Winter Braids Lecture Notes

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Around 3-manifold groups

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Abstract

This text is an expanded version of the minicourse given at the session Winter Braids VIII. The goal is to present some basic properties of 3-manifold groups and to give an overview of some of the major progress made in their study this last decade. It is mostly of expository nature and does not intend to cover the subject. I thank the Winter Braids organizers for their invitation and their kind patience whilst these notes were completed, and the referee for his careful reading and his suggestions which greatly improved the exposition.

1. Introduction

The last decade has seen spectacular progress in our understanding of the algebraic properties of the fundamental groups of 3-manifolds due mainly to G. Perelman's geometrisation theorem and the works of I. Agol and D. Wise. We know that except for graph manifolds, each closed, irreducible 3-manifold virtually fibers over the circle and thus its fundamental group has a finite index subgroup which is an extension of $\mathbb{Z}$ by a surface group. Necessary and sufficient conditions for a given group to be isomorphic to a closed 3-manifold group have been given in terms of group presentations (see [Gon75],[Tur84]), however, no intrinsic algebraic characterisation of 3-manifold groups is currently known. The following question will be the guideline of these lectures.

Question 1.1. *Which finitely presented groups can or cannot occur as the fundamental group of a compact orientable 3-manifold M^3?*

John Stallings [Sta63] showed that this question is algorithmically undecidable.

Theorem 1.2. *Given any non-empty class $\mathcal{M}$ of compact connected 3-manifolds, there is no algorithm for deciding whether or not a finite presentation of a group defines a group isomorphic to the fundamental group of an element of $\mathcal{M}$.*

The proof reduces to showing that being a 3-manifold group is a Markovian property, and therefore is undecidable by the Adian-Rabin's Theorem.

Recall that a property P of finitely presented groups, which is preserved under isomorphisms, is a Markovian property if:

1. there exists a finitely presented group with the property P.
2. there exists a finitely presented group which cannot be embedded into any finitely presented group with the property P.

Stallings showed that the free abelian group $\mathbb{Z}^4$ cannot be isomorphic to a subgroup of any 3-manifold group. The proof of this fact is not trivial since it needs the Sphere theorem (cf. Theorem 3.3, section 3).

Recently D. Groves, J. F. Manning and H. Wilton [GMW12] proved that the class of fundamental groups of closed, geometric 3-manifolds is algorithmically recognizable provided that a solution to the word problem is given. Geometric manifolds are 3-manifolds which carry one of the eight homogeneous 3-dimensional geometry listed by Thurston (see Theorem 6.1, section 6). This result implies that geometric 3-manifold groups can be recognized in some classes of groups as, for example, linear groups or residually finite groups.

Throughout these lectures we work in the smooth category and, unless otherwise stated, manifolds will be assumed to be connected, compact and orientable, possibly with boundary. Basic references for 3-manifold topology are the books [Hem76], [Jac80], [Sta71].

2. Finite presentation

A group G is *finitely generated* if it can be generated by finitely many elements $g_1, \cdots, g_n$. In this case G is a quotient of the free group F_n of rank n by a normal subgroup $R \trianglelefteq F_n$. If R is normaly generated by finitely many elements $r_1, \cdots, r_k$, called relators, the group G is said to be *finitely presented*. We write $G = \langle g_1, \cdots, g_n \mid r_1, \cdots, r_k \rangle$, a presentation of the group G with n generators and k relators. The discrepancy $n-k$ is called the *deficiency* of the presentation.

If G is finitely generated the minimal number of generators is called the rank $rk(G)$ of G. If G is finitely presentable, the *deficiency* $\mathrm{def}(G)$ of G is the maximal deficiency over all presentations of G. Here is a classical majoration for the deficiency due to D. Epstein [Ep61]:

Lemma 2.1. *For a finitely presentable group G, $def(G) \leq b_1(G) - rk(H_2(G,\mathbb{Z}))$.*

Given a finite presentation $1 \to R \to F_n \to G \to 1$ for G, the proof of this lemma is based on the Hopf's formulas:

$$H_1(G,\mathbb{Z}) \cong F_n/[F_n,F_n]R \text{ et } H_2(G,\mathbb{Z}) \cong (R \cap [F_n,F_n])/[R,F_n]$$

Examples

1. $\mathrm{def}(F_n) = n$.
2. $\mathrm{def}(\mathbb{Z}^2) = 1$ and $\mathrm{def}(\mathbb{Z}^3) = 0$, while it is negative for $n \geq 4$.
3. For a finite group G, $\mathrm{def}(G) \leq 0$ since a group of deficiency ≥ 1 has infinite abelianization.

A finitely presentable group is *efficient* if its deficiency is realized on a finite presentation. Abelian groups, free groups, surface groups are efficient. It follows that most finitely generated abelian groups have deficiency < 0. The only finitely generated abelian groups with deficiency 0 are $\mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$, $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}^3$.

For a closed orientable surface S of genus g, Lemma 2.1 shows that $\mathrm{def}(\pi_1(S)) \leq 2g-1$ and the equality is realized on the canonical one-relator presentation of $\pi_1(S)$.

On the other hand there are torsion free, finitely presentable groups which are not efficient. Martin Lustig [Lus95] gave the first example with the group $G = \langle a,b,c \mid a^2b^{-3}, [a,c], [b,c] \rangle \cong \pi_1(S^3 \setminus K) \times \mathbb{Z}$, where $K \subset S^3$ is the trefoil knot.

A finite dimensional compact manifold admits a cellular decomposition with one 0-cell and finitely many 1-cells and 2-cells. Any such decomposition gives rise to a finite presentation of its fundamental group $\pi_1(M)$ where the generators correspond to the 1-cells and the relators to the 2-cells. So $\pi_1(M)$ is finitely presentable. In his famous article on the Analysis Situs, after having introduced the notion of fundamental group attached to a manifold, Poincaré raises the following questions (see [Poin95]):

"Il pourrait être intéressant de traiter les questions suivantes.

1) Etant donné un groupe G défini par un certains nombres d'équivalences fondamentales, peut-il donner naissance à une variété fermée à n dimensions?

2) Comment doit-on s'y prendre pour former cette variété?

3) Deux variétés d'un même nombre de dimensions qui ont même groupe G sont-elles toujours homéomorphes?

Ces questions exigeraient de difficiles études et de longs développements. Je n'en parlerai pas ici."

It is known since M. Dehn [Deh10, Deh12], see [DeSt87, papers 3 and 4] or [dlH10, section 7], that any finitely presented group can be realized as the fundamental group of a closed orientable 4-manifold. This fact shows the impossibility of classifying 4-manifolds since it is impossible to classify finitely presented groups. However the construction does not give in general an aspherical manifold (i.e. a manifold whose fundamental group is the only non-trivial homotopy group). In particular the cohomology of this manifold does not coincide with the cohomology of the group in general (cf. Section 9).

Theorem 2.2. *Any finitely presented group is the fundamental group of a* 2*-complex and also of a 4-dimensional closed orientable 4-manifold.*

Here is a sketch of the proof, see [dlH10] for more details. Given a finite presentation of a group $G = \langle g_1, \cdots, g_n \mid r_1, \cdots, r_k \rangle$ one can associate a 2-complex X with one 0-cell, n 1-cells (one for each generator) and k 2-cells (one for each relator) which are attached to the 1-skeleton according to the relator words. One can always subdivide this 2-complex X to get a finite simplicial 2-complex. A finite simplicial 2-complex can always be embedded in $\mathbb{R}^5$: this is done for example by sending the vertices to distinct points on the parametrized curve (t, t^2, t^3, t^5) which has the property that no 6 distinct points lie on a common hyperplane. It follows then that two distinct 3-simplices meet only at common faces, edges or vertices. One can endow $\mathbb{R}^5$ with a simplicial struture which makes X a subcomplex. The union W of the interiors of the simplexes in $\mathbb{R}^5$ which contain a vertex of X in the barycentric subdivision of X defines a neighborhood of X which deformation retracts on X; hence $G = \pi_1(X) \cong \pi_1(\overline{W})$. Moreover, since the codimension of X in $\overline{W}$ is 3, $\pi_1(\overline{W}) \cong \pi_1(\overline{W} \setminus X)$, and $\overline{W} \setminus X$ deformation retracts on $\partial\overline{W}$. It follows that $G = \pi_1(X) \cong \pi_1(\overline{W}) \cong \pi_1(\overline{W} \setminus X) \cong \pi_1(\partial\overline{W})$, where $M = \partial\overline{W}$ is a closed orientable 4-manifold. □

In contrast, closed 3-manifolds admits topological and geometrical properties that put constraints on their fundamental groups. For example, since Heegaard and Poincaré one knows that every smooth, closed, orientable 3-manifold splits along an embedded surface into two handlebodies. One way to get such a splitting, called a Heegaard decomposition, is to consider the boundary of a regular neighborhood of the 1-skeleton of a triangulation of the closed 3-manifold. The existence of a Heegaard decomposition shows that the fundamental group of a closed 3-manifold admits a balanced, presentation which means a presentation with equal number of generators and relators, that is to say of deficiency 0.

Corollary 2.3. *Let M be a closed, orientable 3-manifold, then def*$(\pi_1(M)) \geq 0$.

In fact one has the following more general and precise result, see [Ep61], which uses the Sphere Theorem (Theorem 3.3, section 3).

Proposition 2.4. *Let M be a compact orientable 3-manifold, then the following holds:*

(i) *If* $\partial M \neq \emptyset$*, then def*$(\pi_1(M)) \geq 1 - \chi(M)$.

(ii) *If* $\partial M = \emptyset$*, then def*$(\pi_1(M)) \geq 0$*. Moreover if M is irreducible, def*$(\pi_1(M)) = 0$.

3. Prime decomposition

An orientable 3-manifold M is *irreducible* if any embedded 2-sphere in M bounds a 3-ball. Otherwise it is said reducible. By Alexander's theorem [Al24] irreducibility holds for S^3 and $\mathbb{R}^3$:

Theorem 3.1 (Alexander's Theorem)**.** *Every embedded 2-sphere in S^3 or $\mathbb{R}^3$ bounds a 3-ball.*

Here is a useful criterion for irreducibility, see [Jac80]

Lemma 3.2. *Let M be an orientable 3-manifold and $p : \bar{M} \to M$ be a covering. If $\bar{M}$ is irreducible then M is irreducible.*

To a manifold M one can associate higher homotopy groups $\pi_n(M)$, $n > 1$ generated by homotopy classes of pointed applications $f : (S^n, \star) \to (M, \star)$. These groups are abelian, while the fundamental group is usually not. A manifold is said *aspherical* whenever its higher homotopy groups $\pi_n(M) = \{0\}$ for $n \geq 2$. The following theorem of C. Papakyriakopoulos [Papa57] is a fundamental result for the study of 3-manifolds and their fundamental groups (cf. [Hem76],[Jac80] [Sta71]).

Theorem 3.3 (Sphere Theorem)**.** *Let M be an orientable 3-manifold such that $\pi_2(M) \neq \{0\}$. Then M contains an embedded sphere S^2 which does not bound a (homotopy) ball in M. In particular M is reducible.*

Corollary 3.4. *Let M be a compact, orientable, irreducible 3-manifold, then :*
(i) $\pi_2(M) = \{0\}$.
(ii) $\pi_1(M)$ *is infinite if and only if* $\pi_3(M) = \{0\}$. *In this case* $\pi_1(M)$ *is torsion free and* M *is aspherical.*
(iii) $\pi_1(M)$ *is finite if and only if* $\pi_3(M) \neq \{0\}$.

Assertion (i) is a direct consequence of Theorem 3.3. Assertions (ii) and (iii) follow from the fact that M and its universal cover $\widetilde{M}$ have the same higher homotopy groups π_n, for $n \geq 2$, and that $\pi_1(M)$ is infinite if and only if $\widetilde{M}$ is non compact, by using the Hurewicz theorem.

As a corollary we get Proposition 2.4(ii)

Corollary 3.5. *Let M be a closed, orientable, irreducible 3-manifold, then:*
$def(\pi_1(M)) = 0$.

For a closed orientable 3-manifold $\mathrm{def}(\pi_1(M)) \geq 0$ by Corollary 2.3. Hence it is sufficient to show that $\mathrm{def}(\pi_1(M)) \leq 0$ when M is irreducible. When $\pi_1(M)$ is finite, this follows from the fact that the deficiency of a finite group is ≤ 0. When $\pi_1(M)$ is infinite it is a consequence of Lemma 2.1 together with the facts that $H_2(\pi_1(M), \mathbb{Z}) \cong H_2(M, \mathbb{Z})$ and $rk(H_2(M, \mathbb{Z})) = b_2(M) = b_1(M) = b_1(\pi_1(M))$ by Poincaré duality. □

Corollary 3.6. *The possible abelian fundamental groups for a closed orientable 3-manifold are $\pi_1(S^3) = \{1\}$, $\pi_1(S^1 \times S^2) = \mathbb{Z}$, $\pi_1(T^3) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(L(p, q) = \mathbb{Z}/p\mathbb{Z}$.*

The *connected sum* of two orientable 3-manifolds is the orientable 3-manifold obtained by removing the interior of a 3-ball in each manifold and gluing the remaining parts together by an orientation reversing homeomorphism of the boundary spheres. A 3-manifold M is *prime* if it cannot be decomposed as a non-trivial connected sum of two manifolds, i.e. if $M = M_1 \sharp M_2$, then M_1 or M_2 is the 3-sphere.

The next theorem shows that any compact orientable 3-manifold can be split along a finite collection of essential embedded spheres into prime manifolds. It is due to H. Kneser[Kne29], see also J. Milnor [Mil62] for the uniqueness (cf [Hem76], [Jac80]).

Theorem 3.7 (Prime decomposition)**.** *Every compact, orientable 3-manifold is a connected sum of finitely many 3-manifolds that are either homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$ or irreducible. Moreover, the connected summands are unique up to ordering and orientation-preserving homeomorphism.*

As a corollary one gets the following free factorization of the fundamental group of a compact orientable 3-manifold:

Corollary 3.8. *For a compact orientable 3-manifold M, $\pi_1(M) \cong F_k \star G_1 \star \cdots \star G_n$, where F is a free group and $G_i \cong \pi_1(M_i) \neq \{1\}$, with M_i a compact, irreducible, orientable 3-manifold with (possibly empty) incompressible boundary. This decomposition as a free product is unique.*

The notion of incompressibility for a surface in a 3-manifold M is defined in the next section. The assertion about the incompressibility of the boundaries of the 3-manifolds M_i relies on the Loop Theorem (see Theorem 4.1, section 4).

A topological converse to the algebraic decomposition of the fundamental group of a 3-manifold is given by Stalling's solution [Sta71] of the Kneser's conjecture, see also [Hem76, chapter 7]:

Theorem 3.9. *Let M be a compact orientable 3-manifold. Any non-trivial decomposition $\pi_1(M) \cong A \star B$ with $A \neq \{1\}$ and $B \neq \{1\}$ can be realized by a connected sum $M = M_1 \sharp M_2$ with $\pi_1(M_1) \cong A$ and $\pi_1(M_2) \cong B$.*

The idea is to realize the non-trivial groups A and B by complexes K_A, K_B with vanishing π_2, and join them by an edge to get a complex K with fundamental group $A \star B$ and still vanishing π_2. Then the isomorphism $\pi_1(M) \cong A \star B$ can be realized by a map $f : M \to K$ by starting from a triangulation of M and defining f first on the 1-skeleton by using the generators, then on the 2-skeleton by using the relations and then extending it to the 3-simplices by using the fact that $\pi_2(K)$ vanishes. By making f piecewise linear and taking the preimage of a regular value on the edge between K_A and K_B, one gets a bicollared 2-dimensional submanifold of M. Then by using the Loop Theorem (see Theorem 4.1) one can produce a surface Σ such that $\pi_1(\Sigma) = \{1\}$. So Σ is a separating 2-sphere yielding the desired connected sum. □

To get a full converse to Corollary 3.8 one needs the solution of the Poincaré Conjecture due to G. Perelman, see [Per03b], [MoT07]:

Theorem 3.10 (Perelman)**.** *Let M be a closed 3-manifold, then $\pi_1(M) = \{1\}$ if and only if M is homeomorphic to S^3.*

A groupe G is *freely indecomposable*, if it is neither trivial, nor infinite cyclic, nor isomorphic to the free product of two nontrivial groups.

Corollary 3.11. *A compact orientable 3-manifold is irreducible, with a possibly empty incompressible boundary, if and only if $\pi_1(M)$ is freely indecomposable.*

4. incompressible surfaces

A surface will always be assumed to be compact and orientable. Incompressible surfaces plays a key part in the study of 3-manifolds and their fundamental groups.

A properly embedded surface $(F, \partial F) \subset (M, \partial M)$ is called incompressible if the morphism $\pi_1(F) \to \pi_1(M)$ induced by the inclusion is injective and F does not bound a 3-ball. Otherwise Σ is called compressible.

The surface $(\Sigma, \partial\Sigma) \subset (M, \partial M)$ is called essential if it is incompressible and does not cobound a product region with a subsurface of ∂M.

The proof of the Dehn Lemma by C. Papakyriakopoulos [Papa57] has been a crucial step for the classification of 3-manifods and in particular the study of embedded surfaces in 3-manifolds. We state below the Loop Theorem, which is a stronger version due to J. Stallings [Sta60, Sta71]. It shows that the Euler characteristic of a compressible surface can be increased by cutting the surface along some embedded disk whose interior is disjoint from the surface.

Theorem 4.1 (Loop Theorem)**.** *Let M be a 3-manifold and let $F \subset \partial M$ be a boundary component. If* $\ker\{\pi_1(F) \to \pi_1(M)\} \neq \{1\}$*, then there exists a properly embedded disk $(D, \partial D) \hookrightarrow (M, \Sigma)$ such that $1 \neq [\partial D] \in ker\{\pi_1(\Sigma) \to \pi_1(M)\}$.*

Corollary 4.2. *Let M be a compact 3-manifold. Then $\pi_1(M) \cong \mathbb{Z}$ if and only if $M = S^1 \times S^2$ or $S^1 \times D^2$.*

Here is a useful way of getting a properly embedded incompressible surface:

Proposition 4.3. *Let M be a compact orientable 3-manifold. Let $\varphi : \pi_1(M) \to \mathbb{Z}$ be a non-trivial epimorphism. Then there exists a properly embedded essential surface F in M such that $\pi_1(F)$ is a subgroup of* $\ker \phi$.

Since S^1 is aspherical the epimorphism $\varphi : \pi_1(M) \to \mathbb{Z}$ can be realized by a homotopically non trivial smooth map $f : M \to S^1$. In particular f cannot be homotoped off any point in S^1. By Sard's theorem there is a point $x \in S^1$ which corresponds to a regular value of f and such that $F = f^{-1}(x)$ is an orientable codimension one properly embedded submanifold of M. Then by using the Loop Theorem one can perform a surgery on the surface F till it becomes π_1- injective. The process must stop since the Euler characteristic $\chi(F)$ increases after each compression and the surface must survive because the map f is not homotopically trivial. □

Corollary 4.4. *Let M be a compact orientable 3-manifold. If $H_1(M; \mathbb{Q}) \neq \{0\}$, then M contains a properly embedded essential surface. This is in particular true if ∂M is non empty and has a component of genus ≥ 1.*

If ∂M contains a component of genus ≥ 1, then there are two closed curves γ and γ' on ∂M which meet transversally in a single point. If say γ represents a trivial class in $H_1(M; \mathbb{Q})$, then it bounds a relative 2- cocycle in $H_2(M, \partial M; \mathbb{Q})$ whose intersection number with γ' is ± 1. Hence γ' cannot represent a trivial class in $H_1(M; \mathbb{Q})$. Therefore one of the curves γ or γ' must represent a non trivial class in $H_1(M; \mathbb{Q})$. □

4.1. Gluing along surfaces

Let $(X, \partial X)$ and $(Y, \partial Y)$ be two compact orientable 3-manifolds. Let $F_1 \subset \partial X$ and $F_2 \subset \partial Y$ be two boundary components. If there is a diffeomorphism $\psi : F_1 \to F_2$, one can glue F_1 and F_2:

$$M = X \cup_\psi Y = X \sqcup Y / \{x \in F_1 \sim \psi(x) \in F_2\}$$

If the morphisms $\pi_1(F_1) \to \pi_1(X)$ and $\pi_1(F_1) \to \pi_1(Y)$ are injective, then Seifert-van Kampen theorem implies that:

$$\pi_1(M) = \pi_1(X) \star \pi_1(Y) / \langle \psi_*(\pi_1(F_1)) = \pi_1(F_2) \rangle$$

Hence $\pi_1(M)$ is the free product of $\pi_1(X)$ and $\pi_1(Y)$ with amalgamation along the subgroups $\pi_1(F_1)$ and $\psi_*(\pi_1(F_1))$, which is usually noted $\pi_1(M) = \pi_1(X) \star_A \pi_1(Y)$, where A is identified with $\pi_1(F_1)$ in $\pi_1(X)$ and $\psi_*(\pi_1(F_1))$ in $\pi_1(Y)$.

Conversely a splitting of the group allows to produce an essential surface:

Proposition 4.5. *Let M be a compact orientable 3-manifold such that $\pi_1(M)$ is isomorphic to a non trivial amalgamated product $A \star_C B$ with $A \neq \{1\}$ and $B \neq \{1\}$ or to a non trivial HNN-extension $A \star_C$. Then M contains a properly embedded essential surface F such that $\pi_1(F) \subset C$, after conjugation.*

A quick proof consists in considering the proper action of $\pi_1(M)$ on the Bass-Serre tree associated to the splitting of $\pi_1(M)$ and to build an equivariant map f from the universal cover $\widetilde{M}$ of M to this Bass-Serre tree T. The preimage of a regular value on one edge of T gives an equivariant surface in $\widetilde{M}$ which projects to an embedded surface in M which can be compressed using the Loop Theorem to get an incompressible surface F whose fundamental group belongs to the stabilizer of the edge and thus is conjugate to a subgroup of C. Another method is to mimick Stalling's proof of the Kneser conjecture by using aspherical complexes. □

4.2. Dehn filling

Let X be a compact, orientable 3-manifold such that a component of ∂X is a torus T. Choose a simple closed curve $\alpha \subset T$ which does not bound on T and consider the manifold:

$$X(\alpha) = X \bigcup_{\alpha=\{\star\}\times\partial D^2} S^1 \times D^2$$

One says that $X(\alpha)$ is obtained from X by Dehn filling of the boundary component T along the curve α. The topological type of $X(\alpha)$ depends only of the homology class $[\alpha] \neq 0 \in H_1(T;\mathbb{Z})$, called the filling slope. A presentation of the fundamental group $\pi_1(X(\alpha))$ is obtained from a presentation of $\pi_1(X)$ by adding a relator corresponding to the element $\alpha \in \pi_1(X)$.

Dehn filling is a fundamental construction in 3-manifold topology, see [Gor95]. It can be used to produce many 3-manifolds M, with the homology of S^3. The exterior $E(K) = S^3 \setminus \mathcal{N}(K)$ of a knot $K \subset S^3$ is a compact orientable 3-manifold whose fundamental group $\pi_1(E(K))$ is normally generated by a single peripheral element, called a meridional element. The first homology group $H^1(E(K);\mathbb{Z}) \cong \mathbb{Z}$ is generated by the image of this meridional element. Let $X = E(K)$ and choose a simple closed curve $\alpha \subset \partial X$ which meets the boundary of a Seifert surface in a single point. Then $H_i(X(\alpha);\mathbb{Z}) = \{0\}$ for $i = 1, 2$. Moreover $X(\alpha)$ is homeomorphic to S^3 if and only if the curve α is homologous to the meridian on ∂X by P. Kronheimer and T. Mrowka's solution of the property P conjecture see [KrMr04].

4.3. Mapping tori

Let F be a compact, orientable surface and $\varphi : F \to F$ be an orientation preserving diffeomorphism, then one defines the mapping torus :

$$M = F \rtimes_\varphi S^1 := F \times [0,1]/\{(x,0) \sim (\varphi(x),1)\}$$

The surface F is a properly embedded, essential, non-separating surface in M. The homeomorphism type of M depends only on the isotopy class of the monodromy φ in the mapping class group $\pi_0 Diff(F)$ of the surface. Moreover M is aspherical unless F is the 2-sphere S^2.

The group $\pi_1(M)$ is isomorphic to the semi-direct product $\pi_1(F) \rtimes_{\varphi_*} \mathbb{Z}$, which corresponds to the split exact sequence:

$$1 \to \pi_1(F) \to \pi_1(M) \to \mathbb{Z} \to 1,$$

The action of $\mathbb{Z}$ on $\pi_1(F)$ is given by the automorphism $\varphi_* \in Aut(\pi_1(F))$ induced by the monodromy φ. Moreover the isomorphism type of $\pi_1(M)$ depends only on the class of φ_* in $\mathrm{Out}(\pi_1(F))$. For example $F \rtimes_\varphi S^1 \cong F \times S^1$ if and only if φ_* is an inner automorphism, which is equivalent for φ to be isotopic to the identity.

Because of the virtual fibering theorem (see Theorem 7.8 in section 7), mapping tori became, up to taking a finite cover, predominant in in the study of compact orientable, irreducible 3-manifolds with zero Euler characteristic.

5. Finitely generated subgroups

A *surface group* is a group isomorphic to the fundamental group of a closed orientable surface. Surface groups have been central in the study of 3-manifold groups. A group is said to be *coherent* if every finitely generated subgroup is finitely presentable. Coherence is a fundamental property of 3-manifold groups due to Peter Scott's compact core theorem [Sco73a, Sco73b], see also [RuSw90]:

Theorem 5.1 (Compact core theorem)**.** *A non compact 3-manifold M with finitely generated fundamental group contains a compact core, that is to say a 3-dimensional compact submanifold N such that the inclusion map $N \hookrightarrow M$ induces an isomorphism on fundamental groups. In particular $\pi_1(M)$ is finitely presentable.*

Since subgroups correspond to fundamental groups of covers, one obtains:

Corollary 5.2 (Coherence)**.** *Any finitely generated subgroup of a (possibly non compact)* 3-*manifold group is fintely presentable, that is to say 3-manifold groups are coherent.*

There are examples of closed hyperbolic 4-manifolds whose fundamental group is not coherent see [BoMe94] and [Pot94]. The direct product of two non cyclic free groups is not coherent. In particular such a group cannot be a 3-manifold subgroup.

The following result shows that the existence of a finitely generated normal subgroup of infinite index in a 3-manifold group corresponds to rather special topological situations. In its full generality it is obtained as a combination of results by J. Stallings [Sta62], J. Hempel-W. Jaco [HeJa72], P. Scott ([Sco83b], G. Mess [Mes01] (see also [Mai01, Mai03]), P. Tukia [Tuk88], D. Gabai [Ga92], A. Casson-D. Jungreis [CaJu94] and Perelman's Theorem 3.10.

Theorem 5.3. *Let M be an orientable compact 3-manifold. Let $K \trianglelefteq \pi_1(M)$ be a non trivial, finitely generated, normal subgroup of infinite index. Then one of the following cases occurs:*

(*i*) *$K \ncong \mathbb{Z}$ is the fundamental group of a compact surface, $\pi_1(M)/K \cong \mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$, and either M or a 2-fold cover of M is a surface bundle over S^1.*

(*ii*) *$K \cong \mathbb{Z}$, $\pi_1(M)/N$ is virtually the fundamental group of a compact surface, and M is virtually a S^1-bundle (i.e. a Seifert fibered manifold).*

Corollary 5.4. *Let M be a compact orientable 3-manifold. If $\pi_1(M)$ is isomorphic to a non trivial direct product then M is homeomorphic to a product $F \times S^1$ of a compact surface by a circle.*

The following classical lemma together with Corollary 3.8 shows that M, as above, is prime with possibly a non empty incompressible boundary. The only prime, non irreducible, compact orientable 3-manifold is $S^1 \times S^2$. Since K is a non trivial, infinite index, normal subgroup, $\pi_1(M)$ cannot be isomorphic to $\mathbb{Z}$ and therefore M must be irreducible.

Lemma 5.5. *A finitely generated normal subgroup $K \neq \{1\}$ of a non trivial free product $G_1 \star G_2$ has finite index.*

Cohomological computations, using the Hochschild-Serre spectral sequence, show that either $K \cong Z$ or the quotient $\pi_1(M)/K$ is virtually infinite cyclic.

The case (*i*) where $\pi_1(M)/K \cong \mathbb{Z}$ is due to Stallings [Sta62]. One first shows that K is the fundamental group of a non separating properly embedded essential surface F in M by applying the construction given in Corollary 4.4 to the epimorphism onto $\pi_1(M)/K \cong \mathbb{Z}$ and the fact that by Lemma 5.5 the only finitely generated normal subgroup of infinite index in a free group is the trivial group. Then one shows that the closure of the manifold obtained by cutting M along the surface F is homeomorphic to the product $F \times [0,1]$ by using the infinite cyclic cover of M associated to the epimorphism onto $\pi_1(M)/K \cong \mathbb{Z}$ and the following result (see [Sta62]):

Proposition 5.6. *Let M be a compact, orientable, irreducible 3-manifold and let $F \subset \partial M$ be a compact 2-manifold not homeomorphic to a sphere or a disk. If the inclusion map $F \hookrightarrow M$ induces an isomorphism on fundamental groups, then there is a homeomorphism between M and $F \times [0,1]$ which sends F to $F \times 0$.*

The case (*ii*) where $K \cong \mathbb{Z}$ corresponds to the characterization of Seifert manifolds. The first step (see G. Mess [Mes01] and also Maillot [Mai01]) consists in showing that the quotient group $\pi_1(M)/K$ is quasi-isometric to a complete, quasi- homogeneous Riemannian plane. The second step follows from work of Casson-Jungreis [CaJu94] and Gabai[Ga92] on convergence groups on the circle which show that $\pi_1(M)/K$ is virtually a surface group.

Remark. A statement analogous to Theorem 5.3 holds in the setting of 3-dimensional Poincaré duality groups, see section 9.

5.1. Simple loop conjecture

One possible way to try to produce surface groups in 3-manifold groups would be to prove an algebraic analogue of the Loop Theorem 4.1.

Conjecture 5.7 (Simple Loop Conjecture)**.** *Let $h : \pi_1(S) \to \pi_1(M)$ be a homomorphism from a closed orientable surface S of genus $g \geq 1$ to a closed orientable 3-manifold M. If $\ker(h) \neq \{1\}$, then there is an essential simple closed curve $\gamma \subset S$ which belongs to $\ker h$.*

The conjecture is true for a torus, but remains open for surface of genus $g \geq 2$. If the conjecture is true in general, a 2-sided map $f : S \to M$ which does not induce an injection on fundamental groups could be replaced by a 2-sided map of a surface of lower genus by performing a surgery on S and on the map f. After finitely many such steps one would get a π_1-injective map of a closed surface into M, but this surface could be a 2-sphere which may or may not bound a ball in M.

When the target is a surface the simple loop conjecture was proved by David Gabai [Ga85]. The simple loop conjecture holds when the target M carries a foliation by circles (Seifert fibered manifold) by J. Hass [Has99] and more generally for a graph manifold by H. Rubinstein and S. Wang [RuWa98] (the definition of a gaph manifold is given in the next section). More recently it has been established by Zemke [Zem16] when M carries the geometry SOL (geometric manifolds are defined in the next section).

Let S be a closed orientable surface of genus $g \geq 2$ and G be a group. A homomorphism $h : \pi_1(S) \to G$ admits an *essential factorization* through a non trivial free product $A \star B$ if $h = h_1 \circ \theta$ where $h_1 : \pi_1(S) \to A \star B$, $\theta : A \star B \to G$ are homomorphisms and the image $h_1(\pi_1(S))$ is not conjugate into one of the factors A or B. The following result of Stallings [Sta66] gives an alternative statement of the Simple Loop Conjecture.

Lemma 5.8. *Let $h : \pi_1(S) \to G$ be a homomorphism from the fundamental group of a closed orientable surface S of genus $g \geq 2$ into a group G. The kernel $\ker h$ contains the class of an essential simple closed curve if and only if h admits an essential factorization through a non trivial free product. In particular, if $\ker h$ does not contain a simple closed curve, the image $h(\pi_1(S))$ is freely indecomposable.*

Conjecture 5.9 (Essential factorization)**.** *Let S be a closed orientable surface of genus $g \geq 2$ and M be a closed orientable 3-manifold. Every homomorphism $h : \pi_1(S) \to \pi_1(M)$ is injective or admits an essential factorization through a free product.*

A result of T. Delzant [Del95] implies that, up to conjugacy, there are at most finitely many homomorphisms of a surface group in the fundamental group of a closed, orientable, hyperbolic 3-manifold, which do not admit an essential factorization through a free product. Hence:

Corollary 5.10. *Given a closed orientable surface S and a closed orientable hyperbolic 3-manifold M, up to conjugacy, there are at most finitely many homomorphisms $h : \pi_1(S) \to \pi_1(M)$ such that $\ker h$ does not contain a simple loop.*

This last result gives some evidence that the conjecture may be true for M a hyperbolic 3-manifold. Recently V. Markovic has announced a proof of the Simple Loop Conjecture for a hyperbolic 3-manifold.

5.2. Kleinian groups

A major result concerning fundamental groups of hyperbolic 3-manifolds (so called kleinian groups) is the proof of Marden's tameness conjecture by Agol [Ag04] and Calegary-Gabai [CaGa06], see [Can08] for a survey.

Theorem 5.11. *(Tameness Conjecture) Let M be an orientable hyperbolic 3-manifold. If* $\pi_1(M)$ *is finitely generated, then M is homeomorphic to the interior of a compact 3-manifold.*

Here is an important consequence of the tameness theorem together with works of F. Bonahon [Bon86], D. Canary [Can96] and W. Thurston [Thu79]:

Corollary 5.12. *Let M be a finite volume hyperbolic 3-manifold and* $\Gamma \subset \pi_1(M)$ *be a finitely generated subgroup. Then one of the following possibilities occures:*

(i) Γ *is the fundamental group of a* π_1*-injective immersed surface which lifts to a fiber in a finite cover which is a surface bundle (i.e.* Γ *is a virtual fiber).*

(ii) Γ *is a geometrically finite subgroup, which means that its Nielsen core has finite volume. For a surface subgroup it is equivalent to being quasi-fuchsian (i.e. the limit set of* Γ *is a circle).*

Let $\mathrm{Com}_{\pi_1(M)}(\Gamma) = \{g \in \pi_1(M) \mid \Gamma \cap g\Gamma g^{-1}$ is of finite index in $\Gamma\}$ be the *commensurator* of Γ in $\pi_1(M)$. An equivalent formulation of properties (i) and (ii) of Corollary 5.12 is as follows:
Property (i) is equivalent to the fact that $\mathrm{Com}_{\pi_1(M)}(\Gamma)$ has finite index in $\pi_1(M)$.
Property (ii) is equivalent the fact that Γ has finite index in its commensurator $\mathrm{Com}_{\pi_1(M)}(\Gamma)$.

6. Geometric decomposition

A *geometry* is a homogeneous, simply-connected, unimodular Riemannian manifold. A manifold is *geometric* if it is diffeomorphic to the quotient of a geometry by a discrete subgroup of its isometry group. We recall first the classification of the eight 3-dimensional geometries with maximal isometry group, see [Thu79, Thu97] [Sco83a], and also [Bon02], [BMP03].

Theorem 6.1 (Classification of 3-dimensional geometries)**.** *Up to equivalence there are exactly eight maximal geometries in dimension* 3*:*

1- Three isotropic geometries modeled on $\mathbb{S}^3$, $\mathbb{E}^3$, *and* $\mathbb{H}^3$*;*

2- Four anisotropic geometries whith isotropy subgroup $SO(2)$*, modeled on* $\mathbb{S}^2 \times \mathbb{E}^1, \mathbb{H}^2 \times \mathbb{E}^1$*, Nil and* $\widetilde{S}L(2,\mathbb{R})$*.*

3- The geometry SOL with trivial isotropy subgroup, modeled on the simply-connected 3*-dimensional sovable Lie group which is not nilpotent.*

A fundamental result of the beginning of this century is the full proof of Thurston's geometrization conjecture by Gregori Perelman [Per02, Per03a, Per03b], see also [KL08], [MoT07, MoT14], [CaZu06], [B3MP10].

Theorem 6.2 (Geometric decomposition)**.** *Any compact, orientable, irreducible 3-manifold splits along a finite collection of essential, pairwise disjoint and non parallel, embedded tori into geometric submanifolds.*

Such a decomposition with a minimal number of tori is unique up to isotopy and isometries of the geometric pieces. It is called the geometric decomposition of M.

When the geometric decomposition involves only Seifert fibered pieces, the manifold is called a *graph 3-manifold*. Such a manifold is obtained by gluing along some boundary components finitely many elementary pieces homeomorphic to a solid torus $S^1 \times D^2$ or a composite space $S^1 \times \{\text{punctured disk}\}$. In the case of a Haken 3-manifolds (i.e. when the manifold is irreducible and contains a properly embedded essential surface) which is not a torus bundle, the topological splitting along tori underlying the geometric decomposition is due to W. Jaco - P. Shalen [JaS79] and K. Johannson [Joh79] and is called the JSJ-splitting of the manifold.

The geometry of a compact, orientable, geometric 3-manifold with zero Euler characteristic can be characterized in term of its fundamental group.

Corollary 6.3 (Geometric manifolds)**.** *Let M be a compact, orientable* 3*-manifold with zero Euler characteristic.*

(i) *M is hyperbolic if and only if* $\pi_1 M$ *is infinite, freely indecomposable, not virtually cyclic and any* $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$ *is peripheral of infinite index.*

(ii) *M is elliptic (i.e. a finite quotient of* S^3*) if and only if* $\pi_1(M)$ *is finite.*

(iii) *M is euclidean if and only if* $\pi_1(M)$ *contains a* $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ *subgroup. Such a subgroup is normal of finite index.*

(iv) *M is nilpotent (i.e. carries the geometry Nil) if and only if* $\pi_1(M)$ *is infinite, nilpotent, but not virtually abelian.*

(v) *M carries the Seifert fibered geometry* $\widetilde{S}L(2, \mathbb{R})$ *if and only if* $\pi_1(M)$ *contains an infinite cyclic normal subgroup and is not nilpotent, nor virtually a product.*

(vi) *M carries the product geometry* $\mathbb{H}^2 \times \mathbb{E}^1$ *if and only if* $\pi_1(M)$ *is vrtually a non abelian product.*

(vii) *M carries a product geometry* $\mathbb{S}^2 \times \mathbb{E}^1$ *if and only if* $\pi_1(M)$ *is virtually infinite cyclic.*

($viii$) *M is a Solv manifold (i.e. carries the geometry Sol) if and only if* $\pi_1(M)$ *contains a normal* $\mathbb{Z} \oplus \mathbb{Z}$ *subroup but no normal* $\mathbb{Z}$*, that is to say* $\pi_1(M)$ *is solvable but not nilpotent.*

The geometries (ii) to (vii) correspond to the Seifert fibered manifolds. As a corollary one gets the following characterization of Seifert fibered orientable manifolds.

Corollary 6.4. *A compact, orientable 3-manifold M admits a Seifert fibration if and only if* $\pi_1(M)$ *is finite or admits an infinite cyclic normal subgroup.*

The geometric decomposition reduces many problem on 3-manifolds to the case of geometric manifolds and combination theorems. In particular it implies that every closed orientable aspherical 3-manifold is determined, up to homeomorphism, by its fondamental group. This is a special case of the so-called Borel conjecture.

Conjecture 6.5 (Borel)**.** *Two closed aspherical n-manifolds M and N with isomorphic fundamental groups are homeomorphic.*

The Borel conjecture is true in dimension 2 by J. Nielsen, using elementary homotopy theory. In dimension 3, it follows from Corollary 6.3, Mostow's rigidity theorem [Mos73] and Waldhausen's work [Wal68] for the case of closed, irreducible 3-manifolds containing an incompressible surface (so called Haken manifolds).

Theorem 6.6. *The Borel conjecture is true in dimension 3.*

Another consequence of the geometric decomposition is that the homeomorphism problem for closed orientable triangulated 3-manifolds is solvable, see [Kup17], see also [AFW15b], [B3MP10], [Mat03], [ScSh14].

Proposition 6.7 (Homeomorphism problem)**.** *The homeomorphism problem for closed orientable triangulated* 3*-manifolds is solvable.*

The geometric decomposition implies that the fundamental group of a compact orientable 3-manifold is isomorphic to the fundamental group of a graph of groups:

- The vertex groups are fundamental groups of geometric manifolds.
- The edge groups are trivial or isomorphic to $\mathbb{Z}^2$.

This graph of group structure has many important consequences. For example it allows us to extend to 3-manifold groups some interesting properties of fundamental groups of geometric 3-manifolds.

Corollary 6.8. *Let M be a compact orientable 3-manifolds. Then:*
(i) $\pi_1(M)$ *is residually finite (see* [Hem87]*).*

(ii) $\pi_1(M)$ *satifies the Tits alternative: any finitely generated subgroup contains a rank two free group or is virtually solvable.*

There are aspherical closed n-manifolds, $n \geq 4$, the fundamental groups of which are not residually finite see [Mes90].

Another consequence of this graph of group structure is that some classical decision problems formulated by Dehn are solvable in the class of 3-manifold groups:

(a) The *word problem* asks for an algorithm to decide whether or not a word in the generators represents the trivial element. Its solution for compact 3-manifold group follows from the property that 3-manifolds groups are residually finite, see for example [AFW15b].

(b) The *conjugacy problem* asks for an algorithm to decide whether or not a pair of words in the generators are conjugate. Its solution for compact 3-manifold groups is due to J-P. Préaux [Pre06, Pre16], see also [AFW15b].

(c) The *isomorphism problem* in a class of groups asks for an algorithm to decide whether or not two finite presentations of groups in the given class present isomorphic groups. The isomorphism problem for closed orientable 3-manifold groups is solvable, see [AFW15b], [ScSh14], [Sel95],

7. Some virtual properties of 3-manifold groups

7.1. Surface subgroups

In the following a surface group is the fundamental group of a closed orientable surface. There are many closed, irreducible, orientable 3-manifolds which do not contain any essential closed surface. Such 3-manifolds are called *small*. But it was a long standing conjecture (the *surface subgroup conjecture*) that the fundamental group of every closed, irreducible 3-manifold with infinite fundamental group contains a non trivial surface subgroup. Because of the geometric decomposition a small 3-manifold must be geometric, and thus must be hyperbolic or carry a Seifert fibered geometry. An irreducible Seifert fibered 3-manifold with infinite fundamental group always contains a subgroup $\mathbb{Z} \oplus \mathbb{Z}$. Hence the surface subgroup conjecture reduces to the case of hyperbolic 3-manifolds. It has been solved by J. Kahn and V. Markovic [KaM12]:

Theorem 7.1 (Surface subgroup theorem)**.** *A closed orientable hyperbolic 3-manifold M contains a dense set of π_1-injective, immersed, quasi-fuchsian, orientable, closed surfaces.*

Here dense means that every pair of distinct points in the sphere at infinity of the universal cover $\mathbb{H}^3$ can be separated by the limit set of a π_1-injective, immersed, quasi-fuchsian, orientable, closed surface in M.

A π_1-injective, immersed, closed surface is called quasi-fuchsian if its limit set is a circle. By the solution of the tamenes conjecture, a π_1-injective, immersed, closed surface in a closed orientable hyperbolic 3-manifold is either quasi-fuchsian or a virtual fiber, see Corollary 5.12

Theorem 7.1 has been a key step towards the proof of the Virtual Fibering Theorem for a closed hyperbolic 3-manifold M, see Theorem 7.8. Other key ingredients for the proof are the notions of non-positively curved special cube complex and of right-angled Artin groups (RAAG).

7.2. Right angled Artin groups (RAAG)

Let Γ be a finite , non empty, simple graph (i.e. without loops or multiple edges) and let $\{s_1, \cdots, s_k\}$ be its vertices. One associates to Γ the *right-angled Artin group* $A(\Gamma)$ with generators $\{s_1, \cdots, s_k\}$ and relations $[s_i, s_j] = 1$ when there is an edge between the vertices s_i and s_j. If Γ is the disjoint union of two graphs Γ_1 and Γ_2, then $A(\Gamma) \cong A(\Gamma_1) \star A(\Gamma_2)$. For a detailled introduction to RAAG see [Cha07].

There are very strong constraints for a RAAG to be virtually a 3-manifold group: it only happens when each connected component of Γ is a tree or a triangle. See [Dro87], [HeMe99], [Gor04] for the following result based on the fact that most RAAG are incoherent while 3-manifold groups are coherent by Theorem5.1:

Theorem 7.2 (RAAG versus 3-manifold group)**.** *A RAAG $A(\Gamma)$ is virtually a 3-manifold group if and only if one of the following cases occures:*

(*i*) *Each component of Γ is a tree and $A(\Gamma)$ is isomorphic to $\pi_1(S^3 \setminus L)$ where $L \subset S^3$ is a disjoint union of connected sums of copies of the Hopf link;*

(*ii*) *Each component is a triangle and $A(\Gamma)$ is isomorphic to $\pi_1(T^3 \sharp \cdots \sharp T^3)$.*

On the other hand deep results of I. Agol [Ag13] and D. Wise [Wi09, Wi12, Wi17] show that the fundamental group of a closed hyperbolic 3-manifold is virtually a subgroup of a RAAG. By a result of F. Haglund and D. Wise [HaW08] the fundamental group of a compact non-positively curved special cube complex is virtually a subgroup of a RAAG. Works of N. Bergeron-D. Wise [BeW10] and M. Sageev [Sag95] together with Theorem 7.1 show that the fundamental group of a closed hyperbolic 3-manifold M is isomorphic to the fundamental group of a compact non-positively curved cube complex. Next Agol [Ag13], based on works of Wise [Wi09, Wi12, Wi17] succeeded in showing that $\pi_1(M)$ is virtually the fundamental group of a compact non-positively curved special cube complex, and therefore virtually embeds into a RAAG. This result has been then extended to the case of any compact, irreducible, orientable 3-manifold with zero Euler characteristic which is not a graph manifold by P. Przytycki and D. Wise [PW14, PW18], see also Y. Liu [Liu13] for the graph manifold case.

Theorem 7.3 (Virtual embedding into RAAG)**.** *Let M be a compact, orientable, irreducible 3-manifold with zero Euler characteristic and which is not a closed graph manifold. Then $\pi_1(M)$ contains a finite index subgroup which is a subgroup of a RAAG.*

This virtual embedding result has some important algebraic consequences for 3- manifold groups:

Corollary 7.4. *Let M be a compact, orientable, irreducible 3-manifold with zero Euler characteristic and which is not a closed graph manifold. Then:*

(*i*) *$\pi_1(M)$ is large, i.e. for every $n \in \mathbb{N}^*$ there is a finite index subgroup which surjects onto a free group of rank n. In particular M has infinite virtual first betty number.*

(*ii*) *$\pi_1(M)$ is linear over $\mathbb{Z}$, that is to say it admits a faithful representation into $GL_k(\mathbb{Z})$ for some integer $k \geq 2$, which a priori depends of M.*

Property (*ii*) follows from the fact that RAAG groups are commensurable with right angled Coxeter groups which are known to be $\mathbb{Z}$-linear, see [HaW10]. It raises the following question:

Question 7.5. *With respect to property* (*ii*) *is there a uniform dimension k for a linear representation over $\mathbb{Z}$, or even over $\mathbb{C}$?*

Fundamental groups of nilpotent manifolds, Solv manifolds or aspherical Seifert fibered manifolds which do not carry a product geometry do not virtually embed into a RAAG, but they are linear. Moreover by [Liu13] one knows precisely which closed orientable graph manifolds have a fundamental group which virtually embeds into a RAAG, and hence are linear too. So one could expect a positive answer to the following question:

Question 7.6. *Are closed graph manifold groups linear ?*

The *subgroup separability property* of a closed hyperbolic 3-manifold group is a key property to study virtual properties of hyperbolic 3-manifolds. It is equivalent to the condition that finitely generated groups are closed in the profinite topology of $\pi_1(M)$. This is the topology where the finite index subgroups form a fundamental system of neighborhoods for the trivial element of $\pi_1(M)$ (see Section 8).

Theorem 7.7. *The fundamental group of a closed orientable hyperbolic 3-manifold is subgroup separable or LERF (locally extended residually finite), that is to say every finitely generated subgroup is the intersection of finite index subgroups.*

There are examples of graph-manifold groups which are not LERF [BKS87]. Being LERF is not a common property for finitely generated groups. Free groups are LERF, but for example the group $F_2 \times F_2$ is not LERF, see [AG73]. There are also examples of non LERF groups which are free products of LERF groups with an amalgamated cyclic subgroup [Ri90].

7.3. Virtual Fibering Theorem

After Perelman's proof of the existence of a geometric decomposition, one of the most important achievements in 3-manifold topology is the Virtual Fibering Theorem of Agol [Ag08, Ag13] and Przytycki-Wise [PW14, PW18]. It shows that most compact, irreducible, orientable 3-manifolds with zero Euler characteristic virtually fiber over the circle.

Theorem 7.8 (Virtual fibering theorem)**.** *Any compact, orientable, irreducible 3-manifold with zero Euler characteristic that is not a closed graph manifold admits a finite cover which is a surface bundle over* S^1.

This theorem is a consequence of the fact that the fundamental group of such a manifold is virtually a subgroup of a RAAG (Theorem 7.3) and of Agol's virtual fibration criterion [Ag08]. The key property is the RFRS (Residually Finite Rationally Solvable) property which is verified by RAAG and hence by its subgroups. According to [Ag08] a group π is RFRS if there is a cofinal chain $\pi = \pi_0 \supset \pi_1 \supset \pi_2 \supset \cdots$ of finite index normal subgroups π_i of π such that for each i the epimorphism $\pi_i \to \pi_i/\pi_{i+1}$ factors through $H_1(\pi_i; \mathbb{Q})$. See [FrKi14], [AFW15a] for more details.

It follows from Theorem 7.8 that virtually the fundamental group of a closed, aspherical, orientable 3-manifold has a rather special form:

Corollary 7.9. *The fundamental group of a closed, aspherical, orientable 3-manifold contains a finite index subgroup of one of the following types:*

(i) *An extension of* $\mathbb{Z}$ *by a surface group;*

(ii) *An extension of a surface group by* $\mathbb{Z}$*;*

(iii) *The fundamental group of a graph of groups where each vertex group is a product of a free group by* $\mathbb{Z}$ *and each edge group is* $\mathbb{Z} \oplus \mathbb{Z}$.

Given a prime integer p, a group is called residually p-finite if the intersection of its p-power index normal subgroups is trivial. It is called virtually residually p-finite if it has a finite index subgroup which is residually p-finite. Free groups and Surface groups are residually p-finite for every prime p. M. Aschenbrenner and S. Friedl [AF13] have proved that every 3-manifold group is virtually residually p-finite for all but finitely many primes p. Right angled Artin groups (RAGG) have the stronger property to be residually p-finite for every prime p. So 3-manifold groups which virtually embeds into a RAAG have a finite index subgroup which is residually p-finite for every prime p.

8. Profinite properties of 3-manifold groups

8.1. Profinite completion of a group

A standard way to study residually finite infinite groups is through their finite quotients. The question of how much of the topology of a 3-manifold can be detected from the finite quotients of its fundamental group has attracted some attention recently, see [Re15, Re18]. To a group π we associate the inverse system $\{\pi/K\}_K$ where K runs over all finite index normal subgroups of π. The profinite completion $\widehat{\pi}$ of π is then defined as the inverse limit of this system, i.e.

$$\widehat{\pi} = \varprojlim \pi/K.$$

When each finite group π/K is endowed with the discrete topology, we equip $\widehat{\pi}$ with the finest topology such that all the epimorphisms $\widehat{\pi} \to \pi/K$ are continuous.

Here is a more explicit way, to define the topology on the profinite completion $\widehat{\pi}$. Consider the set of all finite index normal subgroups K of π and equip each finite quotient π/K with the discrete topology. Then the product

$$\prod_K \pi/K$$

is a compact group. The diagonal map $g \in \pi \to \{gK\}_K$ defines a homomorphism:

$$i_\pi \colon \pi \to \prod_K \pi/K.$$

This homomorphism $i_\pi \colon \pi \to \widehat{\pi}$ is injective since π is residually finite. The profinite completion of π can be defined as the closure :

$$\widehat{\pi} = \overline{i_\pi(\pi)} \subset \prod_K \pi/K.$$

By construction $\widehat{\pi}$ is a compact, totally disconnected, topological group. A subgroup $U \subset \widehat{\pi}$ is open if and only if it is closed and of finite index. A subgroup $H \subset \widehat{\pi}$ is closed if and only if it is the intersection of all open subgroups of $\widehat{\pi}$ containing it. The induced topology on π is called the profinite topology: a neighbourhood basis for the trivial element is given by the set of finite index normal subgroups of π.

When π is finitely generated, a deep result of N. Nikolov and D. Segal [NS07] states that every finite index subgroup of $\widehat{\pi}$ is open. The proof uses the classification of finite simple groups. It means that $\widehat{\widehat{\pi}} = \widehat{\pi}$. In particular the map $K \subset \pi \to \overline{K} \subset \widehat{\pi}$ gives a one-to-one correspondence between the subgroups with the same finite index in π and in $\widehat{\pi}$. Moreover $\overline{K} = \widehat{K}$ in this case. The inverse map is given by $H \subset \widehat{\pi} \to H \cap \pi \subset \pi$.

A group homomorphism $\varphi \colon \pi_1 \to \pi_2$ between two groups π_1 and π_2 induces a continuous homomorphism $\widehat{\varphi} \colon \widehat{\pi}_1 \to \widehat{\pi}_2$. Moreover if φ is an isomorphism, then so is $\widehat{\varphi}$. On the other hand, a homomorphism $\phi \colon \widehat{\pi}_1 \to \widehat{\pi}_2$ is not necessarily induced by a homomorphism $\varphi \colon \pi_1 \to \pi_2$. If π_1 and π_2 are finitely generated it follows from [NS07] that any homomorphism between $\widehat{\pi}_1$ and $\widehat{\pi}_2$ is continuous.

Finitely generated groups with isomorphic profinite completions have the same set of finite quotients. The converse is true see [DFPR82], [RZ10, Corollary 3.2.8]

Lemma 8.1. *Two finitely generated groups have isomorphic profinite completions if and only if they have the same set of finite quotients.*

This follows from the fact that one can define the profinite topology of a finitely generated group π by using the cofinal collection of characteristic subgroups:

$$\pi(n) = \bigcap_{[\pi:K] \leq n} K.$$

Therefore studying properties of residually finite and finitely generated groups with isomorphic profinite completions is equivalent to the study of properties or invariants which are detected by their finite quotients.

8.2. 3-manifold groups

M will be a compact orientable aspherical 3-manifold with empty or toroidal boundary. The geometric decomposition implies that fundamental groups of 3-manifolds are residually finite, see [Hem87]. Hence $\pi_1(M)$ injects into its profinite completion $\widehat{\pi_1(M)}$.

Following [Re15, Re18] an orientable compact 3-manifold M is called *profinitely rigid* if $\widehat{\pi_1(M)}$ distinguishes $\pi_1(M)$ from all other 3-manifold groups. Otherwise it is called *profinitely flexible*

It follows from the geometric decomposition that a compact orientable aspherical 3-manifold which does not contain any essential properly embedded annulus is determined, up to homeomorphism, by its fundamental group, see [Joh79]. Hence profinite rigidity for such a manifold implies that it is determined, up to homeomorphism, by the set of finite quotients of its fundamental group. However there are examples of closed 3-manifolds which are not profinitely rigid. All the examples known at the moment are graph manifolds, they include:

(a)— Infinitely many Seifert fibered manifolds which are surface bundles with periodic monodromy (J. Hempel [Hem14]).

(b)— Infinitely many Sol manifolds (L. Funar,[Fun13], P. Stebe [Ste72]).

(c)— Infinitely many graph manifolds with a non trivial geometric decomposition (G. Wilkes [Wil18a])

G. Wilkes [Wil17] showed that the Hempel's examples are the only profinitely flexible closed Seifert fibered 3-manifolds. Later in [Wil18a] he gave necessary and sufficient conditions on the geometric decomposition of a (non Seifert fibered and non Solv) graph manifold for the manifold to be profinitely rigid. The case of Solv manifolds has been recently handled by G. Nery in [Ner18].

As a consequence of his work G. Wilkes [Wil18a] obtained:

Theorem 8.2. *Two orientable graph 3-manifolds M_1 and M_2 such that $\widehat{\pi_1(M_1)} \cong \widehat{\pi_1(M_2)}$ are commensurable.*

This result raises the following question:

Question 8.3 (Commensurability)**.** *Given two orientable aspherical 3-manifold M and N does $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$ imply that $\pi_1(M)$ and $\pi_1(N)$ are commensurable?*

One could ask for a stronger property which holds true for graph manifolds by [Wil18a]:

Question 8.4 (Virtual rigidity)**.** *Does any compact orientable aspherical 3-manifold admit a profinitely rigid finite sheeted cover?*

In particular what about hyperbolic 3-manifolds? Once-punctured torus bundles over the circle are profinitely rigid by M. Bridson-A. Reid-H. Wilton [BRW17]. Moreover G. Gardam [Gar18] showed that finite volume hyperbolic 3-manifolds are distinguished by the finite quotients of their fundamental groups among the snaPea census of 72942 finite volume hyperbolic manifolds. Since no examples of profinitely flexible hyperbolic 3-manifolds are known so far, these result gives some support for a positive answer to the following question:

Question 8.5 (Rigidity)**.** *Is a complete, finite volume, hyperbolic 3-manifold profinitely rigid?*

A weaker version of profinite rigidity would be to ask for finiteness instead of uniqueness, see [AFW15a, p. 138]:

Conjecture 8.6 (Finiteness)**.** *Given a compact, orientable, aspherical 3-manifold M, there are only finitely many compact, orientable, irreducible 3-manifolds N with $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$.*

By [Wil17, Wil18a] and [Ner18], this conjecture holds true for graph 3-manifolds. It would be true for complete, finite volume, hyperbolic 3-manifolds M provided that the profinite completion $\widehat{\pi_1(M)}$ determines the volume. This raises the question:

Question 8.7. *Which invariants or properties of a compact orientable irreducible 3-manifold M are determined by $\widehat{\pi_1(M)}$?*

They are called profinite invariants or properties. It is conjectured that the volume of a hyperbolic 3-manifold is a profinite invariant:

Conjecture 8.8. *The sum of the volumes of the hyperbolic pieces of the geometric decomposition of a compact orientable aspherical 3-manifold is a profinite invariant.*

Since most compact aspherical 3-manifolds with zero Euler characteristic are virtually fibered over the circle, it is natural to ask whether this fiberness property is a profinite property. This has been proved by A. Jaikin-Zapirain [JZ17].

Theorem 8.9 (Fiberness). *Let G be an extension of $\mathbb{Z}$ by a surface group or a finitely generated free group. Any compact, orientable 3-manifold M such that $\widehat{\pi_1(M)} \cong \widehat{G}$ is a surface bundle over the circle.*

8.3. Cohomological properties

A group π is called *good* if for any finite abelian group A and any representation $\alpha\colon \pi \to \mathrm{Aut}_{\mathbb{Z}}(A)$ the inclusion $\iota\colon \pi \to \widehat{\pi}$ induces an isomorphism of twisted cohomology groups:

$$\iota^* : H^j_\alpha(\widehat{\pi}; A) \to H^j_\alpha(\pi; A),\ \forall j.$$

This notion has been introduced by J.P. Serre [Ser97]. It is crucial to transfer cohomological informations via profinite completion. For example if π is good of finite cohomological dimension then $\widehat{\pi}$ is torsion free.

The proof of the following theorem [Cav12] uses the fact that surface groups are good and the virtual fibration Theorem 7.8

Theorem 8.10. *The fundamental group of any compact aspherical* 3*-manifold is good.*

A first corollary states:

Corollary 8.11. *For a compact aspherical* 3*-manifolds:*

(i) *The property of being closed is a profinite property.*

(ii) *The Euler characteristic $\chi(M)$ is a profinite invariant.*

Assertion (i) follows from the fact that an aspherical 3-manifold is closed if and only if the third cohomology group with $\mathbb{Z}_2$-coefficients is non-zero. Asserion (ii) is a direct consequence of the computation of the Euler characteristic.

8.4. Geometries

The profinite completion distinguishes the hyperbolic geometry among Thurston's geometries because hyperbolic manifold groups are residually non-abelian simple, while the other geometric manifold groups contain non-trivial normal subgroups. Much deeper results of H. Wilton and P.Zaleskii [WZ17a, WZ17b] and G. Wilkes [Wil18a, Wil18b] state that the profinite completion detects the eight geometric structures and the geometric decomposition of a closed orientable aspherical 3-manifold:

Theorem 8.12. *Let M and N two closed orientable aspherical 3-manifolds such that $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$, then:*

(1) *If M admits a geometric structure then N admits the same geometric structure.*

(2) *The underlying graph of the geometric decompositions of M and N are isomorphic and the corresponding vertex groups have isomorphic profinite completion, hence the same geometry.*

A key ingredient for the proof of these results is the *efficiency* of the geometric decomposition of a compact, aspherical, orientable 3-manifold M: it means that the profinite topology of $\pi_1(M)$ induces the profinite topology of the edge and vertex groups of the corresponding decomposition of $\pi_1(M)$ as a graph of groups. Moreover the edge and vertex groups are closed in the the profinite topology of $\pi_1(M)$. The closed case follows from [WZ14, Theorem A], and the case with toroidal boundary can be deduced from it by gluing some hyperbolic pieces onto each boundary component in order to get a closed manifold whose geometric decomposition is given by the geometric decomposition of the original manifold together with the attached hyperbolic manifolds.

Another ingredient of the proof is the use of an analogue of the classical Bass-Serre theory in the setting of finite graphs of profinite groups. Like in the classical case one can define the fundamental group of a finite graph of profinite groups and build a profinite Bass-Serre tree on which this group acts, see [Rib17]. A consequence of the efficiency of the geometric decomposition of an aspherical, compact, orientable 3-manifold M is that the profinite completion of the fundamental group of the corresponding graph of groups is the fundamental group of the finite graph of profinite groups obtained from the JSJ-graph of M by replacing each vertex and edge group by its profinite completion and each monomorphism from an edge group into a vertex group by its extension to the profinite completions. The associated profinite Bass-Serre tree is the inverse limit of the Bass-Serre trees of the graph of groups decompositions of the finite index, normal subgroups of $\pi_1(M)$ induced by the geometric decomposition of the corresponding finite cover of M. The profinite completion $\widehat{\pi_1(M)}$ acts continuously on the profinite Bass-Serre tree with quotient the graph of the geometric decomposition of M. Moreover the Bass-Serre tree associated to the geometric decomposition of M embeds and is dense in the profinite Bass-Serre tree.

Theorem 8.12 and the profinite Bass-Serre theory are crucial to establish a weaker version of profinite rigidity for 3-manifold groups called Grothendieck rigidity.

8.5. Grothendieck rigidity

The following question of Grothendieck [Gr70] initiated the study of the profinite rigidity of residually finite, finitely presented groups:

Grothendieck problem *Let $f : \pi_1 \to \pi_2$ be a homomorphism of finitely presented, residually finite groups for which the extension $\widehat{f} : \widehat{\pi_1} \to \widehat{\pi_2}$ is an isomorphism. Is f an isomorphism?*

Bridson-Grunewald [BG04] answered negatively this question in 2004. Finitely generated exemples were previously given by Platonov-Tavgen [PT86] in 1986.

According to [LR11] a residually finite group G is said to be *Grothendieck rigid* if for every finitely generated proper subgroup $H \subset G$ the inclusion induced map $\widehat{H} \to \widehat{G}$ is not an isomorphism.

For instance free groups and surface groups are Grothendieck rigid. The case of closed or geometric 3-manifold groups follows from [Cav12] and [LR11], and the general case with boundary from [BoFr17].

Theorem 8.13. *The fundamental group of any irreducible, orientable, compact, connected 3-manifold with empty or toroidal boundary is Grothendieck rigid.*

Scott's compact core Theorem 5.1 allows to show that the Grothendieck rigidity for a 3-manifold group is equivalent to the following topological statement [BoFr17]

Theorem 8.14. *A map $f : M \to N$ between two aspherical, orientable, compact, connected 3-manifold with empty or toroidal boundary induces an isomorphism $\widehat{f_*} : \widehat{\pi_1(M)} \to \widehat{\pi_1(N)}$ if and only if it is a homotopy equivalence.*

The map f is not assumed to be proper, that is to say to send ∂M in ∂N. The Grothendieck rigidity shows that the profinite flexibility of a 3-manifold group cannot be induced by a map. This might lead one to believe that profinite flexibility among 3-manifold groups is in some sense exotic.

In [BrWi14] M. Bridson and H. Wilton showed that given an arbitrary pair of finitely presented, residually finite groups $f : \pi_1 \hookrightarrow \pi_2$ there is no algorithm to decide whether or not the induced map $\widehat{f} : \widehat{\pi}_1 \to \widehat{\pi}_2$ is an isomorphism. Also there does not exist either algorithms that can decide whether the profinite map $\widehat{f}$ is surjective, or whether $\widehat{\pi}_1$ and $\widehat{\pi}_2$ are isomorphic.

8.6. Knot groups

Knot groups occupy a special place among 3-manifold groups and thus it is natural to study their profinite completions. The following theorem summarises the main results obtained so far about profinite completions of knot groups see [BoFr15], [BrRe15], [Uek18], [Wil18c]

Theorem 8.15. *Let K_1 and K_2 be two knots in S^3 such that $\widehat{\pi_1(E(K_1))} \cong \widehat{\pi_1(E(K_2))}$. Then the following results hold:*

1. *K_1 and K_2 have the same Seifert genus;*

2. *K_1 is fibered if and only if K_2 is fibered;*

3. *K_1 and K_2 have the same Alexander polynomial;*

4. *If K_1 is an iterated torus knot, then $K_1 = K_2$;*

5. *If K_1 is the figure-eight knot, then $K_1 = K_2$..*

The proof that the Seifert genus of a knot is a profinite invariant relies on the goodness property of knot groups and the computation of the Seifert genus via the homology of the knot group with coefficient twisted by finite linear representations.

The profinite completion $\widehat{\pi_1(E(K))}$ detects the figure-eight complement and the iterated torus knot complements among all compact connected 3-manifolds, see [BrRe15], [Wil18c]. Thus one may wonder to what extend the profinite completion allows to characterise knot groups among 3-manifold groups.

One may also expect knot groups to be profinitely rigid among knot groups. Prime knots with isomorphic groups have homeomorphic complements by W. Whitten [Wh87], hence one can expect:

Conjecture 8.16. *A prime knot $K \subset S^3$ is determined, up to homeomorphism, by $\widehat{\pi_1(E(K))}$.*

9. Poincaré duality groups

When G is the fundamental group of a closed, aspherical n-manifold, its homology and cohomology with coefficients in a $\mathbb{Z}G$-module A satisfy a form of Poincaré duality between $H^i(G;A)$ and $H_{n-i}(G;A)$. A group with this property is called an *n-dimensional Poincaré duality group* or a *PD(n)-group*.

A group G is a PD(n)-group if it acts freely, properly discontinuously and cocompactly on a contractible cell complex X with $H^*_c(X,\mathbb{Z}) \cong H^*_c(\mathbb{R}^n,\mathbb{Z})$. This property is equivalent to the following algebraic conditions (see [Bro82, Chapter VIII]):

(1) G is of type *FP*: there is a projective $\mathbb{Z}[G]$-resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ which is finitely generated in each dimension and 0 in all but finitely many dimensions;

(2) $H^n(G;\mathbb{Z}G) \cong \mathbb{Z}$ and $H^i(G;\mathbb{Z}G) = \{0\}$ for all $i \neq n$.

The *FP* condition implies that PD(n)-groups are torsion-free and finitely generated. The second one implies that for $n \geq 2$, PD(n)-groups are 1-ended and therefore indecomposable by Stallings theorem (cf. [Sta71, 4.A.6.6])

A PD(n)-group G is said *orientable* if the action of G on $H^n(G;\mathbb{Z}G)$ is trivial, and non-orientable otherwise. Each PD(n)-group contains an orientable PD(n)-group of index 1 or 2.

We list below some basic properties of a PD(n)-group G, see [Bro82, Chapter VIII]:

(*i*) G is finitely generated.

(*ii*) G has finite cohomological dimension $cd(G) = n$.

(*iii*) The following three properties are equivalent for a subgroup $H \subset G$:

1. H is a PD(n)-group

2. H is a finite index subgroup of G

3. $cd(H) = n$

(iv) A subgroup H has infinite index in G if and only if $cd(H) < n$.

In the 60's Wall asked whether this duality property is sufficient to characterise the fundamental groups of closed aspherical n-dimensional manifolds. For each $n \geq 4$, M. Davis [Dav98, Theorem C] produced examples of PD(n)-groups which are not finitely presentable, and thus cannot be the fundamental group of a closed, aspherical n-dimensional manifold. This leads to the following conjecture:

Conjecture 9.1 (Wall)**.** *A finitely presented PD(n)-group is isomorphic to the fundamental group of a closed, aspherical n-dimensional manifold.*

B. Eckmann, P. Linnell and H. Müller solved the 2-dimensional case, see [Ec87].

Theorem 9.2. *A PD(2)-group is a surface group.*

In dimension 3 the question remains widely open, though spectacular progress in the understanding of the algebraic properties of 3-manifolds groups have been made. In the remainder of this section PD(3)-group are assumed to be orientable, since one can reduce the Wall conjecture to the orientable case. It is worth remarking that by the Sphere Theorem 3.3 and Perelman's Theorem 3.10 a PD(3)-group can only be the fundamental group of a closed, aspherical 3-manifold. Results similar to some of the fundamental results in 3-manifold theory, have been established in the setting of PD(3)-groups, see [Hil19], [Wa03, Wa04], [Tho95]. They give some evidence towards Wall's conjecture in dimension 3. For a detailled exposition of results in this direction, we refer to J. Hillman's books [Hil02, Hil19].

B. Bowditch [Bow04] verified Wall Conjecture for a $PD(3)$ groups which contains a non-trivial, normal cyclic subgroup, see also [Hil85] for the case of positive first Betti number. Combining results of B. Bowditch [Bow04], J. Hillman[Hil85, Hil87] and C.B. Thomas [Tho84] allows us to establish a PD(3)-version of Theorem 5.3:

Theorem 9.3. *A PD(3)-group G contains a non trivial finitely generated normal subgroup K of infinite index such that G/K is finitely presentable if and only if G is isomorphic to the fundamental group of a* 3*-manifold which is either a surface bundle or Seifert fibered.*

By a surface bundle we mean either a bundle over S^1, or the union of two twisted I-bundles along their boundaries (often called semi-bundle).

An important special case of Theorem 9.3 states:

Corollary 9.4. *A PD(3)-group G is isomorphic to the fundamental group of a surface bundle over S^1 if and only if it admits an epimorphism onto $\mathbb{Z}$ with finitely generated kernel.*

In the 3-manifold case the finitely presentable assumption for the quotient G/K in Theorem 9.3 is not needed since 3-manifold groups are coherent by Scott's Theorem 2.2, hence G and K are finitely presentable and so is G/K. This raises the following questions:

Question 9.5. *Is a PD(3)-group finitely presentable? Is it coherent? Is it almost coherent?*

Remark. Theorem 9.3 remains true under the weaker assumption that the quotient G/K is *almost finitely presentable* or FP_2. A group G is called FP_n if there is a projective $\mathbb{Z}[G]$-resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ which is finitely generated in degrees n or less. The condition FP_1 is equivalent to being finitely generated. Finitely presentable groups are FP_2, but the converse does not hold. Since $PD(3)$-groups are FP, they are finitely generated and almost finitely presentable. A group G is *almost coherent* if every finitely generated subgroups is FP_2. It leads to a stronger, homological version of Theorem 9.3 which shows the importance of the FP_2 condition.

Theorem 9.6. *A PD(3)-group G contains a non-trivial FP_2 normal subgroup of infinite index if and only if G is isomorphic to the fundamental group of a* 3*-manifold which is either a surface bundle or Seifert fibered.*

Several other deep results illustrate the strong correlation between the algebraic properties of $PD(3)$ groups and those of 3-manifold groups. For example M. Dunwoody and E. Swenson [DuSw00] proved the following version of the torus theorem:

Theorem 9.7 (Torus Theorem)**.** *If an orientable $PD(3)$ group G contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup then either G splits over some subgroup $\mathbb{Z} \oplus \mathbb{Z}$ or G is a Seifert fibered 3-manifold group.*

More generally an orientable PD(3)-group admits a canonical splitting along free abelian groups of rank 2 with vertex groups isomorphic to either Seifert manifold groups or atoroidal $PD(3)$ pairs in the sense of [Cas07, Definition 1(2)]. This splitting as a graph of groups is analogous to the one induced on the fundamental group of a closed orientable aspherical 3-manifold by the JSJ-splitting. See [DuSw00] and [Wa03, Theorem 10.8] or [Wa04, Theorem 4.2] for the case of finitely presentable $PD(3)$ groups, and [Cas07] or [Kro90] and [Hil06] for the general case which avoids the finitely presentable assumption (cf. [Hil19] for more details).

These results show that one of the main open cases of the Wall Conjecture 9.1 is the case of atoroidal PD(3)-groups, that is when the PD(3)-group does not contain any subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. A possible approach in this case would be to split the Wall Conjecture into two conjectures with a more geometric flavour: the weak hyperbolization conjecture and the Cannon conjecture.

The weak hyperbolization conjecture is a generalization of the hyperbolization conjecture in the setting of geometric group theory. It holds true for a CAT(0) PD(3)-group by [KaK07].

Conjecture 9.8 (Weak hyperbolization conjecture)**.** *The Caley graph of an atoroidal finitely presented PD(3)-group is Gromov-hyperbolic.*

It follows from [BeM91] that a torsion free group whose Caley graph is Gromov-hyperbolic is a PD(3)-group if and only if its boundary is homeomorphic to S^2. Then the solution of the geometrization conjecture implies that for word hyperbolic PD(3)-groups the Wall Conjecture is equivalent to the Cannon Conjecture

Conjecture 9.9 (Cannon Conjecture)**.** *A torsion free, infinite group whose Caley graph is Gromov-hyperbolic and whose boundary is homeomorphic to S^2 is a Kleinian group.*

V. Markovic [Mar13], see also [Hai14], has showed that the Cannon Conjecture for a group G is equivalent to the existence of enough quasi-convex surface subgroups in G to separate every pair of points in the boundary of G. This raises the question of the existence of surface subgroups in a PD(3)-group.

Conjecture 9.10. *An atoroidal PD(3)-group always contains a PD(2)-group.*

A weaker interesting step in this direction would be to show that an atoroidal PD(3)-group always contains a FP_2 subgroup of infinite index. This would imply that an atoroidal PD(3)-group always contain a non-abelian free group by the following result due to M. Kapovich and B. Kleiner [KaK05, Corollary 1.3(2)]

Proposition 9.11. *An infinite index FP_2 subgroup of an orientable $PD(3)$ group either contains a surface subgroup or is free.*

This, in turn, would be sufficient to show that PD(3)-groups verfy the Tits alternative, since by the torus theorem [DuSw00] a non atoroidal PD(3)-group is a Seifert fibered 3-manifold group or splits over a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup. The Tits alternative for PD(3)-groups is still an open question, see [BoB19], [Hil03], [Hil19] for a detailed discussion of this question.

The fact that PD(3)-groups are torsion free and the geometrization of finite group actions on a closed orientable 3-manifold implies that a $PD(3)$ group is isomorphic to a 3-manifold group if and only if it is virtually a 3-manifold group. (See [GMW12, Theorem 5.1].) This result together with the virtual fibration Theorem 7.8 and Theorem 9.3 leads to the following criterion for an atoroidal PD(3)-group to be a 3-manifold group:

Corollary 9.12. *An atoroidal PD(3)-group is the fundamental group of a closed 3-manifold if and only if it virtually splits as an extension of $\mathbb{Z}$ by a finitely generated group.*

Recently D. Kielak generalised Agol's virtual fibration criterion [Ag08] to the setting of finitely generated groups. His main result [Kie18, Thm 5.3] together with Theorem 7.8 and Corollary 9.12 gives the following more algebraic criterion:

Corollary 9.13. *An atoroidal PD(3)-group G is the fundamental group of a closed 3-manifold if and only if G is virtually RFRS and its first L^2-Betti number $\beta_1^{(2)}(G) = 0$.*

The criterion given in Corollary 9.13 shows that it may be of interest to study profinite properties of PD(3)-groups. Few results are known in this direction, even concerning some basic properties which seem difficult to establish without geometry:

Question 9.14. *Let G be a PD(3)-group:*

(1) *Does G admit a non trivial finite quotient?*

(2) *Is the profinite completion of G infinite?*

(3) *Is G residually finite?*

(4) *Is G a good group in the sense of Serre?*

For 3-manifold groups these properties are consequences of the existence of a geometric decomposition (Theorem 6.2) and of the virtual fibering theorem (Theorem 7.8). It is also worth to mention that G. Mess [Mes90] has exhibited PD(n)-groups which are not residually finite for $n \geq 4$. Moreover M. Bridson and H. Wilton [BrWi15] have showed that there is no algorithm to determine whether or not a finitely presented group has a non-trivial finite quotient.

So the following weaker version of Wall Conjecture 9.1 is of interest:

Conjecture 9.15. *A PD(3)-group whose profinite completion is isomorphic to the profinite completion of a 3-manifold group is isomorphic to a 3-manifold group.*

It is even not known whether the assumption that the profinite completion of the PD(3)-group G is isomorphic to the profinite completion of a 3-manifold group does imply that G is residually finite or good.

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