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Abstract

This note is based on the lectures that I have given during the winter school *Winter Braids IV, School on algebraic and topological aspects of braid groups* held in Dijon on 10 - 13 February 2014. The aim of series of three lectures was to give an overview of geometrical and topological properties of 4-dimensional Lefschetz fibrations. Meanwhile, I could briefly introduce real Lefschetz fibrations, fibrations which have certain symmetry, and could present some interesting features of them.

This note will be yet another survey article on Lefschetz fibrations. There are excellent lecture notes/ survey papers/ book chapters on Lefschetz fibrations. You can, for example, look at [3], [11], [14], [20] among many others. In this note I intent to take my time on real Lefschetz fibrations as much as on Lefschetz fibrations in order not to repeat what was already done perfectly.

1. Lefschetz fibrations

Let X^4 be a smooth, connected, compact, oriented 4-manifold; likewise let B be a smooth, connected, compact, oriented 2-manifold, (in this note B will mostly be taken as S^2 or D^2). A very rough definition of a Lefschetz fibration would be that it is a complex Morse function. Precisely, a Lefschetz fibration on X is a smooth projection $\pi : X^4 \to B$ with $\pi(\partial X) = \partial B$, and such that there are finitely many critical points in the interior of X with distinct images around which one can find complex charts $(U \subset X, U' \subset \mathbb{C}^2, \phi_U : U \to U')$ and $(V \subset B, V' \subset \mathbb{C}, \Phi_V : V \to V')$ such that $\phi_V \circ \pi \circ \phi_U^{-1}(z_1, z_2) = z_1^2 + z_2^2$. Two Lefschetz fibrations will be considered to be *isomorphic* if there exist orientation preserving diffeomorphisms $H : X \to X$ and $h : B \to B$ such that $\pi \circ H = h \circ \pi$.

By definition fibers over the regular values (called *regular fibers*) are compact closed connected oriented surfaces of some fixed genus g (this follows basically from the inverse mapping theorem). Sometimes we use the term *genus-g Lefschetz fibration* to refer to Lefschetz fibrations whose regular fibers are of genus g. We call the fibers over critical values the *singular fibers*, (Figure 1.1 schematically illustrates a Lefschetz fibration).

Before proceeding further we note that the assumption of distinct critical values is insignificant as we can always perturb (small enough) the projection so that π becomes injective on the critical set.

Remark 1.1. As it will be useful in the study of local model, let us note that the notion of Lefschetz fibration can be slightly generalized to cover the case of fibrations whose fibers have non-empty boundary. In this case the boundary of X becomes naturally divided into two parts: the *vertical boundary* which is the inverse image $\pi^{-1}(\partial B)$, and the *horizontal boundary* which forms the trivial bundle $\partial F \times B$ where F denotes a generic fiber.

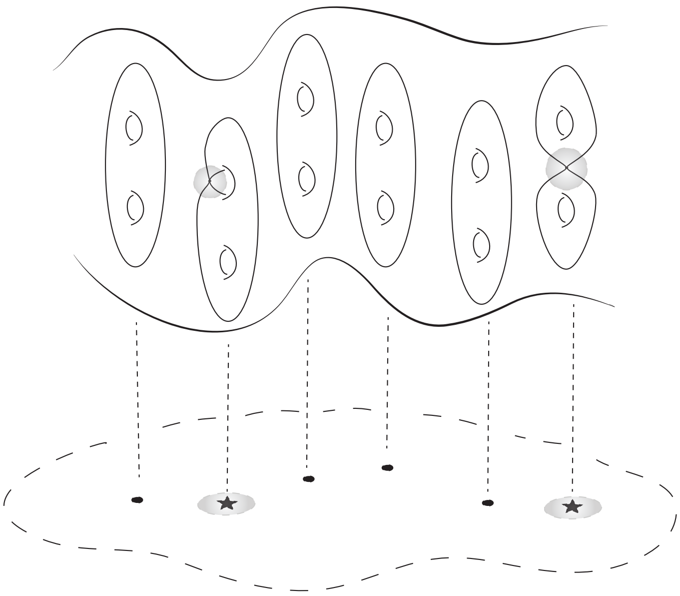

Figure 1.1

1.1. Lefschetz fibrations around singular fibers

As there is a fixed local model for a Lefschetz fibration around its critical points, we first want to understand what can be observed by looking at the local model and how the local model describes Lefschetz fibrations around singular fibers.

Without loss of generality, we will consider $U' = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1^2 + z_2^2| \leq \epsilon\}$ and $V' = \{z \in \mathbb{C} : |z| \leq \epsilon^2, 0 < \epsilon < 1\}$ so that around the critical point and the corresponding critical value the projection looks like

$$\begin{array}{cccc} \pi' : & U' & \to & V' \\ & (z_1, z_2) & \to & z_1^2 + z_2^2. \end{array}$$

This fibration has one critical point at $(0,0) \in U'$ over $0 \in V'$. The fiber over 0 defined locally by the equation $z_1^2 + z_2^2 = (z_1 + iz_2)(z_1 - iz_2) = 0$. This equation defines two complex planes intersecting transversally at one point. Thus, locally we see a node as a singularity. Meanwhile, over any other point $r' \in \mathbb{C} \setminus \{0\}$, the fiber is defined by the quadratic equation $z_1^2 + z_2^2 = r'$ which defines a cylinder, see Figure 1.2.

As we all know monodromy is an essential information to understand the fibration around its singular fiber. The monodromy of a fibration $\pi : X \to B$ is the morphism

$$\mu_\pi : \pi_1(B \setminus \Delta, b) \to Map(F_b)$$

where Δ is the set of critical values and $Map(F_b)$ is the mapping class group, the group of isotopy classes of orientation preserving diffeomorphisms of the fiber F_b over a regular value b. (Let us note that in order the monodromy map to be a homomorphism, on $Map(F_b)$ we consider the product operation defined as $fg = g \circ f$.)

By fixing a diffeomorphism $\psi : \Sigma_g \to F_b$ where Σ_g is an abstract surface of genus-g, one can consider the monodromy homomorphism in terms of Σ_g. In this case we consider

$$\begin{array}{cccc} \psi_* : & Map(\Sigma_g) & \to & Map(F_b) \\ & f & \to & \psi \circ f \circ \psi^{-1} \end{array}$$

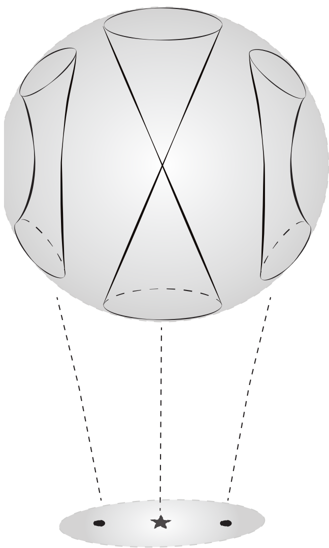

Figure 1.2

so that the monodromy map is defined as

$$\psi_*^{-1} \circ \mu_\pi : \pi_1(B \setminus \Delta, b) \to Map(\Sigma_g).$$

There is a simple way to understand the monodromy of π using $\pi' : U' \to V'$ and by considering a branched covering description of the fibers of π' using an auxiliary projection $p_1 : \mathbb{C}^2 \to \mathbb{C}_{z_1}$ $p_1 : (z_1, z_2) \to z_1$. For such a projection, let us define the fiber $F'_{w'}$, $w' \in \mathbb{C}$ of π' as the double branch cover $z_2^2 = z_1^2 - w'$ of $\mathbb{C}_{z_1}$ with branching set satisfying $z_1^2 - w' = 0$. As the only critical value is 0, we need to understand what happens to the fiber as we go around once along a simple closed curve surrounding the origin positively. Let us consider the loop $\gamma(\theta) = \{w' = \delta e^{i\theta} \in \mathbb{C} : \theta \in [0, 2\pi], 0 < \delta << 1\}$. It is easy to see that when θ moves from 0 to 2π, the two points satisfying $z_1^2 = w'$ switch their places by turning on the positive direction on $\mathbb{C}_{z_1}$ (that by the way corresponds to the generator of the braid group with 2 strings as it defines a diffeomorphism of $\mathbb{C}_{z_1} \setminus \{2 \text{ points}\}$) whose support is a sufficiently small disk $D \in \mathbb{C}_{z_1}$ containing the branching points (points satisfying $z_1^2 = w'$). To see the effect of this on the fiber $F'_{w'}$, $w' \in \mathbb{C} \setminus \{0\}$, let us denote by $\alpha' \subset F'_{w'}$ the inverse image by p_1 of the line in $D \in \mathbb{C}_{z_1}$ connecting the two branching points (see Figure 1.3) and let $\nu(\alpha') = p_1^{-1}(D)$, a sufficiently small closed neighborhood $\alpha' \in F'_{w'}$. We identify $\nu(\alpha')$ with $S^1 \times [0, 1]$. When θ make a full turn what we observe on $S^1 \times [0, 1]$ is the diffeomorphism (which is identity on the boundary $S^1 \times \{0, 1\}$) that we refer as *(positive) local Dehn twist*:

$$\begin{array}{ccc} S^1 \times [0,1] & \to & S^1 \times [0,1] \\ (\theta, t) & \to & (\theta + 2\pi t, t). \end{array}$$

Now, let $\alpha = \Phi_U^{-1}(\alpha')$ and $\nu(\alpha) \subset F_b$ denote the inverse image of the neighborhood $\nu(\alpha') \subset F'_{b'}$ by the chart map Φ_U. Then the *(positive) Dehn twist* on F_b is defined as the extension by identity of the local Dehn twist to the outside of $\nu(\alpha)$ (such an extension is meaningful as the local Dehn twists is the identity on the boundary of $\nu(\alpha)$). We will use the notation t_α for the positive Dehn twist along the curve α, and negative Dehn twist along the same curve will then be the inverse, t_α^{-1}.) It might be useful to note here that Dehn twists are important diffeomorphisms since their isotopy classes generate the mapping class group [8]. (Please note also that we will keep the same notation for both the honest diffeomorphism and its

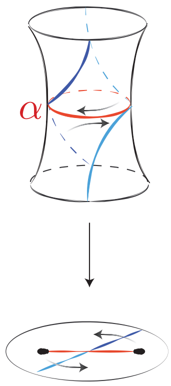

Figure 1.3

isotopy class. Recall that when isotopy classes are considered, the product notation will be used instead of composition.)

Remark 1.2. If the curve α bounds a disk on the surface, the Dehn twist along α is isotopic to identity; hence, it is the identity element of the mapping class group. This is because every diffeomorphism of D^2 fixed on $\partial D^2 = S^1$ is isotopic to identity. In this case, the singular fiber which is obtained by collapsing the vanishing cycle to a point has an embedded sphere. It is not hard to see that this sphere has self-intersection -1 (the calculation is very similar to the one we make for framing in Section 1.3). Fibrations without such phenomenon are called *relatively minimal* in the literature. We will always assume that fibrations are relatively minimal and do not use the terminology. This assumption does not cause any loss of generality. Indeed, a special care is needed only for the classification results, discussions of the next two sections apply for all Lefschetz fibrations.

By definition Dehn twists do not depend on the particular choice in the isotopy class of α. Therefore, it is determined by the isotopy class of α. Thus, to understand possible models of a neighborhood of a singular fiber of a Lefschetz fibration, we are interested in the classification (up to diffeomorphism) of the isotopy classes of essential simple closed curves on an oriented surface.

Topologically there are two types of simple closed curves: separating, non-separating. Separating (respectively non-separating) means that the complement of the curve has two pieces (respectively one piece). Up to diffeomorphism and isotopy there exists only one non-separating curve on an oriented surface. The number of separating ones, however, is determined by how the curve separates the genus of the surface. Thus, there are $\lfloor \frac{g}{2} \rfloor$ (integral part of $\frac{g}{2}$) separating curves. Hence, there are $[\frac{g}{2}]+1$ possible models for a neighborhood. Indeed each such model is realized. There are many ways to construct corresponding fibrations, one of which will be discussed later.

1.2. Factorization of the monodromy and Hurwitz equivalence classes

Our aim is to understand the image of the monodromy morphism for fibrations over D^2. For this purpose we want to choose a basis $(\gamma_1, \ldots, \gamma_n)$ for $\pi_1(D^2 \setminus \Delta, b)$ where $n = |\Delta|$ and b is the marked regular value.

Let us consider the set of rays r_i from the base point b passing through a critical value. Generically each ray will pass through at most one critical value (if not, we can always perturb the fibration slightly so that this would be the case.) The rays r_i are enumerated by the order at b considered in the positive direction. Now, around critical values we choose a set of sufficiently small disjoint loops, each encircling positively a unique critical value. Let γ_i be the homotopy class of the loop obtained by conjugated concatenation of the small disjoint loops with the portion of the ray between b and the point of the loop nearest to b (see Figure 1.4). By construction $\gamma_1 * \gamma_2 * \ldots * \gamma_n$ is homotopic to ∂D^2 with positive orientation.

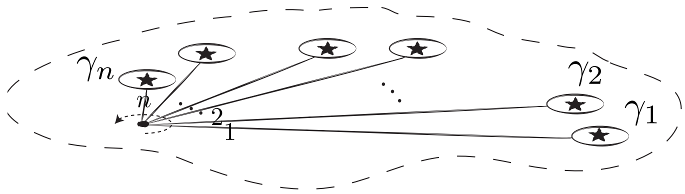

Figure 1.4

A basis obtained as above is called as *geometric basis* or *Artin basis*. By the construction, the choice of geometric basis is not unique. It is a classical result due to E. Artin [1] that two geometric bases are related by an element of the braid group $B_n = \langle \sigma_i, i = 1, \ldots n-1 : \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle$. The action is given by (see also Figure 1.5):

$$(\ldots, \gamma_i, \gamma_{i+1}, \ldots) \xrightarrow{\sigma_i} (\ldots, \gamma_i * \gamma_{i+1} * \gamma_i^{-1}, \gamma_i, \ldots).$$

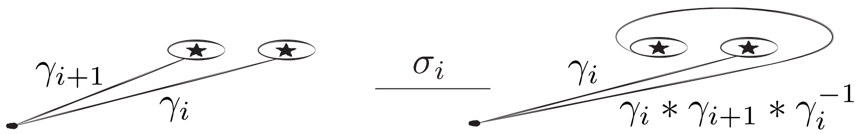

Figure 1.5

So by the above discussions, to each basis $(\gamma_1, \ldots, \gamma_n)$, we assign an n-tuple $(t_{\alpha_1}, \ldots, t_{\alpha_n})$ of Dehn twists such that $t_{\alpha_1}t_{\alpha_2}\ldots t_{\alpha_n} = \mu_\pi(\partial D^2)$. We call such an n-tuple a *monodromy factorization* or *n-factorization* of $\mu_\pi(\partial D^2)$. The ambiguity in choosing the basis $(\gamma_1, \ldots, \gamma_n)$ reads on n-factorization as

$$(\ldots, t_{\alpha_i}, t_{\alpha_{i+1}}, \ldots) \xrightarrow{\sigma_i} (\ldots, t_{\alpha_i}t_{\alpha_{i+1}}t_{\alpha_i}^{-1}, t_{\alpha_i}, \ldots).$$

We call the above change a *Hurwitz moves* and consider two factorizations *strong Hurwitz equivalent* if they are related by finite sequence of Hurwitz moves; we consider them *weak Hurwitz equivalent* if in addition we make a global conjugation by an orientation preserving diffeomorphism.

Theorem 1.3 (Kas, 1980, [15]). *There is bijection between the set of isomorphism classes of Lefschetz fibrations over D^2 and the set of (weak) Hurwitz equivalence classes of factorization of the monodromy along ∂D^2.*

Note that one direction of this theorem is clear from the above discussions. To give the idea for the other direction, we will look at the local model with a slightly different point view which we present in the next section. Theorem of A. Kas asserts the importance of the classification of Hurwitz equivalence classes of the monodromy factorization. However, classification of Hurwitz equivalence classes is a subtle question and the answer is not known in general. The case of $g = 1, n = 2$ is studied in [7] in which an explicit characterization of elements admitting 2-factorization is given as well as a complete classification of Hurwitz equivalence classes of product of two Dehn twists. According to main results, elements which admit 2-factorization have at most two Hurwitz equivalence classes and there is unique example in which one weak equivalence class constitutes two strong equivalence classes. In all the other cases weak and strong classes match [7].

1.3. Handle decomposition description

We now look at the fiber $F'_{t'} := \{z_1^2 + z_2^2 = t'\} \subset \mathbb{C}^2$ where t' is taken to be in $\mathbb{R} \subset \mathbb{C}$ and consider $\mathbb{R}^2 \cap \{z_1^2 + z_2^2 = t'\} \subset \mathbb{C}^2$. This intersection is the curve $x_1^2 + x_2^2 = t'$ where $x_j = Re(z_j), j = 1, 2$ and it bounds a disk $D_{t'}$ on $\mathbb{R}^2 \subset \mathbb{C}^2$. When t' approaches 0, this curve shrinks to a point creating the corresponding singular fiber. The curve is called the *vanishing cycle*. The vanishing cycle is exactly the curve $\alpha' \subset F'_{t'}$ described above and thus the monodromy around the corresponding singular fiber is the Dehn twist along the vanishing cycle.

The disk $D_{t'} \subset \mathbb{R}^2 \subset \mathbb{C}^2$ bounded by the vanishing cycle $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = t'\} \subset F'_{t'}$ is called *thimble*. A neighborhood of the thimble $D_{t'}$ can be viewed as $D^2 \times D^2 \subset \mathbb{R} + i\mathbb{R}^2 \subset \mathbb{C}^2$. We can interpret a copy of $D^2 \times D^2$ as a 4-dimensional 2-handle. As a matter of fact, a neighborhood of a singular fiber of a Lefschetz fibration can be constructed abstractly from $\Sigma_g \times D^2$ by attaching a 4-dimensional 2-handle $D^2 \times D^2$ on $\partial(\Sigma_g \times D^2)$, where Σ_g is an abstract surface diffeomorphic to the fiber over the base point chosen for describing the monodromy.

The attachment of the 2-handle $D^2 \times D^2$ is performed by choosing an embedding $\partial D^2 \times D^2 \to \partial(\Sigma_g \times D^2)$. Such an embedding can be encoded by an embedding $\varphi_0 : S^1 \times \{0\} \hookrightarrow \partial(\Sigma_g \times D^2)$ with its trivial normal bundle and an identification of the normal bundle $\nu(\varphi_0(S^1 \times \{0\}))$ with $S^1 \times \mathbb{R}^2$. The choice of the identification can be seen as an element of $\pi_1(SO(2), id)$ torsor, i.e. the difference of two choices is an element of $\pi_1(SO(2), id)$, so it can be identified by an element of $\mathbb{Z}$ (cf. [14]). Such a number is usually referred to as *framing*.

To understand how the framing should be chosen we look back to the local model. For this purpose, consider $\varphi_0 : S^1 \times \{0\} \hookrightarrow \partial(F'_{b'} \times D^2)$ such that $\varphi_0\left(S^1 \times \{0\}\right) = \alpha' \times \{b'\}$, and fix a basis $\langle (1,0,0,0), (0,1,0,0), (0,0,i,0), (0,0,0,i) \rangle$ of $\mathbb{R}^2 \times i\mathbb{R}^2$. Consider a point $p = (\sqrt{t'}\cos(\theta), \sqrt{t'}\sin(\theta), 0, 0)$ on $\alpha' \subset F'_{t'}$. Consider the vector field $\xi_p = (-\sin(\theta), \cos(\theta), 0, 0)$ tangent to $F'_{t'}$. Note that $i\xi_p$ is a vector field tangent to $F'_{t'}$ and normal to $T_p\alpha'$, thus $i\xi_p = (0, 0, -i\sin(\theta), i\cos(\theta))$ is a vector in the normal space of $\varphi_0(S^1 \times \{0\})$ at p. As dimension of the normal space is 2, it suffices for us to compare $i\xi_p$ with either one of the constant vector fields $(0,0,i,0)$ or $(0,0,0,i)$. Namely, we need the winding number of $i\xi_p$ with respect to, for example, the constant vector field $(0,0,0,i)$ which is easily calculated as 1. Thus, the framing is 1 with respect to the orientation of $\mathbb{R}^2 \times i\mathbb{R}^2$. However, as we have chosen orientation preserving charts, the orientation we are interested is the same as the one of $\mathbb{C}^2$ which is opposite to the orientation of $\mathbb{R}^2 \times i\mathbb{R}^2$. Thus, the framing we need to consider is indeed -1. (This calculation, as it is, is not my own idea, you can find a version of it in [20]).

As a conclusion, we see in this section that given an ordered set $(\alpha_1, \ldots, \alpha_n)$ of simple closed curves on $\Sigma_g \to F_b$, one can construct a Lefschetz fibration over D^2 by successive 2-handle attachments along α_i with framing -1 relative to the product framing on $\partial(\Sigma_g \times D^2)$. For certain choice of a basis of $\pi_1(D^2 \setminus \Delta, b)$, the fibration obtained this way has the monodromy factorization $(t_{\alpha_1}, \ldots, t_{\alpha_n})$.

1.4. Classification results of genus-g Lefschetz fibrations over S^2

Let us start with discussing the case of $g = 0$. As all simple closed curves on S^2 bound a disk (and we assume that vanishing cycles do not bound a disk on the fiber), all genus-0 Lefschetz fibrations are indeed S^2 bundles over S^2. It is worth mentioning here that there are two types of such bundles: the product bundle and the twisted product $S^2 \tilde{\times} S^2$. This is a classical result from [29] in which the classification of r-sphere bundles over s-sphere is given.

As for the higher genus fibrations we first note that the theorem of A. Kas can be modified to cover the case of Lefschetz fibrations over S^2. First of all if we have a fibration over S^2, we can chose a regular fiber, trivialize a small neighborhood of this fiber and consider the complement of this trivialize neighborhood to obtain Lefschetz fibration over D^2 whose monodromy along ∂D^2 is the identity. Conversely, if the monodromy along ∂D^2 of a fibration over D^2 is the identity, then we can extend this fibration to a fibration over S^2 by gluing a trivial fibration $\Sigma_g \times D^2$ along the boundary, $\Sigma_g \times S^1$. Such a gluing can be performed by a choice of S^1 family of orientation preserving diffeomorphisms $\Sigma_g \to \Sigma_g$. At a chosen point of S^1, one can fix the diffeomorphism to be the identity, so that the family can be interpreted as an element in $\pi_1(Diff_+(\Sigma_g), id)$ where $Diff_+(\Sigma_g)$ denotes the space of orientation preserving diffeomorphisms of Σ_g. Thus the different choice of the extension is determined by the homotopic properties of the space $Diff_+(\Sigma_g)$. Note that if such an extension is unique, then the Hurwitz equivalence classes of the factorization of the identity classify genus-g Lefschetz fibrations over S^2.

Indeed, we know:

Theorem 1.4 (Earle-Eells, 1969, [10])**.** *If $g \geq 2$, then the components of $Diff_+(\Sigma_g)$ are contractible. If $g = 1$, the identity component of $Diff_+(T^2)$ contains T^2 as a deformation retract.*

Therefore, in the case of $g > 1$ the extension is unique. Despite the fact that a special care is needed in the case of $g = 1$, it is the only case for which an explicit classification is known, that will be the subject of the next section.

1.4.1. Elliptic Lefschetz fibrations

In the sequel, genus-1 Lefschetz fibrations will be called an *elliptic Lefschetz fibrations* in reference to the elliptic surfaces that they are related to. We have the following classification result.

Theorem 1.5 (Moishezon-Livné, 1977, [18])**.** *Elliptic Lefschetz fibrations over S^2 are classified by the number of the critical values, which can only be a multiple of 12. Let $E(1)$ denote the isomorphism class of fibrations with 12 critical values. Then, the class $E(k)$ of fibrations with $12k$ critical values can be taken as the k-fold fiber sum $E(1)\sharp_F E(1)\cdots\sharp_F E(1)$.*

This theorem is presented in the book "Complex surfaces and connected sums of complex projective planes" by B. Moishezon [18]. The book has an appendix by R. Livné in which he present a classification of Hurwitz equivalence classes of n-factorization of the identity in $\mathrm{PSL}(2,\mathbb{Z})$. B. Moishezon modify this result to the group $\mathrm{SL}(2,\mathbb{Z})$. Indeed, the classes, $E(k)$, correspond exactly the n-factorization of the identity. The conclusion is that n can only be a multiple of 12, and for each n, there is a unique class. The theorem, thus, also implies that the problem of non-uniqueness of the extension of fibrations aver D^2 to fibrations over S^2 that we have discussed in the previous section is indeed not essential and there is a way to get rid of it. It is basically because of the fact that the monodromy group (by which we understand the image of μ_π) generates $Map(T^2)$, (see Lemma 7, [18]). As it will be used later, it is worth mentioning here that when the monodromy group generates the mapping class group, the fibration is said to have a *transitive monodromy*.

Now let us construct an elliptic fibration with only 12 critical values.

Consider two generic homogenous polynomials P_0 and P_1 of degree 3 defining two non-singular cubic curves C_0 and respectively C_1 in $\mathbb{C}P^2$. Topologically C_0 and C_1 are oriented surfaces of genus $1 = \frac{(3-1)(3-2)}{2}$ and generically they intersect at 9 distinct points. Consider the pencil of cubics defined by $t_0P_0 + t_1P_1$ where $t_0, t_1 \in \mathbb{C}$ and are not simultaneously zero. This pencil covers $\mathbb{C}P^2$ and defines a projection $\mathbb{C}P^2 \setminus \{C_0 \cap C_1\} \to \mathbb{C}P^1$ by $t_0P_0+t_1P_1 \to [t_0, t_1]$. Each fiber of this projection is a cubic curve punctured at 9 points of common intersection. By blowing up these nine points we get $\mathbb{C}P^2 \sharp 9\overline{\mathbb{C}P^2}$ together with the projection $\pi : \mathbb{C}P^2 \sharp 9\overline{\mathbb{C}P^2} \to$

$\mathbb{C}P^1$. For generic choice of P_i, singular fibers of π are of nodal type as prescribed in the local model of Lefschetz fibrations.

Let us now count the number of critical values by means of Euler characteristics of the concerning spaces. We have $\chi(\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}) = 12$ while regular fibers being tori have $\chi(T^2) = 0$. So regular fibers have no effect on the count of Euler characteristic. Singular fibers are obtained from tori by pinching a simple closed to a point. Topologically they are spheres with two distinct points identified, so they have Euler characteristic 1. Therefore, $\pi : \mathbb{C}P^2\#9\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ has 12 singular fibers. This fibration will be the model we have in mind when we refer to the class $E(1)$.

To find a representative of the class $E(k)$, we perform k-fold fiber sum of $E(1)$. Recall that the fiber sum is a connected sum which respects the fiber structure. To perform a fiber sum we take out a sufficiently small neighborhood of a regular fiber which can be identified with $T^2 \times D^2$ considered with its projection $T^2 \times D^2 \to D^2$. Then we glue along the boundaries with a fiber preserving orientation reserving diffeomorphism.

As mentioned above elliptic Lefschetz fibrations can be considered as special elliptic surfaces. In that sense it might be interesting to investigate whether or not there is an algebraic realization or not. The theorem of B. Moishezon and R. Livné implicitly implies that each class of elliptic Lefschetz fibration can indeed be realized algebraically as we can find explicit model for each class. Consider, for example, a generic homogeneous polynomial $Q_k(z_0, z_1, z_2, t_0, t_1)$ of bi-degree $(3, k)$ in variable $(z_0, z_1, z_2, t_0, t_1,)$ homogeneous of degree k in (t_0, t_1) and of degree 3 in (z_0, z_1, z_2). For $k = 1$, for example, we have $Q_1 = t_0P_0(z_0, z_1, z_2) + t_1P_1(z_0, z_1, z_2)$ where $P_i, i = 1, 2$ homogenous polynomials of degree 3. The polynomial Q_k defines an algebraic subset of $\mathbb{C}P^2 \times \mathbb{C}P^1$ which we consider as the total space of $E(k)$. The projection is, then, obtained from the restriction of the projection to the second factor $\mathbb{C}P^2 \times \mathbb{C}P^1 \to \mathbb{C}P^1$. Generically, this projection defines a genus-g Lefschetz fibration with $12k$ singular fibers.

1.4.2. Genus-$g \geq 2$ Lefschetz fibrations

Although, in the literature some interesting features of genus-$g > 2$ Lefschetz fibrations can be found (e.g. [2], [16]), the complete classification is not yet known. Meanwhile, the case of $g = 2$ is quite well-understood by works of B. Siebert and G. Tian [25, 26] as well as K. N. Chakiris [5] and I. Smith [27] .

Let us first consider the following relations in the mapping class group of a genus-2 surface (note also that these relations can be generalized to any genus).

$$\begin{aligned} &(A) \quad (t_1t_2t_3t_4t_5t_5t_4t_3t_2t_1)^2 = \mathrm{id}, \\ &(B) \quad (t_1t_2t_3t_4t_5)^6 = \mathrm{id}, \\ &(C) \quad (t_1t_2t_3t_4)^{10} = \mathrm{id}, \end{aligned}$$

where $t_i, i = 1, \ldots, 5$ denotes the Dehn twist along the curves shown in Figure 1.6.

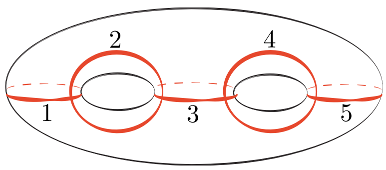

Figure 1.6:

By the theorem of A. Kas, the above relations correspond to a Lefschetz fibration over S^2. These fibrations are indeed building blocks of certain class of genus-2 Lefschetz fibrations by the following theorems.

Namely we have:

Theorem 1.6 (Siebert-Tian, 2005, [26])**.** *Let $\pi : X \to S^2$ be a genus-2 Lefschetz fibration with transitive monodromy. If there is no separating vanishing cycle then π is isomorphic to a holomorphic Lefschetz fibration.*

The first version of the below theorem appeared in [5] and later rediscovered by I. Smith [27] independently.

Theorem 1.7 (Chakiris, 1983 [5]; Smith, 1999 [27])**.** *Assume that a holomorphic genus-2 fibration has no separating vanishing cycle. Then, it is a fiber sum of the shape $A^m B^n = 1$ or $C^r = 1$ where m, n, r are non-negative integers and where A, B, C are the words presented above.*

Combining the above two theorems, we conclude that genus-2 Lefschetz fibrations with transitive monodromy and without separating vanishing cycles can be written as fiber sum of the three fixed models given above. The condition that there is no separating vanishing cycle is essential as it is known that there are infinitely many non-holomorphic genus-2 Lefschetz fibrations (see [19],[28]).

Note that in the case of $g = 1$ there is no essential separating curve and the monodromy happens to be always transitive, in that sense we may interpret the above conclusion as the generalization of the theorem of B. Moishezon and R. Livné.

2. Real Lefschetz fibrations

Intuitively, real structures are topological generalizations of the complex conjugation on complex algebraic varieties defined over the reals. On an oriented 4-manifold, a real structure is an orientation preserving involution whose fixed point set is of dimension 2 -if it is not empty- while on an oriented surface it is an orientation reversing involution (and its fixed point set is either of dimension 1 or empty). We consider real structures up to conjugation by an orientation preserving diffeomorphism.

Real Lefschetz fibrations are Lefschetz fibrations where the total and base space have real structures compatible with the fibration. Namely, for real structures $c_X : X \to X$ and $c_B : B \to B$ the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "c_X"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\
B \arrow[r, "c_B"] & B.
\end{tikzcd}$$

Real Lefschetz fibrations appear, for instance, as blow-ups of pencils of hyperplane sections of complex projective algebraic surfaces defined by real polynomial equations. For example, in the construction of $E(1)$ presented previously if the polynomials P_0 and P_1 are chosen to have real coefficients, the construction yield a real $E(1)$.

We consider two real Lefschetz fibrations $\pi : X \to B$ and $\pi' : X' \to B'$ to be *isomorphic* if there exist orientation preserving diffeomorphisms $H : X \to X'$ and $h : B \to B'$ such that the following diagram is commutative.

$$\begin{tikzcd}
& X \arrow[rr, "H"] \arrow[dl, "c_X"'] \arrow[dd, "\pi"' near start] & & X' \arrow[dl, "c_{X'}"'] \arrow[dd, "\pi'"] \\
X \arrow[rr, crossing over, "H" near start] \arrow[dd, "\pi"'] & & X' & \\
& B \arrow[rr, "h" near start] \arrow[dl, "c_B"'] & & B' \arrow[dl, "c_{B'}"'] \\
B \arrow[rr, "h"] & & B' \arrow[from=uu, crossing over, "\pi'" near start] &
\end{tikzcd}$$

By definition the sets of critical points and of critical values are invariant under the action of the corresponding real structure. One observes two types of critical points/ values: the ones fixed by the real structure (that we call *real critical points/ values*), and those which come as a pair of critical values interchanged by the real structure. Fibers over the real points of the (B, c_B) are fixed under the action of c_X, and thus, inherit a real structure from the real structure c_X of the total space. We call such fibers the *real fibers*.

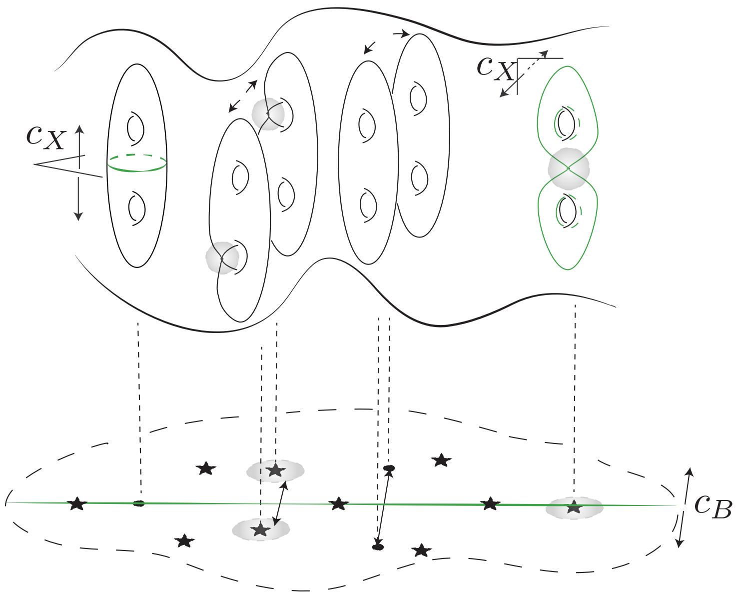

Figure 2.1

2.1. Real Lefschetz fibrations around singular fibers

In this section we investigate real Lefschetz fibrations around a real singular fiber.

It will be convenient for us to interpret a neighborhood of a real singular fiber as a real Lefschetz fibration $\pi : X \to D^2$ with only one critical value. The real structure on D^2 is taken as the standard reflection (for example, the complex conjugation for $D^2 \subset \mathbb{C}$). Note that the critical value and the critical point are necessarily real.

By definition we can find equivariant local charts (U, ϕ_U), (V, ϕ_V) around the critical point and the critical value such that U and V are closed discs and $\pi|_U : (U, c_U) \to (V, conj)$ is equivariantly isomorphic (via ϕ_U and ϕ_V) to either of $\pi'_\pm : (U'_\pm, conj) \to (V', conj)$, where

$$U'_\pm = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq \sqrt{\epsilon}, \left|z_1^2 \pm z_2^2\right| \leq \epsilon^2\}$$

and

$$V' = \{z \in \mathbb{C} : |z| \leq \epsilon^2, 0 < \epsilon < 1\}$$

with $\pi'_\pm(z_1, z_2) = z_1^2 \pm z_2^2$.

The above real local models, $\pi'_\pm : U'_\pm \to V'$, can be seen as two real structures on the neighborhood of a critical point. It is easy to see that these two real structures are not equivalent as the difference already appear at the level of the singular fibers, see Figure 2.2.

To understand the action of the real structures on the regular real fibers of $\pi'_\pm$, we can use the branched covering defined by the projection $(z_1, z_2) \to z_1$. In the case of π'_+, there are two types of real regular fibers; the fibers $F'_{t'}$ with $t' < 0$ have no real points, their vanishing cycles have invariant representatives, and in this case, c acts on the invariant vanishing

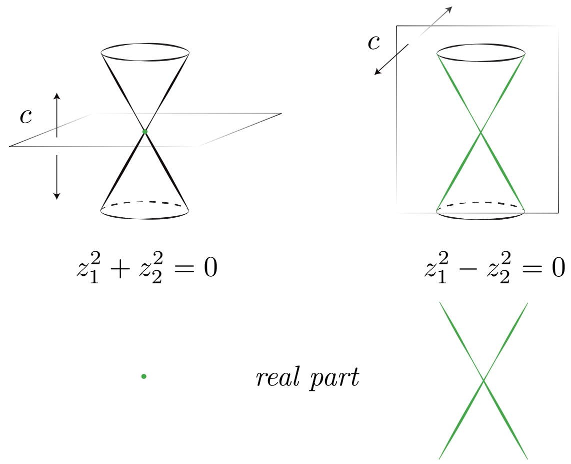

Figure 2.2:

cycles as an antipodal involution; the fibers $F'_{t'}$ with $t' > 0$ have a circle as their real part and this circle is an invariant (pointwise fixed) representative of the vanishing cycle. In the case of π'_-, all the real regular fibers are of the same type and the real part of such a fiber consists of two arcs each having its endpoints on the two different boundary components of the fiber; the vanishing cycles have invariant representatives, and c acts on them as a reflection.

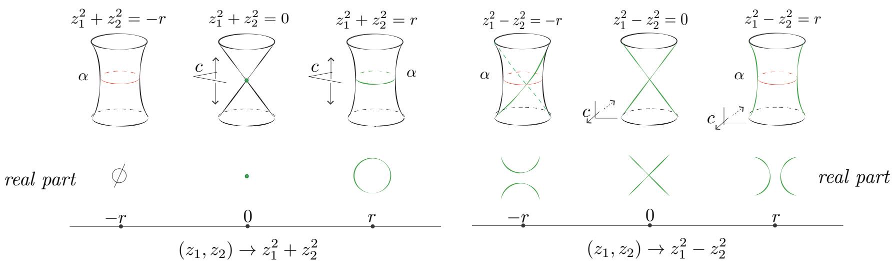

Figure 2.3

Using the ramified covering $(z_1, z_2) \to z_1$, we observe that the horizontal boundary, of the fibration $\pi'_\pm$ is equivariantly trivial and has a distinguished equivariant trivialization. Moreover, since the complement of U in $\pi^{-1}(V)$ does not contain any critical point, X can be written as union of two real Lefschetz fibrations with boundary: one of them, $U \to V$, is isomorphic to $\pi'_\pm : U'_\pm \to V'$, and the other one is isomorphic to the trivial real fiber bundle $T \to V'$ whose real fibers are equivariantly diffeomorphic to the complement of an open regular neighborhood of the vanishing cycle $\alpha \subset F_b$. The action of the complex conjugation on the boundary components of the real fibers of $T \to V'$ determines the type of the model, $\pi'_\pm : E_\pm \to V'$ glued to $T \to V'$: in the case of π'_+ it switches the boundary components while in the case of π'_- boundary components are preserved (and the complex conjugation acts as a reflection on each of them).

Therefore, around a real singular fiber we have two different real regular fibers (defined up to isotopy): one on the left and the other on the right of the singular fiber. (Here right/ left makes sense once we fixed the orientation on the real part of $(D^2, conj)$. We call fibrations with fixed orientation on the real part of the base *directed*). Let us denote the real structure inherited on a real fiber lying on the left of the singular fiber by c_-, respectively we denote by c_+ the one appearing on a real fiber lying on the right of the singular fiber. From the equivariant local model we see that we can find a representative of the vanishing cycle α such that $c_-(\alpha) = c_+(\alpha) = \alpha$. One also observes a very important property of the monodromy: $t_\alpha = c_+ \circ c_- = c_- c_+$. Therefore, two of the three variants are essential information in describing the neighborhood of a real singular fiber. Let us fix the pair (c_-, α).

Conversely, given a couple (c, α) where c is a real structure on a surface and α is a simple closed curve such that $c(\alpha) = \alpha$, one can construct a real Lefschetz fibration over D^2 with a unique singular fiber obtained by pinching the curve α on a nearby chosen real regular fiber. A priori there is an ambiguity of the choice of real structure $c = c_-$ or $c = c_+$, such an ambiguity can be omitted by keeping track of the orientation of the real part of D^2 (for detailed and more rigorous explanation you can look at [21, 24]). The upshot is that the diffeomorphism type of the pair (c, α) (by which we mean that the pair (c, a)) can be replaced by $(f \circ c \circ f^{-1}, f(\alpha))$ where f is an orientation preserving diffeomorphism) classifies neighborhoods of a real singular fiber of directed real Lefschetz fibrations. There is a way to enumerate such pairs. As before one should consider the cases when the curve α is non-separating and separating one at a time. When α is non-separating, there are 6 classes if $g = 1$; $8g - 3$ classes if $g > 1$ and is odd; $8g - 4$ classes otherwise. The formula concerning the case when α is separating is a bit more complicated and can be found in [21, 24].

Remark 2.1. Indeed the property that monodromy decomposes into product of two real structures can be observed along any loop on which the real structure acts as reflection. For example, along the boundary of D^2 for real Lefschetz fibrations over D^2. It is, thus, interesting to investigate elements of $Map(\Sigma_g)$ which have such decomposition property. In [22] a characterization of elements of $Map(T^2)$ admitting a decomposition into product of two real structures is given.

2.2. Classification of real elliptic Lefschetz fibrations

In this section we give a real version of the theorem of B. Moishezon and R. Livné for certain class of real Lefschetz fibrations.

Let $\pi : X \to S^2$ be a directed real elliptic Lefschetz fibration. We consider the restriction, $\pi_{\mathbb{R}} : X_{\mathbb{R}} \to S^1$, of π to the real part $X_{\mathbb{R}}$ of X. By definition, fibers of $\pi_{\mathbb{R}}$ are the real parts of the real fibers of π. The base S^1 is oriented, whereas the total space $X_{\mathbb{R}}$ is either empty or a surface not necessarily oriented nor connected.

By definition of real Lefschetz fibrations, the map $\pi_{\mathbb{R}}$ is an S^1-valued Morse function on $X_{\mathbb{R}}$ whose regular fibers can be S^1, $S^1 \amalg S^1$ or the empty set (this is a direct implication of the classification of real structures on T^2). On the other hand, singular fibers are either a wedge of two circles (this occurs in the case when the critical point is of index 1) or a disjoint union of S^1 with an isolated point or just an isolated point (these cases occur when the critical point is of index 0 or 2). As an immediate consequence, we note that the real part $X_{\mathbb{R}}$ is not empty if π has a real critical value.

For the sake of simplicity, we first focus on fibrations which admit a real section. By a real section, we understand a section $s : S^2 \to X$ commuting with the real structures. Existence of a real section ensures that fibers of $\pi_{\mathbb{R}}$ are never empty, so, in this case, there are only two possible topological types for regular fibers: S^1 or $S^1 \amalg S^1$.

We now introduce a decoration on the base S^1 of $\pi_{\mathbb{R}} : X_{\mathbb{R}} \to S^1$ as follows. First, label the critical values of $\pi_{\mathbb{R}}$ by "$\times$" or "$\circ$" according to the parity of indices of the corresponding critical points. Namely, if the corresponding critical point is of index 1, label the critical value by "$\times$", otherwise by "$\circ$". (Note that $\pi_{\mathbb{R}}$ has critical values as long as π has real critical values.) Second, consider a labeling on the set of *regular intervals*, $S^1 \setminus \{\text{critical values}\}$. Over each regular interval the topology of the fibers of $\pi_{\mathbb{R}}$ is fixed; moreover, it alternates as we pass through a critical value. We label regular intervals over which fibers have two components by doubling the interval, see Figure 2.4. Intervals over which the fibers are a copy of S^1 remain unlabeled.

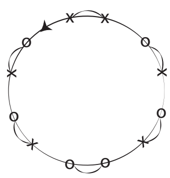

Figure 2.4

We, now, consider some “standard" pieces out of which all possible configurations can be built. It is convenient for us to take pieces with two consecutive critical values (in order to avoid the problem of matching real structures). Let us choose a regular value on S^1 (for some later use we choose the point on an unlabeled regular interval). With respect to this point and the orientation of S^1, we have 4 instances for a pair of two critical values. In order to simplify the decoration, for each instance we introduce a new notation as shown in Figure 2.5.

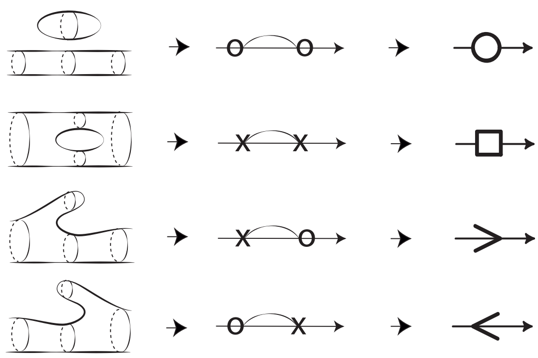

Figure 2.5

The oriented S^1 decorated using elements of the set $S = \{\bigcirc, \square, >, <\}$ is called an *oriented necklace diagram* (an example is shown in Figure 2.6). We call the elements of the set S *(necklace) stones* and the pieces of the circle between the stones *(necklace) chains*. Two oriented necklace diagrams are considered identical if they contain the same types of stones going in the same cyclic order.

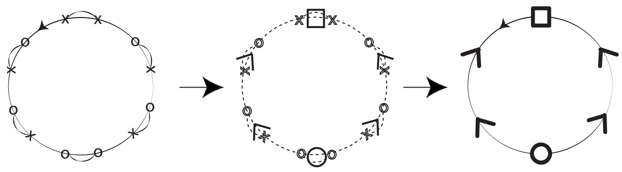

Figure 2.6

It is obvious from the construction that oriented necklace diagrams are invariants of directed real elliptic Lefschetz fibrations. For non-directed real Lefschetz fibrations, we do not have a preferable orientation on the necklace diagram. Non-directed fibrations, hence, determine a pair of oriented necklace diagrams related by a mirror symmetry in which ○-type and □-type stones remain unchanged, while >-type and <-type stones interchanged.

Now we want to assign a *monodromy* to each necklace stone. First recall that $Map(T^2) = \mathrm{SL}(2,\mathbb{Z})$, due to the fact that every diffeomorphism $f : T^2 \to T^2$ is isotopic to a linear diffeomorphism. The latter diffeomorphisms by definition are induced on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ by a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by a matrix $A \in \mathrm{SL}(2,\mathbb{Z})$. Note that we can naturally identify $T^2 = H_1(T^2,\mathbb{R})/H_1(T^2,\mathbb{Z})$ and interpret matrix A as the induced automorphism f_* in $H_1(T^2,\mathbb{Z})$. Let a denote the simple closed curve on T^2 represented by the equivalence class of the horizontal interval $I \times 0 \subset \mathbb{R}^2$, and b is similarly represented by the vertical interval $0 \times I$. We have the intersection number (a, b) is equal to 1; hence, the homology classes represented by these curves are integral generators of $H_1(T^2,\mathbb{Z})$. Using Picard-Lefschetz formula we get

$$A = t_{a*} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = t_{b*} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

We have $\mathrm{SL}(2,\mathbb{Z}) = \langle A, B : ABA = BAB, (AB)^6 = id\rangle$. Also, it is convenient to consider another presentation of $\mathrm{SL}(2,\mathbb{Z}) = \langle X, Y : X^2 = Y^3, X^4 = id\rangle$ obtained by replacing $X = ABA$, and $Y = AB$.

Now, note that each real structure, $c : T^2 \to T^2$, induces an isomorphism c_* on $H_1(T^2,\mathbb{Z}) \cong \mathbb{Z}a \oplus \mathbb{Z}b$ that defines two rank 1 subgroups $H^c_\pm(T^2) = \{\gamma \in H_1(\mathrm{T}^2,\mathbb{Z}) : c_*\gamma = \pm\gamma\}$. (If c has one real component, then $H_1(T^2,\mathbb{Z})/\langle H^c_+, H^c_-\rangle = \mathbb{Z}/2\mathbb{Z}$, otherwise, $H_1(T^2,\mathbb{Z}) = H^c_+ \oplus H^c_-$.) Moreover, a real structure c on a real fiber F defines a pair of bases $\pm(a, b)$ so that $a \pm b$ generates $H^c_\pm$. This defines a canonical identification of $H_1(F,\mathbb{Z})$ with $H_1(T^2,\mathbb{Z})$.

To each decoration around a critical value q, we assign the transition matrix P_q from a basis of $H^c_+ \oplus H^c_- \subset H_1(T^2,\mathbb{Z})$ to a basis of $H^{c'}_+ \oplus H^{c'}_- \subset H_1(T^2,\mathbb{Z})$ where c, c' are left and, respectively, right real structures on the real fibers near F_q. Note that the choice of the basis is determined up to sign ; the matrices we obtained are indeed in $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\pm I = \langle X, Y : X^2 = Y^3 = id\rangle$. Matrices assigned to stones will then be the product of two matrices associated to the corresponding decoration. Because we choose the marked point on an unlabeled interval, the matrices have coefficients in $\frac{1}{2}\mathbb{Z}$. However, they are conjugate (by a fixed matrix) to the following matrices in $\mathrm{PSL}(2,\mathbb{Z})$. As it is convenient for us to work with $\mathrm{PSL}(2,\mathbb{Z})$ we fixed the monodromies of stones as the following elements in $\mathrm{PSL}(2,\mathbb{Z})$.

$\mathbb{P}_\square = YXY$
$\mathbb{P}_\bigcirc = XYXYX$
$\mathbb{P}_> = Y^2X$
$\mathbb{P}_< = XY^2$

The matrices $\mathbb{P}_\bigcirc, \mathbb{P}_\square, \mathbb{P}_>, \mathbb{P}_<$ are called *monodromies of stones*. The product (with respect to a chosen marked point and to the orientation) of monodromies of necklace stones is called the *monodromy relative to the marked point* of the oriented necklace diagram. The *monodromy* of an oriented necklace diagram is, thus, defined as the conjugacy class of its monodromy relative to a marked point.

We will now focus on fibrations whose critical values are all fixed by the action and call such fibrations *totally real*. It is not hard to see that in this case the monodromy of the necklace diagram becomes the identity element in $\mathrm{PSL}(2,\mathbb{Z})$.

We are ready now to state:

Theorem 2.2 (S., 2007, [21, 23])**.** *There exists a one-to-one correspondence between the set of oriented necklace diagrams with* $6k$ *stones and with the identity monodromy and the set of isomorphism classes of directed totally real elliptic Lefschetz fibrations* $\mathrm{E}(k)$*,* $k \in \mathbb{N}$*, that admit a real section.*

Corollary 2.3 (S., 2007, [21, 23])**.** *There exists a bijection between the set of symmetry classes of non-oriented necklace diagrams with* $6k$ *stones and with the identity monodromy and the set of isomorphism classes of non-directed totally real* $\mathrm{E}(k)$*,* $k \in \mathbb{N}$ *which admit a real section.*

Thus, each necklace diagram defines a decomposition of the identity in PSL$(2,\mathbb{Z})$ into a product of $6k$ elements that are chosen from the set of monodromies of necklace stones. There is a simple algorithm to find all necklace diagrams associated with E(k). Applying the algorithm, we obtain the complete list of necklace diagrams of E(1).

The following theorem concerns $n = 1$.

Theorem 2.4 (S., 2007, [21, 23])**.** *There exist precisely 25 isomorphism classes of non-directed totally real* E(1) *admitting a real section. These classes are characterized by the non-oriented necklace diagrams presented in Figure 2.7.*

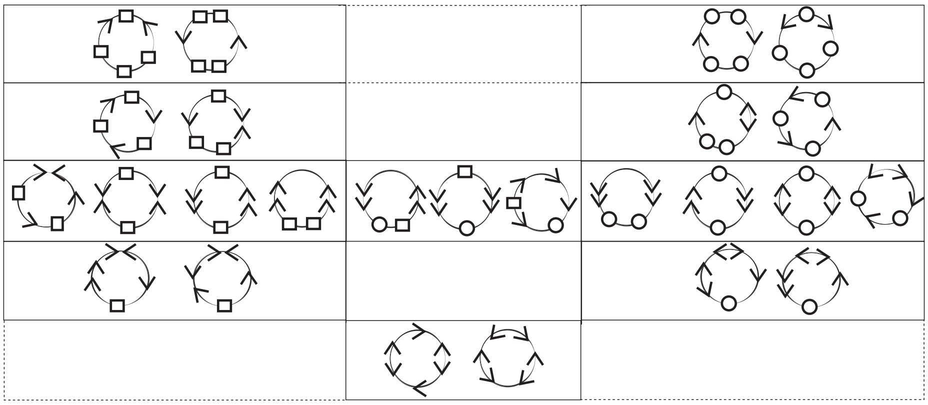

Figure 2.7

The assumption of admitting a real section is not imperative. The classification without this assumption can be done by means of *refined necklace diagrams*. A refinement of a necklace diagram is obtained by replacing each ○-type stone with one of the following refined stones, ○, ⊘, ◌, ⊘. If the refined necklace diagram is identical to the underlying necklace diagram then the corresponding real Lefschetz fibration admits a real section.

It is clear that if a real structure on a fiber of π has no real component, then the nearby critical values can only be of type "∘". In other words, existence or lack of a real section influences only ○-type necklace stones. Both of the refined stones of type ○, ⊘ correspond to the case where the real structure on the real fibers over the interval between the two critical values has 2 real components. Already the real part distinguishes the cases of ○ and ⊘, see Figure 2.8. (In Figures 2.8 and 2.9 below, the dotted part depicts the traces of the curves on which the inherited real structure acts as the antipodal map.)

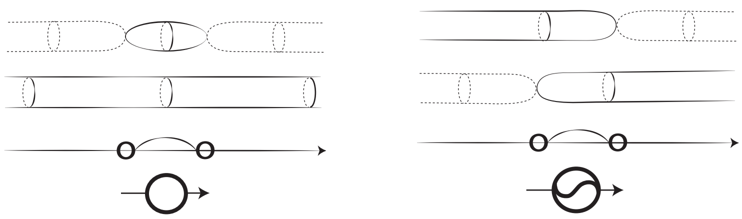

Figure 2.8

The refined stones ◌ and ⊘ correspond to the case where the real structure on the real fibers of π has no real component. As depicted in Figure 2.9, the real part of $X_{\mathbb{R}}$ does not distinguish the two situations associated with ◌, ⦸. The difference between ◌ and ⦸ (as well as between ○ and ⊘) can indeed be conceived by comparing the equivariant isotopy classes of the two vanishing cycles corresponding to the two critical values of the necklace stone (by *equivariant isotopy*, we mean an isotopy which commutes with the real structure). In the case of ◌ (respectively, ○) the equivariant isotopy classes of the two vanishing cycles are the same, while in the case of ⦸ (respectively, ⊘) the two vanishing cycles are of different equivariant classes.

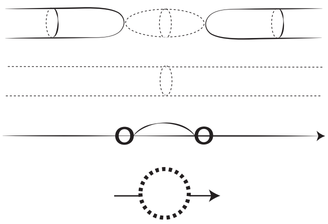

Figure 2.9

There is no difference between real structures with 2 real components and real structures with no component on the homological level. As a consequence, the calculation of the monodromy is not affected by the refinement. Thus we have:

Theorem 2.5 (S., 2007, [21, 23])**.** *There is a one-to-one correspondence between the set of oriented refined* 6*k-stone necklace diagrams whose monodromy is the identity and the set of isomorphism classes of directed totally real* $\mathrm{E}(k)$*,* $k \in \mathbb{N}$*.*

It is worth mentioning here that unlike *usual* elliptic Lefschetz fibrations, real elliptic Lefschetz fibrations are not always fiber sum. For example the fibration corresponding to the necklace diagram shown in Figure 2.10 cannot be written as the fiber sum of two real Lefschetz fibrations [21, 23]. This comes from the analysis of possible division of this necklace diagram into necklace diagrams of 6 stones.

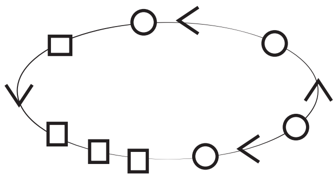

Figure 2.10

As in the case of Lefschetz fibrations we can investigate if there is algebraic realization of the real fibrations. In the case of real Lefschetz fibrations there are classes which cannot be realized algebraically. For examples real $E(1)$ corresponding to necklace diagrams shown in Figure 2.11 are not algebraically realized, [21, 23]. However, we show that totally real elliptic fibrations admitting a real section with $\chi(X_{\mathbb{R}}) = \chi(X)$ are all algebraic (recall that in general one has $\chi(X_{\mathbb{R}}) \le \chi(X)$, known as *Thom-Smith inequality*). In [7], we generalize this result to fibrations which are not necessarily totally real using generalized necklace diagrams, the so-called *pendant necklace diagrams*. The Idea of pendant necklace diagrams is based on the fact that such fibrations can be considered in three pieces: neighborhood of the real locus (determined by its necklace diagram), a symmetric pair of Lefschetz fibrations over a disk with r critical values (determined by the Hurwitz equivalence classes of r-factorizations, called r-pendants).

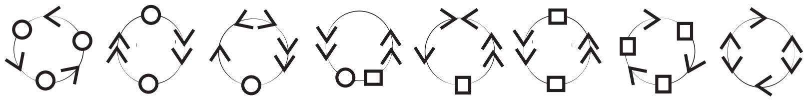

Figure 2.11

3. Geometric properties of Lefschetz fibrations

3.1. Digression: symplectic structures

In this section I will make a quick review of symplectic vector spaces. For further reading you can look at [4], [17].

Let V be a vector space over $\mathbb{R}$. A *symplectic structure* on V is a skew-symmetric non-degenerate bilinear form Ω and we call the pair (V, Ω) a *symplectic vector space*. The non-degeneracy condition on the skew-symmetric form implies that the dimension of V is indeed even.

Once we have a bilinear form Ω we can consider the subspace (the so-called *symplectic complement*) $W^\Omega = \{v \in V : \Omega(v, w) = 0, \forall w \in W\}$. A subspace W is then called *isotropic* if $W \subset W^\Omega$; called *coisotropic* if $W^\Omega \subset W$; *symplectic* if $W \cap W^\Omega = \{0\}$; Lagrangian if $W = W^\Omega$.

Now consider a smooth manifold X of even dimension. If every tangent space $T_pX, \forall p \in X$ has a symplectic structure defined by a global closed 2-form ω, then we call the pair (X, ω) a *symplectic manifold*. In other words, a symplectic structure on X is a closed non-degenerate differential 2-form ω. A submanifold $Y \subset X$ is called *symplectic* if $\omega|_Y$ is non-degenerate, and it is called *Lagrangian* if it is the maximum possible set satisfying $\omega|_Y \equiv 0$, in which case the dimension is half of the dimension of X.

There are natural examples of symplectic manifolds like cotangent bundles of manifolds. A very standard example of a symplectic manifold can be $(\mathbb{R}^{2k}, \omega_{std} = \sum dx_i \wedge dy_i)$. (The subspaces $\mathbb{R}^s \times \mathbb{R}^s = (x_1, \ldots x_s, 0, \ldots, 0, y_1, \ldots, y_s, 0, \ldots, 0), 1 \leq s \leq k$ are symplectic while $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^k$ are Lagrangian.) This example is crucial by the theorem of G. Darboux [6] which states that every symplectic manifold locally looks like $(\mathbb{R}^{2k}, \omega_{std})$. Note that symplectic manifolds are naturally orientable as ω^k defines a volume form on X. Thus, when $k = 1$, every orientable manifold admits a symplectic structure by a choice of volume form.

From now on, we will be focusing on the case $2k = 4$. Let us note that not all (orientable) 4-manifolds are symplectic. For instance $S^1 \times S^3$ cannot admit a symplectic structure just because $H^2(S^1 \times S^3) = 0$. Symplectic 4-manifolds form a class larger than complex manifolds. In the next section we will see that 4-dimensional Lefschetz fibrations are indeed topological counter part of symplectic 4-manifolds.

3.2. Relation to Lefschetz fibrations

Let us first start with introducing theorems relating Lefschetz fibrations to symplectic 4-manifolds.

Theorem 3.1 (Gompf, 1999, [13, 14])**.** *Let $\pi : X \to S^2$ be a Lefschetz fibration with $[F] \neq 0 \in H_2(X, \mathbb{R})$. Then there exists a symplectic structure ω on X such that fibers are symplectic submanifolds.*

The idea of the proof of this theorem is based on the article of W. P. Thurston [30] in which he gives examples of symplectic but non-Kähler manifolds by constructing a symplectic structure on certain T^2 bundles over T^2. Indeed, his idea generalizes to every manifold X^{2k} which fibers over a symplectic manifold B^{2k-2} with the property that the fundamental class of the fiber F is not homologous to zero in $H_2(X, \mathbb{R})$. His idea is based on the fact that when $[F] \neq 0 \in H_2(X, \mathbb{R})$, one can construct a closed 2-form β which is non-degenerate on the fibers. By assumption, the base has a symplectic form ω_B so one considers $\omega = \beta + K\pi^*\omega_B$ where $\pi : X \to B$ is the projection. The form ω then becomes non-degenerate on X for sufficiently large values of $K \in \mathbb{R}$.

In the case of Lefschetz fibration this idea applies. A special care is needed around the singularities but singularities of Lefschetz fibration have a simple fixed local model. As for the converse, from symplectic manifolds to Lefschetz fibrations, we have the following important theorem due to S. K. Donaldson.

Theorem 3.2 (Donaldson,1999, [9])**.** *Let (X, ω) be a compact symplectic manifold with integral class $[\omega]$. Then for large $k \in \mathbb{N}$, X carries a Lefschetz pencil such that the fibers are symplectic submanifolds representing the Poincaré dual of $k[\omega]$. Thus a blow-up of X carries a symplectic Lefschetz fibrations over S^2.*

To give a very rough idea of the proof of the theorem S. K. Donaldson, it is convenient to look at the construction of $E(1)$ in an other manner that allow us to generalize the construction. Namely, we interpret homogeneous polynomials of degree 3 as holomorphic sections of the bundle $O(3) \to \mathbb{C}P^2$ and follow the construction as it is. (Recall that $O(3)$ is the 3-fold tensor product of $O(1)$ which is the dual of the canonical line bundle of $\mathbb{C}P^2$.)

In a more general setting one considers a very ample line bundle $L \to X$ (L being very ample line bundle implies that there exist enough global sections to set up an embedding of X into a projective space). Say there exists $s_0, s_1 : X \to L$ with $\{s_i = 0\} = Y_i \subset X$. Then, consider the following well-defined projection

$$\begin{array}{rcl} X \setminus \{Y_0 \cap Y_1\} & \to & \mathbb{C}P^1 \\ x & \to & [s_0(x) : s_1(x)]. \end{array}$$

Generically the intersection $Y_0 \cap Y_1$ is transverse and is a discrete set of cardinality say m. By blowing up at the intersection $Y_0 \cap Y_1$ we can extend this projection to $X \# m\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ that, hence, provides a Lefschetz fibration over $S^2 \cong \mathbb{C}P^1$. In the case of symplectic manifolds, S. K. Donaldson shows that one can find enough *approximately holomorphic sections* which makes the construction of a Lefschetz fibration possible.

It is worth mentioning here that the real analogue of the theorem of S. K. Donaldson is given by D. Gayet [12]. Namely, he showed that

Theorem 3.3 (Gayet, 2008, [12])**.** *Let (X, ω, c_X) be a compact real symplectic manifold (i.e. $c_X^*(\omega) = -\omega$) with $[\omega]$ integral, then a blow-up of X carries a real Lefschetz fibrations over S^2 with fibers symplectic submanifolds.*

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