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Christoffel and Minkowski problems in Minkowski space

CHRISTOFFEL AND MINKOWSKI PROBLEMS IN MINKOWSKI SPACE

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ABSTRACT. —   For convex sets in the Lorentzian Minkowski space bounded by space-like hyperplanes, it is possible to define area measures, similarly to the classical definition for convex bodies in the Euclidean space. Here the measures are defined on the hyperbolic space rather than on the round sphere. We are particularly interested by convex sets invariant under the action of isometries groups of the Minkowski space, so that the measures can be defined on compact hyperbolic manifolds. We can then look at the Christoffel and the Minkowski problems (i.e. particular measures are prescribed) in a general setting. In dimension $(2+1)$, the Christoffel problem include a famous construction by G. Mess. In this dimension, the smooth version of the Minkowski problem already had a positive answer, and we show that this is a specificity of dimension $(2+1)$, while the general problem has a solution in all dimensions.

CONTENTS

1. A Lorentzian Fenchel–Jensen formula

For details concerning this section, we refer to [6, 7, 13].

Let K be a convex body in $\mathbb{R}^{d+1}$ and ω be a Borel set of the sphere $\mathbb{S}^d$, seen as the set of unit vectors of the Euclidean space $\mathbb{R}^{d+1}$. Let $B_\epsilon(K, \omega)$ be the set of points p which are at distance as most ϵ from their metric projection $\bar{p}$ onto K and such that $p - \bar{p}$ is collinear to a vector belonging to ω. The Fenchel–Jensen formula [11] says that the volume of $B_\epsilon(K, \omega)$ is a polynomial with respect to ϵ:

(I) $$V(B_\epsilon(K, \omega)) = \frac{1}{d+1} \sum_{i=0}^{d} \epsilon^{d+1-i} \binom{d+1}{i} S_i(K, \omega) \ .$$

Each $S_i(K, \cdot)$ is a finite positive measure on the Borel sets of the sphere, called the *area measure of order i*. $S_0(K, \cdot)$ is only the Lebesgue measure of the sphere $\mathbb{S}^d$, and $S_d(K, \omega)$ is the d-dimensional Hausdorff measure of the pre-image of ω for the Gauss map.

A well-known particular case of (I) is when $\omega = \mathbb{S}^2$, the *Steiner formula*.

The problem of prescribing the area measure of order d is the (generalized) Minkowski problem, and the one of prescribing the area measure of order 1 is the (generalized) Christoffel problem (each problem having a smooth and a polyhedral version).

We will introduce suitable convex sets in the Minkowski space, such that an analogous theory can be done. Of course, convexity and volume (here the Lebesgue measure on $\mathbb{R}^{d+1}$) do not depend on the ambient metric, but the orthogonal projection does.

The *Minkowski space* is $\mathbb{R}^{d+1}$ endowed with the bilinear form

$$\langle x, y \rangle_- := x_1 y_1 + \cdots + x_d y_d - x_{d+1} y_{d+1} \ .$$

A vector v is *space-like* if $\langle v, v \rangle_- > 0$, and *future time-like* if $\langle v, v \rangle_- < 0$ and its last coordinate is positive. The orthogonal P of a future time-like vector v is space-like, in the sense that the restriction of $\langle \cdot, \cdot \rangle_-$ onto P is positive definite, see Figure 1.1. The *future side* of P is the side containing v. We denote by $I^+(p)$ the set of future time-like vectors based at a point p. It is a convex cone with apex p.

In all the text, we will identify the hyperbolic space with the set

$$\mathbb{H}^d := \{x \in I^+(0) | \langle x, x \rangle_- = -1\}$$

endowed with the induced metric. It plays a role analogue to the round sphere in the Euclidean space.

Let K be the intersection of the closure of the future side of space-like hyperplanes. By construction, K is a convex set, and one sees easily that $\forall p \in K, I^+(p) \subset K$. In particular, K is unbounded and with non-empty interior. Moreover, the support planes of K (hyperplanes meeting K and
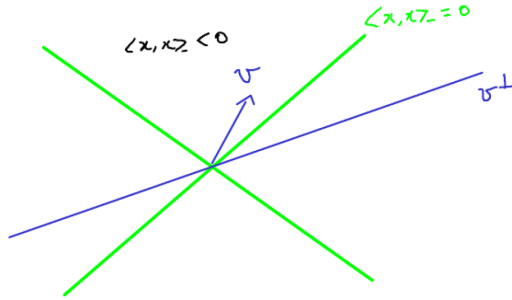
Figure 1.1. Orthogonality in Minkowski space.

such that K is on one side) are space-like or light-like (i.e. the induced bilinear form is degenerate).

For example, $I^+(0)$ is the intersection of the future side of all the space-like vector hyperplanes. It also has as support planes all the light-like vector hyperplanes.

Let us denote by $K(\mathbb{H}^d)$ the convex set bounded by $\mathbb{H}^d$. It is the intersection of the future side of all the planes tangent to $\mathbb{H}^d$.

We will denote by $\partial_s K$ the subset of the boundary of K of points contained in a space-like support plane. For example, $\partial_s(I^+(p)) = \{p\}$. Actually, $\partial_s K$ determines K because $K = \bigcup_{k \in \partial_s K} \overline{I^+(k)}$.

Conversely to the Euclidean case, the orthogonal projection is well-defined from the interior of K. For any point x in the interior of K, there exists a unique point $r_K(x) \in \partial_s K$ which maximizes the Lorentzian distance from x on $\partial_s K$. More precisely, $r_K(x)$ is in the intersection of ∂K and the past cone of x. This intersection is a compact set. See Figure 1.2.

The Lorentzian distance between x and $r_K(x)$ is called the *cosmological time* of x:

$$T_K(x) = \langle x - r_K(x), x - r_K(x) \rangle_- \ .$$

Moreover, $x - r_K(x)$ is a future time-like vector orthogonal to a support plane of K at $r_K(x)$. So $(x - r_K(x))/|T_K(x)| \in \mathbb{H}^d$.

The *Gauss map* G_K of K is the set-valued map which associates to $x \in \partial_s K$ the subset of $\mathbb{H}^d$ of vectors orthogonal to the space-like support planes of K at x. It is a well-defined map if and only if $\partial_s K$ is C^1. Otherwise we consider it as a set-valued map, see Figure 1.3.
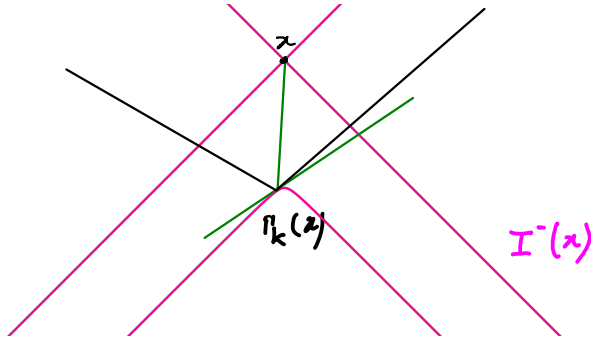
Figure 1.2. The orthogonal projection onto the complementary of a
space-like convex set in Minkowski space.
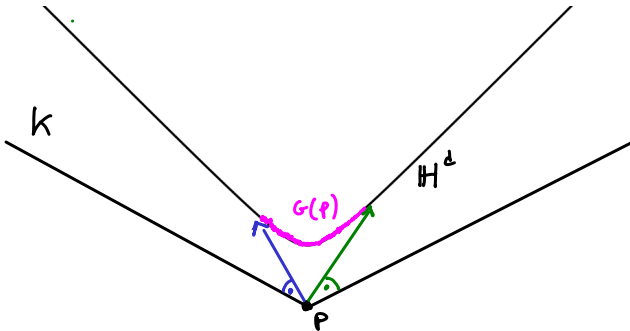
Figure 1.3. The Gauss map at a non-smooth point.

DEFINITION 1.1. — K is a F-convex set if G_K is surjective onto $\mathbb{H}^d$, i.e.
any future time-like vector is orthogonal to a support plane of K, i.e. any
space-like hyperplane is parallel to a support plane of K.

See Figure 1.4. We will restrict our attention to F-convex sets. First, this
is a natural assumptions as the Euclidean Gauss map of convex bodies is
surjective onto $\mathbb{S}^d$. Second, for $t > 0$, if K_t is the set of points at distance
t from $\partial_s K$, then K_t is C^1, and G_{K_t} is proper: if $\omega \subset \mathbb{H}^d$ is compact, then
$G_{K_t}^{-1}(\omega)$ is compact [6].

Figure 1.4. *Some examples of the total image of the Gauss map of space-like convex sets. The future cone of a point is the only one being a F-convex set.*

We can now state the Lorentzian analogue of the Fenchel–Jensen formula. In the following, V is the volume, i.e. the Lebesgue measure of $\mat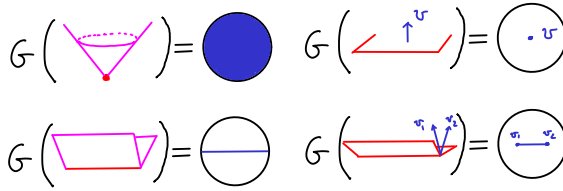hbb{R}^{d+1}$. Note that isometries of the Minkowski space preserve the volume (they are composed by a linear part living in $O(1,d)$, of determinant 1, and by a vector of $\mathbb{R}^{d+1}$ acting by translation).

THEOREM 1.2. — *There exist Radon measures $S_0(K,\cdot),\dots,S_d(K,\cdot)$ on $\mathbb{H}^d$ such that, for any compact set ω of $\mathbb{H}^d$, $S_i(K,\omega)$ satisfy the following formula. Let K be an F-convex set in $\mathbb{R}^{d+1}$ and $\omega \subset \mathbb{H}^d$ be compact. Then*

$$(1.1) \qquad V(\cup_{0<t<\epsilon} G_{K_t}^{-1}(\omega)) = \frac{1}{d+1} \sum_{i=0}^{d} \epsilon^{d+1-i} \binom{d+1}{i} S_i(K,\omega) \ .$$
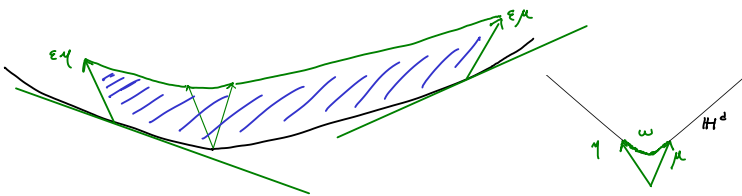
See Figure 1.5 and Figure 1.6.

Figure 1.5. *The volume of the dashed part is $V(\cup_{0<t<\epsilon} G_{K_t}^{-1}(\omega))$. For a given ω, it is a degree $(d+1)$ polynomial in ϵ.*

Recall that as $\mathbb{H}^d$ is σ-compact, a Borel measure, finite on compact sets, has the inner regularity property, and hence is a Radon measure. Radon

measures are the measures given by the Riesz representation theorem. $S_i(K, \cdot)$ is called the *area measure of order i* of K. We have that $S_0(K, \cdot)$ is the volume form of $\mathbb{H}^d$. We will mention in Section 5 that $S_d(K, \omega)$ can be seen as a derivative of the volume of an ϵ neighbourhood, so it is the area of the boundary of the convex set, in the Minkowski sense.

See the end of Section 3 for an idea of the proof of Theorem 1.2.
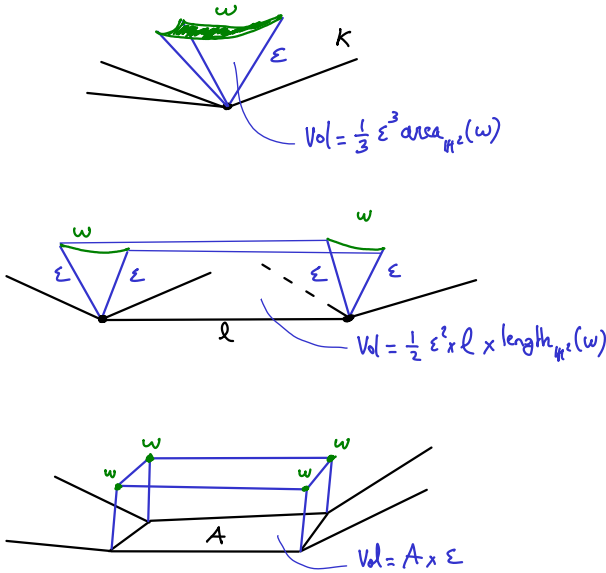
Figure 1.6. *Illustration of the Lorentzian Fenchel–Jensen formumla in the polyhedral case for $d = 2$. In particular, in those cases, from the top to the bottom, $S_0(K)(\omega)$ is the hyperbolic volume of ω, $S_1(K)(\omega)$ is the hyperbolic length of ω weighted by the length of the edge, and $S_2(K)(\omega)$ is the area of the face.*

Let us mention basic operations on the space of F-convex sets. It is invariant under *(Minkowski) sum*

$$K + K' = \{k + k' | k \in K, k' \in K'\}$$

and positive homotheties: for $\lambda > 0$

$$\lambda K = \{\lambda k | k \in K\} \ .$$

The closure of the future cone $I^+(p)$ of a point p is a F-convex set which plays a role analoguous to the one of single points in the theory of convex bodies, in the sense that $K + I^+(p)$ is a translation of K by the vector p.

2. Analytical point of view

For details concerning this section, we refer to [7, 13].

Let K be a F-convex set. The *support function* H_K of K is the map

$$H_K : I^+(0) \to \mathbb{R}, \eta \mapsto \max\{\langle \eta, k \rangle_-, k \in K\} \ .$$

For example, $H_{I^+(p)} = \langle \cdot, p \rangle_-$. The map H_K is sublinear: 1-homogeneous and subadditive

$$H_K(\lambda\eta) = \lambda H_K(\eta), \ \lambda > 0, \ H_K(v + w) \leqslant H_K(v) + H_K(w) \ .$$

In particular, H_K is convex. Classical results in convex geometry say that any sublinear function on $I^+(0)$ is the support function of a unique F-convex set. Moreover, $H_{K+K'} = H_K + H_{K'}$ and $H_{\lambda K} = \lambda H_K$, $\lambda > 0$.

By homogeneity, H_K is determined by its restriction to any subspace which meets any future time-like line exactly once. Two of them have particular interest:

- h_K, the restriction of H_K to the intersection of $I^+(0)$ and the horizontal plane at height 1. This intersection is $B^d \times \{0\}$, with B^d the open unit ball centred at 0 of $\mathbb{R}^d$. We will consider h_K as a map on B^d. It can be checked that support functions of F-convex sets are exactly the convex functions on B^d.
- $\overline{h}_K$, the restriction of H_K on $\mathbb{H}^d$.

Although the geometric interpretation of the support function on $\mathbb{H}^d$ is clear (see Figure 2.1), there is no straightforward intrinsic characterisation of functions on $\mathbb{H}^d$ which are support functions. But the support functions on $\mathbb{H}^d$ of C_+^2 F-convex sets ($\partial_s K$ is a C^2 hypersurface and the Gauss map is a C^1 diffeomorphism) are exactly the C^2 maps $\overline{h} : \mathbb{H}^d \to \mathbb{R}$ such that

(2.1) $$\nabla^2 \overline{h}_K - \overline{h}_K g$$

is positive definite, where g is the hyperbolic metric and ∇^2 the hyperbolic Hessian.

If h is a support function on the open ball B^d, it can be extended to ∂B as a lower semi-continuous convex function, but the extension may have infinite values. If the extension has a finite value at a point $\ell \in \partial B$, this
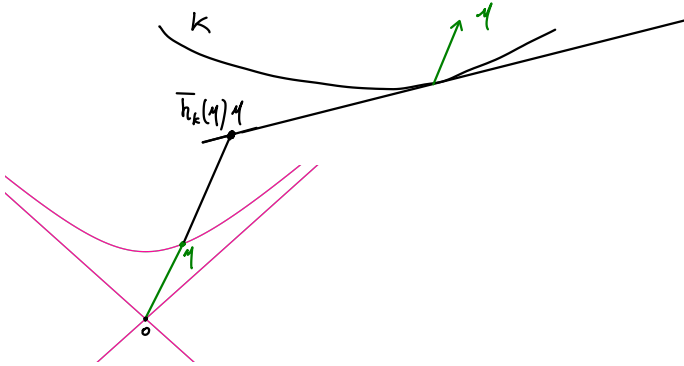
Figure 2.1. The support function on $\mathbb{H}^d$ is the (Lorentzian) distance from the origin to the support plane of K orthogonal to η.

has the following meaning: in the Minkowski space, ℓ is a light-like vector, and there is a light-like hyperplane orthogonal to ℓ which is a support plane of the F-convex set with support function h. Here support plane is employed in a general sense, as the light-like support plane may not meet the boundary of the F-convex set. For example, the support function of $K(\mathbb{H}^d)$ is zero on the boundary of B^d, but none of the light-like plane passing through the origin meets the hyperboloid.

In the C_+^2 case, the eigenvalues of (2.1) are positive real numbers, the *principal radii of curvature*. They are the inverse of the principal curvatures (note that the principal curvatures are defined on the hypersurface, and the principal radii of curvature are defined on $\mathbb{H}^d$, but in this case the Gauss map is a diffeomorphism). In the $d = 1$ case, it can be checked that the radius of curvature is the radius of the osculating hyperbola of the curve.

In the C_+^2 case, $S_i(K) = \phi_i \, \mathrm{d} \, \mathbb{H}^d$, and ϕ_i is the ith elementary symmetric function of the radii of curvature of K. In particular:

$$(2.2) \qquad \phi_1 = \frac{1}{d} \operatorname{trace}(\nabla^2 \overline{h}_K - hg) = \frac{1}{d}(\nabla - \operatorname{Id})\overline{h}_k$$

with ∇ the hyperbolic Laplacian. Following an idea of [4], one can prove that for any F-convex set K, $S_1(K)$ is equal to (2.2) in the sense of distribution: $\forall f \in C_c^\infty(\mathbb{H}^d)$,

$$(S_1(\overline{h}), f) = \int_{\mathbb{H}^d} f\left(\frac{1}{d}\Delta - 1\right)\overline{h}\mathrm{d}\mathbb{H}^d := \int_{\mathbb{H}^d} \overline{h}\left(\frac{1}{d}\Delta - 1\right) f\mathrm{d}\mathbb{H}^d.$$

For $i = d$, in the C_+^2 case, ϕ_d is the determinant of (2.1). It is the product of the principal radii of curvature, so the inverse of the product of the principal curvatures, and this last product is known as the Gauss–Kronecker curvature. On the ball B^d,

$$(2.3) \qquad\qquad S_d(K) = \lambda \det(\mathrm{Hess}\,\overline{h})\mathcal{L}$$

with $\lambda = \sqrt{1 - \|\cdot\|}$, $\|\cdot\|$ the Euclidean norm on B^d and $\mathcal{L}$ the Lebesgue measure on B^d. Here we identify Borel sets of $\mathbb{H}^d$ and Borel sets of B^d via the central projection from the origin in the ambient Minkowski space. In the general case, the above formula holds, with at the place of $\det(\mathrm{Hess}\,\overline{h})\mathcal{L}$, the *Monge–Ampère measure* associated to h. We refer to [7, 14] for a precise definition.

3. Group action

For details concerning this section, we refer to [1, 6, 7, 12, 13, 16].

One of the main motivations for the definition of F-convex sets is the following class of examples. Let Γ be a group of isometries of $\mathbb{H}^d$ such that $\mathbb{H}^d/\Gamma$ is a compact manifold. We also denote by Γ the corresponding group of linear isometries in Minkowski space. A Γ-convex set is a space-like future convex set setwise invariant under the action of Γ. The trivial example is the closure of $I^+(0)$. A Γ-convex set is either a F-convex sets contained in $I^+(0)$ or the closure of $I^+(0)$. Another example is $K(\mathbb{H}^d)$. So the closure of $I^+(0)$ is the maximal Γ-convex set (for the inclusion). If we consider the quotient of $I^+(0)$ by Γ, we get a flat Lorentzian spacetime (a Lorentzian connected time-orientable manifold equipped with a time-orientation) of a particular kind that we now describe.

Let us recall some general definitions. A Lorentz manifold M is globally hyperbolic if it admits a Cauchy surface, i.e. a spacelike hypersurface which intersects every inextensible time-like path at exactly one point. A classical result of R. Geroch states that the existence of a single Cauchy surface implies the existence of a foliation by such hypersurfaces. A globally hyperbolic spacetime is said to be spatially compact if its Cauchy surfaces are compact. A globally hyperbolic spacetime (M, g) is maximal if every isometric embedding of M in another globally hyperbolic spacetime of the

same dimension and which sends Cauchy surfaces to Cauchy surfaces is onto. For short, MGHC stands for "maximal globally hyperbolic spatially compact". Hence, $I^+(0)/\Gamma$ is a flat (future complete) MGHC. Note that it is not past complete.

In the converse direction, if M is a flat (future complete) MGHC, it is known that if S is a Cauchy surface of M, then up to a finite covering S can be equipped by a metric locally isometric to $\mathbb{H}^k \times \mathbb{R}^{d-k}$ for some $0 \leqslant k \leqslant d$. We consider only the case where S is of hyperbolic type (that is $k = d$). Let M be such a manifold. Then its universal cover isometrically embeds into Minkowski space, and its image is a F-convex set, invariant under a representation of $\pi_1(M)$ into the isometries of Minkowski space. The image of $\pi_1(M)$ is an affine deformation Γ_τ of a cocompact lattice Γ.

More precisely, let $C^1(\Gamma, \mathbb{R}^{d+1})$ be the space of 1-*cochains*, i.e. the space of maps

$$\tau : \Gamma \to \mathbb{R}^{d+1} \ .$$

For $\gamma_0 \in \Gamma$, we will denote $\tau(\gamma_0)$ by τ_{γ_0}. The space of 1-*cocycles* $Z^1(\Gamma, \mathbb{R}^{d+1})$ is the subspace of $C^1(\Gamma, \mathbb{R}^{d+1})$ of maps satisfying

$$(3.1) \qquad\qquad \tau_{\gamma_0 \mu_0} = \tau_{\gamma_0} + \gamma_0 \tau_{\mu_0} \ .$$

For any $\tau \in Z^1$ we get a group Γ_τ of isometries of Minkowski space, with linear part Γ and with translation part given by τ: for $x \in \mathbb{R}^{d+1}$, $\gamma \in \Gamma_\tau$ is defined by

$$\gamma x = \gamma_0 x + \tau_{\gamma_0} \ .$$

The cocycle condition (3.1) expresses the fact that Γ_τ is a group. In other words, Γ_τ is a group of isometries which is isomorphic to its linear part Γ. Of course, $\Gamma_0 = \Gamma$.

The space of 1-*coboundaries* $B^1(\Gamma, \mathbb{R}^{d+1})$ is the subspace of $C^1(\Gamma, \mathbb{R}^{d+1})$ of maps of the form $\tau_{\gamma_0} = \gamma_0 v - v$ for a given $v \in \mathbb{R}^{d+1}$. It is easy to check that if τ and τ' differ by a 1-coboundary, then $\gamma x = f \gamma' f^{-1} x$, with f a translation. The names come from the usual cohomology of groups, and $H^1(\Gamma, \mathbb{R}^{d+1}) = Z^1(\Gamma, \mathbb{R}^{d+1})/B^1(\Gamma, \mathbb{R}^{d+1})$ is the 1-cohomology group. As we will deal only with 1-cocycles and 1-coboundaries, we will call them cocycles and coboundaries respectively.

A τ-*convex set* is a future space-like closed convex set setwise invariant under the action of Γ_τ. They are F-convex sets, and for any cocycle τ, there exist τ-convex sets. They are all included into a single τ-convex set, denoted by Ω_τ (if $\tau = 0$, then $\Omega_\tau = I^+(0)$). The quotient of the interior of Ω_τ by Γ_τ is a future complete flat MGHC, and they are all obtained in this way.

Support functions of τ-convex sets on $I^+(0)$ are *τ-equivariant*:

(3.2) $$H(\gamma_0\eta) = H(\eta) + \langle \gamma_0^{-1}\tau_{\gamma_0}, \eta \rangle_- \ .$$

So a support function can be defined on $\mathbb{H}^d/\Gamma$, only if $\tau = 0$. But if G is the Gauss map of a τ-convex set, then $G(\gamma x) = \gamma_0 G(x)$. From this it follows that the area measures of τ-convex sets are Γ-invariant.

A crucial property is that, given a cocycle τ, all the support functions on B^d of τ-convex sets have a same continuous extension on ∂B^d. If $\tau = 0$, this extension is the constant function equal to 0.

If $\tau \neq 0$, the sets Ω_τ are much more complicated than a convex cone. Roughly speaking, they are the future of a space-like real tree, instead of a single point. In the case of the cone, specific tools are available (radial function, duality, ...) that makes the study of Γ-convex sets very similar to the one of convex bodies, where the compactness is replaced in some sense by cocompactness.

Let us sketch the proof of Theorem 1.2. Formula (1.1) is first proved in the Γ-convex case, where the proof mimics the one for convex bodies. Then it is proved that any compact part of the boundary of a F-convex set can be seen as a part of the boundary of a suitable Γ-convex set, for a suitable Γ (up to a translation) — note the local nature of (1.1). Then the existence of the area measures is obtained using Riesz representation theorem.

4. Solutions to the Christoffel problem

For details concerning this section, we refer to [3, 13].

4.1. Equivariant solutions

A natural question is to know, given a Radon measure μ on $\mathbb{H}^d$, if there exists a F-convex set with μ as area measure of order i. Form the analytical point of view, one can first solve an equation (of the kind (2.2) if $i = 1$ and (2.3) if $i = d$), and then check if the solution is actually the support function of a convex set. From the point of view of convex geometry, the prescription of the area measure of order d is the most relevant one, as in some sense the class of support function if the natural one in which one can solve (2.3). This equation is the subject of the next section. Any solution to (2.2) doesn't need to be a support function, but the problem

has still many interests. Actually it is the simplest one as (2.2) is linear in the support function.

From this, a solution to the Christoffel problem (find K such that μ is the area measure of order one of K) can be found as follows. For simplicity, we consider first the case where $\mu = \phi \, \mathrm{d} \, \mathbb{H}^d$, and ϕ has compact support. Hence one wants to find h such that, given ϕ on $\mathbb{H}^d$ with compact support,

$$\frac{1}{d} \Delta h - h = \phi \; .$$

A particular solution is given by the function $h_\phi \in C^\infty(\mathbb{H}^d)$ defined as

$$(4.1) \qquad h_\phi(x) = d \int_{\mathbb{H}^d} G(x,y)\phi(y) \mathrm{d}\mathbb{H}^d(y) \; ,$$

where we use the following notations: $k : (0, +\infty) \to (-\infty, 0)$ is

$$k(\rho) = \frac{\cosh \rho}{v_{d-1}} \int_{+\infty}^{\rho} \frac{\mathrm{d}t}{\sinh^{d-1}(t) \cosh^2(t)} \; ,$$

with v_{d-1} the area of $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. Note that k is solution of the ODE

$$\ddot{k}(\rho) + \frac{\dot{A}(\rho)}{A(\rho)} \dot{k}(\rho) - dk(\rho) = 0 \; ,$$

where

$$A(\rho) = \int_{\partial B_\rho(x)} \mathrm{d}A_\rho = v_{d-1} \sinh^{d-1} \rho$$

is the area of the (smooth) geodesic sphere

$$\partial B_\rho = \{y \in \mathbb{H}^d : d_{\mathbb{H}^d}(x,y) = \rho\}$$

centred at any point $x \in \mathbb{H}^d$ and $\mathrm{d}A_\rho$ is the $(d-1)$-dimensional volume measure on ∂B_ρ. Finally, the kernel function $G : \mathbb{H}^d \times \mathbb{H}^d \to \mathbb{R} \cup \{\infty\}$ is given by

$$G(x,y) = k(d_{\mathbb{H}^d}(x,y)) \; .$$

A useful remark is that, if instead of being with compact support, the function ϕ is Γ-invariant, then the solution h_ϕ in (4.1) is also Γ invariant. So in the Γ-invariant case, a complete solution to the Christoffel problem can be given, very similar to the solution of the (Euclidean) convex bodies case. We don't give the statement here (see [13] and the reference therein), because it is tedious to write it explicitly (for the analytical solution (4.1) to be a support function, one has to write explicitly that its 1-homogeneous extension is sublinear).

The linearity of the Christoffel problem gives a solution in the τ-equivariant case. Indeed, note that any τ-equivariant function can be written as the sum of a τ-equivariant function with a Γ invariant function. Moreover,

one can prove that for any τ there exists a smooth function h, τ-equivariant, such that $\frac{1}{d}\Delta h - h = 0$.

Moreover, for a given τ, as all the support functions on B^d of τ-convex sets have the same continuous extension to the boundary, so the solution is unique.

4.2. Analysis versus geometry

Let us consider a Radon measure μ on $\mathbb{H}^d$, which is supported on a totally geodesic hypersurface P, and is the volume form of P for its intrinsinc metric with a weight a. It is easy to construct a F-convex set with μ as area measure of order one. In Minkowski space, P is the intersection of $\mathbb{H}^d$ with a time-like vector hyperplane. Let v be a unit space-like vector orthogonal to P. Then the F-convex set which is the union of the future cones of points on the space-like geodesic segment from the origin to av has μ as area measure of order 1. This process can be easily generalized if the measure is supported by a a finite number of non intersecting weighted hyperplanes, see Figure 4.1.
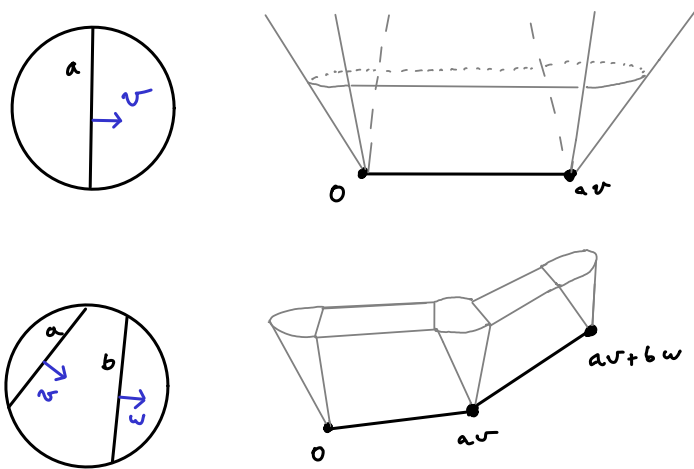

Figure 4.1. The simplest case of solution of the Christoffel problem ($d = 2$), and a straightforward generalization.

Actually, if $d > 2$, the hyperplanes can meet, but some conditions must be given on the weights. In the $d = 2$ case, let us consider a finite number of closed, simple, weighted, non intersecting, geodesics on a compact hyperbolic surface $\mathbb{H}^2/\Gamma$. Then one easily extend the construction described above to the measure on $\mathbb{H}^2$ supported by the lifting of the closed geodesics to the universal cover. When constructing the F-convex set, one can associate an isometry of the Minkowski space to any deck transformation, and at the end this gives (up to translations) a $\tau \in H^1(\Gamma, \mathbb{R}^3)$. The F-convex set so obtained is Ω_τ, the largest τ-convex set on the interior of which Γ_τ acts freely and properly discontinously.

Generalizing the process from weighted multi curves to measured geodesic laminations is the way used in [16] to associate a cocycle to any lamination. We have for example that the total length of the lamination is equal to $S_1(\Omega_\tau)(\mathbb{H}^2/\Gamma)$ (as $S_1(\Omega_\tau)$ is Γ invariant, we consider it as a measure on $\mathbb{H}^2/\Gamma$).

It is not easy to see what will play the role of the measured geodesic laminations in higher dimensions (see [6]). But for any cocycle and any d, the measure $S_1(\Omega_\tau)(\cdot)$ is well-defined. On can prove that $S_1(\Omega_\tau)(\mathbb{H}^d/\Gamma)$ is an asymmetric norm on $H^1(\Gamma, \mathbb{R}^{d+1})$, that corresponds to a norm of Thurston in dimension $(2+1)$ [3].

Moreover it has the following geometric interpretation. Given a cocycle τ, one gets a *future* F-convex set Ω_τ. On also gets a *past* convex set Ω_τ^-, which is a Γ_τ invariant convex set which contains all the past Γ_τ invariant convex sets. Ω_τ and Ω_τ^- intersect if and only if $\tau = 0$. Then $S_1(\Omega_\tau)(\mathbb{H}^d/\Gamma)$ is the average distance between Ω_τ and Ω_τ^- [13].

5. Solutions to the Minkowski problem

For details about those whole section, we refer to [7].

5.1. Some results

THEOREM 5.1 ([7]).
(1) *For any Radon measure μ on $\mathbb{H}^d/\Gamma$, for any $\tau \in H^1(\Gamma, \mathbb{R}^{d+1})$, there exists a τ-convex set K with $S_d(K) = \mu$.*
(2) *K is unique.*
(3) *If*
    a) $\mu = f\mathrm{d}\mathbb{H}^d, f > 0, f \in C^\infty(\mathbb{H}^d/\Gamma)$,

      b) *the solution given by* (1) *is* Cauchy, *i.e. it does not meet the boundary of* Ω_τ,

  then

$(*)$ $\partial_s K$ *is a space-like* C^∞ *hypersurface*

(4) *a)* $\Rightarrow$ *b)* $\Rightarrow$ $(*)$ *in the following cases*
  - $d = 2$
  - $\tau = 0$

(5) *There exist* d, Γ, τ, μ *such that* $[a) \Rightarrow (*)]$ *is false.*

(6) *There exists* $c \geqslant 0$, *depending on* τ, *such that if* f *satisfies* a) *and* $f > c$, *then* $(*)$ *is true.*

Let us do some remarks about this statement.

- Statements 2 and 3 are straightforward translations of classical results about Monge–Ampère equation. 4, 5, 6 also comes from Monge–Ampère results, but with some more work. 1 is obtained by geometric methods and is totally independent from Monge–Ampère theory, see 5.2.

- Of course, for 3 one gets finer regularity results.

- The case $\tau = 0$ in 4 was proved in [17]. The case $d = 2$ was obtained in [2] by totally different methods than in [7]. They use dimensional specificities of the dimension 2 from the point of view of geometry and topology of surfaces. In [7], the dimensional specificity comes from a classical result in Monge–Ampère theory, known as Alexandrov–Heinz theorem. Also in [7] are given conditions on Ω_τ such that a) $\Rightarrow$ b) $\Rightarrow$ $(*)$ holds.

- For 5, an explicit counter example is constructed for $d = 3$, from a counter-example of Pogorelov for the regularity of solution of Monge–Ampère equation. The Pogorelov example is a function on the 3-ball with Monge–Ampère measure larger than a positive constant times the Lebesgue measure, which is linear (i.e. not strictly convex) on a segment joining two points of the sphere. From this Pogorelov example, one has to construct the support function of a τ-convex set in the Minkowski space, using explicit Γ and τ found in [9]. For this τ, we then obtain the following fact:

      there exists a positive constant α such that there does not exist any smooth τ-convex set with constant Gauss–Kronecker curvature α.

- Recently the smooth version of Minkowski problem in dimension
  (2+1) was extended to space-like convex surfaces (no group action)
  in [8].

5.2. The covolume

Finally let us say a word about the proof of the first item of Theorem 5.1. The first remark is that, for $\tau, \tau' \in H^1(\Gamma, \mathbb{R}^{d+1})$, $\lambda > 0$, if K is τ-convex, then λK is $(\lambda\tau)$-convex. And if K' is τ'-convex, then $K + K'$ is $(\tau + \tau')$-convex. In particular, the set $\mathcal{C}(\tau)$ of τ-convex sets is convex.

We introduce a functional on this set, the *covolume*. If K is a τ-convex set, it is the volume of the intersection of $\Omega_\tau \setminus K$ with a fundamental domain for the action of Γ_τ (this does not depend on the fundamental domain).

The main point is that

   the covolume is convex on $\mathcal{C}(\tau)$

(one can check that it is not convex on the union of the $\mathcal{C}(\tau)$ for all $\tau \in H^1(\Gamma, \mathbb{R}^{d+1})$). The proof lies on those two facts:

(1) there exists a convex fundamental domain for the action of Γ_τ in any τ-convex set with C^1 boundary (we are not able to prove that there exists a convex fundamental domain in Ω_τ, but using a limit argument, this suffices for our purpose),

(2) if we consider the intersection C of K, a fundamental domain, and the past of a space-like hyperplane such that it contains the intersection of $\Omega_\tau \setminus K$ with the fundamental domain, then C is a convex body of a particular kind, a *convex cap*, see Figure 5.1. The covolume of K is equal to a constant minus the volume of C. And a direct application of Fubini theorem shows that the volume of convex caps (with a fixed basis) is concave [5].

Then the idea is to minimize the following functional on $\mathcal{C}(\tau)$ (where τ-convex sets are identified with their support functions on $\mathbb{H}^d$, and $\bar{h}_\tau$ is the support function of Ω_τ):

$$(5.1) \qquad L_\mu(\bar{h}) = \text{covolume}(\bar{h}) - \int_{\mathbb{H}^d/\Gamma} \bar{h}_\tau - \bar{h} \, d\mu \ .$$

We prove that this convex functional is coercive, that a minimum is reached and that this minimum is a solution (i.e. gives a τ-convex set with area measure μ). Heuristically, the proof of this last assertion follows from the fact that the area measure is the Gâteaux gradient of the covolume.
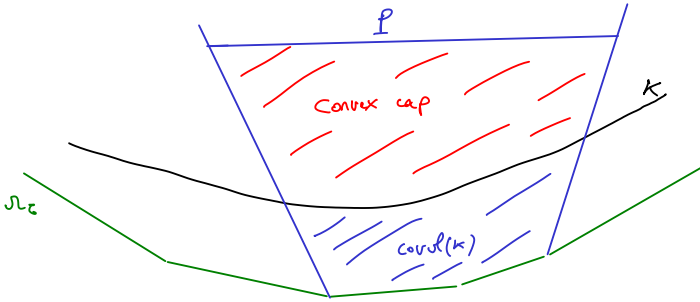
Figure 5.1. *To the intersection of any τ-convex set with a convex fundamental domain, one can associate a* convex cap*: a convex body which projects bijectively onto a convex set P in a hyperplane.*

However this is a bit cumbersome to prove it in full generality in our context (for the convex bodies case, see [10]). But one manages to pass over to prove that the minimum is a solution.

Let us finish by two remarks in the case $\tau = 0$, i.e. when the convex sets are Γ-invariant:

(1) The covolume of a Γ-convex set with support function $\bar{h}$ on $\mathbb{H}^d$ writes explicitly as

$$-\frac{1}{d+1} \int_{\mathbb{H}^d/\Gamma} \bar{h} \, \mathrm{d} \, S_d(K)$$

(there is no explicit formula for the covolume for a non-zero τ). With this, one can check that functional (5.1) introduced by I. Bakelman to give a variational proof of the Monge–Ampère problem [18].

(2) $\mathcal{C}(0)$ is a convex cone. Moreover,

$$\mathrm{covol}(\lambda K) = \lambda^{d+1} \, \mathrm{covol}(K) \ .$$

So (on the vector space spanned by the cone of support functions), the covolume can be polarized as a *mixed-covolume*, and a theory parallel to the theory of mixed-volume for convex bodies can be developed [12]. Actually, without mention of the support functions, we only need notions of (co)volume and (co)convexity, two non-metric notions, so this can be done without referring to Lorentzian geometry [15].

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