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CORRUGATION PROCESS AND ϵ-ISOMETRIC MAPS

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ABSTRACT. — Convex Integration is a theory developed in the '70s by M. Gromov. This theory allows to solve families of differential problems satisfying some convex assumptions. From a subsolution, the theory iteratively builds a solution by applying a series of *convex integrations*. In a previous paper [6], we proposed to replace the usual convex integration formula by a new one called *Corrugation Process*. This new formula is of particular interest when the differential problem under consideration has the property of being *of Kuiper*. In this paper, we consider the differential problem of ϵ-isometric maps and we prove that it is Kuiper in codimension 1. As an application, we construct ϵ-isometric maps from a short map having a conical singularity.

RÉSUMÉ. — L'intégration convexe est une théorie développée dans les années 70 par M. Gromov. Cette théorie permet de résoudre des familles de problèmes différentiels vérifiant certaines hypothèses de convexité. A partir d'une sous-solution, elle construit itérativement une solution en appliquant une succession *d'intégrations convexes*. Dans un précédent article [6], on a proposé une formule, appelés *procédé de corrugation*, alternative à la formule d'intégration convexe. Cette nouvelle formule est particulièrement intéressante dans le cas où le problème différentiel considéré possède la propriété d'être *de Kuiper*. Ici on considère le problème différentiel des applications ϵ-isométriques et on prouve qu'il vérifie la propriété d'être de Kuiper en codimension 1. A titre d'application, nous montrons comment construire directement des applications ϵ-isométriques à partir d'applications courtes ayant une singularité conique.

1. General introduction

1.1. The Nash–Kuiper Theorem

A map $f : (M,g) \to \mathbb{E}^n$ between a Riemannian manifolds (M,g) and the Euclidean space $\mathbb{E}^n = (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ is said to be isometric if $g = f^*\langle\cdot,\cdot\rangle$. It is said to be *strictly short* if $g - f^*\langle\cdot,\cdot\rangle$ is positive definite (as usual f^*h denotes the pullback of the metric h by f). In other words, the length

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of the image of any curve in M by a strictly short map is shorter than the length of the curve in M. The C^1 embedding theorem of Nash and Kuiper states that close to every strictly short map lies a C^1-isometric map:

THEOREM 1.1 ([3, 4]). — *Let (M^m, g) be a compact Riemannian manifold and let $f_0 : (M^m, g) \to (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, with $n > m$, be a strictly short embedding. Then for any $\epsilon > 0$ there exists a C^1-isometric embedding $f : (M^m, g) \to (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ such that,* $\sup_x \|f(x) - f_0(x)\| < \epsilon$.

The proof considers an increasing sequence of metrics g_k converging toward g and a decreasing sequence ϵ_k converging toward 0. A sequence of maps $f_1, \ldots, f_k, \ldots$ is then iteratively built such that, for each k, f_k is an ϵ_k-isometric map from (M, g_k) to $\mathbb{E}^n$ i. e.

$$\|g_k - f_k^* h\| < \epsilon_k.$$

Parameters of the construction are chosen to insure the C^1 convergence of the sequence $(f_k)_k$ so that the limit map f_∞ is isometric.

1.2. Differential relations

We now introduce the formalism of Gromov's Convex Integration Theory [2]. This theory can be seen as a wide generalization of Nash's approach. It provides a powerful tool to solve a large family of differential constraints. We denote by

$$J^1(M, W) := \{(x, y, L) \,|\, x \in M,\ y \in W,\ L : T_x M \to T_y W \quad \text{a linear map}\}.$$

the 1-jet space of C^1 maps between M and W. Every C^1-map f gives rise to a section $j^1 f : x \mapsto (x, f(x), df_x)$ of $J^1(M, W)$ called the *1-jet* of f. For any section $\mathfrak{S}_0 : x \mapsto (x, f_0(x), L_x)$ we denote by $f_0 = \text{bs} \quad \mathfrak{S}_0$ its *base map*.

A *differential relation* $\mathcal{R}$ is any subset of the 1-jet space $J^1(M, W)$. For instance, the ϵ-isometric condition defines the differential relation $\mathcal{I}s(\epsilon)$ of ϵ-isometric maps:

$$\mathcal{I}s(\epsilon) := \{(x, y, L) \mid \|g_x - L^* h_y\| < \epsilon\}$$

where $L^* h$ denotes the pullback by L of the metric h. Observe that, as a topological subspace, $\mathcal{I}s(\epsilon)$ is open.

DEFINITION 1.2. — *Let $\mathfrak{S} : M \to J^1(M, W)$ be a section. We say that $\mathfrak{S}$ is a* formal solution *of $\mathcal{R}$ if the image of $\mathfrak{S}$ lies in $\mathcal{R}$. Moreover, if there exists a C^1-map $f : M \to W$ such that $j^1 f = \mathfrak{S}$, we say that $\mathfrak{S}$ is a* holonomic solution *of $\mathcal{R}$.*

For instance, building an ϵ-isometric map f is equivalent to finding a holonomic section $j^1 f$ of $\mathcal{I}s(\epsilon)$.

Under some topological and convex assumptions on $\mathcal{R}$, the Convex Integration Theory allows to deform a formal solution $\mathfrak{S}_0$ to a holonomic one. Each convex integration modifies the 1-jet of a formal solution (x_0, f_0, L_0) in a given direction u to obtain a new formal solution $\mathfrak{S} = (x, f, L)$ such that $L(v) = df(v)$. Loosely speaking $\mathfrak{S}$ is "partially" holonome in the direction u. Here is how it works for $M = [0,1]^m$ and $W = \mathbb{E}^n$. In this case a formal solution writes

$$\mathfrak{S}_0 : x \mapsto \big(x, f_0(x), v_1(x), \ldots, v_m(x)\big) \in \mathcal{R}$$

where we have identified the 1-jet space with the product

$$J^1([0,1]^m, \mathbb{R}^n) = [0,1]^m \times \mathbb{R}^n \times (\mathbb{R}^n)^m$$

and $\mathfrak{S}_0$ is holonomic if there exists a map f such that

$$\mathfrak{S}_0 = j^1 f : x \mapsto \big(x, f(x), \partial_1 f(x), \ldots, \partial_m f(x)\big).$$

From a formal solution $\mathfrak{S}_0$, the Convex Integration Theory builds a finite sequence of formal solutions $\mathfrak{S}_k$ such that for every $k \in \{1, \ldots, m\}$ we have

$$\mathfrak{S}_k : x \mapsto (x, f_k(x), \partial_1 f_k(x), \ldots, \partial_k f_k(x), v_{k+1}(x), \ldots, v_m(x)) \in \mathcal{R}.$$

In particular

$$\mathfrak{S}_m = j^1 f_m : x \mapsto (x, f_m(x), \partial_1 f_m(x), \ldots, \partial_m f_m(x))$$

is a holonomic solution of $\mathcal{R}$.

1.3. Corrugation Process

To build the sequence $\mathfrak{S}_k$, we propose in [6] to replace the usual formula of the Convex Integration Theory by another one, called Corrugation Process:

Definition 1.3. — *Let $f_0 : [0,1]^m \to \mathbb{R}^n$ be a map, ∂_j be a direction, $\gamma : [0,1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ be a loop family and $N \in]0, +\infty[$. We define the map $f_1 : [0,1]^m \to \mathbb{R}^n$ by*

$$f_1(x) := f_0(x) + \frac{1}{N} \int_{s=0}^{N x_j} \gamma(x,s) - \overline{\gamma}(x) ds \tag{1.1}$$

where $\overline{\gamma}(x) = \int_0^1 \gamma(x,s)ds$ denotes the average of the loop $t \mapsto \gamma(x,t)$. We say that f_1 is obtained from f_0 by a Corrugation Process *in the direction ∂_j and we denote $f_1 = CP_\gamma(f_0, \partial_j, N)$.*

The Corrugation Process satisfies the following three properties which are at the basis of the Convex Integration Theory [5].

Proposition 1.4 ([6]). — *The map* $f_1 = CP_\gamma(f_0, \partial_j, N)$ *satisfies*

(P_1) $\|f_0 - f_1\|_{C^0} = O(1/N)$,

(P_2) $\|\partial_i f_0 - \partial_i f_1\|_{C^0} = O(1/N)$ *for every* $i \neq j$.

Moreover if $\forall\, x \in [0,1]^m$ *we have* $\partial_j f_0(x) = \overline{\gamma}(x)$ *then*

(P_3) $\partial_j f_1(x) = \gamma(x, Nx_j) + O(1/N)$ *for all* $x \in [0,1]^m$.

Provided that N is large enough, this proposition shows that the Corrugation Process allows to modify the j^{th} partial derivative while keeping the other derivatives under control. Consequently, this formula performs the deformations required to build the sequence $(\mathfrak{S}_k)_k$ provided γ has values in a well chosen region. For more details and for a proof of this proposition see [6]. We give below a coordinate free expression of the Corrugation Process:

Definition 1.5. — *Let* $f_0 : U \to (W, h)$ *be a map from an open set* $U \subset M$, $\pi : U \to \mathbb{R}$ *be a submersion and* $\gamma : U \times \mathbb{R}/\mathbb{Z} \to f_0^* TW$ *be a loop family such that* $\gamma(x, .) : \mathbb{R}/\mathbb{Z} \to f_0^* TW_x$ *for every* $x \in U$. *The map defined by Corrugation Process is defined by*

$$f_1 = CP_\gamma(f_0, \pi, N) : x \mapsto \exp_{f_0(x)}\left(\frac{1}{N}\int_{t=0}^{N\pi(x)} \gamma(x,t) - \overline{\gamma}(x)dt\right)$$

where $\exp : TW \to W$ *is the exponential map induced by the metric* h.

1.4. Subsolutions

Subsolutions are a refinement of the notion of formal solution. This refinement is needed to ensure the existence of a loop family γ whose its values is chosen in an appropriate region and whose its average is the partial derivative $\partial_j f_0$ under consideration (see property (P_3) of Proposition 1.4).

Let $\mathcal{R}$ be a differential relation, $\sigma = (x, y, L) \in \mathcal{R}$ and $(\lambda, u) \in T_x^* M \times T_x M$ such that $\lambda(u) = 1$. We set

$$\mathcal{R}(\sigma, \lambda, u) := Conn_{L(u)} \{v \in T_y W \mid (x, y, L + (v - L(u)) \otimes \lambda) \in \mathcal{R}\}$$

where $Conn_a A$ denotes the path connected component of A that contains a. We say that $\mathcal{R}(\sigma, \lambda, u)$ is the *slice* of $\mathcal{R}$ over $\mathfrak{S}$ with respect to (λ, u). Note that the linear map $L + (v - L(u)) \otimes \lambda$ coincides with L over $\ker \lambda$ and maps u to v. We then denote by IntConv $\mathcal{R}(\sigma, \lambda, u)$ the interior of the convex hull of $\mathcal{R}(\sigma, \lambda, u)$.

DEFINITION 1.6. — *Let $U \subset M$, $\pi : U \to \mathbb{R}$ be a submersion and $u : U \to TM$ be a vector field such that $d\pi_x(u_x) = 1$. Let $x \mapsto \mathfrak{S}(x) = (x, f_0(x), L(x))$ be a formal solution of $\mathcal{R}$ over U. If for every x in U the base map $f_0 = \mathrm{bs}\ \mathfrak{S}$ satisfies*

$$df_0(u_x) \in \mathrm{IntConv}\ \mathcal{R}(\mathfrak{S}(x), d\pi_x, u_x)$$

then the formal solution $\mathfrak{S}$ is called a subsolution of $\mathcal{R}$ with respect to $(d\pi, u)$.

In the case where $U = [0,1]^m$, $W = \mathbb{R}^n$, $\pi(x) = x_j$ and $u = \partial_j$, the condition of the definition means that $\partial_j f_0(x)$ lies in the interior of the convex hull of

$$\begin{aligned}&\mathcal{R}(\sigma, dx_j, \partial_j)\\ &\quad = Conn_{v_j}\left\{t \in \mathbb{R}^n \mid (x, f_0(x), v_1, \ldots, v_{j-1}, t, v_{j+1}, \ldots, v_m) \in \mathcal{R}\right\}.\end{aligned}$$

From a subsolution $\mathfrak{S}$ of $\mathcal{R}$ with respect to $(d\pi, u)$ the convex integration theory builds a map f_1 whose derivative along u_x lies in the slice $\mathcal{R}(\mathfrak{S}(x), d\pi_x, u_x)$:

LEMMA 1.7. — *Let $\mathcal{R}$ be an open differential relation and let $\mathfrak{S}$ be a subsolution of $\mathcal{R}$ with respect to $(d\pi, u)$ and with base map $f_0 = \mathrm{bs}\ \ \mathfrak{S}$. Then there exists a loop family γ such that for every $x \in U$ we have $\overline{\gamma}(x) = df_0(u_x)$ and for every $(x,t) \in U \times \mathbb{R}/\mathbb{Z}$ the image of γ lies in $\mathcal{R}(\mathfrak{S}(x), d\pi_x, u_x)$. If we set $f_1 := CP_\gamma(f_0, \pi, N)$ for this loop family γ, we have*

$$\forall\, x \in U, \quad df_1(u_x) \in \mathcal{R}(\mathfrak{S}(x), d\pi_x, u_x)$$

for N large enough.

Proof. — The existence of γ follows the Integral Representation Lemma of the Convex Integration Theory of Gromov ([2, p. 169] or [5, p. 29]). The property on $df_1(u_x)$ is a direct consequence of the point (P_3) of Proposition 1.4. □

1.5. Kuiper relations

In the usual approach, the family of loops γ is constructed a posteriori once the subsolution $\mathfrak{S}$ given. However the construction of a holonomic solution often requires to repeat the Corrugation Process in several directions ∂_j and consequently needs to re-build at each step the loop family γ on a different subsolution at each time. In [6], we propose to simplify this approach by constructing a bigger loop family $\tilde{\gamma}$ that could be used indifferently regardless of the subsolution. This simplification leads to introduce the notion of surrounding loop family and then the notion of Kuiper relation.

Basically, a surrounding family is a family of loops lying inside $\mathcal{R}$ which is double indexed by its base point σ and its average w and where (σ, w) are allowed to vary in the largest possible space, that is, inside

$$\mathrm{IntConv}(\mathcal{R}, d\pi, u) := \left\{(\sigma, w) \in p_y^* TW \;\middle|\; w \in \mathrm{IntConv}\ \mathcal{R}(\sigma, d\pi_x, u_x)\right\}.$$

In that definition, $p_y^* TW$ is the bundle over $\mathcal{R}$ induced by the projection $p_y : \mathcal{R} \to W$, $\sigma = (x, y, L) \mapsto y$.

Definition 1.8. — *Let $\mathcal{R}$ be a differential relation of $J^1(U, W)$. We say that a loop family*

$$\begin{aligned} \widetilde{\gamma} : \mathrm{IntConv}(\mathcal{R}, d\pi, u) &\longrightarrow C^0(\mathbb{R}/\mathbb{Z}, TW) \\ (\sigma, w) &\longmapsto \widetilde{\gamma}(\sigma, w)(\cdot) \end{aligned}$$

is surrounding with respect to $(d\pi, u)$ *if for every* (σ, w) *we have*

1. $t \mapsto \widetilde{\gamma}(\sigma, w)(t)$ *is a loop in* $\mathcal{R}(\sigma, d\pi_x, u_x)$,
2. *the average of* $t \mapsto \widetilde{\gamma}(\sigma, w)(t)$ *is* w,
3. *there exists a continuous homotopy* $H : \mathrm{IntConv}(\mathcal{R}, d\pi, u) \times [0,1] \to TW$ *such that* $H(\sigma, w, 0) = \widetilde{\gamma}(\sigma, w)(0)$, $H(\sigma, w, 1) = L(u_x)$ *and* $H(\sigma, w, t) \in \mathcal{R}(\sigma, d\pi_x, u_x)$ *for all* $t \in [0, 1]$.

Note that point (3) is a homotopic property needed to state a potential h-principle for $\mathcal{R}$.

Then for any subsolution $\mathfrak{S} = (x, f_0, L)$ we choose the loop family

$$\gamma(x, t) := \widetilde{\gamma}\big(\mathfrak{S}(x), df_0(u_x)\big)(t) \in \mathcal{R}(\sigma, d\pi_x, u_x)$$

for every $(x, t) \in U \times \mathbb{R}/\mathbb{Z}$, and we write $CP_{\widetilde{\gamma}}(\mathfrak{S}, \pi, N) := CP_\gamma(f_0, \pi, N)$.

We would like to ensure that all loops $\widetilde{\gamma}(\sigma, w)$ share the same pattern.

Definition 1.9. — *Let $p, q > 0$ be two natural numbers and $A \subset \mathbb{R}^q$ be a parameter space. A family of 1-periodic curves $c : A \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^p$ is said to be* a pattern.

We denote by $E \to W$ the fiber bundle over W with fiber $\mathcal{L}(\mathbb{R}^p, T_y W) = (\mathbb{R}^p)^* \otimes T_y W$ and we consider its pull back by the projection

$$q : \mathrm{IntConv}(\mathcal{R}, d\pi, u) \to W,\ (\sigma, w) \mapsto y.$$

A section ψ of $q^* E$ defines a family of linear maps $\psi(\sigma, w) : \mathbb{R}^p \to T_y W$.

Definition 1.10. — *Let c be a loop pattern. If there exist a surrounding loop family γ : $\mathrm{IntConv}(\mathcal{R}, d\pi, u) \to C^0(\mathbb{R}/\mathbb{Z}, TW)$ with respect to $(d\pi, u)$, a section ψ of $q^*E \to \mathrm{IntConv}(\mathcal{R}, d\pi, u)$ and a map $\mathbf{a}$: $\mathrm{IntConv}(\mathcal{R}, d\pi, u) \to A$ such that, for all $((\sigma, w), t) \in \mathrm{IntConv}(\mathcal{R}, d\pi, u) \times \mathbb{R}/\mathbb{Z}$,*

$$\gamma(\sigma, w)(t) = \psi(\sigma, w) \circ c(\mathbf{a}(\sigma, w), t)$$

we then say that $\mathcal{R}$ is a Kuiper relation with respect to $(c, d\pi, u)$.

If $(c_1, \ldots, c_p)$ denote the components of c in the standard basis of $\mathbb{R}^p$ and if $\mathbf{e}_1, \ldots, \mathbf{e}_p$ denote the image of this basis by ψ, the above definition writes

$$\gamma(\sigma, w)(t) = \sum_{i=1}^{p} c_i(\mathbf{a}(\sigma, w), t)\, \mathbf{e}_i(\sigma, w).$$

We denote the periodic primitive of the c_i's by

$$C_i(a, t) = \int_{s=0}^{t} c_i(a, s) - \overline{c_i}(a) ds.$$

Proposition 1.11. — *Let c be a loop pattern, $\mathcal{R}$ be an open Kuiper relation with respect to $(c, d\pi, u)$, $\mathfrak{S} = (x, f_0, L_0)$ be a subsolution and γ be a c-shaped surrounding loop family. Then $f_1 = CP_\gamma(\mathfrak{S}, \pi, N)$ has the following analytic expression*

$$f_1(x) = \exp_{f_0(x)} \left(\frac{1}{N} \sum_{i=1}^{p} C_i(a(x), N\pi(x)) e_i(x) \right) \tag{1.2}$$

where $a(x) := \mathbf{a}(\mathfrak{S}(x), df_0(u_x))$, $e(x) := \mathbf{e}(\mathfrak{S}(x), df_0(u_x))$ and $x \in U$. Moreover, if N is large enough, the section

$$x \mapsto \mathfrak{S}_1 := (x, f_1, L_1 = L_0 + (df_1(u_x) - L_0(u_x)) \otimes d\pi)$$

is a formal solution of $\mathcal{R}$.

In the case where $U = [0,1]^m$, $W = \mathbb{R}^n$, $\pi(x) = x_j$ and $u = \partial_j$ the map $f_1 = CP_\gamma(\mathfrak{S}, \partial_j, N)$ is given by

$$f_1(x) = f_0(x) + \frac{1}{N} \left(\sum_{i=1}^{p} C_i(a(x), Nx_j) e_i(x) \right).$$

In [6] the reader will find a proof of the Proposition 1.11 as well as examples of Kuiper relations. In the next section, we prove that the relation of ϵ-isometric maps is Kuiper in codimension one.

2. The relation of ϵ-isometric maps

In this article, we prove the following theorem:

Theorem 2.1. — *Let M and W be orientable Riemannian manifolds such that* $\dim W = \dim M + 1$. *For every $\epsilon > 0$, the relation $\mathcal{I}s(\epsilon)$ is a Kuiper relation.*

The key point of the proof of this theorem is to build a loop family $\hat{\gamma}$ c-shaped for all couples (σ, w) such that σ belongs to $\mathcal{I}s(\epsilon)$ and w belongs to the convex hull of the slice $\mathcal{I}s(\epsilon)(\sigma, \lambda, u)$, for some λ, u. To understand the slice $\mathcal{I}s(\epsilon)(\sigma, \lambda, u)$ and its convex hull, we first present its geometric description and a description of its subsolutions. We then give a proof of Theorem 2.1.

2.1. Geometric description of the relation of isometric maps

The relation of ϵ-isometric maps is a thickening of the relation of isometric maps

$$\mathcal{I}s := \{(x, y, L) \mid g = L^*h\} \subset J^1(M, W)$$

where g is a metric of M and L^*h is the pullback by L of the metric h of W. So in this paragraph we give a geometric description of the relation of isometric maps. Such a description can be found in [2, p. 202], [5, p. 194]. For the sake of completeness we recall this description here in the coordinate-free case and we give some extra details needed for our construction of a surrounding loop family of the relation of ϵ-isometric maps.

Let $\sigma = (x, y, L) \in \mathcal{I}s$. Let $\lambda \in T_x^*M$ and $u \in T_xM$ such that $\lambda(u) = 1$. For every $v \in T_yW$, we set $L_v := L + (v - L(u)) \otimes \lambda$. We have

$$\begin{aligned}\mathcal{I}s(\sigma, \lambda, u) &:= Conn_{L(u)}\{v \in T_yW \mid (x, y, L_v) \in \mathcal{I}s\}\\ &= Conn_{L(u)}\{v \in T_yW \mid g_x = L_v^*h_y\}.\end{aligned}$$

Note that, by the definition of L_v, we have $L_v(u) = v$ and for every $u_0 \in \ker\lambda$ we have $L_v(u_0) = L(u_0)$, in particular $L_v(\ker\lambda) = L(\ker\lambda)$. Let $w_1 = \alpha_1 u + a_1$ and $w_2 = \alpha_2 u + a_2$ with $\alpha_1, \alpha_2 \in \mathbb{R}$ and $a_1, a_2 \in \ker\lambda$. As $g = L^*h$, we have

$$\begin{aligned}&(g - L_v^*h)(w_1, w_2)\\ &= \alpha_1\alpha_2\big(g(u,u) - h(v,v)\big) + \alpha_1 h\big(L(u) - v, L(a_2)\big) + \alpha_2 h\big(L(u) - v, L(a_1)\big).\end{aligned}$$

From this expression it is readily seen that $g = L_v^* h$ if and only if $g(u,u) = h(v,v)$ and $v \in L(u) + L(\ker\lambda)^\perp$. So v lies inside the $(n-1)$-dimensional sphere S_u of radius $\|u\|_g$ and inside the affine $(n-m+1)$-plane

$$P_u := L(u) + L(\ker\lambda)^\perp.$$

Thus $\mathcal{I}s(\sigma,\lambda,u) = S_u \cap P_u$ is a $(n-m)$-dimensional sphere of T_yW and its convex hull is a ball of the same dimension (see Figure 2.1). Since we have assumed $n > m$, the space $\mathcal{I}s(\sigma,\lambda,u)$ is arc-connected. Since $\mathcal{I}s(\sigma,\lambda,u)$ is a $(n-m)$-dimensional sphere, IntConv $\mathcal{I}s(\sigma,\lambda,u)$ is a $(n-m+1)$-ball of P_u.

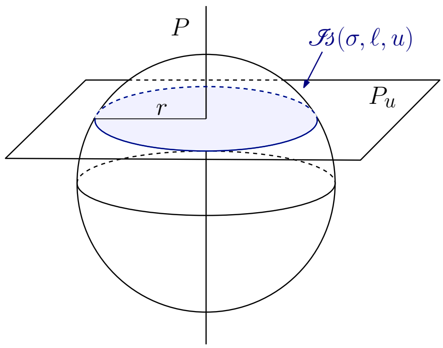

Figure 2.1. The slice $\mathcal{I}s(\sigma,\lambda,u)$ and its convex hull: *the* $(n-m)$-*dimensional sphere in dark blue is* $\mathcal{I}s(\sigma,\lambda,u)$ *and the convex hull* IntConv $\mathcal{I}s(\sigma,\lambda,u)$ *is the* $(n-m+1)$-*ball in light blue.* P *denotes the* $(m-1)$-*plane* $L(\ker\lambda)$

So a slice of the relation of ϵ-isometric maps is a thickening of $\mathcal{I}s(\sigma,\ell,u)$ (see Figure 2.2).

2.2. Characterization of subsolutions of the relation of isometric maps

Let proj_0 be the orthogonal projection on $\ker\lambda$ in T_xM and proj_P be the orthogonal projection on $P = L(\ker\lambda)$ in T_yW. We characterize subsolutions of $\mathcal{I}s$ with respect to $(d\pi, u)$, for a submersion $\pi : U \subset M \to \mathbb{R}$ and a tangent vector field $u : U \to TM$ such that $d\pi(u) = 1$, in the following Proposition 2.2:

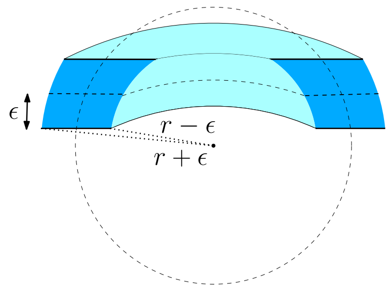

Figure 2.2. Illustration of a slice of $\mathcal{I}s(\epsilon)$: *in blue, a piece of* $\mathcal{I}s(\epsilon)(\sigma,\lambda,u)$*. The slice* $\mathcal{I}s(\epsilon)(\sigma,\lambda,u)$ *is obtained as the intersection of the* ϵ*-thickening of the* $(n-m+1)$*-plane* P_u *and the* ϵ*-thickening of the* $(n-m)$*-sphere* S_u *of radius* r*.*

Proposition 2.2. — *Let* $f_0 : M \to W$ *be a* C^1*-map and* $P := df_0(\ker d\pi)$ *such that* $\dim\ P(x) = m-1$ *for all* $x \in U$*. If* f_0 *satisfies* $g|_{\ker d\pi} = f_0^* h|_{\ker d\pi}$*, then a section*

$$x \mapsto \mathfrak{S}(x) = \Big(x, f_0(x), L_x := (df_0)_x + \big(v_x - (df_0)_x(u_x)\big) \otimes d\pi_x\Big)$$

is a formal solution of $\mathcal{I}s$ *with respect to* $(d\pi, u)$ *if and only if, for every* x*, the vector* v_x *can be written in the form* $v_x = \operatorname{proj}_{P(x)} L_x(u_x) + \tau_x$ *where* $\tau_x \in P(x)^\perp$ *and* $\|\tau_x\|_h = r(x) = \sqrt{\|u_x\|_g^2 - \|\operatorname{proj}_0 u_x\|_g^2}$*.*

Proof. — Recall that $v_x \in \mathcal{I}s(\mathfrak{S}(x), d\pi_x, u_x)$ if and only if $v_x \in S_{u(x)} \cap P_{u(x)}$ i.e.

$$\|v_x\|_h^2 = \|u_x\|_g^2 \quad \text{and} \quad \operatorname{proj}_{P(x)} v_x = \operatorname{proj}_{P(x)} L(u_x).$$

Decomposing v_x in $P(x) \oplus P(x)^\perp$, we have $v_x = \operatorname{proj}_{P(x)} L(u_x) + \widetilde{\tau_x}$, where $\widetilde{\tau_x}$ is a vector of $P(x)^\perp$ of norm $\|\widetilde{\tau_x}\|_h = r(x)$ by definition of r. Now we have to give an expression of the radius r which only depends on u and not to $\mathfrak{S}$. By the Pythagorean theorem we have

$$r(x)^2 = \|\widetilde{\tau_x}\|_h^2 = \|v_x\|_h^2 - \|\operatorname{proj}_{P(x)} L(u_x)\|_h^2.$$

As $\|v_x\|_h = \|u_x\|_g$, we then have $\|\widetilde{\tau_x}\|_h^2 = \|u_x\|_g^2 - \|\operatorname{proj}_{P(x)} L(u_x)\|_h^2$. The space $P = L(\ker\lambda)$ depends on L, so $\mathfrak{S}$. Let $u_x = \operatorname{proj}_0 u_x + (u_x - \operatorname{proj}_0 u_x)$ with proj_0 the orthogonal projection on $\ker\lambda$. Then

$$L(u_x) = L\big(\operatorname{proj}_0 u_x + (u_x - \operatorname{proj}_0 u_x)\big) = L(\operatorname{proj}_0 u_x) + L(u_x - \operatorname{proj}_0 u_x).$$

As L is isometric we have, for any $a \in \ker\lambda$ and $b \in (\ker\lambda)^{\perp}$,

$$\langle a, b\rangle = 0 \Leftrightarrow \langle L(a), L(b)\rangle = 0.$$

In particular, for $b = u - \operatorname{proj}_0 u$, that implies $L(u - \operatorname{proj}_0 u) \in L(\ker\lambda)^{\perp} = P(x)^{\perp}$. Thus $\operatorname{proj}_{P(x)} L(u_x) = L(\operatorname{proj}_0 u_x)$ and

$$\left\|\operatorname{proj}_{P(x)} L(u_x)\right\|_h = \|L(\operatorname{proj}_0 u_x)\|_h = \|\operatorname{proj}_0 u_x\|_g$$

the last equality comes from L is isometric. So

$$r(x)^2 = \|\widetilde{\tau_x}\|_h^2 = \|v_x\|_h^2 - \|\operatorname{proj}_0 u_x\|_g^2 . \qquad \square$$

2.3. Proof of Theorem 2.1

We begin with a preparatory Lemma 2.3, then describe $\operatorname{IntConv} \mathcal{I}s(\epsilon)(\sigma, \lambda, u) \cap P_u(w)$ and define a c-shaped loop family for the relation $\mathcal{I}s(\epsilon)$. We finally construct $\widehat{\gamma}$ and prove that it is surrounding.

Let $\sigma = (x, y, L) \in \mathcal{I}s(\epsilon)$. Let $\lambda \in T_x^*M$, $u \in T_xM$ such that $\lambda(u) = 1$, and let $w \in \operatorname{IntConv} \mathcal{I}s(\epsilon)(\sigma, \lambda, u)$. Note that as $\mathcal{I}s(\epsilon)$ is a thickening of $\mathcal{I}s$ and by definition of σ and w, the distance (for the metric h) between w and P_u is less than 2ϵ, but w does not belong necessarily to P_u. We denote by $P_u(w)$ the affine $(n-m+1)$-plane that contains w and which is a translation of P_u:

$$P_u(w) := \{v \in T_yW \mid \operatorname{proj}_P w = \operatorname{proj}_P v\}$$

where P denotes $L(\ker\lambda)$. Thanks to the following Lemma 2.3, we can assume that w belongs to P_u:

Lemma 2.3. — *Let $(\sigma, w) \in \operatorname{IntConv}(\mathcal{I}s(\epsilon), \lambda, u)$ with $\sigma = (x, y, L)$. There exists a homotopy $\sigma_t = (x, y, L_t)$ such that $\sigma_0 = \sigma$, $\sigma_t \in \mathcal{I}s(\epsilon)(\sigma, \lambda, u)$ for all $t \in [0,1]$, and $\operatorname{proj}_P L_1(u) = \operatorname{proj}_P w$.*

Proof. — We set $v_0 = L_0(u) = L(u)$. We can assume that $\|v_0\|_h \geqslant \|w\|_h$. Indeed, if $\|v_0\|_h < \|w\|_h$ we perform a first homotopy. Let

$$\widetilde{L_t} = L + (\widetilde{v_t} - v_0) \otimes \lambda$$

where

$$\widetilde{v_t} := \operatorname{proj}_P v_0 + \left((1-t) + t\frac{\sqrt{\|w\|_h^2 - \|\operatorname{proj}_P v_0\|_h^2}}{\|v_0 - \operatorname{proj}_P v_0\|_h}\right)(v_0 - \operatorname{proj}_P v_0).$$

This homotopy joins v_0 to $\widetilde{L_1}(u) = \widetilde{v_1}$ where $\|\widetilde{v_1}\|_h = \|w\|_h$. Let $V_0 = v_0$ if $\|v_0\|_h \geqslant \|w\|_h$, and $V_0 = \widetilde{v_1}$ if $\|v_0\|_h < \|w\|_h$. In both cases, we consider the homotopy $L_t = L + (v_t - V_0) \otimes \lambda$ with:

$$v_t := t \operatorname{proj}_P w + (1-t) \operatorname{proj}_P V_0 + \varphi(t)(V_0 - \operatorname{proj}_P V_0)$$

and

$$\varphi(t) = \sqrt{\frac{\|V_0\|_h^2 - \|t \operatorname{proj}_P w + (1-t) \operatorname{proj}_P V_0\|_h^2}{\|V_0 - \operatorname{proj}_P V_0\|_h^2}}.$$

Since $\|V_0\|_h \geqslant \|w\|_h$ the numerator is positive and φ is well defined. By definition of φ, for every t, we have $\|v_t\|_h = \|V_0\|_h$. This property ensures that $\sigma_t = (x, y, L_t) \in \mathcal{I}s(\epsilon)(\sigma, \lambda, u)$ for all $t \in [0,1]$. By the expression of v_t, we have $\operatorname{proj}_P v_1 = \operatorname{proj}_P w$. □

This Lemma 2.3 and Point (3) of Definition 1.8 imply that it is enough to construct the loop family $\hat{\gamma}$ for every couple (σ, w) such that $\operatorname{proj}_P L(u) = \operatorname{proj}_P w$. We assume in the sequel that this last condition is fulfilled together with the fact that the codimention is one.

2.3.1. Description of IntConv $\mathcal{I}s(\epsilon)(\sigma, \lambda, u) \cap P_u(w)$.

By assumption $n = m+1$ therefore the space $P_u(w)$ is a 2-plane. We denote by $D(\rho)$ the open disk of $P_u(w)$ with radius ρ and center $\operatorname{proj}_P(L(u))$ and by $A(\rho_{min}, \rho_{max})$ the open annulus $D(\rho_{max}) \setminus \overline{D(\rho_{min})}$. The intersection of the thickened relation $\mathcal{I}s(\epsilon)(\sigma, \lambda, u)$ with $P_u(w)$ is either an annulus or a disk depending on the value of ϵ. Precisely, let

$$r_{min}^2(\epsilon) := \min\left((\|u\|_g - \epsilon)^2 - \|\operatorname{proj}_P w\|_h^2, 0\right)$$
$$r_{max}^2(\epsilon) := (\|u\|_g + \epsilon)^2 - \|\operatorname{proj}_P w\|_h^2.$$

because the sphere S_u of Paragraph 2.1 is of radius $\|u\|_g$. A computation shows that $\mathcal{I}s(\epsilon)(\sigma, \lambda, u) \cap P_u(w)$ is the annulus $A(r_{min}(\epsilon), r_{max}(\epsilon))$ if $r_{min}(\epsilon) > 0$ and the disk $D(r_{max}(\epsilon))$ if $r_{min}(\epsilon) = 0$. In any case,

$$\operatorname{IntConv} \mathcal{I}s(\epsilon)(\sigma, \lambda, u) \cap P_u(w) = D(r_{max}(\epsilon)).$$

In particular, we have $w \in D(r_{max}(\epsilon))$ and $L(u) \in A(r_{min}(\epsilon), r_{max}(\epsilon))$. We want to build a c-shape loop family inside $A(r_{min}(\epsilon), r_{max}(\epsilon))$, for that we define a disk which will support $\hat{\gamma}$ and such that a neighborhood of this disk will be in $A(r_{min}(\epsilon), r_{max}(\epsilon))$ too. Let $D(\widetilde{r})$ a disk where

$$\widetilde{r} = \max\left(\sqrt{\|L(u)\|_h^2 - \|\operatorname{proj}_P L(u)\|_h^2}, \sqrt{\|w\|_h^2 - \|\operatorname{proj}_P w\|_h^2} + \frac{1}{3} d_1(w)\right)$$

where

$$d_1(w) := dist\Big(w, \partial\big(\text{IntConv } \mathcal{I}s(\epsilon)(\sigma,\lambda,u)\big)\Big)$$

is the distance between w and the boundary of the convex hull of $\mathcal{I}s(\epsilon)$ (σ,λ,u). Moreover we have $w \in D(\widetilde{r})$ and $\partial D(\widetilde{r}) \subset A(r_{min}(\epsilon), r_{max}(\epsilon))$.

2.3.2. Parametrization of $D(\widetilde{r})$.

Let ν be th unique unit normal vector of $L(T_xM)$ induced by the orientation of M and W. We see $P_u(w)$ as the complex plane $\mathbb{C}$ by identifying the base $(\nu, (L(u) - \text{proj}_P L(u))/\|L(u) - \text{proj}_P L(u)\|_h)$ with $(1, i)$ and we define a parametrization of $\overline{D(\widetilde{r})}$ by

$$\begin{aligned} b : [0,\pi]\times[0,1] &\longrightarrow \overline{D(\widetilde{r})} \\ (\theta,\beta) &\longmapsto \text{proj}_P L(u) + \beta\widetilde{r}e^{i\theta} + (1-\beta)\widetilde{r}e^{-i\theta}. \end{aligned}$$

This parametrization is 1-to-1 except over points of the form $(0,\beta)$ and (π,β). It maps the boundary of the square $[0,\pi]\times[0,1]$ onto the circle $\partial D(\widetilde{r})$.

2.3.3. The shape

We first define the parameter space A to be

$$A := \left\{ (\eta,\theta,\beta) \in \left]0, \frac{1}{2}\right[\times [0,\pi]\times[0,1] \;\middle|\; \eta \leqslant \beta \leqslant 1-\eta \right\}.$$

and then the shape $c : A \times \mathbb{R}/\mathbb{Z} \to \mathbb{C}\times\mathbb{R}$ by

$$c(\eta,\theta,\beta,t) := \big(\exp(ig_{\theta,\beta}(t)) + \eta\cos\theta, 1\big).$$

The image of $c(\eta,\theta,\beta,\cdot)$ is a whole circle of center $(\eta\cos\theta, 1)$ and radius 1. Let $\beta' = \beta - \frac{\eta}{2}$, the angular function $g_{\theta,\beta}$ is the piecewise linear map given by

$$(i) \qquad g_{\theta,\beta}(0) = 0 \text{ and } g_{\theta,\beta}\left(\frac{1}{2}\right) = 2\pi$$

$$(ii) \qquad g_{\theta,\beta}(t) = \theta \text{ on } \left[\frac{\eta\theta}{4\pi}, \frac{\beta'}{2} + \frac{\eta\theta}{4\pi}\right]$$

$$(iii) \qquad g_{\theta,\beta}(t) = 2\pi - \theta \text{ on } \left[\frac{\beta'}{2} + \frac{\eta(2\pi-\theta)}{4\pi}, \frac{1}{2} - \frac{\eta\theta}{4\pi}\right]$$

on $[0,\frac{1}{2}]$ and such that $g_{\theta,\beta}(t) = g_{\theta,\beta}(1-t)$ for all $t \in [0,\frac{1}{2}]$ (see its graph on Figure 2.3). A computation shows that

$$\overline{c(\eta,\theta,\beta)} = \big(\beta e^{i\theta} + (1-\beta)e^{-i\theta}, 1\big).$$

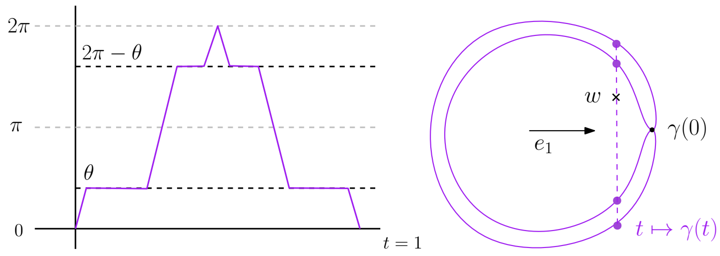

Figure 2.3. Proof of Theorem 2.1*: Left: the graph of the function $g_{\theta,\beta}$, Right: the image of the loop γ in the affine plane $P_u(w)$, the two circles visualise the round-trip of the loop.*

2.3.4. The loop family

Since b induces a bijection between $]0,\pi[\times]0,1[$ and $D(\widetilde{r})$, there exists a unique couple $(\theta,\beta)\in\,]0,\pi[\times]0,1[$ such that $b(\theta,\beta)=w$. We define two functions c_1 and c_2 by the equality

$$c(\eta,\theta,\beta,\cdot)=(c_1(\cdot)+ic_2(\cdot),1)$$

(η will be chosen later). We put

$$\mathbf{e}_1 := \widetilde{r}\frac{\nu}{\|\nu\|_h},\quad \mathbf{e}_2 := \widetilde{r}\frac{L(u)-\operatorname{proj}_P L(u)}{\|L(u)-\operatorname{proj}_P L(u)\|_h}\quad\text{and}\quad \mathbf{e}_3 := \operatorname{proj}_P L(u)$$

and we define the loop family $\widetilde{\gamma}$ by

$$\widetilde{\gamma}(\sigma,w)(t) := c_1(t)\mathbf{e}_1 + c_2(t)\mathbf{e}_2 + \mathbf{e}_3.$$

The image of the loop $\widetilde{\gamma}(\sigma,w)$ is the translated circle $\partial D(\widetilde{r})+\widetilde{r}\eta\cos\theta\mathbf{e}_1$ which lies inside the annulus $A(\widetilde{r}(1-\eta),\widetilde{r}(1+\eta))$ of $P_u(w)$. Consequently, to ensure that the image of $\widetilde{\gamma}(\sigma,w)$ is in the relation, it is enough to choose η such that $A(\widetilde{r}(1-\eta),\widetilde{r}(1+\eta))\subset A(r_{min}(\epsilon),r_{max}(\epsilon))=\mathcal{I}s(\epsilon)(\sigma,\lambda,u)\cap P_u(w)$. It is readily checked that the choice

$$\eta := \frac{1}{3}\min\big(d_1(w),d_2(L(u))\big)$$

where $d_2(L(u))=dist(L(u),\partial A(r_{min}(\epsilon),r_{max}(\epsilon)))$ is convenient. It is also straightforward to see that this loop family satisfies the Average Constraint: $\overline{\widetilde{\gamma}}(\sigma,w)=w$. The base point of the loop is $\widetilde{\gamma}(\sigma,w)(0)=(1+\eta\cos\theta)\mathbf{e}_1+\mathbf{e}_3$. The homotopy $H(s):=(\cos s\ \mathbf{e}_1+\sin s\ \mathbf{e}_2)+\eta\cos\theta\mathbf{e}_1+\mathbf{e}_3$ with $s\in[0,\frac{\pi}{2}]$ connects $\widetilde{\gamma}(\sigma,w)(0)$ with $\eta\cos\theta\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3$ A linear homotopy joins this last point to $L(u)=(\|L(u)-\operatorname{proj}_P L(u)\|_h)\mathbf{e}_2/\widetilde{r}+\mathbf{e}_3$. Consequently,

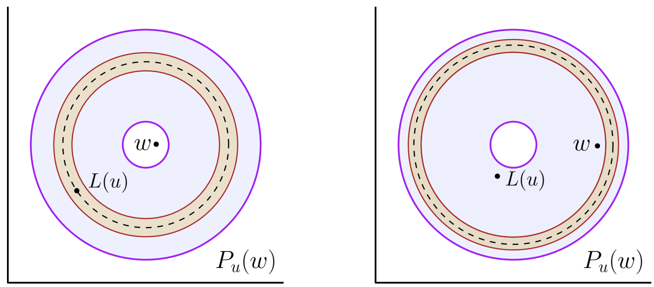

Figure 2.4. Proof of Theorem 2.1*: In purple, the trace of the relation on $P_u(w)$, in black (dashed line) the boundary of disk $D(\widetilde{r})$ and in brown, the annulus $A(\widetilde{r}(1-\eta), \widetilde{r}(1+\eta))$ depending on whether $\sqrt{\|L(u)\|_h^2 - \|\operatorname{proj}_P L(u)\|_h^2} > \sqrt{\|w\|_h^2 - \|\operatorname{proj}_P w\|_h^2} + \frac{1}{3}d_1(w)$ (left) or not (right), see the definition of $\widetilde{r}$.*

the loop family $\tilde{\gamma}$ is c-shaped and surrounding (see Definition 1.8). This proves that $\mathcal{I}s(\epsilon)$ is a Kuiper relation.

3. An application: desingularization of a cone to a surface ϵ-isometric to a flat cylinder

Proposition 1.11 together with the Kuiper property of the relation of ϵ-isometric maps are the reason of the absence of integrals in the formula proposed in [1, 3] to solve $\mathcal{I}s(\epsilon)$. The approach developed here also allows to apply the h-principle in its full generality for $\mathcal{I}s(\epsilon)$. Indeed, in the above cited references, the formulas only make sense when the base map f_0 is an immersion but in the framework of the h-principle this hypothesis is not required: provided that $\mathfrak{S}$ is a subsolution, any base map f_0, singular or not, is convenient.

Here, we illustrate this point with a basic example. We consider a singular map sending a flat cylinder onto a cone and we use the Kuiper property of $\mathcal{I}s(\epsilon)$ to build an ϵ-isometric map arbitrarily closed (in the C^0 sense) to the initial singular map.

3.1. Formal solution

We identify the flat cylinder of height $\frac{1}{20}$ and radius $\frac{1}{2\pi}$ with the space $Cyl = \mathbb{R}/\mathbb{Z} \times [-10^{-1}, 10^{-1}]$ endowed with the Euclidean metric. We define our formal solution to be

$$\mathfrak{S}_0 : (x,y) \longmapsto \big((x,y), f_0(x,y), v_1(x,y), \partial_2 f_0(x,y)\big)$$

where f_0 is a parametrization of a cone:

$$f_0(x,y) = \frac{1}{\sqrt{2}}\Big(y\cos(2\pi x),\ y\sin(2\pi x), y\Big)$$

and v_1 is such that $\partial_1 f_0(x,y) = \sqrt{2}\pi y v_1(x,y)$. Precisely:

$$v_1(x,y) = \Big(-\sin(2\pi x), \cos(2\pi x), 0\Big).$$

Observe that for every (x,y) we have

$$\begin{cases} \|v_1(x,y)\| = 1 \\ \|\partial_2 f_0(x,y)\| = 1 \\ \langle v_1(x,y), \partial_2 f_0(x,y)\rangle = 0 \end{cases}$$

so the section $\mathfrak{S}_0$ is a formal solution of the relation of ϵ-isometric maps for every $\epsilon > 0$.

3.2. Subsolution

The section $\mathfrak{S}_0$ fails to be holonomic only in its v_1-component. To obtain a holonomic section, we thus intend to apply a Corrugation Process in the direction ∂_1. To do so, we need to check that $\mathfrak{S}_0$ is a subsolution with respect to ∂_1. As v_1 and $\partial_2 f_0$ are orthogonal, the slice $\mathcal{I}s(\mathfrak{S}_0, \partial_1)$ lies inside the plane spanned by v_1 and the normal vector

$$n(x,y) = v_1(x,y) \wedge \partial_2 f_0(x,y) = \frac{1}{\sqrt{2}}\Big(\cos(2\pi x), \sin(2\pi x), -1\Big)$$

(see the proof of Theorem 2.1). This slice is a circle a radius 1. The section $\mathfrak{S}_0$ is a subsolution if and only if the derivative $\partial_1 f_0$ lies in the convex hull of $\mathcal{I}s(\mathfrak{S}_0, \partial_1)$. This condition is equivalent to $|y| < (\sqrt{2}\pi)^{-1}$. Since $y \in [-10^{-1}, 10^{-1}]$, this last inequality is fulfilled. This shows that $\mathfrak{S}_0$ is subsolution with respect to ∂_1 of $\mathcal{I}s(\mathfrak{S}_0, \partial_1)$, and thus of $\mathcal{I}s(\epsilon)(\mathfrak{S}_0, \partial_1)$ for every $\epsilon > 0$.

3.3. Corrugation Process

We consider the shape $c : Cyl \times \mathbb{R}/\mathbb{Z} \to \mathbb{C} \times \mathbb{R}$ defined in subsection 2.1 where $\mathbb{C}$ is identified with the plane spanned by (v_1, n):

$$c(\eta, \theta, \beta, t) := \big(\exp(i g_{\theta,\beta}(t)) + \eta\cos\theta, 1\big).$$

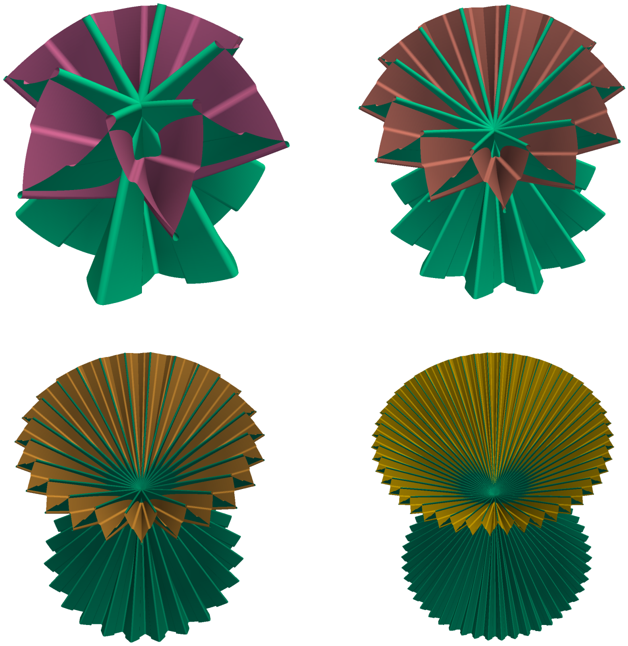

Figure 3.1. Corrugation Process applied from a cone*: Several images of $f_1(Cyl)$ with $\eta = 0.2$ and, from left to right and up to down, $N = 6, 12, 24, 48$. Observe the C^0-density property (see Proposition 1.4 $(P1)$) : the larger N, the closer the surface to the cone.*

In that expression, β and θ are defined by the relation

$$\partial_1 f_0 = \beta e^{i\theta} + (1-\beta)e^{-i\theta}.$$

Since v_1 is collinear to $\partial_1 f_0$ the coefficient β is constant equal to 1/2 and $\theta = \arccos(\langle \partial_1 f_0, v_1 \rangle) = \arccos(\sqrt{2}\pi y)$. For short we denote g for $g_{\theta,\beta}$. The loop family γ is thus given by

$$\gamma(x,y,t) = \Big(\cos(g(x,y,t)) + \eta \left\| \partial_1 f_0(x,y) \right\| \Big) v_1(x,y) + \sin(x,y,t) n(x,y).$$

Observe that $\gamma(x,y,t) \in \mathcal{I}s(\eta)(\mathfrak{S}_0, \partial_1)$. The Corrugation Process generates a map $f_1(x,y) = f_0(x,y) + \frac{1}{N}\Gamma(x,y,Nx)$ with

$$\Gamma(x,y,t) := \int_{s=0}^{t} \gamma(x,y,s) - \overline{\gamma(x,y)} ds.$$

Recall that, from property (P_3) of Proposition 1.4, we have

$$\partial_1 f_1(x,y) = \gamma(x,y,Nx) + O(1/N)$$

Let $\epsilon > 0$ be given. To insure $\partial_1 f_1 \in \mathcal{I}s(\epsilon)(\mathfrak{S}_0, \partial_1)$ we have to choose $\eta < \epsilon$ and N large enough.

3.4. Numerical implementation

We use the analytical expression of Proposition 1.11 together with the above expression of γ to implement the Corrugation Process. The images reveal corrugations whose shape varies from a small loop to the one of a roof. A closer look to the surface shows that the shape of the corrugations changes precisely when passing the vertex of the cone. The reason of this behavior is that v_1 the invariant by vertical translation (as opposed to the invariance by central symmetry of the cone and of $\partial_1 f_0$). When η decreases toward zero the map g tends towards a piecewise constant map. Each loop in the family γ stays at the two points $\cos\theta\, v_1 \pm \sin\theta\, n$ for a duration of $\frac{1-\eta}{2}$ each. At the limit, γ is a discontinuous map whose image is two points. As a consequence, when η is small, the image $f_1(Cyl)$ looks like a piecewise linear surface.

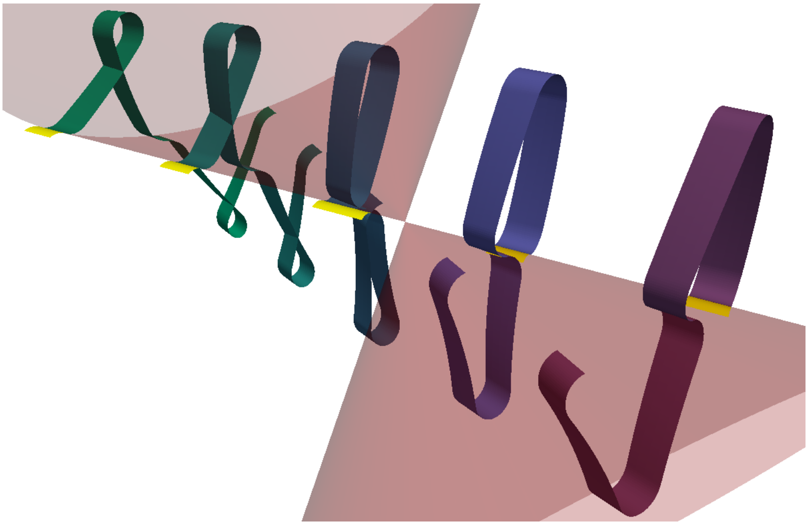

Figure 3.2. The change of the shape of a corrugation when passing the conical singularity*: Here* $N = 6$ *and* $\eta = 0.4$*.*

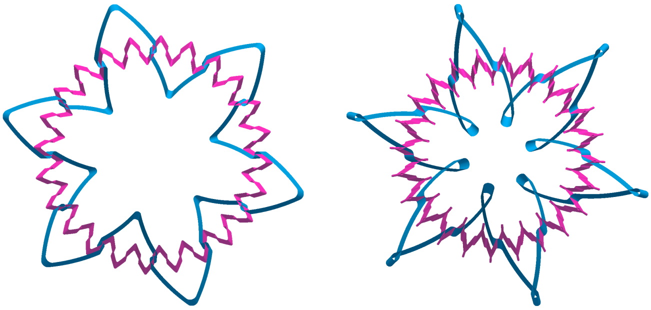

Figure 3.3. Lengths and Corrugations*: In blue, two slices of the image $f_1(Cyl)$ for $N = 6$, in pink, two slices for $N = 24$. On the right the slices are above the horizontal plane passing though the vertex of the cone. They are below this plane on the left. In all cases $\eta = 0.2$.*

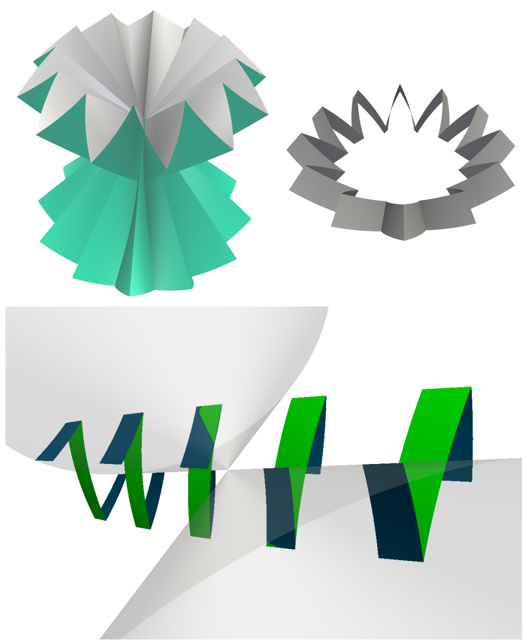

Figure 3.4. PL behavior*: Here $\eta = 0.001$ and $N = 12$. Despite appearances, the map f_1 is still a C^1 immersion. See the close up of Figure 3.5.*

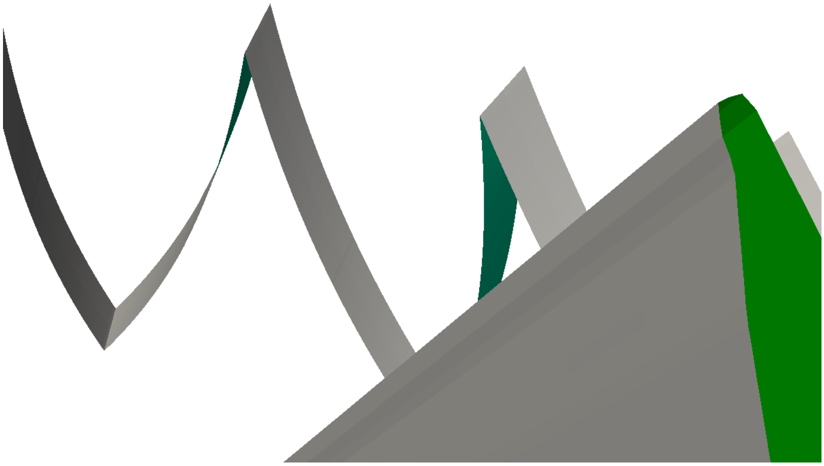

Figure 3.5. Zoom on the peak of a corrugation*: The peak is not a folding. For N large enough, the corrugations are immersed. A close-up shows the roundness of the peak (in the foreground). The angles that appear are artefact due to the discretisation step.*

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