluation of the rational function $\ominus$ is nonzero and has no pole at A_0.

6.5. The reduced skein module

We now consider the $\mathbb{C}$-vector space $\mathcal{S}_{A_0}(M)$ obtained by evaluating at a root of unity A_0 distinct from ± 1 and $\pm i$; recall that $r = r(A_0) \geqslant 2$ is the smallest integer such that $A_0^{4r} = 1$ or, equivalently, such that $[r]_{A_0} = 0$. More explicitly $A_0 = \exp(\frac{\pi i s}{2r})$ with $(s, 2r) = 1$.

Definition 6.24. The *reduced skein* $\mathcal{S}^{\text{red}}_{A_0}(M)$ of a 3-manifold M is the quotient of $\mathcal{S}_{A_0}(M)$ by the relations that kill every skein containing a portion as in Fig. 6.8, i.e. by the subvector space generated by colored graphs in M which are A_0-definable but not A_0-admissible.

The crucial point here is that by killing the skeins in Fig. 6.8 we do not affect the skein module of $\mathbb{S}^3$: indeed every skein in $\mathbb{S}^3$ containing one of the portions in Fig. 6.8 is already zero hence $\mathcal{S}^{\text{red}}_{A_0}(\mathbb{S}^3) = \mathcal{S}_{A_0}(\mathbb{S}^3) = \mathbb{C}$:

Lemma 6.25. *If $s \in \mathcal{S}_{A_0}(\mathbb{S}^3)$ is a skein containing one of the portions in Fig. 6.8 then $s = 0$.*

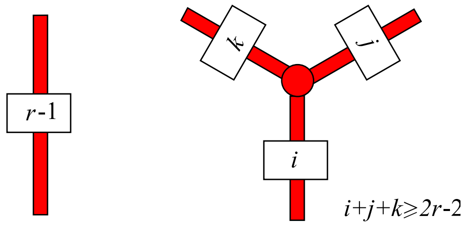


Figure 6.8: The reduced skein vector space $\mathcal{S}^{\rm red}_A(M)$ is constructed by quotienting $\mathcal{S}_{A_0}(S)$ by the span of the elements containing one of these two skeins. Concerning the right triple (i,j,k), note that it is defined only when $i,j,k \leqslant r-1$, and that we quotient only by the three-uples (i,j,k) with $i+j+k \geqslant 2r-2$.

Proof. We give a sketchy proof and refer the reader to [34, Lemma 14.7] for details : the skein can be represented as a skein in $\mathcal{S}_{A_0}(D^2)$, thus the statement is a local one. First of all using the Kauffman relations express s as a linear combination of skeins in $\mathcal{S}_{A_0}(D^2)$ which are planar outside the portion. Then if the portion is as in the left part of the figure then using Equation (6.1) one sees that all the skeins in this combination which contain arcs whose endpoints are in the same sides of the portion are zero ("no self-retour"). Thus s is a multiple of an unknot colored by $r(A_0)-1$ whose evaluation is $(-1)^{r(A_0)-1}[r(A_0)]=0$. Similarly if the portion is as in the right part of the figure, then s is a combination of planar skeins and each time there is a self retour these graphs are zero (because of equation (6.1)). So s is actually a multiple of the theta graph colored by a,b,c whose evaluation is zero by Formula (6.3) (see Remark 6.13). 6.25

It is important that however the statement of the lemma is not true for a general 3-manifold.

Theorem 6.26 (Reduced skein rules)**.** *The statements of Theorems 6.14 and 6.16 remain valid in $\mathcal{S}^{red}_{A_0}(M)$ provided one takes $s \in \mathcal{S}^{red}_{A_0}(M)$, lets the colorings of s' vary over the $r(A_0)$-admissible colorings (recall Definition 6.10) and evaluates the coefficients in the formulas at $A=A_0$.*

Proof. It is clear that the only change in the statement of Theorem 6.16 is to replace $A \to A_0$ (there is nothing to prove as by hypothesis the coloring of s and hence of s' is $r(A_0)$-admissible). The proof of the "reduced version" of Theorem 6.14 is more complicated; to simplify the notation let from now on $r=r(A_0)$. Observe that if in Figure 6.6 one of a,b,d,e is 0 the statement is true: there is only one term in the sum of the r.h.s. and it suffices to check that its coefficient is 1; we leave this to the reader.

Now suppose that one of a,b,d,e is 1, say $a=1$. In this case $c=b\pm 1$ and $f=e\pm 1$; there are then four coefficients to compute:

$$\begin{Bmatrix} 1 & b & c \\ d & e & f \end{Bmatrix} =$$

f\c	c=b-1	c=b+1
f=e-1	$-\frac{[\frac{1+e+d-b}{2}]}{[e+1]}$	$\frac{[\frac{e+b+d+3}{2}][\frac{e+b-d+1}{2}]}{[e+1][b+1]}$
f=e+1	1	$\frac{[\frac{b+d-e+1}{2}]}{[b+1]}$

One can check that these coefficients have no poles at A_0 as $e+1, b+1 < r$ (by hypothesis on s). So all the terms of the Whitehead moves are evaluable at A_0 and, after possibly deleting the terms with $f=r-1$ (which are zero in $\mathcal{S}^{red}_{A_0}(M)$ by definition), one gets the claim.

Let's now perform an induction on $min(a,b,d,e)$; observe that if s contains an a-colored edge (so necessarily $a<r-1$), then we can insert in the middle of it a bigon colored by

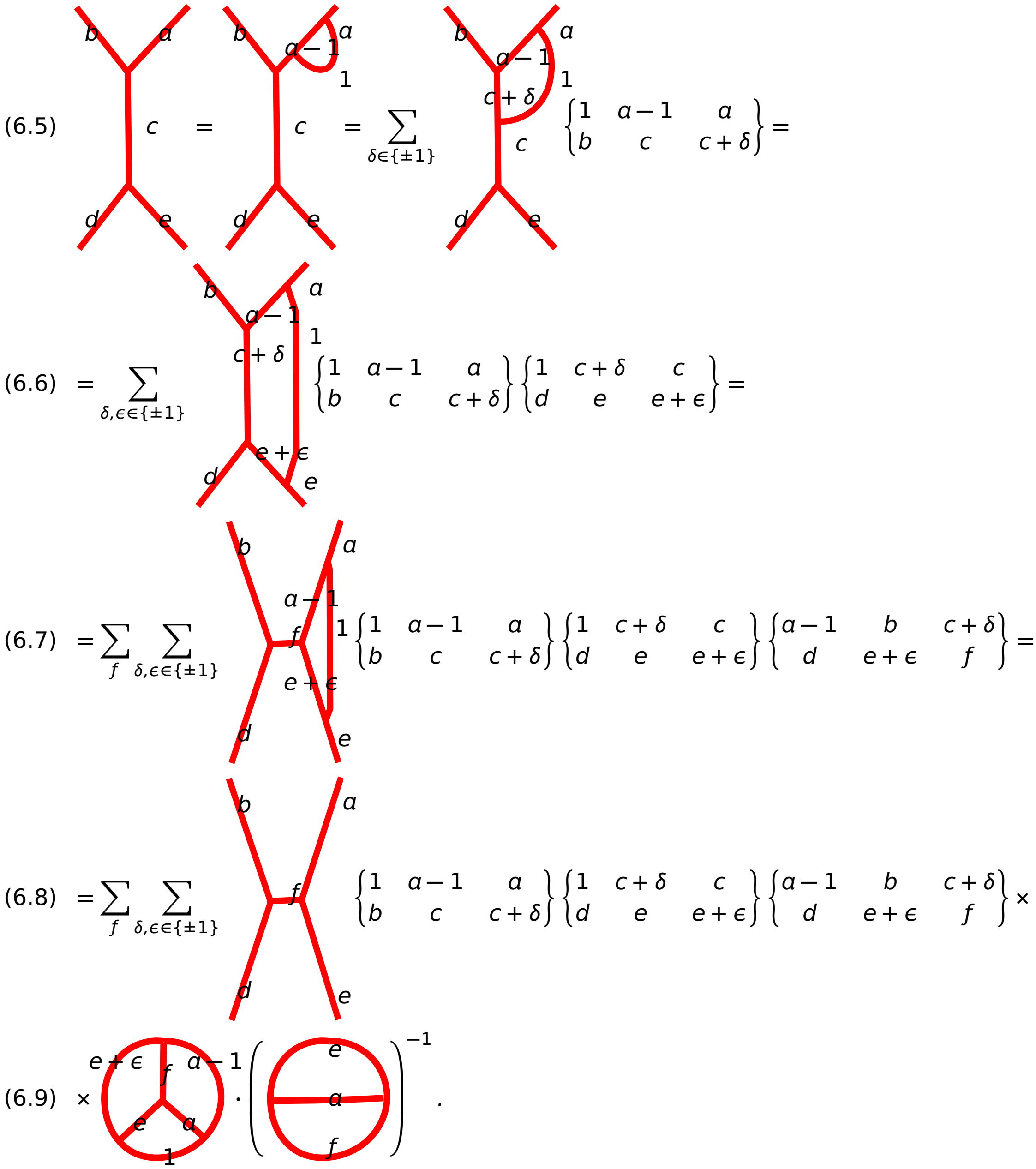


Figure 6.9: The sequence of moves in $\mathcal{S}^{red}_{A_0}(M)$ or in $\mathcal{S}_{\mathbb{Q}}(M)$ used in the proof of Theorem 6.26. Remark that both in $\mathcal{S}^{red}_{A_0}(M)$ and in $\mathcal{S}_{\mathbb{Q}}(M)$, f ranges over a finite set of values, but in the $\mathcal{S}^{red}_{A_0}(M)$ the set of values of f can be smaller than in the $\mathcal{S}_{\mathbb{Q}}(M)$ case.

$a-1$ and 1 without changing the class of s in $\mathcal{S}^{red}_{A_0}(M)$; the same holds in $\mathcal{S}_{\mathbb{Q}}(M)$. Now we apply twice the Whitehead moves to slide the 1-colored edge we just created first over the a-colored edge and then over the c-colored one. The sequence of moves we are applying is depicted in the upper part of Figure 6.9. In the lower part we apply the induction hypothesis to proceed. Finally in the last step of the computation we used the statement of Exercise 6.4.

The argument now goes as follows : the above computation can be performed both in $\mathcal{S}_{\mathbb{Q}}(M)$ or in $\mathcal{S}_{A_0}(M)$; in $\mathcal{S}_{\mathbb{Q}}(M)$ for each fixed value of f there will be 4-terms in the sum (according to the values of δ, ϵ) ending with the coloring of s' containing the color f. In $\mathcal{S}_{A_0}(M)$, by induction, one drops all of these terms in which at least one of the graphs appearing in the sequence of moves is nor r-admissibly colored (we will call this sequence a “dropped sequence"). Collecting the terms associated to dropped and non dropped sequences we can write:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}^{dropped} + \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}^{non\ dropped} \in \mathbb{Q}(A).$$

By induction one has immediately that s can be re-expressed as a linear combination of admissible colorings on s' (i.e. those for which $f \leq r-2, f+a+d \leq 2r-4$ and $f+b+e \leq 2r-4$) and we are left to check the following equality between the evaluations at A_0 :

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_{A_0} = \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}^{non\ dropped}_{A_0} \in \mathbb{C}.$$

A direct inspection on Formula 6.4) shows the following:

1. If a colored tetrahedron (or theta graph or unknot) is A_0-definable, the rational function $_A \in \mathbb{Q}(A)$ (resp. $_A$ or $_A \in \mathbb{Q}(A)$) has no pole at A_0 and is zero if furthermore the coloring is not A_0-admissible.

2. If a colored theta graph is A_0-admissible then the rational function $_A \in \mathbb{Q}(A)$ has no zero at A_0. If it is A_0-definable but not A_0-admissible then it has a zero of order 1 at A_0 (see Remark 6.13).

3. As a consequence, using the expression of $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$ provided in Theorem 6.14, if the colorings in the l.h.s. and r.h.s. of Figure 6.6 are both r-admissible the function $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$ has no pole at A_0.

Then if $f \leq r-2$, $f+a+d \leq 2r-4$ and $f+b+e \leq 2r-4$ both

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \text{ and } \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}^{non\ dropped}$$

are evaluable at A_0 and so also their difference, $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}^{dropped}$ is; in particular it has no pole at A_0. We are left to check that it has a zero there. The reasons why a term has been dropped can be:

1. $c+1 = r-1$, so $\delta = 1$ (dropped after the first Whitehead move);

2. $c+1 < r-1$ but $e+1 = r-1$, so $\epsilon = 1$ (dropped after the second Whitehead move);

3. $c+1 < r-1$, $e+1 < r-1$ but $d+c+e+2 > 2r-4$, so $\delta = \epsilon = 1$ (dropped after the second Whitehead move).

A direct computation using Formula (6.4) shows that the following holds:

$$\begin{Bmatrix} 1 & a-1 & a \\ b & c & c+\delta \end{Bmatrix} \begin{Bmatrix} 1 & c+\delta & c \\ d & e & e+\epsilon \end{Bmatrix} =$$

$\delta \backslash \epsilon$	$\epsilon = -1$	$\epsilon = 1$
$\delta = 1$	$-\frac{[\frac{a+b-c}{2}][\frac{e+d-c}{2}]}{[a][e+1]}$	$\frac{[\frac{a+b-c}{2}]}{[a]}$
$\delta = -1$	$\frac{[\frac{a+b+c+2}{2}][\frac{a-b+c}{2}]}{[c+1][a]} \frac{[\frac{d+e+c+2}{2}][\frac{c+e-d}{2}]}{[e+1][c]}$	$\frac{[\frac{a+b+c+2}{2}][\frac{a-b+c}{2}]}{[c+1][a]} \frac{[\frac{c+d-e}{2}]}{[e+1]}$

so that this part of the coefficients have no pole at A_0 as $c, e \le r-2$ by the hypotheses $s \in \mathcal{S}^{red}_{A_0}(M)$. So we need to prove that in each of the above cases, the remaining coefficient, which is the product:

$$\begin{Bmatrix} a-1 & b & c+\delta \\ d & e+\epsilon & f \end{Bmatrix} \cdot \frac{☮}{\ominus}$$

(the tetrahedron and the theta graph being colored as in Figure 6.9) has a zero at A_0.

Case 1. In this case $\begin{Bmatrix} a-1 & b & c+\delta \\ d & e+\epsilon & f \end{Bmatrix} = \frac{☮\bigcirc}{\ominus\ominus}$ (see Figure 6.6 for the correct attribution of the colors to the symbols in the r.h.s.) contains a null numerator and its denominator is the product of two non-zero theta graphs colored respectively by $f, b, a-1$ and $f, d, e+\epsilon$. The term $☮ \cdot \ominus^{-1}$ has no pole as both ☮ and $\ominus$ are admissibly colored so they have no pole at A_0 and furthermore the $\ominus$ is non zero (see points 1) and 2) in the above list of remarks).

Case 2. If $c < r-2$ and $e = r-2$ so $\epsilon = 1$ the coefficient

$$\begin{Bmatrix} a-1 & b & c+\delta \\ d & e+\epsilon & f \end{Bmatrix} = \frac{☮\bigcirc}{\ominus\ominus}$$

(where the graphs on the right are suitably colored) is the ratio of two functions which are null at A_0: indeed both the numerator and the denominator are null by Lemma 6.25 as they contain an $r-1$-colored edge; furthermore the denominator contains only a simple zero in the evaluation of a theta-graph colored by $e+\epsilon = r-1, f, a-1$. So the overall ratio can be evaluated at A_0 but maybe is non-zero. The coefficient $☮\ominus^{-1}$ is null because of the term ☮ which is zero as it contains a $r-1$-colored edge and the $\ominus$ term is non zero by the point 2) in the above list of remarks.

Case 3. In this last case the coefficient

$$\begin{Bmatrix} a-1 & b & c+\delta \\ d & e+\epsilon & f \end{Bmatrix} = \frac{☮\bigcirc}{\ominus\ominus}$$

(where the graphs on the right are suitably colored) is the ratio of two functions of which the numerator is null at A_0 by Lemma 6.25 but the denominator is non-zero as it is the product of two theta graphs colored respectively by $f, a-1, b$ and $f, e+\epsilon, d$ which are both r-admissible colorings (by recursion). Finally the last coefficient $☮\ominus^{-1}$ can be evaluated at A_0 as the coloring of $\ominus$ is r-admissible. 6.26

Theorem 6.26 is the key to perform all the skein calculus even at the level of the reduced skein module $\mathcal{S}^{red}_{A_0}(M)$. We will apply it from now on without citing it systematically.

Proposition 6.27 ([33] Theorem at page 347)**.** *Let H_g be a handlebody of genus g and $\Gamma \subset H_g$ be a framed trivalent ribbon graph over which H_g collapses. Then the set $\{\Gamma_c\}$ where c ranges over all the A_0-admissible colorings on Γ forms a basis for $\mathcal{S}^{red}_{A_0}(H_g)$.*

Proof. Cut H_g along embedded discs dual to Γ in order to get a ball. By Theorem 6.26 every skein in H_g intersecting H_g can be reduced via a sequence of fusions to a ribbon graph intersecting the discs along a k-colored edge with $k < r-1$. Once the skein intersects each disc in a single point along an arc colored by a color in $\{0, 1, \ldots r-2\}$, we are left to reduce the remaining skein to a linear combination of colorings of Γ. But then this is a computation in B^3 where it can be seen that every ribbon graph with three endpoints in ∂B^3 colored by a, b, c is a multiple of the framed graph represented by a Y-shaped graph colored by a, b, c. 6.27

Definition 6.28 (Kirby color)**.** The r^{th}- Kirby color is defined as follows :

$$\Omega := \sum_{j=0}^{r-2} (-1)^j [j+1] T_j \in \mathcal{S}^{red}_{A_0}(A \times [-1, 1]).$$

where T_j is the j-colored core of the annulus (recall Exercise 6.6). If $i : L \hookrightarrow \mathbb{S}^3$ is a framed link let $J_\Omega(L) := i_*(\Omega) \in \mathcal{S}_{A_0}(\mathbb{S}^3) = \mathbb{C}$ (where we “color" each component by Ω).

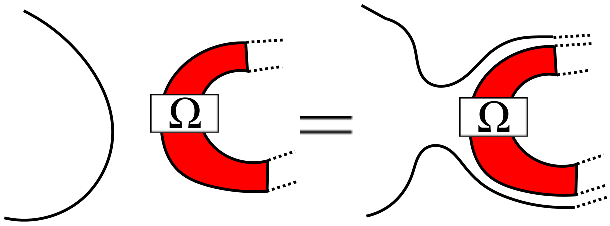


Figure 6.10: The sliding move (or banded sum): the black strand is colored by a color a.

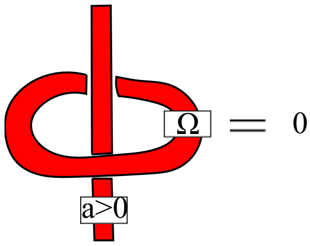


Figure 6.11: The encirclement lemma.

The following proposition is the key property of the Kirby color:

Proposition 6.29. *Let M^3 be a compact oriented three manifold and s be a skein containing a Ω-colored component L and another component T colored by an admissible color a (see Figure 6.10). Let also s' be the skein obtained from s by replacing T with the band connected sum of T and L colored by a. Then $s = s'$ in $\mathcal{S}^{red}_{A_0}(M)$.*

Proof. It is sufficient to prove the statement for $a = 1$ as the Jones-Wenzl idempotents are linear combinations of colorings by parallel strands. To prove the equality apply a first fusion using Theorem 6.26 (as in Figure 6.8) to connect T and L then undo the fusion “from the other side of L": the fusion replaces $T \cup L$ by a trivalent graph which naturally contains a subgraph formed by two segments (with disjoint interiors) s, s' such that $L = s \cup s'$. The presence of the coefficients in $\Omega = \sum_{j=0}^{r-2}(-1)^j[j+1]T_j$ coloring L allows to realize that the result of the fusion is symmetric: making the fusion of the left part of Figure 6.10 on s has the same outcome as making the fusion of the right part of the figure on s'. Let us detail how. To specify that the color of s is c and that of s' is c' we write $T^s_c T^{s'}_{c'}$: so for instance the color of L before the fusions is $\Omega = \sum_{j=0}^{r-2}(-1)^j[j+1]T^s_j T^{s'}_j$. After the fusion on s the colors of s and s' are the following :

$$\sum_{j=0}^{r-3}(-1)^j[j+1]\frac{(-1)^{j+1}[j+2]}{\theta(j,j+1,1)}T^s_{j+1}T^{s'}_j + \sum_{j=1}^{r-2}(-1)^j[j+1]\frac{(-1)^{j-1}[j]}{\theta(j,j-1,1)}T_{j-1}T^{s'}_j.$$

“Looking from the other side of L" boils down to consider s' as the result of a fusion. So set in the first sum $j' = j+1$ and in the second $j' = j-1$, then we get :

$$\sum_{j'=1}^{r-2}(-1)^{j'-1}[j']\frac{(-1)^{j'}[j'+1]}{\theta(j'-1,j',1)}T^s_{j'}T^{s'}_{j'-1} + \sum_{j=0}^{r-3}(-1)^{j'+1}[j'+2]\frac{(-1)^{j'}[j'+1]}{\theta(j'+1,j',1)}T^s_{j'}T_{j'+1}$$

which is exactly the result of a fusion made on s'. Then we can undo the fusion. 6.29

Lemma 6.30 (Encirclement lemma, [31], Chapter 12 Lemma 22)**.** *If $s \in \mathcal{S}(M)$ is a skein containing a Ω-colored 0-framed unknot then $[s] = 0 \in \mathcal{S}^{red}_{A_0}(M)$ if the disc bounded by the unknot intersects s in exactly one point colored by a non-zero color (see Figure 6.11).*

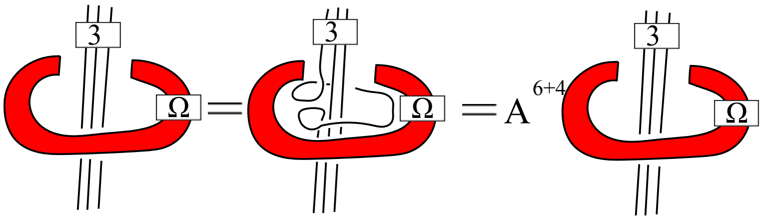


Figure 6.12: An instance of the proof of the encirclement lemma: this is the case when the color of the encircled strand is 3.

Proof. Suppose $a > 0$ and recall that "coloring by a color a" means cabling a component of the skein by a linear combination of parallel strands, with coefficients given by those appearing in the construction of the a^{th}-Jones Wenzl idempotent. Applying Proposition 6.29 to one of these strands as shown in Figure 6.12 so that it loops around all the other strands and applying Kauffman relations to all the crossings in the figure, we see that if there are a strands in total then the so obtained skein is a linear combination of skeins all of which contain at least one strand whose both endpoints are connected to the box representing the Jones-Wenzl idempotent and of a single copy of the skein represented by all vertical strands, whose coefficient is $A^{6+2(a-1)}$. By Equation (6.1) the former skeins are zero, thus we get that the equation depicted in Figure 6.12 and since $a < r-1$ then $A^{2(a-1)+6} \neq 1$. This implies the thesis. 6.30

6.6. The Reshetikhin-Turaev invariants

From now on we will fix $r \geq 3$ and let $A = A_0 = \exp(\frac{is\pi}{2r})$ with $(s,r) = 1$[1]. If k is a framed knot colored by n and k^f is the same knot with a framing twisted f times then by Theorem 6.16 it holds $J_n(k^f) = (-1)^{fn} A^{fn(n+2)} J_n(k)$.

Exercise 6.31. Let u^0 be the unknot and let $D^2 = J_\Omega(u^0)$ then it holds : $D^2 = \frac{r}{2\sin(\frac{\pi s}{r})^2}$.

Solution 6.32. $J_\Omega(u^0) = \sum_{j=0}^{r-2}[j+1]^2 = \frac{1}{(A^2-A^{-2})^2}\sum_{j=0}^{r-2} A^{4j+4} + A^{-4j-4} - 2 =$
$= \frac{1}{(A^2-A^{-2})^2}(A^4\frac{A^{4r-4}-1}{A^4-1} + A^{-4}\frac{A^{-4r+4}-1}{A^{-4}-1} - 2(r-1)) = \frac{-2r}{(2i\sin(\frac{\pi k}{r}))^2} = \frac{r}{2\sin(\frac{\pi k}{r})^2}$.

Proposition 6.33. *Let $u^\pm$ be the unknot with framing ± 1 colored by Ω and let $J_\Omega(u^\pm)$ the value of the skein it represents in $\mathcal{S}(\mathbb{S}^3)$. Then it holds $J_\Omega(u^+) = \overline{J_\Omega(u^-)}$ and $J_\Omega(u^\pm) = \frac{i(1+i^s)}{\sqrt{2}}\left(\frac{4r}{s}\right)A^{-r^2-2r-3}D$ (where $\left(\frac{4r}{s}\right) \in \{\pm 1\}$ is 1 iff $4r$ is a quadratic residue modulo s and D is the positive real number defined as in Exercice 6.31). In particular $J_\Omega(u^\pm) = \rho D$ where ρ is a root of unity whose order divides $4r$.*

Proof. The first statement is a direct consequence of the fact that the skein evaluation of the mirror image of a knot k is obtained by replacing $A \to A^{-1}$, i.e. $[k]_A = [\overline{k}]_{A^{-1}} \in \mathcal{S}(\mathbb{S}^3)$, and A

[1] For the experts: this choice corresponds to working in the SU(2)-theory with $p = 2r$ as opposed to setting $A = \exp(\frac{is\pi}{p})$ with p odd for the $SO(3)$ theory

here is a root of unity. We now prove directly the last claim:

$$J_\Omega(u^+) = \sum_{j=0}^{r-2}(-1)^j A^{j^2+2j}[j+1]^2 = \sum_{j=1}^{r-1}(-1)^{r-1-j}A^{(r-1-j)^2+2(r-1-j)}[r-j]^2 = \tag{6.10}$$

$$= \frac{1}{2}\sum_{j=-(r-1)}^{r-1}(-1)^{r-1-j}A^{(r-1-j)^2+2(r-1-j)}[r-j]^2 = \frac{1}{2}\sum_{j=0}^{2r-1}(-1)^j A^{j^2+2j}[j+1]^2 = \tag{6.11}$$

$$= \frac{1}{4}\sum_{j=0}^{4r-1}(-1)^j A^{j^2+2j}[j+1]^2 = \tag{6.12}$$

$$= \frac{1}{4(A^2-A^{-2})^2}\sum_{j=0}^{4r-1}\left((-1)^j A^{j^2+6j+4} - (-1)^j A^{j^2+2j}2 + (-1)^j A^{j^2-2j-4}\right) \tag{6.13}$$

$$= \frac{1}{4(A^2-A^{-2})^2}\sum_{j=0}^{4r-1}\left(A^{j^2+(6+2r)j+4} - 2A^{j^2+(2r+2)j} + A^{j^2+(2r-2)j-4}\right) \tag{6.14}$$

$$= \frac{A^{4-(r+3)^2}+A^{-4-(r-1)^2}-2A^{-(r+1)^2}}{4(A^2-A^{-2})^2}\sum_{k=0}^{4r-1}A^{k^2} = \frac{2A^{-r^2-5-2r}-2A^{-r^2-2r-1}}{4(A^2-A^{-2})^2}\sum_{k=0}^{4r-1}A^{k^2} = \tag{6.15}$$

$$= \frac{-A^{-r^2-2r-3}}{2(A^2-A^{-2})}\sum_{k=0}^{4r-1}A^{k^2} = \frac{-A^{-r^2-2r-3}}{2(A^2-A^{-2})}\left(\frac{4r}{s}\right)(1+i^s)\sqrt{4r} = \tag{6.16}$$

$$= \frac{-A^{-r^2-2r-3}}{2i\sin(\frac{\pi s}{r})}\left(\frac{4r}{s}\right)(1+i^s)\sqrt{r} = -\left(\frac{4r}{s}\right)\frac{(1+i^s)}{\sqrt{2}i}A^{-r^2-2r-3}D. \tag{6.17}$$

Where we used the following facts : in the first equality we reparametrized the summation, in the second we observed that the $(r-1-j)^{th}$ and the $(r-1+j)^{th}$ term are equal and that $[r]=0$; the third is a reparametrization; in the fourth equality we observed that the j^{th}-term and the $(j+2r)^{th}$ are equal; in the sixth and in the following ones we used the fact that $A^{2r}=-1$ many times and finally we used the Gauss sum formula $\sum_{k=0}^{4r-1}A^{k^2} = \left(\frac{4r}{s}\right)(1+i)\sqrt{4r}$ which holds as soon as $A=\exp(\frac{i\pi s}{2r})$ with (s,r) coprime (see for instance [9], Theorem 1.5.4). 6.33

6.7. Some basic facts about surgery presentations and Kirby calculus

Let $k \subset \mathbb{S}^3$ be a knot and remark that the tubular neighborhood $N(k)$ of k is well defined up to isotopy and diffeomorphic do $D^2 \times \mathbb{S}^1$ (a solid torus). Yet such diffeomorphism is not unique (not even up to isotopy) unless one fixes a framing on k. One canonical way of fixing a framing on k is to use its *Seifert framing*, obtained as follows : 1) orient arbitrarily k and choose a Seifert surface for it i.e. an oriented surface $S \subset \mathbb{S}^3$ such that $\partial S = k$ (it is a nice exercice to check that it exists); 2) the *longitude* of k is the unoriented curve $\lambda = S \cap \partial N(k)$ (up to isotopy we can suppose λ to be a simple closed curve). Since $\partial N(k)$ is a torus and λ is a simple closed curve it can be checked that it is well defined up to isotopy in $\partial N(k)$ and thus it provides a well defined framing on k; furthermore if S' is another Seifert surface for k then the associated curve λ' is isotopic to λ : indeed it holds $[\lambda] = [\lambda'] \in H_1(\partial N(k);\mathbb{Z})$ (as they both generate the kernel of the inclusion $i_* : H_1(\partial N(k);\mathbb{Z}) \to H_1(\mathbb{S}^3 \setminus N(k);\mathbb{Z})$) and two homologous simple closed curves in a torus are isotopic.

Using the Seifert framing on k we can then fix an isotopy class of diffeomorphisms $\phi : D^2 \times \mathbb{S}^1 \to N(k)$ by stipulating that $\phi(\{0\}\times\mathbb{S}^1) = k$ and $\phi(\{1\}\times\mathbb{S}^1) = \lambda$. More in general if we pick any other framing on k it will be obtained from the Seifert framing by "twisting" it an integer number f of times, i.e. by pre-composing ϕ with the self-diffeomorphism $t_f : D^2 \times \mathbb{S}^1 \to D^2 \times \mathbb{S}^1$ defined as $t_f(x,\theta) = (e^{if\theta}x, \theta)$, $\forall x \in D^2, \forall \theta \in \mathbb{S}^1$ (where we parametrize D^2 as the unit disc in $\mathbb{C}$ and $\mathbb{S}^1 = [-\pi,\pi]/-\pi \sim \pi$). It can be checked that each framing on k is isotopic to one obtained this way, so we can canonically speak of the "framing $f \in \mathbb{Z}$" on k. More in general, if $L \subset \mathbb{S}^3$ is a framed link, one can identify its tubular neighborhood $N(L)$ with $D^2 \times \mathbb{S}^1 \times \pi_0(L)$

and for each component of L we have an integer telling us how many times the framing of the component is twisted with respect to its Seifert framing.

Definition 6.34 (Surgery along a link)**.** The 3-manifold obtained by surgery along a framed knot k, denoted also $\mathbb{S}^3_k$ is

$$\mathbb{S}^3_k := \left(\mathbb{S}^3 \setminus \mathring{N}(k)\right) \sqcup_\phi N(k)$$

where $\phi : \partial N(k) \to \partial N(k)$ is the diffeomorphism defined by $\phi(\theta,\alpha) = (\alpha, -\theta), \forall(\theta,\alpha) \in \partial D^2 \times \mathbb{S}^1$ (where we parametrize ∂D^2 and $\mathbb{S}^1$ via $[-\pi,\pi]/-\pi \sim \pi$) . More in general if L is a framed link, S^3_L is obtained from $\mathbb{S}^3$ by simultaneously surgering over all the components of L.

Example 6.35. Let u^f be the unknot in $\mathbb{S}^3$ equipped with the framing obtained by twisting the Seifert framing by f full twists. Then we have :

1. $\mathbb{S}^3_{u^0} = \mathbb{S}^2 \times \mathbb{S}^1$.
2. $\mathbb{S}^3_{u^1} = \mathbb{S}^3$.
3. $\mathbb{S}^3_{u^2} = \mathbb{RP}^3 = L(2,1) = SO(3)$.
4. $\mathbb{S}^3_{u^p} = L(p,1),\ \forall p \in \mathbb{N}$, one of the so-called Lens spaces.

By the extension of isotopies, if L_1 and L_2 are two links in $\mathbb{S}^3$ which are isotopic, then $\mathbb{S}^3_{L_1}$ and $\mathbb{S}^3_{L_2}$ are diffeomorphic. Furthermore it is easy to check that if L_1 and L_2 are two framed links in $\mathbb{S}^3$ which contained in two disjoint balls then $\mathbb{S}^3_{L_1 \sqcup L_2} = \mathbb{S}^3_{L_1} \# \mathbb{S}^3_{L_2}$, so that, by the above example $\mathbb{S}^3_{L_1 \sqcup u^{\pm 1}} = \mathbb{S}^3_{L_1}$. It is less evident to see that if L and L' are two framed links in $\mathbb{S}^3$ which differ as in Figure 6.10 then $\mathbb{S}^3_L = \mathbb{S}^3_{L'}$ (forget about the colors of the components for the purpose of this paragraph). We will not prove this statement, but the reader should think that the manifold $\mathbb{S}^3_L$ is the boundary of the 4-manifold obtained from B^3 by glueing some 2-handles $D^2 \times D^2$ along $N(L) \subset \mathbb{S}^3 = \partial B^4$. Then the claimed diffeomorphism corresponds the fact that the manifolds obtained by surgery on L and L' are the boundary of a same 4-manifold of which one is considering two handle decompositions which differ by a handle slide.

As proved by Rokhlin [39] (see the extremely concise proof of this fact due to Colin Rourke [42]), each closed oriented 3-manifold is the boundary of a 4-manifold as above, thus it admits a surgery presentation. The above discussion also shows that such a presentation is far from being unique, but it presents the list of basic “moves" which allow to relate any two surgery presentations of a same manifold. The content of Kirby’s theorem on surgery presentations of 3-manifolds is precisely to state that these moves are sufficient to relate any two presentations (there are plenty of good references for understanding this theorem, one instance is [25] Theorem 5.3.6 and the following comments):

Theorem 6.36. *Let M be a closed, oriented 3-manifold. Then M can be presented as surgery over a framed link $L \subset \mathbb{S}^3$ and if L, L' are two links such that $M = \mathbb{S}^3_L = \mathbb{S}^3_{L'}$, then they can be connected to each other via a finite sequence of the following modifications :*

1. *“blow up/down"-moves: consisting in replacing $L \longleftrightarrow L \sqcup u^{\pm 1}$ where $u^{\pm 1}$ is an unknot with framing ± 1 and contained in a ball disjoint from L;*
2. *“handle slides": depicted in Figure 6.10 (forget about the coloring of the components for the purpose of this statement);*
3. *isotopies.*

6.8. Reshetikhin-Turaev invariants via surgery

We are now ready to state the main theorem defining Reshetikhin-Turaev invariants, for which we will use the normalization defined in [7] Section 2.

Theorem 6.37 (Reshetikhin-Turaev)**.** *Let (M,T) be a closed oriented 3-manifold containing a framed colored link T colored by a coloring c with values in $\{0,1,\dots r-2\}$. Let $L \subset \mathbb{S}^3$ be a m-components framed link presenting by surgery M (so that $T\subset \mathbb{S}^3\setminus L$ and $M=\mathbb{S}^3_L$) and let $(b_+,b_-)\in\mathbb{N}\times\mathbb{N}$ be the signature of the linking matrix of L. The following is an invariant up to diffeomorphism of (M,T) :*

$$RT_r(M,T) := D^{-b_0(M)-b_1(M)}\frac{J_{\Omega\cup c}(L\cup T)}{(J_\Omega(u^+))^{b_+}(J_\Omega(u^-))^{b_-}} = D^{-b_0(M)-m}\rho^{-\operatorname{sign}(L)}J_{\Omega\cup c}(L\cup T)$$

where ρ is the unit complex number defined in Proposition 6.33 and D the positive real number defined in Exercice 6.31.

Remark 6.38.
- If M is not connected then L is a set of links in $\mathbb{S}^3$ and $J_\Omega(L)$ is the product of the evaluations of each such links. Stated differently one can restrict to $b_0(M)=1$ (i.e. M connected) and extend the above definition to non-connected manifolds multiplicatively.
- The formulation of the invariants we provided above is the same as that of the invariant denoted by $\langle M\rangle_{2r}$ in [7] section 2 (where we take the zero p_1-structure). To make the correspondence between the notations, compare the value of $J_\Omega(u^+)$ given in Proposition 6.33 with that of formula $(*)$ in [7] : our D is η^{-1} and our ρ is k^3 in [7]. This normalization differs from Reshetikhin-Turaev's Theorem 3.3.3 in [43] : in our definition $RT_r(\mathbb{S}^3)\neq 1$.

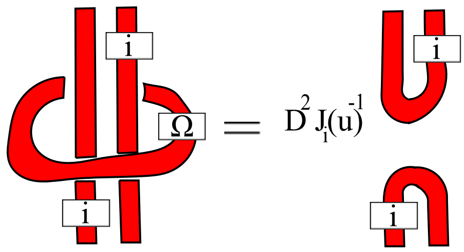
Proof. We give a sketchy proof, we refer to [6] (Theorem B) and [7] Section 2 (for what concerns the renormalization we chose) for details. By Kirby's theorem two framed links in $\mathbb{S}^3$ presenting (M,T) by surgery can be connected by a finite sequence of handle slides, "blow up/down" (corresponding to adding/removing a u^+ or u^-) and isotopies. Invariance under blow up/down is straightforward while under handle slide it is precisely the statement of Proposition 6.29. Invariance under isotopy is automatic by the definition of the skein module of $\mathbb{S}^3$. The last equality in the statement is a direct consequence of Proposition 6.33. 6.38

Remark 6.39. In the proof we actually used a stronger form of the theorem allowing the presence of a non empty link T in M; this was already present in Reshetikhin-Turaev's Theorem 3.3 [43]. The necessary topological result allowing Kirby calculus in this case has been proved by Justin Roberts [41].

Example 6.40. Observe that T may be empty and in that case it is easy to check that $\overline{RT_r(M)} = RT_r(\overline{M})$.

1. $RT_r(\mathbb{S}^3) = D^{-1}$
2. $RT_r(\mathbb{S}^2\times\mathbb{S}^1) = 1$.
3. If $(M,T)=(M_1,T_1)\#(M_2,T_2)$ where the sum is taken along a ball disjoint from T_i, by taking presentations of (M_i,T_i) and putting them in disjoint balls in $\mathbb{S}^3$ we get a presentation of (M,T) and a proof of the equality
$$RT_r(M,T) = \frac{RT_r(M_1,T_1)RT_r(M_2,T_2)}{D} = RT_r(\mathbb{S}^3)RT_r(M_1,T_1)RT_r(M_2,T_2).$$

Lemma 6.41. *The following local identity holds in $\mathcal{S}^{red}_{A_0}(M)$:*

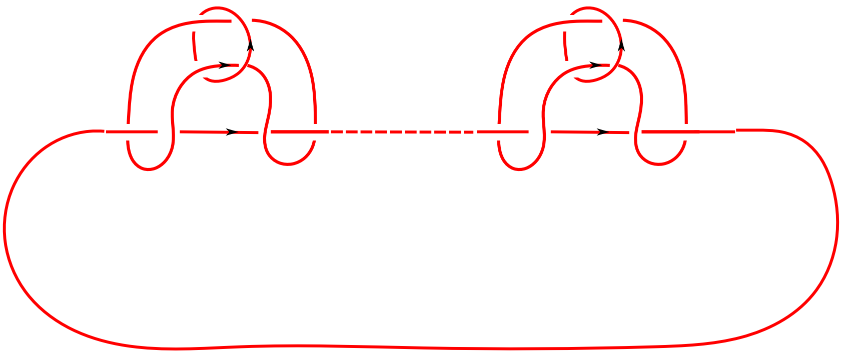


Proof. It is a consequence of the fusion rule in the reduced skein module (Theorem 6.26) and of Lemma 6.30. 6.41

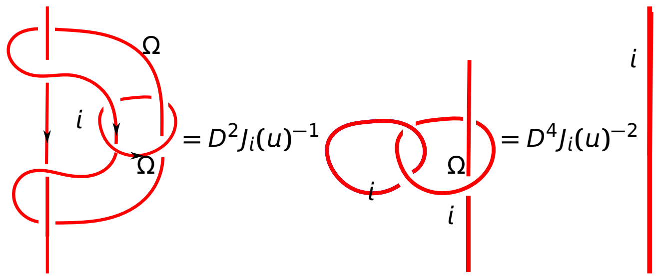
Proposition 6.42 (Verlinde formula)**.** *It holds*

$$RT_r(\Sigma_g \times \mathbb{S}^1) = \frac{r^{g-1}}{2^{g-1}} \sum_{j=1}^{r-1} \frac{1}{\sin(\frac{j\pi}{r})^{2g-2}}. \tag{6.18}$$

Proof. A surgery presentation of $\Sigma_g \times \mathbb{S}^1$ is given by the following diagram (see for instance [25] Section 6.1 and in particular Figure 6.4) :



where g copies of the "handles" are intended. Then applying to each handle twice Lemma 6.41 as follows we get :



So that repeating this procedure for all the handles, summing over all the colors i of the central knot, and taking into account that $\Omega = \sum_i J_i(u) T_i$ then we get :

$$RT_r(\Sigma_g \times \mathbb{S}^1) = D^{(-1-2g-1)} D^{4g} \sum_{i=0}^{r-2} J_i(u)^{-2g+2} = D^{(2g-2)} \sum_{i=0}^{r-2} J_i(u)^{-2g+2} = \tag{6.19}$$

$$= \frac{r^{g-1}}{2^{g-1}\sin(\frac{\pi}{r})^{2g-2}} \sum_{j=0}^{r-2} \frac{\sin(\frac{\pi}{r})^{2g-2}}{\sin(\frac{(j+1)s\pi}{r})^{2g-2}} = \frac{r^{g-1}}{2^{g-1}} \sum_{j=1}^{r-1} \frac{1}{\sin(\frac{j\pi}{r})^{2g-2}} \tag{6.20}$$

where in the last equality we used the hypothesis $(s, r) = 1$ to reorder the terms. 6.42

Remark 6.43. Although it is absolutely not evident from Formula (6.18), $RT_r(\Sigma_g \times \mathbb{S}^1)$ are always natural numbers! Here are some examples :

$$RT_5(\Sigma_2 \times \mathbb{S}^1) = 20, RT_5(\Sigma_3 \times \mathbb{S}^1) = 120, RT_6(\Sigma_3 \times \mathbb{S}^1) = 35, RT_6(\Sigma_3 \times \mathbb{S}^1) = 329\ldots$$

The interested reader may consult Don Zagier's paper [50] on the Verlinde formula to find many striking identities about it.

7. Extending RT_r to a TQFT.

In this section we apply the universal construction to the Reshetikhin-Turaev invariants to get a TQFT. After a first failed attempt we will modify our category Cob_n by decorating suitably the surfaces and provide a proof that one has a TQFT for this new category.

7.1. A negative result

According to the integrality of Formula (6.18) one may hope that the invariants RT_r are actually the phenomenon of the existence of an underlying TQFT. But if one applies the universal construction to Cob_3 he gets the following negative result :

Theorem 7.1 (Gilmer-Wang,[24])**.** *If $r \geq 3$ the result of the universal construction applied to the invariants RT_r is not a TQFT as the vector space associated to a torus is not finite dimensional.*

Proof. Fix a copy of T^2 embedded in the standard way in $\mathbb{S}^3$. We will exhibit manifolds $Z_i, i \in \mathbb{N}$ bounded by $\mathcal{T}^2$ from the inside (i.e. elements of $V_{2r}(T^2)$) and $W_j,\ j \in \mathbb{N}$ bounded by T^2 from the outside (i.e. elements of $V'_r(T^2)$) indexed by the natural numbers and show that the $\mathbb{N}\times\mathbb{N}$ matrix whose $(i,j)^{th}$ entry is $RT_r(W_j \circ Z_i)$ has infinite rank thus proving the thesis. Let Z_i be the manifold obtained by surgery along the $4ri$-framed core of the "inside solid torus" bounded by T^2. And let W_j be the manifold obtained by surgery along the link formed by $4rj$ parallel (and unlinked) copies of the core of the "outside solid torus" each of which is framed by $+1$. A surgery presentation of $W_j \circ Z_i$ is then given by a link with $4rj+1$-components. Applying $4rj$ times an inverse Kirby move of the first type we may reduce to a presentation with only one unknot with framing $4r(i-j)$. Thus, using Proposition 6.33, Exercise 6.31 and the fact that $A^{4r}=1$ we get if $i>j$:

$$RT_r(W_j \circ Z_i) = \frac{D^{-1-\delta_{i,j}}}{J_\Omega(u^+)}\sum_{k=0}^{r-2}(-1)^k A^{-k(k+2)4r(i-j)}\frac{\sin(\frac{\pi k}{r})^2}{\sin(\frac{\pi}{r})^2} = \frac{D^{1-\delta_{i,j}}}{J_\Omega(u^+)}$$

and if $i<j$ a similar computation gives $RT_r(W_j\circ Z_i) = \frac{D}{J_\Omega(u^-)}$. Finally if $i=j$ then we get 1. Since $|J_\Omega(u^-)| = |J_\Omega(u^+)| = D$ and $J_\Omega(u^-) = \overline{J_\Omega(u^+)}$, letting $\rho = \frac{|J_\Omega(u)|}{J_\Omega(u^-)}$ (ρ turns out to be a root of unity depending on r and different from 1) we see that the overall matrix $M_{i,j} := RT_r(W_j\circ Z_i)$ is then:

$$M = \begin{pmatrix} 1 & \rho & \rho & \rho\cdots \\ \rho^{-1} & 1 & \rho & \rho\cdots \\ \rho^{-1} & \rho^{-1} & 1 & \rho\cdots \\ \rho^{-1} & \rho^{-1} & \rho^{-1} & 1 \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{7.1}$$

Then Gilmer and Wang show that letting M_i be the $i\times i$-submatrix of M formed by the first i columns and rows, then for no $i\geq 1$ it can be true that $\det(M_i) = \det(M_{i+1}) = 0$. They do this by proving that $\det(M_{i+1}) = \det(M_i)(1-\rho^{-1}) + (1-\rho)^{i-1}(\rho^{-1}-1)$ and the term $(1-\rho)^{i-1}(\rho^{-1}-1)$ is non zero as $\rho\neq 1$. 7.1

7.2. The solution of the problem

In the proof of Theorem 7.1 we operated multiple inverse Kirby 1-moves and, by the construction of the invariants, this did not affect the value of RT_r. This is actually what causes that the resulting coupling matrix is that of equation (7.1). Suppose that now we take into account these moves and we “pay" each such move by a factor ρ. Stated more explicitly suppose that instead of $RT_r(W_j \circ Z_i)$ we consider $\rho^{-sign(Link)} \cdot RT_r(W_j \circ Z_i)$ where $sign(Link))$ is the signature of the linking matrix of the link presenting $W_j \circ Z_i$ BEFORE the inverse Kirby moves are applied. Then the resulting matrix will look like :

$$M' = \begin{pmatrix} 1 & \rho\cdot\rho^{-1-4r} & \rho\cdot\rho^{-1-4r\cdot 2} & \rho\cdot\rho^{-1-4r\cdot 3}\dots \\ \rho^{-1}\cdot\rho^{1-4rj} & 1 & \rho\cdot\rho^{-1-4r} & \rho\cdot\rho^{-1-4rj\cdot 2}\dots \\ \rho^{-1}\cdot\rho^{1-4rj\cdot 2} & \rho^{-1}\cdot\rho^{1-4rj} & 1 & \rho\cdot\rho^{-1-4rj}\dots \\ \rho^{-1}\cdot\rho^{1-4rj\cdot 3} & \rho^{-1}\cdot\rho^{1-4rj\cdot 2} & \rho^{-1}\cdot\rho^{1-4rj} & 1 \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{7.2}$$

which, since ρ is a root of unity has finite rank.

Clearly, given a surgery presentation of a manifold M via a framed link $L \subset \mathbb{S}^3$, the quantity $\rho^{-sign(Link)} \cdot RT_r(M)$ is *not* an invariant of M. (just apply a Kirby 1-move). So, following Turaev, we use the following :

Definition 7.2 (Extended manifolds and their invariants)**.** An extended manifold is a pair (M, m) with M a compact (possibly with boundary) oriented 3-manifold and $m \in \mathbb{Z}$. The RT_r invariant of a closed extended manifold is defined to be $RT_r(M) \cdot \rho^{-m}$.

The trick is now to stipulate that a surgery presentation via a framed link L of a manifold M actually yields an extended manifold $(M, m = sign(L))$. At this stage this seems to be purely formal. But now the question is : what is the natural category of cobordisms we should consider if we wanted to use “extended manifolds" instead of “manifolds"?

Definition 7.3. The category $\widetilde{\mathrm{Cob}}$ is the category whose objects are oriented compact surfaces Σ equipped with a lagrangian subspace $\mathcal{L} \subset H_1(\Sigma;\mathbb{R})$ and whose cobordisms are cobordisms of Cob_n equipped with an integer. The composition of two cobordisms

$$(M, \Sigma, f_+, \partial_- M, f_-, m) : \Sigma_- \to \Sigma_0$$

and

$$(N, \partial_+ N, g_+, \partial_- N, g_-, n) : \Sigma_0 \to \Sigma_+$$

is defined as the cobordism

$$(N \sqcup_{g_- \circ f_+^{-1}} M, \partial_+ N, g_+, \partial_- M, f_-, m + n - \mu(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)) \tag{7.3}$$

where in the symplectic vector space $H_1(\Sigma_0;\mathbb{R})$ one considers the Maslov index $\mu(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ with :

1. $\mathcal{L}_1 = \{(f_+)_*^{-1}(x) | \exists a \in \mathcal{L}(\partial_- M)\ s.t.\ x = -a \text{ in } H_1(M)\}$;
2. $\mathcal{L}_2 = \mathcal{L}(\Sigma_0)$;
3. $\mathcal{L}_3 = \{(g_-)_*^{-1}(y) | \exists b \in \mathcal{L}(\partial_+ N)\ s.t.\ b = -y \in H_1(N)\}$.

(The fact that $\mathcal{L}_1$ and $\mathcal{L}_3$ are lagrangians is easy to check and left as an exercise; for full details on the topic we refer to [46] Chapter 4, Section 3).

Remark 7.4. In [46] a more complicated formula is provided involving lagrangians in vector spaces of dimension twice that of $H_1(\Sigma_0)$ (see the definition of the glueing of cobordisms at the beginning of section 9.1, Chapter IV). This is due to the fact that in [46] one is allowed to glue along surfaces equipped with different lagrangians (i.e. such that $f_*^{-1}(\mathcal{L}(\partial_+ M)) \neq g_*^{-1}(\mathcal{L}(\partial_- N))$ in the above notation. In our case we suppose equality (by definition of our category $\widetilde{\mathrm{Cob}}$) and this simplifies the formula of the Maslov index : see Chapter IV formula 3.7 in [46] and the computation of m' in the proof of Theorem 9.2.1.

Remark 7.5. If $\partial_- M = \emptyset$ then $\mathcal{L}_1$ is the kernel of the embedding of $H_1(\partial_+ M)$ in $H_1(M)$. Similarly if $\partial_+ N = \emptyset$.

Lemma 7.6. *Let $(\Sigma_g, \mathcal{L})$ be an extended surface. The extended modular group $\widetilde{Mod}(\Sigma_g, \mathcal{L})$ embeds in $\widetilde{Cob}$ via the map $(f, n) \to C(\tilde{f}) := (\Sigma_g \times [-1, 1], \Sigma_g \times \{1\}, Id, \Sigma_g \times \{-1\}, f, n)$.*

Proof. This is an enhanced version of Lemma 2.5; we need to check that the composition of elements of $\widetilde{Mod}$ is mapped to that of the corresponding cobordisms. So this boils down to check that the term $-\mu(f_*(\mathcal{L}), \mathcal{L}, g_*^{-1}(\mathcal{L}))$ (used in formula (5.5)) is the above defined correction factor to the composition of two cobordisms (see Equation (7.3)). This is indeed the case as by definition of C_f we have $f_+ = Id$, $f_- = f$ and $\mathcal{L}(\partial_-(C_f)) = f_*(\mathcal{L})$ so $\mathcal{L}_1 = (Id)_*^{-1}(f_*(\mathcal{L})) = f_*(\mathcal{L})$, while for C_g we have $g_- = g, g_+ = Id$ and $\mathcal{L}(\partial_+(C_g)) = \mathcal{L}$ so $\mathcal{L}_3 = (g_-)_*^{-1}(\mathcal{L}) = g_*^{-1}(\mathcal{L})$. 7.6

It is actually easier to apply the universal construction to our case if we further extend the category of cobordisms by allowing the datum of “skeins" i.e. linear combinations of isotopy classes of framed links inside the cobordisms:

Definition 7.7. For any $r \geq 3$, the category $\widetilde{Cob}_r$ is the category whose objects are those of $\widetilde{Cob}$ and whose morphisms are pairs (M, T) where M is a cobordism of $\widetilde{Cob}$ and $T \in \mathcal{S}_A^{\text{red}}(M)$.

Before stating the main theorem on the construction of TQFTs let us recall Wall's signature theorem for 4-manifolds. Let W be a compact oriented smooth 4-manifold with boundary and let $\sigma(W)$ be the signature of its intersection form $H_2(W; \mathbb{R}) \times H_2(W; \mathbb{R}) \to \mathbb{R}$. Suppose that W contains a properly embedded 3-manifold W_0 (i.e. $\partial M_0 \subset \partial W$) which splits W into $W_1 \sqcup W_2$ and let $\partial W \setminus \partial M_0 = M_1 \sqcup M_2$. Orient M_1, M_2 so that $\partial W_1 = \overline{M_1} \cup M_0$ and $\partial W_2 = \overline{M_0} \cup M_2$, so that the orientations of M_0, M_1, M_2 induce the same orientation on the surface $\Sigma = \partial M_0 = \partial M_1 = \partial M_2$. Let $\mathcal{M}_i \subset H_1(\Sigma; R)$ be the lagrangian subspaces given by the kernel of the inclusion of $H_1(\Sigma; \mathbb{R})$ into $H_1(M_i; \mathbb{R})$. Then the following holds :

Theorem 7.8 (Wall's theorem)**.** $\sigma(W) = \sigma(W_1) + \sigma(W_2) + \mu(\mathcal{M}_1, \mathcal{M}_0, \mathcal{M}_2)$

We are now ready to state the main theorem on the construction of SU(2) Reshetikhin-Turaev TQFTs and give a sketch of its proof (we refer to [7] for all the details).

Theorem 7.9 ([7] Theorem 1.4)**.** *The universal construction applied to the extended Reshetikhin-Turaev invariants of 3-manifolds and to the category $\widetilde{Cob}_r$ yields a TQFT Z_r : $\widetilde{Cob}_r \to Vect$. Furthermore for each Σ_g the vector space $V_{2r}(\Sigma_g) := Z_r(\Sigma_g)$ is equipped with a $Mod(\Sigma_g)$-invariant Hermitian form $\langle , \rangle$, which, if $A = \exp(\frac{i\pi}{2r})$ is positive definite.*

Remark 7.10. The notation $V_{2r}(\Sigma_g)$ is coherent with the original notation coming from [7].

Proof. Let Σ_g be a surface. Observe that each $M \in \mathscr{V}(\Sigma_g)$ gives rise to $\overline{M} \in \mathscr{V}'(\overline{\Sigma_g})$ and since $RT_r(\overline{W}) = \overline{RT_r(W)}$ for each closed 3-manifold W, we get that the modules $V_{2r}(\Sigma_g)$ and $V'_{2r}(\Sigma_g)$ are isomorphic by the isomorphism obtained by extending $\mathbb{C}$-antilinearly the map $M \to \overline{M}$. Thus the natural pairing between them descends to a hermitian, non-degenerate, bilinear form on $V_{2r}(\Sigma_g)$ by Proposition 3.6.

To prove finite dimensionality of $V_{2r}(\Sigma_g)$ we observe that any $M \in \mathscr{V}(\Sigma_g)$ can be transformed into a connected sum of handlebodies H by a finite sequence surgeries along framed links in M. Each such surgery is translated by the replacement of the surgery link by an Ω-colored framed link in H. Indeed we claim that if H_k is the result of a surgery of H along a framed knot k then in $V_{2r}(\Sigma_g)$ it holds $[H_k] = \lambda[H, k_\Omega]$ for some constant $\lambda \in \mathbb{C}$ depending on the framed knot k and on $\mathcal{L}(\Sigma_g)$ (where by $[H, k_\Omega]$ we denote the vector represented by H containing a copy of k colored by Ω).

Let m, l be the homology classes in $H_1(\partial N(k))$ of the meridian and the longitude of k and let $N(k)$ (resp. $N'(k)$) be the solid torus representing a cobordism from $\emptyset$ to $\partial N(K)$ whose meridian is glued to m (resp. l). Furthermore in N (but not in N') let's cable the core of N with the color Ω. To see N and N' as cobordims we equip $\partial N = \partial N'$ with an arbitrary lagrangian $\mathcal{L}$ and N and N' with weights 0.

To prove our claim it is sufficient to prove that in $\mathscr{V}(\mathbb{S}^1\times\mathbb{S}^1)$ it holds $[N'] = \lambda[N]$. So let $R_i, i=1,2 \in \mathscr{V}'(\mathbb{S}^1\times\mathbb{S}^1)$ be any two manifolds, and let also $S_i = R_i \circ N$ and $S_i' = R_i \circ N'$. Finally let $L_i' = L_i \sqcup k \in \mathbb{S}^3$ be framed links presenting S_i' (so that L_i presents S_i and k is the Ω-colored skein in $N\subset N_i$). Considering L_i' as a surgery presentation of S_i' we see k as part of the surgery link while for S_i we consider it as a skein in S_i: in the latter case it implies that k is not taken into account in the computation of the signature of the presentation. So, by Definition 7.2 we have:

$$RT_r(S_i') = D^{-1}\rho^{-\operatorname{sign}(L_i')+\operatorname{sign}(L_i)-w(S_i')+w(S_i)}RT_r(S_i).$$

Then to prove our claim it is sufficient to prove the following:

$$-\operatorname{sign}(L_1')+\operatorname{sign}(L_1)-w(S_1')+w(S_1) = -\operatorname{sign}(L_2')+\operatorname{sign}(L_2)-w(S_2')+w(S_2).$$

Observe that L_i' gives a 4-dimensional oriented smooth manifold W^i whose signature is $\operatorname{sign}(L_i')$ and such that $\partial W^i = S_i'$. Furthermore the regular neighborhood $N(k)$ of k in the surgery presentation provides a 3-manifold M_0 (a solid torus) properly embedded in W and splitting W^i into two submanifolds : W_1^i and W_2^i of which $\partial W_1^i = S_i$ (and so $\sigma(W_1^i) = \operatorname{sign}(L_i)$) and W_2^i is a 2-handle, hence a 4-ball (and so $\sigma(W_2^i) = 0$). Let now $\mathcal{M}_i$ be the lagrangian induced by R_i on $\mathbb{S}^1\times\mathbb{S}^1 = \partial N(k)$. By Wall's theorem and by Lemma 5.21 it holds

$$\operatorname{sign}(L_i') - \operatorname{sign}(L_i) = \mu(\mathcal{M}_i, m(k), l(k)) = \mu(\mathcal{M}_i, m, \mathcal{L}) + \mu(m, l, \mathcal{L}) - \mu(\mathcal{M}_i, l, \mathcal{L}).$$

Now observe that by antisymmetry of the Maslov index and by the definition of the composition of the cobordisms in $\widetilde{\mathrm{Cob}}$ we have that $\mu(\mathcal{M}_i, m, \mathcal{L}) = -\mu(\mathcal{M}_i, \mathcal{L}, m) = w(S_i)$ and $\mu(\mathcal{M}_i, l, \mathcal{L}) = w(S_i')$. This proves the claim as $-\operatorname{sign}(L_i')+\operatorname{sign}(L_i)+w(N_i)-w(N_i') = \mu(m, l, \mathcal{L})$ does not depend on R_i.

Until now we showed that we can reduce by surgeries along links to vectors in $\mathscr{V}(\Sigma_g)$ represented by skeins in a connected sum of handlebodies H_g. We now want to show that actually we can further split each connected sum to a disjoint union of handlebodies. To do so it is sufficient to show that in $V(\mathbb{S}^2\sqcup\overline{\mathbb{S}}^2)$ the following equality holds :

$$[B^3\sqcup\overline{B}^3] = RT_r(\mathbb{S}^3)[\mathbb{S}^2\times[-1,1]]$$

and this is easily proved by testing against cobordisms $M_i \in \mathscr{V}'(\mathbb{S}^2\sqcup\overline{\mathbb{S}}^2)$ and using the equality : $RT_r(M\#N) = RT_r(\mathbb{S}^3)RT_r(M)RT_r(N)$ (we invite the reader to fill the details, considering also the case when M_i is connected). This equality also implies that each manifold bounded by a disjoint union of surfaces $\Sigma_1\sqcup\Sigma_2$ is equivalent in $V(\Sigma_1\sqcup\Sigma_2)$ to a disjoint union of manifolds, one bounded by Σ_1 and the other bounded by Σ_2 so obtaining that $V(\Sigma_1)\otimes V(\Sigma_2) = V(\Sigma_1\sqcup\Sigma_2)$.

The above two arguments show that $V(\Sigma_g)$ can be entirely represented by skeins in a disjoint union of handlebodies H, one per component of Σ. For simplicity let's assume that Σ is connected from now on (the proof is almost identical else). By Proposition 6.27 the reduced skein module of the handlebody H is generated by r-admissible colorings col of any fixed trivalent spine Y of H; let's denote the vectors represented in $\mathscr{V}(\Sigma)$ by these colored spines by $[H, Y_{col}]$. We are only left at proving that these vectors are actually linearly independent in $V_{2r}(\Sigma_g)$. This is easily done by observing that $[\overline{H}, Y_{col}]$ is a vector of $\mathscr{V}'(\Sigma)$ and that the pairing between these vectors is diagonal and non degenerate, namely :

$$\langle[\overline{H}, Y_{col'}], [H, Y_{col}]\rangle = \delta_{col,col'}\cdot f(col, col')$$

where f is a function of the two colorings which can be easily expressed in terms of products of $\ominus$ evaluations which are easily seen to be non-zero when the colorings are r-admissible. The proof of this claim is straightforward by observing that $\overline{H}\circ H = \#_g\mathbb{S}^2\times\mathbb{S}^1$ and so it admits a surgery presentation in $\mathbb{S}^3$ by surgery over g unlinked 0-framed unknots. Furthermore each such unknot encircles exactly two edges of the graph $Y_{col}\sqcup Y_{col'}$ and applying the encirclement Lemma 6.30 g times one concludes. 7.10

Remark 7.11. Theorem 7.9 provides in particular quantum representations of the central extensions of the mapping class groups considered in Section 5.4. Thus one can see these representations as projective representations of the mapping class groups themselves. Furthermore, since the contribution of the Meyer cocycle is only given by multiplication by $\rho^{-\mu}$ which is a root of unity, one can obtain genuine representations of the mapping class groups by considering the action on $End(V_{2r}(\Sigma_g))$.

8. Some properties of the RT-TQFTs.

In this section we rapidly recall some of the known facts concerning the SU(2)-quantum representations obtained from Theorem 7.9 and for some of these results we provide a sketch of proof. In the last subsection we also provide some comments on the new non semi-simple TQFTs.

8.1. Infiniteness

Let $\gamma \subset \Sigma_g$ be a simple closed curve and T_γ the Dehn-twist along γ. Fix a handlebody H_g bounded by Σ_g such that γ bounds a disc D in H_g and pick a trivalent spine Y of H_G intersecting D in exactly one point along an edge e; recall that $\{[H_g, Y, c] | c : E(Y) \to \{0, 1, \dots r-2\}$ $r-$αdmissible colorings$\}$ form a basis of $V_{2r}(\Sigma_g)$. The following holds :

Lemma 8.1. $T_c([H_g, Y, c]) = (-A)^{c(e)(c(e)+2)}[H_g, Y, c]$.

Proof. By construction T_c extends to H_g and its action on Y is just to add a full twist to the framing of the edge e. Thus the relation is just the framing change relation in the skein module. 8.1

Corollary 8.2. *The order of the action on $V_{2r}(\Sigma_g)$ of each Dehn twist is at most $4r$. In particular the representations are never faithful!*

Because of Lemma 8.1 one may think that the image of the quantum representations of $Mod(\Sigma_g)$ considered as projective representations (see Remark 7.11) is small or finite. It is indeed true that the image of $Mod(\mathbb{S}^1 \times \mathbb{S}^1)$ is finite (proved by Gilmer in [21]). On contrast Funar proved :

Theorem 8.3 (Funar, [20])**.** *The image of the mapping class group $Mod(\Sigma_g)$ under the representation arising in the* SU(2)*-TQFT (in both the BHMV and RT versions) is infinite provided that $g \geq 2, r \neq 2, 3, 4, 6$, and if $g = 2$ also $r \neq 10$.*

Corollary 8.4. *The quotients $Mod(\Sigma_g)/ < \{T_\gamma^{4r} | \gamma \subset \Sigma_g\} >$ are infinite provided that $g \geq 2, r \neq 2, 3, 4, 6$ and if $g = 2$ also $r \neq 10$.*

8.2. Irreducibility

Suppose now that r is an odd prime. Then the following holds :

Lemma 8.5. *A basis for $V_{2r}(\mathbb{S}^1 \times \mathbb{S}^1)$ is formed by k copies of the core with framing* 1 *(in the standard embedding of the torus in $\mathbb{S}^3$) each colored by Ω.*

Proof. We already know that a basis (in general) is given by the vectors $T_i := [D^2 \times \mathbb{S}^1, \{0\} \times \mathbb{S}^1, i]$ with $i \in \{0, 1, \dots r-2\}$. To prove our claim it is sufficient to pair the proposed basis against the basis T_i and check that the pairing matrix is non-degenerate. It easily turns out that, up to a permutation of the columns the resulting matrix is a Vandermonde matrix, thus non-degenerate. 8.5

Since a knot colored by Ω also represents a surgery along the knot we may also think that $V_{2r}(\mathbb{S}^1 \times \mathbb{S}^1)$ is generated by some empty three-manifolds bounded by $\mathbb{S}^1 \times \mathbb{S}^1$. This easily implies that for each Σ_g the same is true. These “special" empty vectors, where used by Gilmer and Masbaum [23] to build a lattice in $V_p(\Sigma_g)$ which is acted upon by $Mod(\Sigma_g)$.

Proposition 8.6 (Roberts,[40])**.** *Let $r \geq 3$ be prime. The $\widetilde{Mod}(\Sigma_g)$-module $V_{2r}(\Sigma_g)$ is irreducible.*

Proof. Let H_g be a fixed handlebody and $Y \subset H_g$ be a trivalent spine of H_g let us denote by c any r-admissible coloring of Y. We know that the vectors $\{[H_g, Y, c]\}$ with c ranging over the r-admissible colorings of Y form a basis of $V_{2r}(\Sigma_g)$. To prove that $V_{2r}(\Sigma_g)$ is irreducible, by Schur's lemma it is sufficient to prove that any endomorphism of $V_{2r}(\Sigma_g)$ commuting with the action of $\widetilde{Mod}$ is λId. Observe that each skein in H_g can be represented as a linear combination of skeins in a neighborhood of $\Sigma - Y$ projecting on Σ_g without crossings and, by Lemma 8.5 each such skein can be replaced by a suitable linear combination of Dehn-surgeries along the same curve (or copies of the same curve). Now observe that the curve with framing 1 colored by Ω represents the action of the Dehn-twist along the curve on the skein module of $\Sigma_g \times [-1, 1]$. Thus all the vectors of $V_{2r}(\Sigma_g)$ are linear combinations of elements of the group algebra $\mathbb{C}[Mod(\Sigma_g)]$ applied to the empty vector $v_0 = [H_g] = [H_g, Y, 0]$. Let then T_i be the Dehn-twists along the curves in $\Sigma - g$ bounding discs D_i in H_g dual to the edges of Y; observe that by Theorem 6.16 it holds $T_i([H_g, Y, c]) = \lambda_i(c)[H_g, Y, c]$ where $\lambda_i(c) = (-A)^{c_i(c_i+2)}$ where c_i is the color of the edge of Y intersecting D_i. Since r is prime the values of $\lambda_i(c)$ are all distinct for different c_i. So if a transformation $\theta : V_{2r}(\Sigma_g) \to V_{2r}(\Sigma_g)$ commutes with the action of $Mod(\Sigma_g)$ it must hold $\theta([H_g, Y, c]) = \lambda_c[H_g, Y, c],\ \forall c$. We only need to prove that λ_c does not depend on c. This is due to the fact that each vector is in $\mathbb{C}[Mod(\Sigma_g)]$ and hence we can write $[H_g, Y, c] = \gamma[H_g, Y, 0]$ for some $\gamma \in \mathbb{C}[Mod(\Sigma_g)]$. But then $\theta([H_g, Y, c]) = \theta \cdot \gamma[H_g, Y, 0] = \gamma \cdot \theta[H_g, Y, 0] = \gamma\lambda_0[H_g, Y, 0] = \lambda_0[H_g, Y, c]$ but also $\theta([H_g, Y, c]) = \lambda_c[H_g, Y, c]$. 8.6

On contrast there are known values of r, g for which $V_{2r}(\Sigma_g)$ is reducible:

Theorem 8.7 (Andersen-Fjelstad, [3])**.** *For all $g \geq 1$ the representations $V_{24}(\Sigma_g), V_{36}(\Sigma_g)$ and $V_{60}(\Sigma_g)$ contain at least three invariant submodules.*

Theorem 8.8 (Korinman, [29])**.**

- *If r is odd prime, then $V_{4r}(\Sigma_2)$ is the direct sum of two irreducible sub-representations.*
- *If r_1, r_2 are two odd primes then $V_{2r_1r_2}(\Sigma_2)$ is irreducible.*

8.3. Detecting pseudo-anosov diffeomorphisms

In [4], the following conjecture (now known as the AMU conjecture) was formulated:

Conjecture 8.9 (AMU)**.** *Let Σ be a compact surface (possibly with boundary) such that $\chi(\Sigma) < 0$ and $\phi \in Diff^+(\Sigma)$ be a pseudo anosov diffeomorphism. The action of ϕ on $V_{2r}(\Sigma_g)$ has infinite order for all but finitely many r.*

In these notes we did not recall the construction of the TQFT vector spaces for punctured surfaces or surfaces with boundary. For the purpose of this section, let us just admit that for each $r \geq 3$ there is an extension of the TQFT to the category whose objects are surfaces with finitely many points (or boundary components) decorated by colors in $\{0, 1, \ldots, r-2\}$. The AMU conjecture has been proven only for some of these cases, namely for the 4-punctured sphere whose punctures are colored by 1 (see [4]) or more in general N (see [44]) and for a once punctured torus whose puncture is colored by N (see [44], actually only for the $SO(3)$ theory, corresponding to taking $A = \exp(\frac{\pi i}{p})$ with p odd).

In the direction of detecting pseudo-anosov diffeomorphisms, let us also mention the following result (which, again, holds only for punctured surfaces) obtained in [14]:

Theorem 8.10. *Let Σ be a punctured surface and $\varphi\colon \Sigma \to \Sigma$ a pseudo-Anosov map with dilatation $\lambda > 1$. Let $A = \exp(\frac{\pi ik}{2r})$ with $(k, 2r) = 1$. If*

$$r > -6\chi(\Sigma)\big(\lambda^{-9\chi(\Sigma)} - 9\chi(\Sigma) - 1\big) + 1$$

then the action of ϕ on $V_{2r}(\Sigma)$ is non trivial for some coloring of the punctures.

8.4. Asymptotic fidelity

The following was proved independently by Andersen [2] and by Freedman, Walker and Wang [19]; other proofs were later found by Marché-Narimanejad [36] and Costantino-Martelli [14] (the latter in the case of punctured surfaces):

Theorem 8.11 (Asymptotic fidelity)**.** *For each $g \geq 1$ the quantum representations are asymptotically faithful :*

$$\bigcap_{r\geq 3} ker\rho_{2r}(Mod(\Sigma_g)) = Z(Mod(\Sigma_g)).$$

Proof. (This proof is taken from Freedman, Walker and Wang). Suppose $h \in Mod(\Sigma_g)$ is not central; then there exists a curve $\gamma \subset \Sigma_g$ such that $\gamma \neq h(\gamma)$. Take then a handlebody H_g bounded by Σ_g in which γ bounds a disc D and let Y be a spine of H_g intersecting D along an edge e. Observe that $\mathcal{S}^{red}_{A_0}(H_g)$ is a module over the algebra $\mathcal{S}^{red}_{A_0}(\Sigma_g)$ (where the action is induced by inclusion). The skein represented by γ acts by a scalar on $[H_g, Y, c]$. To show that $h(\gamma)$ does not act as a scalar, observe that pushing it inside H_g and applying fusion rules one can reduce $h(\gamma)\cdot[H_g, Y, c]$ to a linear combination of $[H_g, Y, c']$ for some colorings c'. Taking r much larger than the maximal color c_{max} one gets in any such fusion then one sees that $h(\gamma)\cdot[H_g, Y, c] = c\cdot[H_g, Y, c_{max}] + l.o.t$ where by "lowest order terms" we mean colorings whose sum of colors is less than that of c_{max}. This implies that the action of $h(\gamma)$ is non trivial (i.e. not a multiple of the 0-colored spine) if r is big enough because these colorings represent linearly independent vectors in $V_{2r}(\Sigma_g)$. 8.11

8.5. The non semi-simple TQFTs

We conclude by citing some of the properties of the "non semi-simple TQFTs" recently constructed in [8] in order to compare them with those if the above "standard" SU(2)-TQFTs.

In [8] a new family of TQFTs was constructed by applying the universal construction to a the "non semi-simple Reshetikhin-Turaev" invariants of closed three-manifolds defined in [13]. These invariants are actually invariants of three-uples (M, T, ω) with M a closed oriented three-manifold, $T \subset M$ a (possibly empty) ribbon graph whose edges are colored by objects of a certain category (generalizing the set of colors considered in the standard RT case) and $\omega \in H^1(M\backslash T; \mathbb{C}/2\mathbb{Z})$ is a cohomology class; these three-uples are subject to some compatibility conditions which are generically satisfied. Clearly, in order to apply the universal construction, one needs to decorate the category Cob_n so to include the datum of the cohomology classes, so that in particular the vector spaces associated to a surface are indexed also by a cohomology class on it : $V(\Sigma, \omega), V'(\Sigma, \omega)$. Furthermore (and more importantly) the fact that the invariants are defined only "generically" implies that in the new category of cobordisms some objects have no duals and that $V(\Sigma, \omega)$ and $V'(\Sigma, \omega)$ although dually paired are different (i.e. no linear or antilinear isomorphism is known between them in general).

Despite these apparent difficulties, the properties of these new TQFTs are promisingly different from those of the standard RT TQFTs :

Theorem 8.12 ([8])**.** *Let $\gamma \neq \gamma' \subset \Sigma$ be non trivial disjoint simple closed curves and suppose that $[\gamma] = [\gamma'] \neq 0 \in H_1(\Sigma; \mathbb{Z})$. The action of the Dehn-twist T_γ along γ on $V(\Sigma, 0)$ has infinite order and the action of $T_\gamma \circ T_{\gamma'}^{-1}$ (which belongs to the Torelli group) on $V(\Sigma, \omega)$ has infinite order for almost all ω.*

As of today, no element in the kernel of these representations is known (compare the above theorem with Corollary 8.2).

Appendix A. Basic facts in category theory

The purpose of this appendix is to recall the basic definitions in category theory which we will use in this work. A good reference for most of the topics recalled here is [27].

Definition A.1 (Categories, functors and natural transformations)**.** A category $\mathcal{C}$ is a collection of objects $Ob(\mathscr{C})$ and for each pair of objects (Σ_-, Σ_+) a collection of "morphisms" $\mathrm{Mor}(\Sigma_-, \Sigma_+)$ such that :

1. for each three tuple of objects there are "composition" maps

$$\circ : \mathrm{Mor}(\Sigma_1, \Sigma_2) \times \mathrm{Mor}(\Sigma_2, \Sigma_3) \to \mathrm{Mor}(\Sigma_1, \Sigma_3)$$

   which are associative in the following sense : $(f \circ g) \circ h = f \circ (g \circ h)$ for all three tuple of morphisms which can be composed.

2. for each object Σ, $\mathrm{Mor}(\Sigma, \Sigma)$ contains a special morphism, called Id_Σ such that $f \circ \mathrm{Id}_\Sigma = f \forall f \in \mathrm{Mor}(\Sigma, \Sigma')$ (for any Σ') and similarly $\mathrm{Id}_\Sigma \circ g = g\ \forall g \in \mathrm{Mor}(\Sigma', \Sigma)$.

A category is *small* if both the objects and the morphisms form sets. The product of two categories $\mathscr{C}, \mathcal{D}$ is the category $\mathscr{C} \times \mathcal{D}$ whose objects are pairs $(\Sigma_1, \Sigma_2) \in Ob(\mathscr{C}) \times Ob(\mathcal{D})$ and whose morphisms $Mor((\Sigma_1, \Sigma_2), (\Sigma_1', \Sigma_2')) = Mor_{\mathscr{C}}(\Sigma_1, \Sigma_1') \times Mor_{\mathcal{D}}(\Sigma_2, \Sigma_2')$.

Definition A.2 (Isomorphisms)**.** A morphism $f \in Mor(\Sigma, \Sigma')$ is epic if for all Σ'' and for all $g, g' \in \mathrm{Mor}(\Sigma', \Sigma'')$ it holds $g \circ f = g' \circ f \implies g = g'$. It is monic if for all Σ'' and for all $g, g' \in \mathrm{Mor}(\Sigma'', \Sigma)$ it holds $f \circ g = f \circ g' \implies g = g'$. It is an isomorphism if it exists $f^{-1} \in Mor(\Sigma',, \Sigma)$ such that $f^{-1} \circ f = Id_\Sigma$ and $f \circ f^{-1} = Id_{\Sigma'}$.

If f is an isomorphism then it is both epic and monic : indeed for instance if $g, g' \in Mor(\Sigma', \Sigma'')$ are such that $g \circ f = g' \circ f$ then $g \circ f \circ f^{-1} = g' \circ f \circ f^{-1} \implies g = g'$. It is not true that if f is monic and epic then it is an isomorphism : consider a category with two objects and a single morphism $f \in Mor(\Sigma, \Sigma')$ and only $Id_\Sigma, \mathrm{Id}_{\Sigma'}$ (no morphism in $Mor(\Sigma', \Sigma)$); then it is clearly epic and monic but not an iso.

Definition A.3 (Functors)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is a map assigning to each object Σ of $\mathcal{C}$ an object $F(\Sigma)$ of $\mathcal{D}$ and to each $f \in \mathrm{Mor}(\Sigma_-, \Sigma_+)$ a morphism $F(f) \in \mathrm{Mor}(F(\Sigma_-), F(\Sigma_+))$ such that $F(g \circ f) = F(g) \circ F(f)$ (whenever $g \circ f$ exists) and $F(Id_\Sigma) = Id_{F(\Sigma)},\ \forall \Sigma$. A functor $F : \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if for each $W \in \mathcal{D}$ there exists $V \in \mathcal{C}$ such that W is isomorphic to $F(V)$. It is *faithful* (resp. *fully faithful*) if for each pair of objects $V, V' \in \mathcal{C}$ the map $F : Mor(V, V') \to Mor(F(V), F(V'))$ is injective (resp. bijective).

Definition A.4 (Natural transformations)**.** A natural transformation between a functor $F : \mathcal{C} \to \mathcal{D}$ and a functor $G : \mathcal{C} \to \mathcal{D}$ is a map $n : Obj(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$ such that $n(\Sigma) \in \mathrm{Mor}(F(\Sigma), G(\Sigma))$ $\forall \Sigma \in Obj(\mathcal{C})$ and $n(\Sigma') \circ F(f) = G(f) \circ n(\Sigma) \forall f \in \mathrm{Mor}(\Sigma, \Sigma')$; it is a *natural isomorphism* if $\eta(\Sigma)$ is an isomorphism for each Σ.

Two categories are *equivalent* if there exist functors $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ such that there exist natural isomorphisms between $F \circ G$ and $Id_{\mathcal{D}}$ and $G \circ F$ and $Id_{\mathcal{C}}$.

Proposition A.5 ([27] Proposition XI.1.5)**.** *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories iff it is essentially surjective and fully faithful.*

A category is *essentially small* if it is equivalent to a small one.

A category $\mathcal{C}$ is an *Ab* category if for each Σ, Σ' the collection $Mor(\Sigma, \Sigma')$ is an abelian group and the composition is bilinear with respect to the group operation; it is k-linear if $Mor(\Sigma, \Sigma')$ is a k-vector space where k is a fixed field.

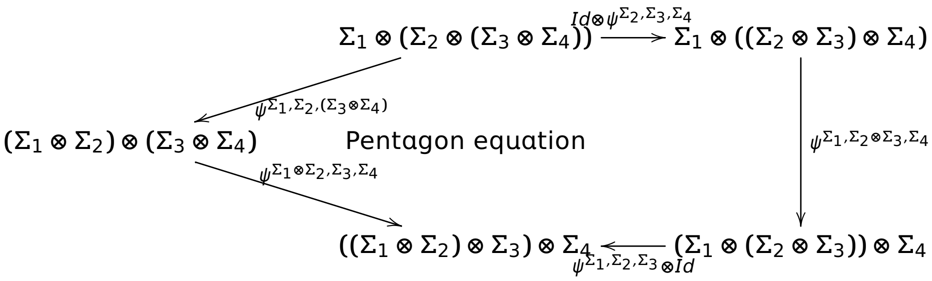
A.1. Monoidal categories and functors

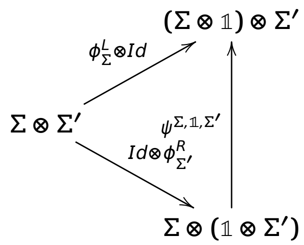
Definition A.6 (Monoidal category)**.** A monoidal category is a category $\mathscr{C}$ equipped with a tensor product bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object denoted $\mathbb{1}$ such that :

1. For each Σ there exists natural isomorphisms $\phi^L_\Sigma : \Sigma \to \Sigma \otimes \mathbb{1}$ and $\phi^R_\Sigma : \Sigma \to \mathbb{1} \otimes \Sigma$;

2. For each objects $\Sigma, \Sigma', \Sigma''$ there exists natural isomorphisms $\psi^{\Sigma,\Sigma',\Sigma''} : \Sigma \otimes (\Sigma' \otimes \Sigma'') \to (\Sigma \otimes \Sigma') \otimes \Sigma''$.

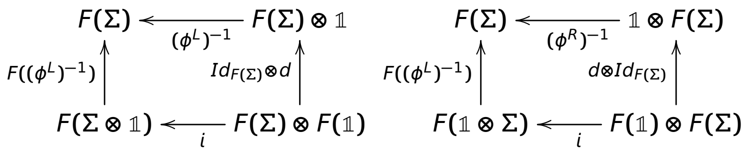
(Here naturality means that for all morphisms $f \in Mor(\Sigma_0, \Sigma), g \in Mor(\Sigma'_0, \Sigma'), h \in Mor(\Sigma''_0, \Sigma'')$ it holds $\phi^R \circ f = (Id \otimes f) \circ \phi^R$, $\phi^L \circ f = (f \otimes Id) \circ \phi^L$, and $\psi^{\Sigma,\Sigma',\Sigma''} \circ (f \otimes (g \otimes h)) = ((f \otimes g) \otimes h) \circ \psi^{\Sigma_0,\Sigma'_0,\Sigma''_0}$.) Such that $\phi^R_{\mathbb{1}} = \phi^L_{\mathbb{1}}$ and for all objects the following pentagon diagrams commute :

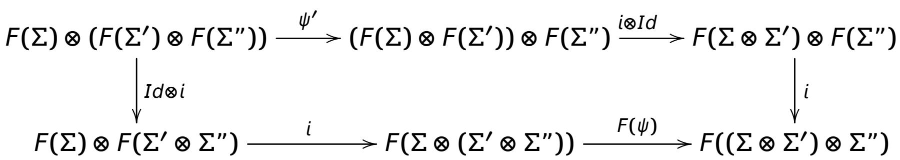




The category is *strict* if $\mathbb{1} \otimes \Sigma = \Sigma = \Sigma \otimes \mathbb{1}$ and $\phi^L_\Sigma = \phi^R_\Sigma = Id_\Sigma$ for all $\Sigma \in Ob(\mathscr{C})$, and finally for each three objects $\Sigma, \Sigma', \Sigma''$ it holds $\Sigma \otimes (\Sigma' \otimes \Sigma'') = (\Sigma \otimes \Sigma') \otimes \Sigma''$ and $\psi^{\Sigma,\Sigma',\Sigma''} = Id_{\Sigma \otimes \Sigma' \otimes \Sigma''}$.

Definition A.7 (Lax monoidal functors)**.** A lax monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between monoidal categories is a functor such that there exist a natural morphism $d : F(\mathbb{1}) \to \mathbb{1}$ and for all objects Σ, Σ' there exist natural morphisms $i_{\Sigma,\Sigma'} : F(\Sigma) \otimes F(\Sigma') \to F(\Sigma \otimes \Sigma')$ which commute with all the associators and identity morphisms, i.e. $\forall \Sigma, \Sigma'$, $\forall f \in \mathrm{Mor}(\Sigma, \Sigma), f' \in \mathrm{Mor}(\Sigma', \Sigma')$ the following holds:





where we denoted ψ (resp. ψ') the associator in $\mathcal{C}$ (resp. in $\mathcal{D}$). A lax monoidal functor F is *monoidal* if d, i are isomorphisms, and it is a *strict monoidal functor* if $F(\mathbb{1}) = \mathbb{1}$ and for each object Σ, Σ' of $\mathscr{C}$ it holds $F(\Sigma \otimes \Sigma') = F'(\Sigma) \otimes F(\Sigma')$ and the corresponding maps d, i are Id.

Definition A.8 (Natural transformations of lax monoidal functors)**.** Let $\mathcal{C}, \mathcal{D}$ be two monoidal categories and $F, F' : \mathcal{C} \to \mathcal{D}$ be two lax monoidal functors. A natural tensor transformation $n : F \to F'$ is a natural transformation $n : F \to F'$ such that the following diagrams commute for every couple of objects $U, V \in \mathcal{C}$:

$$\begin{array}{ccc} F'(\mathbb{1}) & & \\ {\scriptstyle n}\uparrow & \searrow{\scriptstyle d} & \\ F(\mathbb{1}) & \xrightarrow{d'} & \mathbb{1} \end{array} \qquad \begin{array}{ccc} F(U)\otimes F(V) & \xrightarrow{i} & F(U\otimes V) \\ \downarrow{\scriptstyle n\otimes n} & & \downarrow{\scriptstyle n} \\ F'(U)\otimes F'(V) & \xrightarrow{i} & F'(U\otimes V) \end{array}$$

A natural tensor transformation $n : F \to F'$ is a natural tensor isomorphism if it is a natural isomorphism (see the end of Definition A.4). A tensor equivalence F between monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a tensor functor $F : \mathcal{C} \to \mathcal{D}$ such that there exists a tensor functor $G : \mathcal{D} \to \mathcal{C}$ and natural tensor isomorphisms $n : G \circ F \to Id_{\mathcal{C}}$ and $n' : F \circ G \to Id_{\mathcal{D}}$.

From now on, when speaking of functors between monoidal categories we will always mean lax monoidal ones and we will suppress the word “tensor”.

A.2. Braidings

Definition A.9 (Braided category)**.** A braiding on a monoidal category $\mathscr{C}$ is the datum of natural isomorphisms for every pair of objects $\Sigma, \Sigma' \in Ob(\mathscr{C})$ $b_{\Sigma,\Sigma'} : \Sigma \otimes \Sigma' \to \Sigma' \otimes \Sigma$ such that the following diagrams (known as “Hexagon equations") commute :

$$\begin{array}{ccccc} \Sigma\otimes(\Sigma'\otimes\Sigma'') & \xrightarrow{b_{\Sigma,(\Sigma'\otimes\Sigma'')}} & (\Sigma'\otimes\Sigma'')\otimes\Sigma & \xrightarrow{\psi^{-1}} & \Sigma'\otimes(\Sigma''\otimes\Sigma) \\ \downarrow{\scriptstyle\psi} & & & & \downarrow{\scriptstyle Id\otimes b_{\Sigma'',\Sigma}} \\ (\Sigma\otimes\Sigma')\otimes\Sigma'' & \xrightarrow{b_{\Sigma,\Sigma'}\otimes Id} & (\Sigma'\otimes\Sigma)\otimes\Sigma'' & \xrightarrow{\psi^{-1}} & \Sigma'\otimes(\Sigma\otimes\Sigma'') \end{array}$$

$$\begin{array}{ccccc} (\Sigma\otimes\Sigma')\otimes\Sigma'' & \xrightarrow{b_{(\Sigma\otimes\Sigma''),\Sigma''}} & \Sigma''\otimes(\Sigma\otimes\Sigma') & \xrightarrow{\psi} & (\Sigma''\otimes\Sigma)\otimes\Sigma' \\ \downarrow{\scriptstyle\psi^{-1}} & & & & \downarrow{\scriptstyle b_{\Sigma'',\Sigma}\otimes Id_{\Sigma'}} \\ \Sigma\otimes(\Sigma'\otimes\Sigma'') & \xrightarrow{Id_\Sigma\otimes b_{\Sigma',\Sigma''}} & \Sigma\otimes(\Sigma''\otimes\Sigma') & \xrightarrow{\psi} & (\Sigma\otimes\Sigma'')\otimes\Sigma'. \end{array}$$

A braided category is a monoidal category equipped with a braiding. If for each pair of objects $\Sigma, \Sigma' \in \mathcal{C}$ it holds $b_{\Sigma',\Sigma} \circ b_{\Sigma,\Sigma'} = Id_{\Sigma\otimes\Sigma'}$ then the braiding is also called a *symmetry* and $\mathcal{C}$ is a *symmetric monoidal category*.

Remark A.10. As proved in [27], Proposition XIII 1.2, the following diagrams always commute in a braided category :

$$\begin{array}{ccc} \Sigma\otimes\mathbb{1} & \xrightarrow{b_{\Sigma,\mathbb{1}}} & \mathbb{1}\otimes\Sigma \\ {\scriptstyle\phi^L}\uparrow & & \uparrow{\scriptstyle\phi^R} \\ \Sigma & \xrightarrow{Id} & \Sigma \end{array} \qquad \begin{array}{ccc} \mathbb{1}\otimes\Sigma & \xrightarrow{b_{\mathbb{1},\Sigma}} & \Sigma\otimes\mathbb{1} \\ {\scriptstyle\phi^R}\uparrow & & \uparrow{\scriptstyle\phi^L} \\ \Sigma & \xrightarrow{Id} & \Sigma \end{array}$$

.

Furthermore when $\mathscr{C}$ is strict the commutativity of the hexagon diagrams is equivalent to the following equalities :

$$b_{\Sigma,\Sigma'\otimes\Sigma''} = (Id_{\Sigma'} \otimes b_{\Sigma,\Sigma''}) \circ (b_{\Sigma,\Sigma'} \otimes Id_{\Sigma''}) \qquad b_{\Sigma'\otimes\Sigma'',\Sigma} = (b_{\Sigma',\Sigma} \otimes Id_{\Sigma''}) \circ (Id_{\Sigma'} \otimes b_{\Sigma'',\Sigma}).$$

Definition A.11 (Braided functors)**.** A braided functor $F : \mathscr{C} \to \mathcal{D}$ between braided monoidal categories is a lax monoidal functor F such that for all the objects of $\mathscr{C}$ the following diagram commutes :

$$\begin{array}{ccc} F(\Sigma)\otimes F(\Sigma') & \xrightarrow{b_{F(\Sigma),F(\Sigma')}} & F(\Sigma')\otimes F(\Sigma) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ F(\Sigma\otimes\Sigma') & \xrightarrow{F(b_{\Sigma,\Sigma'})} & F(\Sigma'\otimes\Sigma) \end{array}$$

Theorem A.12. *Let $\mathscr{C}$ be a braided category. Then there exists a strict braided category $\mathscr{C}^{str}$ and a monoidal equivalence $F:\mathscr{C}\to\mathscr{C}^{str}$ which is also a braided functor.*

Proof. It is Maclane's coherence theorem. See [27] Proposition XI.5.1 and Exercice XIII.6.5 or [46] Chapter XI, Remark 1.4. A.12

A.3. Pivotal categories

Because of Theorem A.12 we will from now on assume that all the monoidal categories are strict.

Definition A.13 (Left and right duality)**.** A left duality on a strict monoidal category $\mathcal{C}$ is the datum for every object Σ of $\mathcal{C}$ of a *left dual object* Σ^* and morphisms $\overrightarrow{ev}_\Sigma:\Sigma^*\otimes\Sigma\to\mathbb{1}$, $\overrightarrow{coev}:\mathbb{1}\to\Sigma\otimes\Sigma^*$ such that the following "triangular equalities" hold :

$$(Id_\Sigma\otimes\overrightarrow{ev}_\Sigma)\circ(\overrightarrow{coev}_\Sigma\otimes Id_\Sigma)=Id_\Sigma \qquad (\overrightarrow{ev}_\Sigma\otimes Id_{\Sigma^*})\circ(Id_{\Sigma^*}\otimes\overrightarrow{coev}_\Sigma)=Id_{\Sigma^*}.$$

If $f\in\mathrm{Mor}(\Sigma_1,\Sigma_2)$ the *left adjoint* of f, denoted $f^*\in\mathrm{Mor}(\Sigma_2^*,\Sigma_1^*)$ is the morphism defined as:

$$f^*:=(\overrightarrow{ev}_{\Sigma_2}\otimes Id_{\Sigma_1^*})\circ(Id_{\Sigma_2^*}\otimes f\otimes Id_{\Sigma_1^*})\circ(Id_{\Sigma_2^*}\otimes\overrightarrow{coev}_{\Sigma_1}).$$

Similarly a right duality on a strict monoidal category $\mathcal{C}$ is the datum for every object Σ of $\mathcal{C}$ of a *right dual object* ${}^*\Sigma$ and morphisms $\overleftarrow{ev}_\Sigma:\Sigma\otimes({}^*\Sigma)\to\mathbb{1}$, $\overleftarrow{coev}:\mathbb{1}\to({}^*\Sigma)\otimes\Sigma$ such that the following "triangular equalities" hold:

$$(\overleftarrow{ev}_\Sigma\otimes Id_\Sigma)\circ(Id_\Sigma\otimes\overleftarrow{coev}_\Sigma)=Id_\Sigma \qquad (Id_{({}^*\Sigma)}\otimes\overleftarrow{ev}_\Sigma)\circ(\overleftarrow{coev}_\Sigma\otimes Id_{({}^*\Sigma)})=Id_{({}^*\Sigma)}.$$

The right adjoint of $f\in\mathrm{Mor}(\Sigma_1,\Sigma_2)$ is the morphism $({}^*f)\in\mathrm{Mor}({}^*\Sigma_2,{}^*\Sigma_1)$ defined as:

$$({}^*f):=(Id_{({}^*\Sigma_1)}\otimes\overleftarrow{ev}_{\Sigma_2})\circ(Id_{({}^*\Sigma_1)}\otimes f\otimes Id_{({}^*\Sigma_2)})\circ(\overleftarrow{coev}_{\Sigma_1}\otimes Id_{({}^*\Sigma_2)}).$$

If $\mathcal{C}$ has both left and right dualities, then it is called *autonomous*.

Remark A.14. It can be proven (exercise!) that the left (resp. right) dual object, if it exists, is unique up to isomorphism. Furthermore it is important to observe that the existence of a dual object for $\Sigma\in\mathcal{C}$ is a property of V and not an additional structure one defines on $\mathcal{C}$. Finally it can be proven that if $\mathcal{C}$ is autonomous then, each $V\in\mathcal{C}$ is isomorphic to both ${}^*(V^*)$ and $({}^*V)^*$. But in general it is not true that $(V^*)^*$ is isomorphic to V.

Let $\mathcal{C}^{op}$ be the category whose objects are those of $\mathcal{C}$ and morphisms are $Mor^{op}(\Sigma_1,\Sigma_2)=Mor(\Sigma_2,\Sigma_1)$. Equip it with a strict monoidal structure given by $V\otimes^{op}W:=W\otimes V$. Then if $\mathcal{C}$ has a left duality, the "left dual functor" : $L:\mathcal{C}\to\mathcal{C}^{op}$ associating to each object its left dual and to each morphism its left adjoint is a monoidal functor indeed the map $i_{\Sigma_1,\Sigma_2}:L(\Sigma_1)\otimes^{op}L(\Sigma_2)=\Sigma_2^*\otimes\Sigma_1^*\to L(\Sigma_1\otimes\Sigma_2)=(\Sigma_1\otimes\Sigma_2)^*$ is given by:

$$i_{\Sigma_1,\Sigma_2}:=(\overrightarrow{ev}_{\Sigma_2}\otimes Id_{(\Sigma_1\otimes\Sigma_2)^*})\circ(Id_{\Sigma_2^*}\otimes\overrightarrow{ev}_{\Sigma_1}\otimes Id_{\Sigma_2\otimes(\Sigma_1\otimes\Sigma_2)^*})\circ(Id_{\Sigma_2^*\otimes\Sigma_1^*}\otimes\mathrm{coev}_{\Sigma_1\otimes\Sigma_2}).$$

Similarly for the right dual functor $R:\mathcal{C}\to\mathcal{C}^{op}$.

Definition A.15 (Pivotal categories)**.** An autonomous category is pivotal if the left and right duality functors coincide.

A.4. Ribbon categories

Definition A.16. A strict, braided category $\mathcal{C}$ with left duality is *ribbon* if it is endowed with a natural family of isomorphisms $\theta_\Sigma : \Sigma \to \Sigma,\ \forall \Sigma \in Ob(\mathcal{C})$ such that for all $\Sigma_1, \Sigma_2 \in \mathcal{C}$ it holds :

$$\theta_{\Sigma_1 \otimes \Sigma_2} = (\theta_{\Sigma_1} \otimes \theta_{\Sigma_2}) \circ b_{\Sigma_2,\Sigma_1} \circ b_{\Sigma_1,\Sigma_2}$$

and $\theta_{\Sigma^*} = (\theta_\Sigma)^*$. (The naturality of the isomorphisms means that for each $f \in \mathrm{Mor}(\Sigma_1, \Sigma_2)$ it holds $\theta_{\Sigma_2} \circ f = f \circ \theta_{\Sigma_1}$.)

In a ribbon category $\mathcal{C}$ one can define a right duality by stipulating that for each $\Sigma \in Ob(\mathcal{C})$ it holds $({}^*\Sigma) = \Sigma^*$ and defining $\overleftarrow{ev}_\Sigma := \overrightarrow{ev}_\Sigma \circ b_{\Sigma,\Sigma^*} \circ (\theta_\Sigma \otimes Id_{\Sigma^*})$ and $\overleftarrow{coev}_\Sigma := (Id_{\Sigma^*} \otimes \theta_\Sigma) \circ b_{\Sigma,\Sigma^*} \circ \overrightarrow{coev}_\Sigma$ (for a proof that these morphisms do indeed define a right duality on $\mathcal{C}$ see [27] Proposition XIV.3.5). Hence each ribbon category is autonomous; it can actually be proven that it is also pivotal.

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