

Institut Fourier — Université de Grenoble I

Actes du séminaire de
**Théorie spectrale
et géométrie**

Mark GOULD & Artem PULEMOTOV

Existence of homogeneous metrics with prescribed Ricci curvature

Volume 33 (2015-2016), p. 47-54.

<http://tsg.cedram.org/item?id=TSG_2015-2016__33__47_0>

© Institut Fourier, 2015-2016, tous droits réservés.

L'accès aux articles du Séminaire de théorie spectrale et géométrie (<http://tsg.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://tsg.cedram.org/legal/>).

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
<http://www.cedram.org/>

EXISTENCE OF HOMOGENEOUS METRICS WITH PRESCRIBED RICCI CURVATURE

Mark Gould & Artem Pulemotov

ABSTRACT. — Consider a compact Lie group G and a closed subgroup $H < G$. Suppose T is a positive-definite G -invariant $(0,2)$ -tensor field on the homogeneous space $M = G/H$. In this note, we state a sufficient condition for the existence of a G -invariant Riemannian metric on M whose Ricci curvature coincides with cT for some $c > 0$. This condition is, in fact, necessary if the isotropy representation of M splits into two inequivalent irreducible summands. After stating the main result, we work out an example.

1. Introduction

The prescribed Ricci curvature problem consists in finding a metric whose Ricci curvature coincides with a given tensor field. The investigation of this problem is an important segment of geometric analysis with connections to flows, relativity and other areas of research. While many mathematicians made important contributions to the subject, a particularly large number of results were obtained by DeTurck and his collaborators. The papers [10, 11] contain a number of references to, and comments on, the relevant literature.

On a closed manifold, instead of searching for a metric with Ricci curvature equal to a given positive-definite tensor field T , one should search for a metric g and a constant $c > 0$ such that

$$(1.1) \quad \text{Ric } g = cT.$$

We invite the reader to see [10, Introduction] for an explanation and the history of this paradigm. In [10], the second-named author initiated the

Acknowledgements: Mark Gould's research is supported under Australian Research Council's Discovery Projects funding scheme (DP140101492 and DP160101376). Artem Pulemotov is the recipient of an Australian Research Council Discovery Early-Career Researcher Award (DE150101548).

investigation of equation (1.1) on homogeneous spaces. In particular, consider a compact connected Lie group G and a closed connected Lie subgroup $H < G$. Suppose T is a positive-definite G -invariant tensor field on the homogeneous space $M = G/H$. The main result of [10] implies that a metric g and a constant $c > 0$ satisfying (1.1) can be found if H is a maximal connected Lie subgroup of G . In this note, we propose a sufficient condition for the existence of g and c in the case where the maximality assumption does not hold. This condition imposes requirements on both the space M and the tensor field T . It turns out to be sufficient *and* necessary when the isotropy representation of M splits into two inequivalent irreducible summands; see Remark 2.5. In the end of this note, we outline a general procedure for constructing examples that illustrate the applicability of our result. We also work out one such example in some detail.

Our study of the prescribed Ricci curvature problem was largely inspired by the theory of homogeneous Einstein metrics; see, e.g., [3, 4, 12]. It seems likely that our results and the underlying techniques will contribute to the development of that theory. As a more specific application, we expect that the sufficient condition we propose will lead to new existence theorems for the Ricci iteration on homogeneous spaces; cf. [9]. The proofs of our results, along with some extensions, will appear in a forthcoming paper.

2. The main result

Consider a compact connected Lie group G and a closed connected subgroup $H < G$. Assume the homogeneous space $M = G/H$ has dimension 3 or higher, i.e.,

$$\dim M = n \geq 3.$$

Let \mathfrak{g} and \mathfrak{h} be the Lie algebras of G and H . Choose a scalar product Q on \mathfrak{g} induced by a bi-invariant Riemannian metric on G . If \mathfrak{g}' and \mathfrak{g}'' are subspaces of \mathfrak{g} such that $\mathfrak{g}' \subset \mathfrak{g}''$, we use the notation $\mathfrak{g}'' \ominus \mathfrak{g}'$ for the Q -orthogonal complement of \mathfrak{g}' in \mathfrak{g}'' . Define

$$\mathfrak{m} = \mathfrak{g} \ominus \mathfrak{h}.$$

It is clear that \mathfrak{m} is $\text{Ad}(H)$ -invariant. The representation $\text{Ad}(H)|_{\mathfrak{m}}$ is equivalent to the isotropy representation of G/H . We standardly identify \mathfrak{m} with the tangent space $T_H M$. The representation $\text{Ad}(H)|_{\mathfrak{m}}$ is completely reducible.

Let \mathcal{M} be the space of G -invariant Riemannian metrics on M . In what follows, we implicitly identify $g \in \mathcal{M}$ with the bilinear form induced by g

on \mathfrak{m} via the identification of $T_H M$ and \mathfrak{m} . The space \mathcal{M} carries a natural smooth manifold structure; see, e.g., [9, pages 6318–6319]. The properties of \mathcal{M} are explored in [3, Section 4.1].

Our main result imposes the following hypothesis on the homogeneous space M . We will discuss this hypothesis and some examples in Section 3.

HYPOTHESIS 2.1. — *Every Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{s}$ and $\mathfrak{h} \neq \mathfrak{s}$ meets the following requirements:*

- (1) *The representations $\text{Ad}(H)|_{\mathfrak{s}'}$ and $\text{Ad}(H)|_{\mathfrak{s}''}$ are inequivalent for every pair of nonzero $\text{Ad}(H)$ -invariant spaces $\mathfrak{s}' \subset \mathfrak{s} \ominus \mathfrak{h}$ and $\mathfrak{s}'' \subset \mathfrak{g} \ominus \mathfrak{s}$.*
- (2) *The commutator $[\mathfrak{r}, \mathfrak{s}]$ is nonzero for every $\text{Ad}(H)$ -invariant 1-dimensional subspace \mathfrak{r} of $\mathfrak{g} \ominus \mathfrak{s}$.*

Suppose \mathfrak{k} and \mathfrak{k}' are Lie subalgebras of \mathfrak{g} such that

$$(2.1) \quad \mathfrak{g} \supset \mathfrak{k} \supset \mathfrak{k}' \supset \mathfrak{h}.$$

In order to state our main result, we need to introduce some terminology and notation.

DEFINITION 2.2. — *We call (2.1) a simple chain if \mathfrak{k}' is a maximal Lie subalgebra of \mathfrak{k} and $\mathfrak{h} \neq \mathfrak{k}'$.*

Let us emphasise that Definition 2.2 allows the equality $\mathfrak{k} = \mathfrak{g}$ but not $\mathfrak{k}' = \mathfrak{k}$. We denote

$$\mathfrak{j} = \mathfrak{g} \ominus \mathfrak{k}, \quad \mathfrak{j}' = \mathfrak{g} \ominus \mathfrak{k}', \quad \mathfrak{l} = \mathfrak{k} \ominus \mathfrak{k}', \quad \mathfrak{n} = \mathfrak{k}' \ominus \mathfrak{h}.$$

It is obvious that

$$\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{l} \oplus \mathfrak{n} \oplus \mathfrak{h} = \mathfrak{j}' \oplus \mathfrak{n} \oplus \mathfrak{h}, \quad \mathfrak{j}' = \mathfrak{j} \oplus \mathfrak{l}, \quad \mathfrak{k} = \mathfrak{l} \oplus \mathfrak{n} \oplus \mathfrak{h}, \quad \mathfrak{k}' = \mathfrak{n} \oplus \mathfrak{h}.$$

Here and in what follows, the symbol \oplus stands for the Q -orthogonal sum.

Given a bilinear form R on \mathfrak{m} or \mathfrak{g} and a nonzero subspace $\mathfrak{m}' \subset \mathfrak{m}$, we write $R|_{\mathfrak{m}'}$ for the restriction of R to \mathfrak{m}' . Let $\text{tr}_Q R|_{\mathfrak{m}'}$ be the trace of $R|_{\mathfrak{m}'}$ with respect to $Q|_{\mathfrak{m}'}$. Denote

$$\omega(\mathfrak{m}') = \min\{\dim \mathfrak{m}'' \mid \mathfrak{m}'' \in \Omega(\mathfrak{m}')\},$$

where $\Omega(\mathfrak{m}')$ is the class of nonzero $\text{Ad}(H)$ -invariant subspaces $\mathfrak{m}'' \subset \mathfrak{m}'$ such that the representation $\text{Ad}(H)|_{\mathfrak{m}''}$ is irreducible. It is clear that $\omega(\mathfrak{m}')$ always lies between 1 and $\dim \mathfrak{m}'$. In fact, $\omega(\mathfrak{m}')$ equals $\dim \mathfrak{m}'$ if $\text{Ad}(H)|_{\mathfrak{m}'}$ is irreducible.

Given $\text{Ad}(H)$ -invariant subspaces $\mathfrak{u} \subset \mathfrak{m}$, $\mathfrak{v} \subset \mathfrak{m}$ and $\mathfrak{w} \subset \mathfrak{m}$, define a tensor $\Delta(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) \in \mathfrak{u} \otimes \mathfrak{v}^* \otimes \mathfrak{w}^*$ by the formula

$$\Delta(\mathfrak{u}, \mathfrak{v}, \mathfrak{w})(X, Y) = \pi_{\mathfrak{u}}[X, Y], \quad X \in \mathfrak{v}, Y \in \mathfrak{w}.$$

Here, $\pi_{\mathfrak{u}}$ stands for the Q -orthogonal projection onto \mathfrak{u} . Let $\langle \mathfrak{uvw} \rangle$ be the squared norm of $\Delta(\mathfrak{u}, \mathfrak{v}, \mathfrak{w})$ with respect to the scalar product on $\mathfrak{u} \otimes \mathfrak{v}^* \otimes \mathfrak{w}^*$ induced by $Q|_{\mathfrak{u}}$, $Q|_{\mathfrak{v}}$ and $Q|_{\mathfrak{w}}$. The fact that Q comes from a bi-invariant metric on G implies

$$\langle \mathfrak{uvw} \rangle = \langle \mathfrak{vuw} \rangle = \langle \mathfrak{wvu} \rangle = \langle \mathfrak{vuw} \rangle = \langle \mathfrak{uvw} \rangle = \langle \mathfrak{wvu} \rangle.$$

Suppose (2.1) is a simple chain. In order to state our main result, we need to associate a number, denoted $\eta(\mathfrak{k}, \mathfrak{k}')$, to this simple chain. Let B be the Killing form of the Lie algebra \mathfrak{g} . Define $\eta(\mathfrak{k}, \mathfrak{k}')$ by the formula

$$\eta(\mathfrak{k}, \mathfrak{k}') = \frac{2 \text{tr}_Q B|_{\mathfrak{n}} + 2\langle \mathfrak{nj}' \rangle + \langle \mathfrak{nnn} \rangle}{\omega(\mathfrak{n})(2 \text{tr}_Q B|_{\mathfrak{l}} + \langle \mathfrak{lll} \rangle + 2\langle \mathfrak{ljj} \rangle)}.$$

One can show that the denominator cannot equal 0 when Hypothesis 2.1 is satisfied. Moreover, $\eta(\mathfrak{k}, \mathfrak{k}')$ is non-negative. We are now ready to state our main result.

THEOREM 2.3. — *Let Hypothesis 2.1 hold for the homogeneous space M . Assume the tensor field $T \in \mathcal{M}$ satisfies*

$$\frac{\inf\{T(X, X) \mid X \in \mathfrak{n} \text{ and } Q(X, X) = 1\}}{\text{tr}_Q T|_{\mathfrak{l}}} > \eta(\mathfrak{k}, \mathfrak{k}')$$

for every simple chain of the form (2.1). Then there exists a Riemannian metric $g \in \mathcal{M}$ whose Ricci curvature coincides with cT for some $c > 0$.

Remark 2.4. — The quantity

$$\inf\{T(X, X) \mid X \in \mathfrak{n} \text{ and } Q(X, X) = 1\}$$

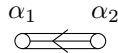
is the smallest eigenvalue of the matrix of $T|_{\mathfrak{n}}$ in a $Q|_{\mathfrak{n}}$ -orthonormal basis of \mathfrak{n} .

Remark 2.5. — Let the isotropy representation of M split into two inequivalent irreducible summands. We refer to [5, 7] for a classification of spaces with this property. Assume Hypothesis 2.1 is satisfied. If it is not, then all the metrics in \mathcal{M} have the same Ricci curvature; see, e.g., [11, Section 4.2]. Theorem 2.3 provides a sufficient condition for the existence of g and c such that (1.1) holds. One can show, with the aid of [10, Proposition 3.1], that this condition is also necessary.

3. An example

In a forthcoming paper, we will prove that generalised flag manifolds satisfy Hypothesis 2.1. Let us outline a procedure for constructing examples of such manifolds; cf. [2, Chapter 7] and [1]. Roughly speaking, the idea is to start with a simple complex Lie algebra $\bar{\mathfrak{g}}$ and produce a subalgebra by removing a node from the Dynkin diagram of $\bar{\mathfrak{g}}$. Passing to compact real forms, then to Lie groups, and taking the quotient, we obtain a homogeneous space. Requirements (1)–(2) of Hypothesis 2.1 are easy to verify. This procedure is closely related to the theory of Lie superalgebras; see, e.g., [8, 6]. We will now consider a concrete example.

Let $\bar{\mathfrak{g}}$ be the exceptional complex Lie algebra \mathfrak{g}_2 . Fix a Cartan subalgebra $\mathfrak{c} \subset \bar{\mathfrak{g}}$. We choose a fundamental root system $\{\alpha_1, \alpha_2\}$ and denote the corresponding set of positive roots by Φ^+ . The Dynkin diagram of $\bar{\mathfrak{g}}$ is



and the formula

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$$

holds true. Consider the decomposition

$$(3.1) \quad \bar{\mathfrak{g}} = L_{-3} \dot{+} L_{-2} \dot{+} L_{-1} \dot{+} L_0 \dot{+} L_1 \dot{+} L_2 \dot{+} L_3.$$

In the right-hand side, the symbol $\dot{+}$ denotes the direct sum. The spaces L_i for $i = -3, \dots, 3$ are given by the formulas

$$\begin{aligned} L_0 &= \mathfrak{c} \dot{+} \mathfrak{g}^{\alpha_2} \dot{+} \mathfrak{g}^{-\alpha_2}, \\ L_1 &= \mathfrak{g}^{\alpha_1} \dot{+} \mathfrak{g}^{\alpha_1 + \alpha_2}, & L_{-1} &= \mathfrak{g}^{-\alpha_1} \dot{+} \mathfrak{g}^{-\alpha_1 - \alpha_2}, \\ L_2 &= \mathfrak{g}^{2\alpha_1 + \alpha_2}, & L_{-2} &= \mathfrak{g}^{-2\alpha_1 - \alpha_2}, \\ L_3 &= \mathfrak{g}^{3\alpha_1 + \alpha_2} \dot{+} \mathfrak{g}^{3\alpha_1 + 2\alpha_2}, & L_{-3} &= \mathfrak{g}^{-3\alpha_1 - \alpha_2} \dot{+} \mathfrak{g}^{-3\alpha_1 - 2\alpha_2}, \end{aligned}$$

where \mathfrak{g}^α is the root space corresponding to the root α . Note that L_1, L_{-1}, L_3 and L_{-3} are 2-dimensional over \mathbb{C} while L_2 and L_{-2} are 1-dimensional over \mathbb{C} . Clearly, L_0 is a Lie subalgebra of $\bar{\mathfrak{g}}$, and \mathfrak{c} is a Cartan subalgebra of L_0 . Moreover, L_0 satisfies the equality

$$L_0 = [L_0, L_0] \dot{+} Z.$$

The space Z in the right-hand side is the 1-dimensional (over \mathbb{C}) center of L_0 . The Lie algebra $[L_0, L_0]$ is simple and isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. One obtains its Dynkin diagram by removing the node corresponding to α_1 from the Dynkin diagram of $\bar{\mathfrak{g}}$.

The space L_i is $\text{ad}(L_0)$ -invariant for every $i = -3, \dots, 3$. The representation $\text{ad}(L_0)|_{L_i}$ is irreducible for i in the set

$$I = \{-3, -2, -1, 1, 2, 3\}.$$

Moreover, $\text{ad}(L_0)|_{L_i}$ and $\text{ad}(L_0)|_{L_j}$ are inequivalent if $i, j \in I$ satisfy $i \neq j$. The decomposition (3.1) is \mathbb{Z} -graded, i.e.,

$$[L_i, L_j] \subset L_{i+j}, \quad i, j = -3, \dots, 3,$$

where we assume $L_{i+j} = \{0\}$ if $|i+j| > 3$. Moreover, the properties of root systems imply the equality

$$(3.2) \quad [L_i, L_j] = L_{i+j}$$

for all $i, j \in I$ such that $i+j \in I$.

Fix a Chevalley basis $\{h_1, h_2; e_\alpha, e_{-\alpha} \mid \alpha \in \Phi^+\}$ of the Lie algebra $\bar{\mathfrak{g}}$. Suppose $*$ is the (antilinear) adjoint operation on $\bar{\mathfrak{g}}$ such that

$$h_1^* = h_1, \quad h_2^* = h_2, \quad e_\alpha^* = e_{-\alpha}, \quad \alpha \in \Phi^+.$$

Let \mathfrak{g} be the set of anti-Hermitian elements of $\bar{\mathfrak{g}}$ with respect to $*$. It is easy to see that \mathfrak{g} is a compact real form of $\bar{\mathfrak{g}}$. We denote by G the connected and simply connected Lie group whose Lie algebra coincides with \mathfrak{g} . The space L_0 is invariant with respect to $*$. The set

$$\mathfrak{h} = \mathfrak{g} \cap L_0$$

is a compact real form of L_0 . We denote by H the connected Lie subgroup of G whose Lie algebra equals \mathfrak{h} . This subgroup is closed. Our next goal is to show that the homogeneous space G/H satisfies Hypothesis 2.1.

Let Q be the negative of the Killing form of \mathfrak{g} . Consider the Q -orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3.$$

For $i = 1, 2, 3$, the space \mathfrak{m}_i is given by the formula

$$(3.3) \quad \mathfrak{m}_i = \mathfrak{g} \cap (L_{4-i} \dot{+} L_{i-4}).$$

The complexification of \mathfrak{m}_i equals $L_{i-4} \dot{+} L_{4-i}$. Using this fact, one can show that \mathfrak{m}_i is $\text{ad}(\mathfrak{h})$ -invariant and the representation $\text{ad}(\mathfrak{h})|_{\mathfrak{m}_i}$ is irreducible. If $j = 1, 2, 3$ is distinct from i , then $\text{ad}(\mathfrak{h})|_{\mathfrak{m}_i}$ and $\text{ad}(\mathfrak{h})|_{\mathfrak{m}_j}$ are inequivalent. According to (3.2) and (3.3), the spaces

$$\mathfrak{k}_1 = \mathfrak{h} \oplus \mathfrak{m}_1 \quad \text{and} \quad \mathfrak{k}_2 = \mathfrak{h} \oplus \mathfrak{m}_2$$

are proper Lie subalgebras of \mathfrak{g} containing \mathfrak{h} as a proper subset. The same formulas imply that no other such subalgebras exist. It is obvious that

$$\mathfrak{k}_1 \ominus \mathfrak{h} = \mathfrak{m}_1, \quad \mathfrak{g} \ominus \mathfrak{k}_1 = \mathfrak{m}_2 \oplus \mathfrak{m}_3, \quad \mathfrak{k}_2 \ominus \mathfrak{h} = \mathfrak{m}_2, \quad \mathfrak{g} \ominus \mathfrak{k}_2 = \mathfrak{m}_1 \oplus \mathfrak{m}_3.$$

These equalities show that \mathfrak{k}_1 and \mathfrak{k}_2 meet requirement (1) of Hypothesis 2.1. By (3.3), the dimensions of \mathfrak{m}_1 , \mathfrak{m}_2 and \mathfrak{m}_3 are even. Therefore, \mathfrak{k}_1 and \mathfrak{k}_2 meet requirement (2) as well.

We can identify G and H with the real exceptional Lie group G_2 and the unitary group $U(2)$, respectively. Thus, the homogeneous space

$$G_2/U(2)$$

satisfies Hypothesis 2.1. Given a tensor field T on this space, Theorem 2.3 provides a sufficient condition for the existence of a Riemannian metric with Ricci curvature equal to cT for some $c > 0$.

BIBLIOGRAPHY

- [1] S. ANASTASSIOU & I. CHRYSIKOS, “The Ricci flow approach to homogeneous Einstein metrics on flag manifolds”, *J. Geom. Phys.* **61** (2011), p. 1587-1600.
- [2] A. ARVANITOEORGOS, *An introduction to Lie groups and the geometry of homogeneous spaces*, Student Mathematical Library, vol. 22, American Mathematical Society, 2003, xvi+141 pages.
- [3] C. BÖHM, “Homogeneous Einstein metrics and simplicial complexes”, *J. Differ. Geom.* **67** (2004), no. 1, p. 79-165.
- [4] C. BÖHM, M. WANG & W. ZILLER, “A variational approach for compact homogeneous Einstein manifolds”, *Geom. Funct. Anal.* **14** (2004), no. 4, p. 681-733.
- [5] W. DICKINSON & M. M. KERR, “The geometry of compact homogeneous spaces with two isotropy summands”, *Ann. Global Anal. Geom.* **34** (2008), no. 4, p. 329-350.
- [6] M. GOULD & R. ZHANG, “Unitary representations of basic classical Lie superalgebras”, *Lett. Math. Phys.* **20** (1990), no. 3, p. 221-229.
- [7] C. HE, “Cohomogeneity one manifolds with a small family of invariant metrics”, *Geom. Dedicata* **157** (2012), p. 41-90.
- [8] V. G. KAC, “Lie superalgebras”, *Adv. Math.* **26** (1977), no. 1, p. 8-96.
- [9] Y. G. NIKONOROV, E. D. RODIONOV & V. V. SLAVSKIĬ, “Geometry of homogeneous Riemannian manifolds”, *J. Math. Sci. (N.Y.)* **146** (2007), no. 6, p. 6313-6390.
- [10] A. PULEMOTOV, “Metrics with prescribed Ricci curvature on homogeneous spaces”, *J. Geom. Phys.* **106** (2016), p. 275-283.
- [11] A. PULEMOTOV & Y. A. RUBINSTEIN, “Ricci iteration on homogeneous spaces”, to appear in *Trans. Am. Math. Soc.*
- [12] M. WANG & W. ZILLER, “Existence and nonexistence of homogeneous Einstein metrics”, *Invent. Math.* **84** (1986), no. 1, p. 177-194.

Mark GOULD
School of Mathematics and Physics
The University of Queensland
St Lucia, QLD 4072 (Australia)
m.gould1@uq.edu.au

Artem PULEMOTOV
School of Mathematics and Physics
The University of Queensland
St Lucia, QLD 4072 (Australia)
a.pulemotov@uq.edu.au