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Carlos MATHEUS

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ON THE DYNAMICS ON THE $SU(2)$-CHARACTER VARIETY OF A ONCE-PUNCTURED TORUS

Carlos Matheus

ABSTRACT. — The natural $SL(2,\mathbb{Z})$-action on the $SU(2)$-character variety of a once-punctured torus respects the level sets of the function $\varkappa$ describing the values $k \in [-2,2]$ of the traces of the matrices associated to a small loop around the puncture.

In 1998, R. Brown used Moser's twisting theorem from KAM theory to show that no element of $SL(2,\mathbb{Z})$ can act ergodically on every level set $\varkappa^{-1}(k)$. As it turns out, Brown's original argument seems to be missing a detail, namely, there is no discussion of the twist condition in his application of Moser's twisting theorem.

In 2002, H. Rüssmann improved Moser's twisting theorem by establishing the stability of (Brjuno) elliptic fixed points of real-analytic area-preserving maps *independently* of twist conditions.

In this note, we observe that Brown's argument can be completed by applying Rüssmann's theorem instead of Moser's twisting theorem.

Let $S_{g,n}$ be a surface of genus $g \geqslant 0$ with $n \geqslant 0$ punctures. Given a Lie group G, the G-character variety of $S_{g,n}$ is the space $X(S_{g,n}, G)$ of representations $\pi_1(S_{g,n}) \to G$ modulo conjugations by elements of G.

The mapping class group $\mathrm{Mod}(S_{g,n})$ of isotopy classes of orientation-preserving diffeomorphisms of $S_{g,n}$ acts naturally on $X(S_{g,n}, G)$.

The dynamics of mapping class groups on character varieties was systematically studied by Goldman in 1997: in his landmark paper [3], he showed that the $\mathrm{Mod}(S_{g,0})$-action on $X(S_{g,0}, SU(2))$ is ergodic with respect to Goldman–Huebschmann measure[1] whenever $g \geqslant 1$.

The ergodicity result above partly motivates the question of understanding the dynamics of individual elements of mapping class groups acting on $SU(2)$-character varieties.

In this direction, Brown [1] studied in 1998 the actions of elements of $SL(2,\mathbb{Z}) = \mathrm{Mod}(S_{1,1})$ on the character variety $X(S_{1,1}, SU(2))$. As it turns

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[1] This nomenclature is not standard: we use it here because Goldman showed in [2] that $X(S_{g,0}, SU(2))$ has a volume form coming from a natural symplectic structure and Huebschmann proved in [4] that this volume form has finite mass.

out, if $\gamma \in \pi_1(S_{1,1})$ is a small loop around the puncture, then the $SL(2,\mathbb{Z})$-action on $X(S_{1,1}, SU(2))$ preserves each level set $\varkappa^{-1}(k)$, $k \in \mathbb{R}$, of the function $\varkappa : X(S_{1,1}, SU(2)) \to \mathbb{R}$ sending $[\rho] \in X(S_{1,1}, SU(2))$ to the trace of the matrix $\rho(\gamma)$. Here, Brown noticed that the dynamics of elements of $SL(2,\mathbb{Z})$ on level sets $\varkappa^{-1}(k)$ with k close to -2 fit the setting of the celebrated KAM theory (assuring the stability of non-degenerate elliptic periodic points of smooth area-preserving maps). In particular, Brown tried to employ Moser's twisting theorem to conclude that no element of $SL(2,\mathbb{Z})$ can act ergodically on all level sets $\varkappa^{-1}(k)$, $k \in [-2, 2]$.

Strictly speaking, Brown's original argument is not complete because Moser's theorem is used without checking the twist condition.

In the sequel, we revisit Brown's work [1] in order to show that his conclusions can be derived once one replaces Moser's twisting theorem by a KAM stability theorem from 2002 due to Rüssmann [7].

1. Statement of Brown's theorem

1.1. $SU(2)$-character variety of a punctured torus

Recall that the fundamental group $\pi_1(S_{1,1})$ of an once-punctured torus is naturally isomorphic to a free group F_2 on two generators α and β such that the commutator $[\alpha, \beta]$ corresponds to a loop γ around the puncture of $S_{1,1}$.

Therefore, a representation $\rho : \pi_1(S_{1,1}) \to SU(2)$ is determined by a pair of matrices $\rho(\alpha), \rho(\beta) \in SU(2)$, and an element $[\rho] \in X(S_{1,1}, SU(2))$ of the $SU(2)$-character variety of $S_{1,1}$ is determined by the simultaneous conjugacy class $(\varphi\rho(\alpha)\varphi^{-1}, \varphi\rho(\beta)\varphi^{-1})$, $\varphi \in SU(2)$, of a pair of matrices $(\rho(\alpha), \rho(\beta)) \in SU(2) \times SU(2)$.

The traces $x = \text{tr}(\rho(\alpha))$, $y = \text{tr}(\rho(\beta))$ and $z = \text{tr}(\rho(\alpha\beta))$ of the matrices $\rho(\alpha)$, $\rho(\beta)$ and $\rho(\alpha\beta)$ provide an useful system of coordinates on $X(S_{1,1}, SU(2))$: algebraically, this is an incarnation of the fact that the ring $\mathbb{R}[SU(2) \times SU(2)]^{SU(2)}$ of invariants of $(A, B) \in SU(2) \times SU(2)$ is freely generated by the traces of A, B and AB.

In particular, the following Proposition 1.1 expresses the trace of $\rho(\gamma)$ $= \rho([\alpha, \beta])$ in terms of $x = \text{tr}(\rho(\alpha))$, $y = \text{tr}(\rho(\beta))$ and $z = \text{tr}(\rho(\alpha\beta))$.

PROPOSITION 1.1. — *Given* $A, B \in SL(2, \mathbb{C})$, *one has*

$$\text{tr}\left(ABA^{-1}B^{-1}\right) = \text{tr}(A)^2 + \text{tr}(B)^2 + \text{tr}(AB)^2 - \text{tr}(A)\,\text{tr}(B)\,\text{tr}(AB) - 2$$

Proof. — By Cayley–Hamilton theorem (or a direct calculation), any $M \in SL(2, \mathbb{C})$ satisfies $M^2 - \mathrm{tr}(M)M + \mathrm{Id} = 0$, i.e., $M + M^{-1} = \mathrm{tr}(M)\,\mathrm{Id}$. Hence, for any $X, Y \in SL(2, \mathbb{C})$, one has

$$XY + Y^{-1}X^{-1} = \mathrm{tr}(XY)\,\mathrm{Id} \quad \text{and} \quad XY^{-1} + YX^{-1} = \mathrm{tr}\left(XY^{-1}\right)\mathrm{Id},$$

so that

$$\mathrm{tr}(XY) + \mathrm{tr}\left(XY^{-1}\right) = \mathrm{tr}(X)\,\mathrm{tr}(Y).$$

It follows that, for any $A, B \in SL(2, \mathbb{C})$, one has

$$\mathrm{tr}\left(ABA^{-1}B^{-1}\right) + \mathrm{tr}\left(ABA^{-1}B\right) = \mathrm{tr}\left(ABA^{-1}\right)\mathrm{tr}(B) = \mathrm{tr}(B)^2$$

and

$$\mathrm{tr}\left(ABA^{-1}B\right) + \mathrm{tr}\left(AB\left(A^{-1}B\right)^{-1}\right) = \mathrm{tr}(AB)\,\mathrm{tr}\left(A^{-1}B\right).$$

Since $\mathrm{tr}(AB(A^{-1}B)^{-1}) = \mathrm{tr}(A^2) = \mathrm{tr}(A)^2 - 2$ and $\mathrm{tr}(A^{-1}B) + \mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B)$, the proof of the Proposition 1.1 is complete. □

1.2. Basic dynamics of $SL(2, \mathbb{Z})$ on character varieties

Recall that the mapping class group $\mathrm{Mod}(S_{1,1})$ is generated by Dehn twists τ_α and τ_β about the generators α and β of $\pi_1(S_{1,1})$. In appropriate coordinates on the once-punctured torus $S_{1,1}$, the isotopy classes of these Dehn twists are represented by the actions of the matrices

$$\tau_\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tau_\beta = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$$

on the flat torus $\mathbb{R}^2/\mathbb{Z}^2$. In particular, at the homotopy level, the actions of τ_α and τ_β on $\pi_1(S_{1,1})$ are given by the Nielsen transformations

(1.1) $\qquad \tau_\alpha(\alpha) = \alpha, \quad \tau_\alpha(\beta) = \beta\alpha, \quad \tau_\beta(\alpha) = \alpha\beta, \quad \tau_\beta(\beta) = \beta.$

Since the elements of $\mathrm{Mod}(S_{1,1}) = SL(2, \mathbb{Z})$ fix the puncture of $S_{1,1}$, they preserve the homotopy class $\gamma = [\alpha, \beta] \in \pi_1(S_{1,1})$ of a small loop around the puncture. Therefore, the $\mathrm{Mod}(S_{1,1})$-action on the character variety $X(S_{1,1}, SU(2))$ respects the level sets $\varkappa^{-1}(k)$, $k \in [-2, 2]$, of the function $\varkappa : X(S_{1,1}, SU(2)) \to [-2, 2]$ given by

$$\varkappa([\rho]) := \mathrm{tr}(\rho(\gamma)).$$

Furthermore, each level set $\varkappa^{-1}(k)$, $-2 < k \leqslant 2$, carries a finite (Goldman–Huebschmann) measure coming from a natural $\mathrm{Mod}(S_{1,1})$-invariant symplectic structure [2, 4].

In this context, the level set $\varkappa^{-1}(2)$ corresponds to impose the restriction $\rho(\gamma) = \mathrm{Id} \in SU(2)$, so that $\varkappa^{-1}(2)$ is naturally identified with the character variety $X(S_{1,0}, SU(2))$.

In terms of the coordinates $x = \mathrm{tr}(\rho(\alpha))$, $y = \mathrm{tr}(\rho(\beta))$ and $z = \mathrm{tr}(\rho(\alpha\beta))$ on $X(S_{1,1}, SU(2))$, we can use Proposition 1.1 (and its proof) and (1.1) to check that

$$(1.2) \qquad \varkappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2$$

and

$$(1.3) \qquad \tau_\alpha(x, y, z) = (x, z, xz - y), \quad \tau_\beta^{-1}(x, y, z) = (xy - z, y, x).$$

Hence, we see from (1.2) that:

- the level set $\varkappa^{-1}(-2)$ consists of a single point $(0, 0, 0)$;
- the level sets $\varkappa^{-1}(k)$, $-2 < k < 2$, are diffeomorphic to 2-spheres;
- the character variety $X(S_{1,1}, SU(2))$ is a 3-dimensional orbifold whose boundary $\varkappa^{-1}(2)$ is a topological sphere with 4 singular points (of coordinates $2(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{-2, 2\}^3$ with $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$) corresponding to the character variety $X(S_{1,0}, SU(2))$.

After this brief discussion of some geometrical aspects of $X(S_{1,1}, SU(2))$, we are ready to begin the study of the dynamics of $\mathrm{Mod}(S_{1,1})$. For this sake, recall that the elements of $\mathrm{Mod}(S_{1,1}) = SL(2, \mathbb{Z})$ are classified into three types:

- $g \in SL(2, \mathbb{Z})$ is called elliptic whenever $|\mathrm{tr}(g)| < 2$;
- $g \in SL(2, \mathbb{Z})$ is called parabolic whenever $|\mathrm{tr}(g)| = 2$;
- $g \in SL(2, \mathbb{Z})$ is hyperbolic whenever $|\mathrm{tr}(g)| > 2$.

The elliptic elements $g \in SL(2, \mathbb{Z})$ have finite order (because $\mathrm{tr}(g) = 0, \pm 1$ and $g^2 - \mathrm{tr}(g)g + \mathrm{Id} = 0$) and the parabolic elements $g \in SL(2, \mathbb{Z})$ are conjugated to $\pm\tau_\alpha^n$ for some $n \in \mathbb{Z}$.

In particular, if $g \in SL(2, \mathbb{Z})$ is elliptic, then g leaves invariant non-trivial open subsets of each level set $\varkappa^{-1}(k)$, $-2 < k \leqslant 2$. Moreover, if $g \in SL(2, \mathbb{Z})$ is parabolic, then g preserves a non-trivial and non-peripheral element $\delta \in \pi_1(S_{1,1})$ and, *a fortiori*, g preserves the level sets of the function $f_\delta : X(S_{1,1}, SU(2)) \to [-2, 2]$, $f_\delta([\rho]) := \mathrm{tr}(\rho(\delta))$. Since any such function f_δ has a non-constant restriction to any level set $\varkappa^{-1}(k)$, $-2 < k \leqslant 2$, Brown concluded that:

PROPOSITION 1.2 ([1, Proposition 4.3]). — *If $g \in SL(2, \mathbb{Z})$ is not hyperbolic, then its action on $\varkappa^{-1}(k)$ is not ergodic whenever $-2 < k \leqslant 2$.*

On the other hand, Brown observed that the action of any hyperbolic element of $SL(2, \mathbb{Z})$ on $\varkappa^{-1}(2)$ can be understood via a result of Katok.

PROPOSITION 1.3 ([1, Theorem 4.1]). — *Any hyperbolic element of* $SL(2,\mathbb{Z})$ *acts ergodically on* $\varkappa^{-1}(2)$.

Proof. — The level set $\varkappa^{-1}(2)$ is the character variety $X(S_{1,0}, SU(2))$. In other words, a point in $\varkappa^{-1}(2)$ represents the simultaneous conjugacy class of a pair $(\rho(\alpha), \rho(\beta))$ of *commuting* matrices in $SU(2)$.

Since a maximal torus of $SU(2)$ is a conjugate of the subgroup

$$T = \left\{ \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} : \theta \in \mathbb{R}/\mathbb{Z} \right\},$$

we have that $X(S_{1,0}, SU(2))$ is the set of simultaneous conjugacy classes of elements of $T \times T$. In view of the action by conjugation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

of the element

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

of the Weyl subgroup of $SU(2)$, we have

$$X(S_{1,0}, SU(2)) = (T \times T)/w.$$

In terms of the coordinates $(\theta, \varphi) \in \mathbb{R}^2/\mathbb{Z}^2$ given by the phases of the elements

$$\left(\begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix}, \begin{pmatrix} e^{2\pi i\varphi} & 0 \\ 0 & e^{-2\pi i\varphi} \end{pmatrix} \right) \in T \times T,$$

the element w acts by $(\theta, \varphi) \mapsto (-\theta, -\varphi)$, so that $X(S_{1,0}, SU(2))$ is the topological sphere obtained from the quotient of $\mathbb{R}^2/\mathbb{Z}^2$ by its hyperelliptic involution ι (and $X(S_{1,0}, SU(2))$ has only four singular points located at the subset $\{0, 1/2\}^2$ of fixed points of the hyperelliptic involution). Moreover, an element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

acts on $T \times T$ by mapping (θ, φ) to $(a\theta + c\varphi, b\theta + d\varphi)$.

In summary, the action of $SL(2,\mathbb{Z})$ on $\varkappa^{-1}(2)$ is given by the usual $SL(2,\mathbb{Z})$-action on the topological sphere $(\mathbb{R}^2/\mathbb{Z}^2)/\iota$ induced from the standard $SL(2,\mathbb{Z})$ on the torus $\mathbb{R}^2/\mathbb{Z}^2$.

By a result of Katok [5], it follows that the action of any hyperbolic element of $SL(2,\mathbb{Z})$ on $\varkappa^{-1}(2)$ is ergodic (and actually Bernoulli). $\square$

1.3. Brown's theorem

The previous two propositions raise the question of the ergodicity of the action of hyperbolic elements of $SL(2, \mathbb{Z})$ on the level sets $\varkappa^{-1}(k)$, $-2 < k < 2$. The following theorem of Brown [1] provides an answer to this question:

THEOREM 1.4. — *Let g be an hyperbolic element of $SL(2, \mathbb{Z})$. Then, there exists $-2 < k < 2$ such that g does not act ergodically on $\varkappa^{-1}(k)$.*

Very roughly speaking, Brown establishes Theorem 1.4 along the following lines. One starts by performing a blowup at the origin $\varkappa^{-1}(-2)$ $= \{(0, 0, 0)\}$ in order to think of the action of g on $X(S_{1,1}, SU(2))$ as a one-parameter family $g_{(k)}$, $-2 \leqslant k \leqslant 2$, of area-preserving maps of the 2-sphere such that $g_{(-2)}$ is a finite order element of $SO(3)$. In this way, we have that $g_{(k)}$ is a non-trivial one-parameter family going from a completely elliptic behaviour at $k = -2$ to a non-uniformly hyperbolic behaviour at $k = 2$. This scenario suggests that the conclusion of Theorem 1.4 can be derived via KAM theory in the elliptic regime.

In the next (and last) section of this note, we revisit Brown's ideas leading to Theorem 1.4 (with an special emphasis on its KAM theoretical aspects).

2. Revisited proof of Brown's theorem

2.1. Blowup of the origin

The origin $\varkappa^{-1}(-2)$ of the character variety $X(S_{1,1}, SU(2))$ can be blown up into a sphere of directions S_{-2}. The action of $SL(2, \mathbb{Z})$ on S_{-2} factors through an octahedral subgroup of $SO(3)$: this follows from the fact that (1.3) implies that the generators τ_α and τ_β of $SL(2, \mathbb{Z})$ act on S_{-2} as

$$\tau_\alpha|_{S_{-2}} (\dot{x}, \dot{y}, \dot{z}) = (\dot{x}, \dot{z}, -\dot{y}), \quad \tau_\beta^{-1}\big|_{S_{-2}} (\dot{x}, \dot{y}, \dot{z}) = (-\dot{z}, \dot{y}, \dot{x}).$$

In this way, each element $g \in SL(2, \mathbb{Z})$ is related to a root of unity

$$\lambda_{-2}(g) \in U(1) = \{w \in \mathbb{C} : |w| = 1\}$$

of order $\leqslant 4$ coming from the eigenvalues of the derivative of $g|_{S_{-2}}$ at any of its fixed points.

Example 2.1. — The hyperbolic element

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \tau_\alpha \tau_\beta$$

acts on S_{-2} via the element $(\dot{x}, \dot{y}, \dot{z}) \mapsto (\dot{z}, -\dot{x}, -\dot{y})$ of $SO(3)$ of order 3.

2.2. Bifurcations of fixed points

An hyperbolic element $g \in SL(2, \mathbb{Z})$ induces a non-trivial polynomial automorphism of $\mathbb{R}^3$ whose restriction to $\varkappa^{-1}([-2, 2])$ describe the action of g on $X(S_{1,1}, SU(2))$. In particular, the set L_g of fixed points of this polynomial automorphism in $\varkappa^{-1}([-2, 2])$ is a semi-algebraic set of dimension < 3.

Actually, it is not hard to exploit the fact that g acts on the level sets $\varkappa^{-1}(k)$, $k \in [-2, 2]$, through area-preserving maps to compute the Zariski tangent space to L_g in order to verify that L_g is one-dimensional (cf. [1, Proposition 5.1]).

Moreover, this calculation of Zariski tangent space can be combined with the fact that any hyperbolic element $g \in SL(2, \mathbb{Z})$ has a discrete set of fixed points in $\mathbb{R}^2/\mathbb{Z}^2$ and, a fortiori, in $\varkappa^{-1}(2) = X(S_{1,0}, SU(2))$ to get that L_g is transverse to $\varkappa$ except at its discrete subset of singular points and, hence, $L_g \cap \varkappa^{-1}(k)$ is discrete for all $-2 \leqslant k \leqslant 2$ (cf. [1, Proposition 5.2]).

Example 2.2. — The hyperbolic element

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \tau_\alpha \tau_\beta$$

acts on $X(S_{1,1}, SU(2))$ via the polynomial automorphism $(x, y, z) \mapsto (z, zy - x, z(zy - x) - y)$ (cf. (1.3)). Thus, the corresponding set of fixed points is given by the equations

$$x = z, \quad y = zy - x, \quad z = z(zy - x) - y$$

describing an embedded pair of curves in $\mathbb{R}^3$.

In general, the eigenvalues $\lambda(p), \lambda(p)^{-1}$ of the derivative at $p \in L_g$ of the action of an hyperbolic element $g \in SL(2, \mathbb{Z})$ on $\varkappa^{-1}(\varkappa(p))$ can be continuously followed along any irreducible component $\ell_g \ni p$ of L_g.

Furthermore, it is not hard to check that λ is not constant on ℓ_g (cf. [1, Lemma 5.3]). Indeed, this happens because there are only two cases: the first possibility is that ℓ_g connects $\varkappa^{-1}(-2)$ and $\varkappa^{-1}(2)$ so that λ varies from $\lambda_{-2}(g) \in U(1)$ to the unstable eigenvalue of g acting on $\mathbb{R}^2/\mathbb{Z}^2$; the second possibility is that ℓ_g becomes tangent to $\varkappa^{-1}(k)$ for some $-2 < k < 2$ so that the Zariski tangent space computation mentioned above reveals that λ varies from 1 (at $\ell_g \cap \varkappa^{-1}(k)$) to some value $\neq 1$ (at any point of transverse intersection between ℓ_g and a level set of $\varkappa$).

2.3. Detecting Brjuno elliptic periodic points

The discussion of the previous two subsections allows to show that the some portions of the action of an hyperbolic element $g \in SL(2, \mathbb{Z})$ fit the assumptions of KAM theory.

Before entering into this matter, recall that $e^{2\pi i \theta} \in U(1)$ is Brjuno whenever θ is an irrational number whose continued fraction has partial convergents $(p_k/q_k)_{k \in \mathbb{Z}}$ satisfying

$$\sum_{k=1}^{\infty} \frac{\log q_{k+1}}{q_k} < \infty.$$

For our purposes, it is important to note that the Brjuno condition has full Lebesgue measure on $U(1)$.

Let $g \in SL(2, \mathbb{Z})$ be an hyperbolic element. We have three possibilities for the limiting eigenvalue $\lambda_{-2}(g) \in U(1)$: it is not real, it equals 1 or it equals -1.

If the limiting eigenvalue $\lambda_{-2}(g) \in U(1)$ is not real, then we take an irreducible component ℓ_g intersecting the origin $\varkappa^{-1}(-2)$. Since λ is not constant on ℓ_g implies that $\lambda(\ell_g)$ contains an open subset of $U(1)$. Thus, we can find some $-2 < k < 2$ such that $\{p\} = \ell_g \cap \varkappa^{-1}(k)$ has a Brjuno eigenvalue $\lambda(p)$, i.e., the action of g on $\varkappa^{-1}(k)$ has a Brjuno fixed point.

If the limiting eigenvalue is $\lambda_{-2}(g) = 1$, we use Lefschetz fixed point theorem on the sphere $\varkappa^{-1}(k)$ with k close to -2 to locate an irreducible component ℓ_g of L_g such that $\{p_k\} = \ell_g \cap \varkappa^{-1}(k)$ is a fixed point of positive index of $g|_{\varkappa^{-1}(k)}$ for k close to -2. On the other hand, it is known that an isolated fixed point of an orientation-preserving surface homeomorphism which preserves area has index < 2. Therefore, p_k is a fixed point of $g|_{\varkappa^{-1}(k)}$ of index 1 with multipliers $\lambda(p_k), \lambda(p_k)^{-1}$ close to 1 whenever k is close to -2. Since a hyperbolic fixed point with positive multipliers has index -1, it follows that p_k is a fixed point with $\lambda(p_k) \in U(1) \setminus \{1\}$ when k is close to -2. In particular, $\lambda(\ell_g)$ contains an open subset of $U(1)$ and, hence, we can find some $-2 < k < 2$ such that p_k has a Brjuno multiplier $\lambda(p_k)$.

If the limiting eigenvalue is $\lambda_{-2}(g) = -1$, then g^2 is an hyperbolic element with limiting eigenvalue $\lambda_{-2}(g^2) = 1$. From the previous paragraph, it follows that we can find some $-2 < k < 2$ such that $\varkappa^{-1}(k)$ contains a Brjuno elliptic fixed point of $g^2|_{\varkappa^{-1}(k)}$.

In any event, the arguments above give the following result (cf. [1, Theorem 4.4]):

THEOREM 2.3. — *Let $g \in SL(2, \mathbb{Z})$ be an hyperbolic element. Then, there exists $-2 < k < 2$ such that $g|_{\varkappa^{-1}(k)}$ has a periodic point of period one or two with a Brjuno multiplier.*

2.4. Moser's twisting theorem and Rüssmann's stability theorem

At this point, the idea to derive Theorem 1.4 is to combine Theorem 2.3 with KAM theory ensuring the stability of certain types of elliptic periodic points.

Recall that a periodic point is called stable whenever there are arbitrarily small neighborhoods of its orbit which are invariant. In particular, the presence of a stable periodic point implies the non-ergodicity of an area-preserving map.

A famous stability criterion for fixed points of area-preserving maps is Moser's twisting theorem [6]. This result can be stated as follows. Suppose that f is an area-preserving C^r, $r \geqslant 4$, map having an elliptic fixed point at origin $(0,0) \in \mathbb{R}^2$ with multipliers $e^{2\pi i \theta}$, $e^{-2\pi i \theta}$ such that $n\theta \notin \mathbb{Z}$ for $n = 1, 2, 3 \ldots, r$. After performing an appropriate area-preserving change of variables (tangent to the identity at the origin), one can bring f into its Birkhoff normal form, i.e., f has the form

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} \xi \cos \left(\sum_{n=0}^{s} \gamma_n \left(\xi^2 + \eta^2 \right)^n \right) - \eta \sin \left(\sum_{n=0}^{s} \gamma_n \left(\xi^2 + \eta^2 \right)^n \right) \\ \xi \sin \left(\sum_{n=0}^{s} \gamma_n \left(\xi^2 + \eta^2 \right)^n \right) + \eta \cos \left(\sum_{n=0}^{s} \gamma_n \left(\xi^2 + \eta^2 \right)^n \right) \end{pmatrix} + h(\xi, \eta)$$

where $s = [r/2] - 1$, $\gamma_0 = 2\pi\theta$, $\gamma_1, \ldots, \gamma_s$ are uniquely determined Birkhoff constants and $h(\xi, \eta)$ denotes higher order terms.

THEOREM 2.4 (Moser twisting theorem). — *Let f be an area-preserving map as in the previous paragraph. If $\gamma_n \neq 0$ for some $1 \leqslant n \leqslant s$, then the origin $(0,0) \in \mathbb{R}^2$ is a stable fixed point.*

The nomenclature "twisting" comes from the fact $\gamma_1 \neq 0$ when f is a twist map, i.e., f has the form $f(r, \theta) = (r, \theta + \mu(r))$ in polar coordinates where μ is a smooth function with $|\mu'(0)| \neq 0$. In the literature, the condition "$\gamma_n \neq 0$ for some n" is called twist condition.

Example 2.5. — The Dehn twist τ_α induces the polynomial automorphism $\tau_\alpha(x, y, z) = (x, z, xz - y)$ on $X(S_{1,1}, SU(2)) = \varkappa^{-1}([-2, 2])$. Each level set $\varkappa^{-1}(k)$, $-2 < k < 2$, is a smooth 2-sphere which is swept out

by the τ_α-invariant ellipses C_{k,x_0} obtained from the intersections between $\varkappa^{-1}(k)$ and the planes of the form $\{x_0\} \times \mathbb{R}^2$.

Goldman [3] observed that, after an appropriate change of coordinates, each C_{k,x_0} becomes a circle where τ_α acts as a rotation by angle $\cos^{-1}(x_0/2)$. In particular, the restriction of τ_α to each level set $\varkappa^{-1}(k)$ is a twist map near its fixed points $(\pm\sqrt{2+k}, 0, 0)$.

In his original argument, Brown [1] deduced Theorem 1.4 from (a weaker version of) Theorem 2.3 and Moser's twisting theorem. However, Brown employed Moser's theorem with $r = 4$ while checking only the conditions on the multipliers of the elliptic fixed point but not the twist condition $\gamma_1 \neq 0$.

As it turns out, it is not obvious to check the twist condition in Brown's setting (especially because it is not satisfied at the sphere of directions S_{-2}).

Fortunately, Rüssmann [7] discovered that a Brjuno elliptic fixed point of a real-analytic area-preserving map is always stable (independently of twisting conditions):

THEOREM 2.6 (Rüssmann). — *Any Brjuno elliptic periodic point of a real-analytic area-preserving map is stable.*

Remark 2.7. — Actually, Rüssmann obtained the previous result by showing that a real-analytic area-preserving map with a Brjuno elliptic fixed point and vanishing Birkhoff constants (i.e., $\gamma_n = 0$ for all $n \in \mathbb{N}$) is analytically linearisable. Note that the analogue of this statement is false in the C^∞ category (as a counterexample is given by $(r, \theta) \mapsto (r, \theta + \rho + e^{-1/r})$).

In any case, at this stage, the proof of Theorem 1.4 is complete: it suffices to put together Theorems 2.3 and 2.6.

Acknowledgments

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Carlos MATHEUS
Centre de Mathématiques Laurent Schwartz,
CNRS (UMR 7640), École Polytechnique,
91128 Palaiseau, (France)
matheus.cmss@gmail.com