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Pierre BÉRARD & Bernard HELFFER

NODAL SETS OF EIGENFUNCTIONS,
ANTONIE STERN'S RESULTS REVISITED

Pierre Bérard & Bernard Helffer

ABSTRACT. — These notes present a partial survey of our recent contributions to the understanding of nodal sets of eigenfunctions (constructions of families of eigenfunctions with few or many nodal domains, equality cases in Courant's nodal domain theorem), revisiting Antonie Stern's thesis, Göttingen, 1924.

1. Introduction

In dimension one, a theorem of C. Sturm [34] says that the zeros (*nodes*) of an eigenfunction associated with the n-th eigenvalue of a self-adjoint Sturm–Liouville problem in a bounded interval divides the interval into n sub-intervals (*nodal domains*).

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (a bounded connected open subset). Consider the Dirichlet eigenvalue problem in Ω,

$$(1.1) \qquad \begin{cases} -\Delta u = \lambda u \text{ in } \Omega\,, \\ \quad u = 0 \text{ on } \partial\Omega\,. \end{cases}$$

Arrange the corresponding eigenvalues in non-decreasing order, with multiplicities,

$$(1.2) \qquad \lambda_1(\Omega) < \lambda_2(\Omega) \leqslant \lambda_3(\Omega) \leqslant \cdots .$$

Given a function u on Ω, the *nodal set* of u is defined to be

$$(1.3) \qquad N(u) := \overline{\{x \in \Omega \mid u(x) = 0\}}.$$

The connected components of $\Omega \setminus N(u)$ are called the *nodal domains* of u. Denote by $\mu(u)$ the number of nodal domains of u.

In 1923, R. Courant [15] gave an upper bound for the number of nodal domains of the eigenfunctions of a self-adjoint eigenvalue problem. In particular, his theorem states that *an eigenfunction associated with the n-th eigenfunction of the eigenvalue problem* (1.1) *has at most n nodal domains.*

Sketch of the proof of Courant's theorem. — Let $\{u_n\}, n \geqslant 1$ be an orthonormal basis of eigenfunctions of problem (1.1), associated with the eigenvalues $\lambda_n(\Omega), n \geqslant 1$. Let v be an eigenfunction, associated with the eigenvalue $\lambda_k(\Omega)$. Assume that v has at least $(k+1)$ nodal domains, $\omega_1, \omega_2, \ldots$. For $1 \leqslant j \leqslant k$, define the function v_j by,

$$(1.4) \qquad v_j(x) = \begin{cases} v(x) & \text{if } x \in \omega_j, \\ 0 & \text{otherwise.} \end{cases}$$

One can find a linear combination $w := \sum_{j=1}^{k} \alpha_j v_j$ such that w is orthogonal to $u_1, \ldots u_{k-1}$, and has L^2-norm 1. Taking into account the definition of v_j, one finds that $\int_{\Omega} |dw|^2 \, dx = \lambda_k(\Omega)$. It follows from the min-max that w is also an eigenfunction associated with $\lambda_k(\Omega)$. Since w vanishes identically on the open set ω_{k+1}, it must be identically zero on Ω, which contradicts the fact that it has norm 1. □

Note that an eigenfunction associated with $\lambda_1(\Omega)$ has exactly one nodal domain (Ω is connected). For orthogonality reasons, an eigenfunction associated with the eigenvalue $\lambda_k(\Omega), k \geqslant 2$, has at least two nodal domains. In particular, an eigenfunction with $\lambda_2(\Omega)$ has exactly two nodal domains.

In his paper, Courant indicates that, for partial differential equations, it is easy to give examples of eigenvalues of higher rank and corresponding eigenfunctions with only two nodal domains. He does not give any detail, but we can guess that he had in mind the case of the square as described in Pockels' book [32, Chap. II.6.b]. For the square $]0, \pi[\times]0, \pi[\subset \mathbb{R}^2$, the Dirichlet eigenvalues are the numbers $\lambda_{m,n} = m^2 + n^2$, and a complete set of orthogonal eigenfunctions is given by the functions $u_{m,n}(x,y) = \sin(mx)\sin(ny)$, where m, n are positive integers. Figure 1.1 displays the nodal sets of some eigenfunctions associated with the eigenvalues $\lambda_2 = \lambda_3$ (top row) and $\lambda_4 = \lambda_5$ (bottom row), corresponding respectively to $\lambda_{1,2} = \lambda_{2,1}$ and $\lambda_{1,3} = \lambda_{3,1}$. Similarly, Figure 1.2 displays some nodal sets for the

eigenvalues $\lambda_6 = \lambda_7$ (top row) and $\lambda_8 = \lambda_9$ (bottom row), corresponding respectively to $\lambda_{2,3} = \lambda_{3,2}$ and $\lambda_{1,4} = \lambda_{4,1}$.
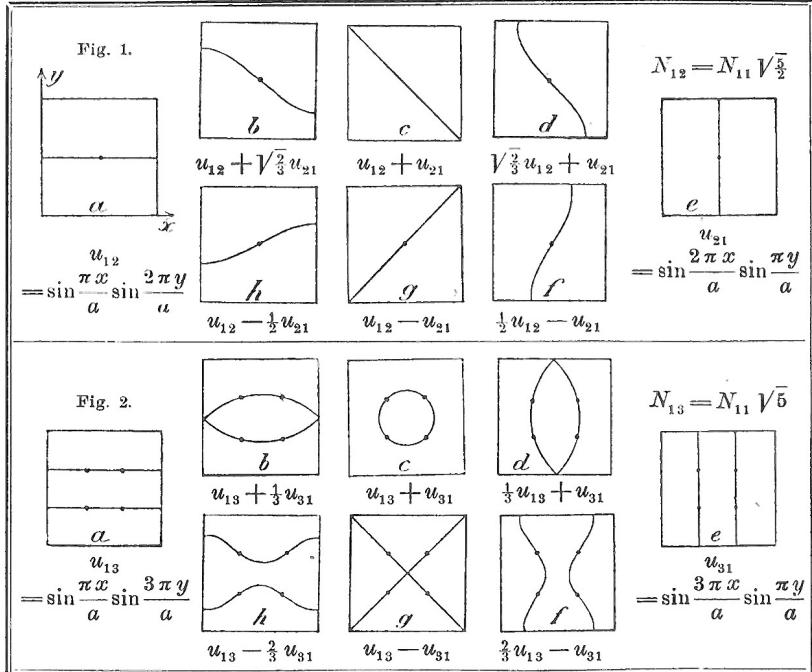

Figure 1.1. Eigenvalues, $\lambda_2 = \lambda_3$ and $\lambda_4 = \lambda_5$, [32, p. 80]

Note that Courant's upper bound is not sharp in the case of a multiple eigenvalue. Indeed, assume that, for some positive integers k and m,

$$\lambda_{k-1}(\Omega) < \lambda_k(\Omega) = \cdots = \lambda_{k+m}(\Omega) < \lambda_{k+m+1}(\Omega).$$

Since any eigenfunction u associated with $\lambda_j(\Omega), k+1 \leqslant j \leqslant k+m$ is also an eigenfunction associated with $\lambda_k(\Omega)$, we have $\mu(u) \leqslant k$, so that Courant's upper bound is definitely not achieved for the eigenvalues $\lambda_j(\Omega), k + 1 \leqslant j \leqslant k+m$.

The following formulation of Courant's theorem takes care of possible multiplicities.

THEOREM 1.1. — Let $\mathcal{N}(\lambda) := \#\{j : \lambda_j < \lambda\}$ be the counting function. For any λ, let $\mathcal{E}(\lambda)$ be the eigenspace associated with λ if λ is an eigenvalue, and $\{0\}$ otherwise. For any λ, let $\mu(\lambda)$ denote the maximum number of nodal domains of an eigenfunction in $\mathcal{E}(\lambda)$, possibly 0 if $\mathcal{E}(\lambda) = \{0\}$. Then, $\mu(\lambda) \leqslant \mathcal{N}(\lambda) + 1$.
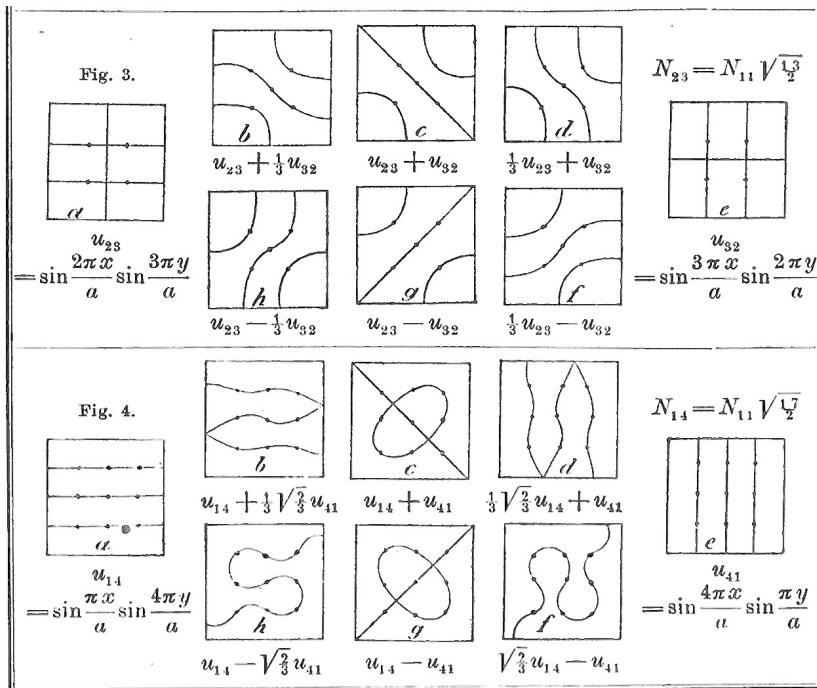
Figure 1.2. Eigenvalues, $\lambda_6 = \lambda_7$ and $\lambda_8 = \lambda_9$, [32, p. 80]

Remark. — Courant's theorem is quite general (no assumption on the degree of the operator nor on the number of variables), the main requirement being that the eigenvalue problem is self-adjoint. One can for example consider the Neumann eigenvalue problem for the Laplacian, or take Ω to be a compact Riemannian manifold, with or without boundary, and replace Δ by the associated Laplace–Beltrami operator (for example the sphere with the spherical Laplacian). One can also consider non-compact cases, for example the isotropic quantum harmonic operator $-\Delta + |x|^2$ in $\mathbb{R}^2$.

NATURAL QUESTIONS. — In view of the preceding discussion, the following questions are natural.

   Question 1 — Is the number 2 the best general lower bound for the number of nodal domains? How often does this phenomenon happen?

   Question 2 — Can one give sharper upper (lower) bounds on the number of nodal domains?

Question 3 — Call the eigenvalue λ *Courant-sharp* whenever $\mu(\lambda) = \mathcal{N}(\lambda) + 1$. Can one characterize Courant-sharp eigenvalues? Can one determine the Courant-sharp eigenvalues of a given domain or manifold?

Question 1 is the main topic of these notes. It was first investigated by Antonie Stern [33] in her 1924 PhD thesis at the university of Göttingen, under the supervision of R. Courant. She considered the eigenvalue problem for the Laplacian in the square membrane with Dirichlet boundary conditions, and for the spherical Laplacian on the 2-sphere. In both cases, she proved that there exists an infinite sequence of eigenvalues, and corresponding eigenfunctions with exactly two nodal domains, see [4] (tags [Q1,K1,K2]) and Section 2 for more details. The spherical case was also studied by H. Lewy [28, 1977]. Courant-sharp eigenvalues of spheres and Euclidean balls are investigated in [22]. An infinite sequence of eigenfunctions with only two nodal domains on the flat square 2-torus is mentioned in [19], with a reference to [23], see Section 7 for more details.

Question 2 has first been tackled by Å. Pleijel [31, 1956]. Using the Faber–Krahn inequality, he proved that for any bounded domain $\Omega \subset \mathbb{R}^2$, the number of nodal domains of a Dirichlet eigenfunction associated with $\lambda_k(\Omega)$ is asymptotically less than $\left(\frac{2}{\mathbf{j}}\right)^2 k < 0.7\,k$, where $\mathbf{j}$ denotes the least positive zero of the Bessel function J_0. This result implies that there are only finitely many Courant-sharp Dirichlet eigenvalues. For more details, we refer to [6] and the survey [10]. See also the recent contributions [11, 12] for the case of the two-dimensional quantum harmonic oscillator, and [13] for Schrödinger operators with radial potentials.

In some cases, it is possible to improve Courant's upper bound by using symmetries. This occurs in particular for the spherical Laplacian on the sphere $\mathbb{S}^2$. The idea is to use the antipodal map, the fact that even and odd spherical harmonics are always orthogonal, and to adapt Courant's proof as sketched above. This also occurs for the isotropic quantum harmonic oscillator $-\Delta + |x|^2$ in $\mathbb{R}^2$. For more details, we refer to Leydold's papers [29, 30] and to [8, 7].

Question 3 is related to spectral minimal partitions. For more details, we refer to the survey [10].

Remarks.

(i) For domains, we have only mentioned the Dirichlet eigenvalues. The same investigations can be made for the Neumann eigenvalues as well. To determine Courant-sharp eigenvalues for the Neumann problem turns out to be more difficult, we refer to [10] for more details and references. Note that, in the Neumann case, it does not seem possible to construct an infinite sequence of eigenfunctions with only two nodal domains [21, p. 36]. More precisely, the following conjecture has been proposed by Helffer and Persson Sundqvist [21, Conjecture 10.7]: For any sequence of eigenfunctions of the Neumann problem in the square associated with an infinite sequence of eigenvalues, the number of nodal domains tends to infinity. Concerning Question 2, C. Léna [27] recently proved that Pleijel's theorem is true for the Neumann case.

(ii) It is claimed in [16, Footnote 1, p. 454] that any linear combination of eigenfunctions associated with eigenvalues less than or equal to λ_n has at most n nodal domains. This assertion was questioned by V. Arnold, and proved to be wrong by O. Viro for linear combinations of spherical harmonics in dimension 3 [1, 24], see also [31, Section 7].

(iii) In higher dimension, one can easily construct high energy eigenfunctions with only two nodal domains on product manifolds of the form $\mathbb{S}^2 \times M$, with the product metric $g_{\mathbb{S}^2} \times \epsilon^2 g_M$. A less trivial example can also be given using Riemannian submersions with totally geodesic fibers on the 3-sphere [9, Corollary 6.7].

The paper is organized as follows. In Section 2, we summarize Stern's results and geometric ideas for the square membrane and for the 2-sphere. In Section 3, we revisit and complete Stern's proofs in the case of the square membrane. In Section 4, we explain Stern's results for the sphere. In Section 5, we explain how Stern's ideas can be applied to construct regular spherical harmonics with many nodal domains. In Section 6, we extend Stern's result to the case of the 2-dimensional isotropic quantum harmonic oscillator. In Section 7, we look at eigenfunctions of the flat rectangular 2-torus.

2. The results of Antonie Stern

Antonie Stern was born in Dortmund in 1892. She defended a PhD thesis in mathematics at Göttingen University on July 30, 1924, under the

supervision of R. Courant. She later worked as a researcher at the Kaiser-Wilhelm-Institut für Arbeitsphysiologie in Dortmund from 1929 to 1933. She emigrated to Israel in 1939, and still lived in Rehovot in 1967. We are grateful to Annette Vogt for providing us with these details, see [35].

2.1. Stern's results

Stern's thesis was published in Göttingen, in a volume containing other theses, and reviewed in the Jahrbuch für Mathematik.

> JFM 51.0356.01
> Stern, Antonie (Stern, Antonie)
> Bemerkungen über asymptotisches Verhalten von Eigenwerten und Eigenfunktionen.
> Math.- naturwiss. Diss. (German) [D] Göttingen, 30 S.
> Published: 1925
> Im 1. Teil wird u. a. gezeigt, einmal, dass es zu jedem Eigenwert von $\Delta u + \lambda u = 0$ auf der Kugel Eigenfunktionen gibt, deren Nullinien die Kugelfläche in zwei oder drei Gebiete teilen; ferner, dass (beim gleichen Problem) bei den Eigenwerten Gebietszahlen von ganz verschiedener Grössenordnung auftreten, so dass ein asymptotisches Gesetz für die Gebietsanzahlen nicht besteht. Im 2. Teil wird (für gewisse Randwertaufgaben bei partiellen Differentialgleichungen 2. Ordnung) ein asymptotisches Gesetz für die Verteilung der Eigenwerte in Abhängigkeit von den Randbedingungen hergeleitet.
> [Haupt, O.; Prof. (Erlangen)]
> Subject heading: Vierter Abschnitt. Analysis. Kapitel 13. Potentialtheorie. Theorie der partiellen Differentialgleichungen vom elliptischen Typus.
> Signatory at SUB: <19>: 1601/Diss. 19 845

Stern's thesis is mentioned in a footnote in the classical book of Courant–Hilbert, see [16, Chap. VI.6, pp. 455-456] and [17, Chap. VI.6, p. 396], together with two figures from her thesis. This citation only concerns Stern's results for the square membrane. To this day, we did not find any reference to Stern's results for the sphere in the book of Courant–Hilbert, nor elsewhere in the literature.

Here is a quotation from the introduction of Stern's thesis [33, Einleitung, p. 3].

> [...] Im eindimensionalen Fall wird nach den Sätzen von Sturm das Intervall durch die Knoten der $n$ten Eigenfunktion in n Teilgebiete zerlegt. Dies Gesetz verliert seine Gültigkeit bei

> mehrdimensionalen Eigenwertproblemen, [...] es läßt sich
> beispielsweise leicht zeigen, daß auf der Kugel bei jedem Eigen-
> wert die Gebietszahlen 2 oder 3 auftreten, und daß bei Ordnung
> nach wachsenden Eigenwerten auch beim Quadrat die Gebiet-
> szahl 2 immer wieder vorkommt.[...]

Stern's thesis is written in a rather discursive style. As far as Question 1 is concerned, the main results of her thesis can be summarized in theorem form as follows.

THEOREM 2.1 (Stern's result for the square membrane). — *For the square* $]0, \pi[\times]0, \pi[\subset \mathbb{R}^2$, *there exists a sequence* $\{v_m\}$ *of Dirichlet eigenfunctions associated with the eigenvalues* $\lambda_{2m,1} = 1 + 4m^2$, $m \geqslant 1$, *such that* $\mu(v_m) = 2$.

THEOREM 2.2 (Stern's results for the sphere). — *On the 2-sphere, for any positive integer* ℓ, *there exists a spherical harmonic* u_ℓ *of degree* ℓ, *such that*

$$(2.1) \qquad \mu(u_\ell) = \begin{cases} 2 & \text{if } \ell \text{ is odd,} \\ 3 & \text{if } \ell \text{ is even.} \end{cases}$$

Upon reading [33], it seemed to us that the proofs lacked some important details although the ideas were nice and geometric. We thought it useful to give complete proofs along these geometric ideas. Doing so, we actually obtained more precise quantitative results, and a better understanding of the bifurcations of nodal sets, see [6, 8]. Our methods also turned out to be useful to determine Courant-sharp Dirichlet eigenvalues of the equilateral, hemiequilateral and isosceles triangles [5]; see [2] for the right-isosceles triangle with Neumann boundary condition, and [3] for 2-rep-tile domains with Neumann boundary condition.

Theorem 2.2 was rediscovered by H. Lewy in 1977. Lewy's proofs [28] are less geometric and more analytic than Stern's proofs, see [8] for a comparison. He also proves that the lower bound 3 for spherical harmonics of even degree is sharp. Lewy was also a student of Courant. He defended his thesis in Göttingen in 1926. His paper does however not mention Stern's thesis.

2.2. Stern's ideas

We now describe Stern's ideas in the case of the square $]0, \pi[\times]0, \pi[\subset \mathbb{R}^2$. Recall the notation

$$(2.2) \qquad u_{\ell,m}(x, y) = \sin(\ell x) \sin(my),$$

where ℓ, m are positive integers, and $x, y \in]0, \pi[$.

For any positive integer r, Stern considers the family of eigenfunctions $u_{2r,1} + \mu\, u_{1,2r}$, associated with the eigenvalue $\lambda_{2r,1} = 4r^2 + 1$, where μ a real parameter close to 1. Stern claims that the nodal set of $u_{2r,1} + u_{1,2r}$ is given by Figure 2.1.

> Wir betrachten die Eigenwerte $\lambda_n = \lambda_{2r,1} = 4r^2 + 1$, $r = 1, 2, \ldots$ und die Knotenlinie der zugehörige Eigenfunktion $u_{2r,1} + u_{1,2r} = 0$, für die sich, wie leicht mittels graphischer Bilder nachgewiesen werden kann, die Figur 7 ergibt.

As a matter of fact, Figure 2.1 illustrates the case $r = 6$. For a proof of this claim, we refer to [6, Prop. 6.6], see also [19].
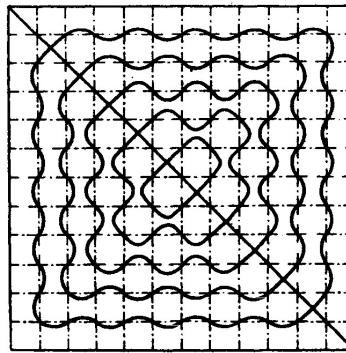

Figure 2.1. Stern [33, Figur 7]

Next, Stern claims that when μ leaves the value 1, all the nodal crossings disappear at the same time and in the same manner, leading to a nodal line as in Figure 2.2. It follows that for any positive integer r, there exist values of μ for which the above eigenfunction has exactly two nodal domains.

> Laßen wir nur μ von $\mu = 1$ aus abnehmen, so lösen sich die Doppelpunkte der Knotenlinie alle gleichzeitig und im gleichem Sinne auf, und es ergibt sich die Figur 8. Da die Knotenlinie aus einem Doppelpunktlosen Zuge besteht, teilt sich das Quadrat in zwei Gebiete und zwar geschieht dies für alle Werte $r = 1, 2, \ldots$, also Eigenwerte $\lambda_n = \lambda_{2r,1} = 4r^2 + 1$.

In order to prove her claim, Stern introduces two geometric ideas. Consider the eigenfunctions $u_{\ell,m}$ and $u_{m,\ell}$ associated with the eigenvalue $\lambda_{\ell,m} = \ell^2 + m^2$. Their nodal sets are the union of segments parallel to the x and y axes. The connected components of the set $]0, \pi[\times]0, \pi[\backslash(N(u_{\ell,m}) \cup N(u_{m,\ell}))$ are open rectangles which can be colored (white/grey) according
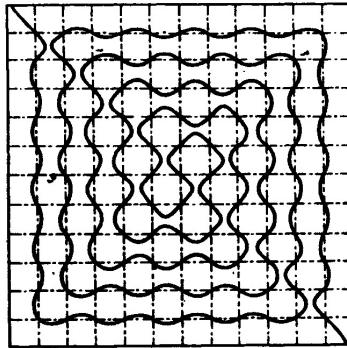
Figure 2.2. Stern [33, Figur 8]

to the sign $-/+$ of the function $u_{\ell,m}\, u_{m,\ell}$, forming a kind of *checkerboard*. The following properties hold.

(1) If $\mu > 0$, then the nodal set of $u_{\ell,m} + \mu\, u_{m,\ell}$ is contained in the white rectangles of the checkerboard.

(2) For all μ, the nodal set $N(u_{\ell,m} + \mu\, u_{m,\ell})$ contains $N(u_{\ell,m}) \cap N(u_{m,\ell})$, the set of *fixed points*.

> Um den typischen Verlauf der Knotenlinie zu bestimmen, [...] Legen wir die Knotenliniensysteme von $u_{\ell,m}$ ($\ell - 1$ Parallelen zur y-Achse, $m-1$ zur x-Achse) und $u_{m,\ell}$ ($m-1$ Parallelen zur y-Achse, $\ell - 1$ zur x-Achse) übereinander, so kann für $\mu > 0$ (< 0) die Knotenlinie nur in den Gebieten verlaufen, in denen beide Funktionen verschiedenes (gleiches) Vorzeichen haben.

> Weiter müssen alle zum Eigenwert $\lambda_{\ell,m}$ gehörigen Knotenlinien durch Schnittpunkte der Liniensysteme $u_{\ell,m} = 0$ and $u_{m,\ell} = 0$, also durch $(\ell - 1)^2 + (m - 1)^2$ feste Punkte hindurchgehen.

Remark. — It is interesting to note that the consideration of the fixed points already appears in the figures in Pockels's book (see also the corresponding picture in the book of Courant–Hilbert, [16, p. 302]).

3. Stern's proofs revisited

As the elements of proof given by Stern did not look sufficient, we looked into the following items.

- Give a precise description of the nodal set of the eigenfunction $u_{1,2r} + u_{2r,1}$.
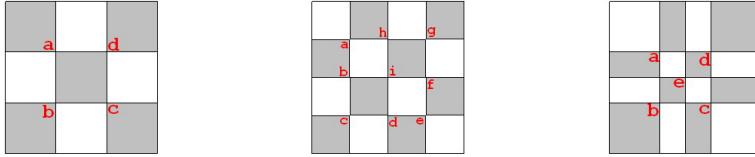
Figure 2.3. Checkerboards and fixed points for $\lambda_{1,3}$,$\lambda_{1,4}$ and $\lambda_{2,3}$

- Establish the regularity of the nodal set of the function $u_{1,2r} + \mu\, u_{2r,1}$ in the interior of the square, for μ close to 1 and different from 1 .
- Prove that the nodal set of $u_{1,2r} + \mu\, u_{2r,1}$ is connected (in particular, that no connected component of the nodal set can avoid all the fixed points).

Recall the definition of the nodal set $N(\Phi)$ of a Dirichlet eigenfunction Φ of the square $]0,\pi[^2$

$$(3.1) \qquad N(\Phi) = \overline{\{(x,y) \in]0,\pi[^2 \mid \Phi(x,y) = 0\}}\,.$$

Given an integer R, introduce the following one-parameter family of eigenfunctions associated with the eigenvalue $\lambda_{1,R} = 1 + R^2$,

$$(3.2) \quad \Phi_{1,R}^\theta(x,y) = \cos\theta\,\sin(x)\,\sin(Ry) + \sin\theta\,\sin(Rx)\,\sin(y)\,,\ \theta \in [0,\pi[\,.$$

It is convenient to factor out $\sin(x)\,\sin(y)$, and to consider the family

$$\phi_{1,R}^\theta(x,y) = \cos\theta\,U_{R-1}(\cos y) + \sin\theta\,U_{R-1}(\cos x)\,,$$

where U_n denotes the Chebyshev polynomial with degree n.

Remark. — When R is even, due to natural symmetries, it suffices to consider the values $\theta \in [0, \frac{\pi}{4}]$.

3.1. New ingredients

To complete Stern's proofs, we introduce two key ingredients, and a technical one:

- the analysis of critical zeros,
- the analysis of the trace of the nodal set on the boundary of the square,

• separation lemmas.

As a by-product, we obtain qualitative versions of Stern's results, and a better understanding of the bifurcations of nodal sets in the family.

A *critical zero* of Φ^θ is a point (x, y) such that $\Phi^\theta(x, y) = 0$ and $d_{(x,y)}\Phi^\theta = 0$. The local structure of the nodal set in a neighborhood of a critical zero is determined by Bers' theorem. In dimension two, up to C^1 diffeomorphism, the nodal set in the neighborhood of a zero of order k is the same as that of a homogeneous harmonic polynomial of degree k, see [14].

Remarks.

(i) In full generality, the preceding result holds at critical zeros of eigenfunctions in the interior of a 2-dimensional domain. Since the Dirichlet eigenfunctions of the square extend to the whole plane, the same property holds at boundary points as well (one has then to include the sides of the square in the nodal set).

(ii) In their books Pockels and Courant–Hilbert do not say anything about the choice of the parameters for the pictures they produce. It is interesting to note that some of the parameters correspond precisely to the occurrence of critical zeros.

(iii) T. Hoffmann-Ostenhof pointed out to us the Master degree thesis (Diplom Arbeit) of J. Leydold [29] which is based on a precise analysis of critical zeros.

The *interior critical zeros* of the function $\Phi^\theta_{1,R}$ are given by the equations,

(3.3)
$$\begin{cases} \cos\theta\, U_{R-1}(\cos y) + \sin\theta\, U_{R-1}(\cos x) = 0\,, \\ \sin\theta\, U'_{R-1}(\cos x) = 0\,, \\ \cos\theta\, U'_{R-1}(\cos y) = 0\,. \end{cases}$$

They correspond to self-intersections of the nodal set. It follows that the only *possible* interior critical zeros are the points of the form (q_i, q_j) where the $\cos q_i$ are the zeros of the polynomial $U'_{R-1}(t)$. Furthermore, a given critical zero is associated with a unique value of the parameter $\theta \in [0, \pi[$.

As a consequence, the function Φ^θ has no interior critical zero, except for a finite set of values of θ.

The trace of the nodal set $N(\Phi^\theta_{1,R})$ on the boundary of the square is determined by the equation

(3.4)
$$\cos\theta\, U_{R-1}(\cos y) + \sin\theta\, U_{R-1}(\cos x) = 0\,,$$

where we have to fix $x = 0$ or $x = \pi$, and solve an equation in y; *resp.* where we have to fix $y = 0$ or $y = \pi$, and solve an equation in x. The points obtained in this manner are points at which an interior nodal arc hits the boundary. They are always critical zeros of order at least two. When they are of order at least three, they correspond to points in the boundary hit by at least two interior nodal arcs. Such higher order zeros are determined by the zeros of $U'_{R-1}(t)$.

The *vertices* of the square are treated separately. Using properties of the Chebyshev polynomials, one can prove the following properties for the functions Φ^θ.

- The vertices are critical zeros of order two or four.
- The boundary critical zeros, away from the vertices, are of order two or three.
- The interior critical zeros are of order two.
- The fixed points are not critical zeros. At these points, the nodal set consists of a regular arc which is transversal to the horizontal and vertical lines.

Based on the preceding results, Figures 3.1–3.3 give the possible nodal patters in the white squares, respectively at the vertices, at the boundary or in the interior. Note that the configurations (Ae), (Ce) and (De) in Figure 3.2, and (A) and (B) in Figure 3.3 are stable under small variations of θ.
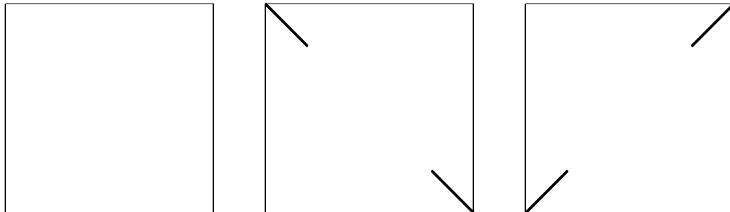

Figure 3.1. R even, nodal patterns at the vertices. From left to right: $\theta \neq \pi/4$ and $3\pi/4$; $\theta = \pi/4$; $\theta = 3\pi/4$

As a consequence there are only finitely many possible forms for the nodal sets of Φ^θ. Changes can only occur for the values of θ corresponding to the existence of critical zeros. Figure 3.4 illustrates the case of the eigenvalue $\lambda_{1,8}$.
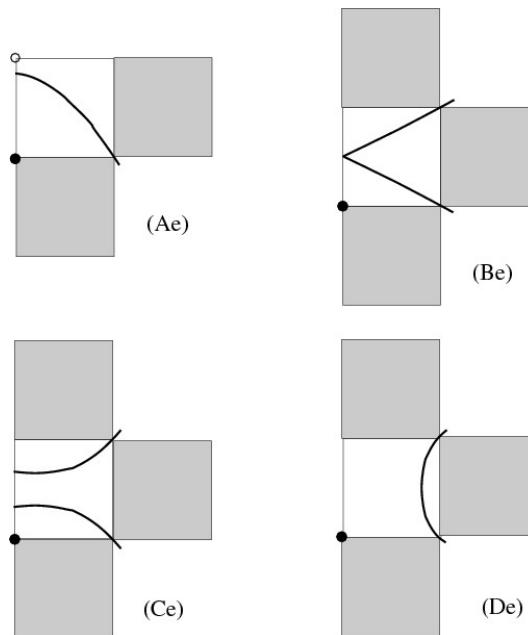
*Figure 3.2.* R even, nodal patterns on the boundary
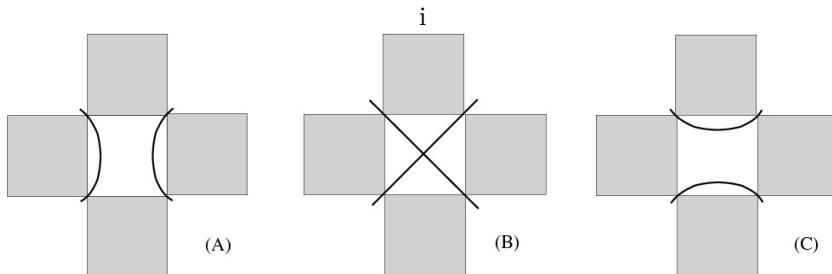
Figure 3.3. Interior nodal patterns

3.2. Detailed proof in the case $\lambda_{1,4}$

In this section, we give a detailed proof in the particular case of $\lambda_{1,4}$. We work with the family of functions,

$$(3.5) \qquad \Phi_{1,4}^{\theta}(x,y) := \cos\theta\,\sin(x)\,\sin(4y) + \sin\theta\,\sin(4x)\,\sin(y)\,,$$

which we shall simply denote by Φ^{θ}.
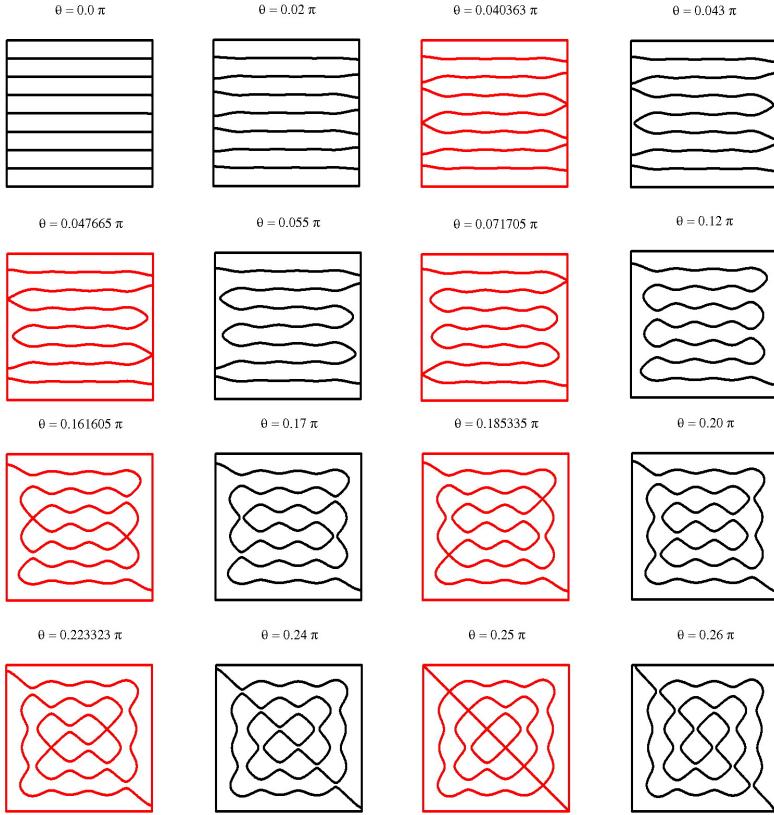
Figure 3.4. Typical nodal sets for $\lambda_{1,8}$

We have the symmetries, $\Phi^{\frac{\pi}{2}-\theta}(x,y) = \Phi^\theta(y,x)$ and $\Phi^\theta(\pi - x, y) = -\Phi^{\pi-\theta}(x,y)$, so that it suffices to restrict to $\theta \in [0, \frac{\pi}{4}]$.

Trace of the nodal set $N(\Phi^\theta)$ on the boundary. — Assuming that $\theta \in [0, \frac{\pi}{4}]$, we find that the points at which the nodal set hits the boundary are located on the sides $\{x = 0\}$, respectively $\{x = \pi\}$, and that they are given by the equations,

$$\cos\theta\, U_3(\cos y) + 4\sin\theta = 0\,,$$

and

$$\cos\theta\, U_3(\cos y) - 4\sin\theta = 0\,,$$

respectively.

There is one critical value of θ, namely $\theta_c = \arccos(\frac{1}{3}\sqrt{\frac{2}{3}})$. When $\theta < \theta_c$, the trace consists of three points on each side (zeros of order 2); when $\theta = \theta_c$, there are two points on each side (one zero of order 2, and one of order 3); when $\theta > \theta_c$, there is one point on each side (a zero of order 2). Figure 3.5 displays the checkerboard, the fixed points and the boundary points in the three cases.
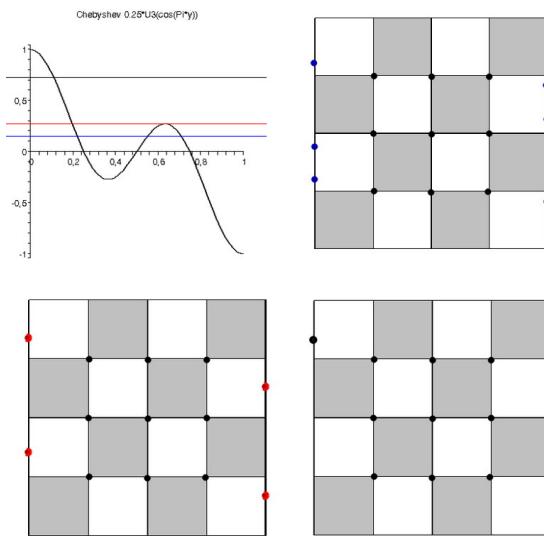

Figure 3.5. Trace of $N(\Phi^\theta)$ on the boundary

Note that the first and third configurations are stable under small variations of θ.

Interior critical zeros. — The only possible interior critical zeros are the points (x, y) such that $x, y = \arccos\sqrt{\frac{1}{6}}$ or $\pi - \arccos\sqrt{\frac{1}{6}}$. They only occur for two critical values of θ, $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. On the boundary these values of y correspond to the critical zeros of order 3. The corresponding critical values of the parameter $\theta \in [0, \frac{\pi}{4}]$ are θ_c (two critical zeros of order 3 on the boundary, no interior critical zero), and $\frac{\pi}{4}$ (no critical zero on the boundary, two critical zeros at the vertices, and two interior critical zeros). Figure 3.6 displays the checkerboard, the fixed points and the possible critical zeros.

It turns out that the dotted horizontal lines through the possible critical zeros provide partial barriers for the nodal set. This is the content of the
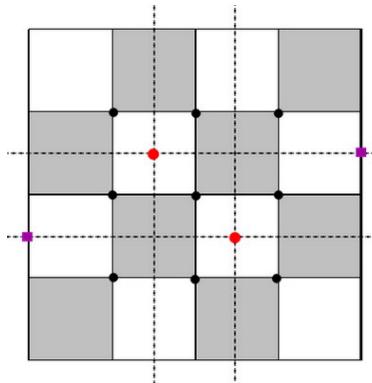
Figure 3.6. Checkerboard, fixed points, critical zeros

following separation result. We look at the intersection of the nodal set
$N(\Phi^\theta)$ with the lines $\{y = \arccos \sqrt{\frac{1}{6}}\}$ and $\{y = \pi - \arccos \sqrt{\frac{1}{6}}\}$. This
amounts to solving the equation

$$U_3(\cos x) = 4 \frac{\tan \theta_c}{\tan \theta}.$$

This equation has no solution when $0 \leqslant \theta < \theta_c$; a unique solution $(x = 0)$,
when $\theta = \theta_c$; a unique solution in $]0, \pi[$, when $\theta_c < \theta < \frac{\pi}{4}$; exactly two
distinct solutions in $]0, \pi[$, when $\theta = \frac{\pi}{4}$ (the critical zeros). From these prop-
erties, one can deduce the course of the nodal sets. The following figures
illustrate these three cases.

Let us for example consider the first case, $\theta < \theta_c$. Recall that a nodal
arc can only cross the edges of the small white squares at the vertices, and
cannot cross the horizontal dotted lines (separation lemma). Start from
the highest boundary point on the left side of the square. The nodal arc
issued from this point has to leave the white smaller square at its south-east
corner. From there, it cannot go below the dotted line, so that it must end
up at the highest boundary point on the right side of the square. Looking
at the arcs which are issued from the other boundary points, we see that
the nodal set contains the three curves which appear in Figure 3.7, right
sub-figure in the bottom row. Any other connected component of the nodal
set would not meet the boundary of the square, would not contain any of
the fixed points, and would therefore be entirely contained within a small
white square. We could therefore find a nodal domain entirely contained in
a small white square, leading to a contradiction by domain monotonicity
of Dirichlet eigenvalues. The cases $\theta = \theta_c$ and $\theta_c < \theta < \frac{\pi}{4}$ can be treated
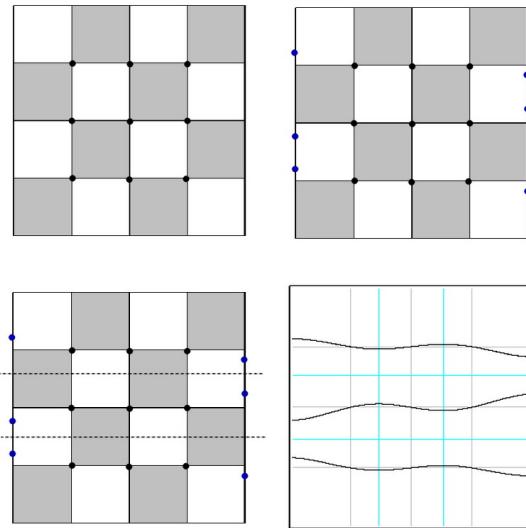
Figure 3.7. Case $\theta < \theta_c$
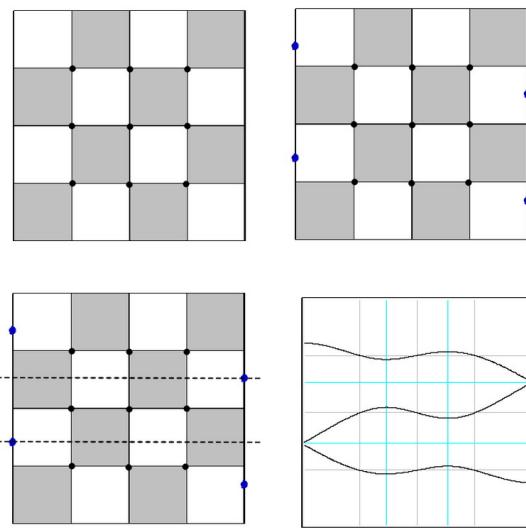
Figure 3.8. Case $\theta = \theta_c$
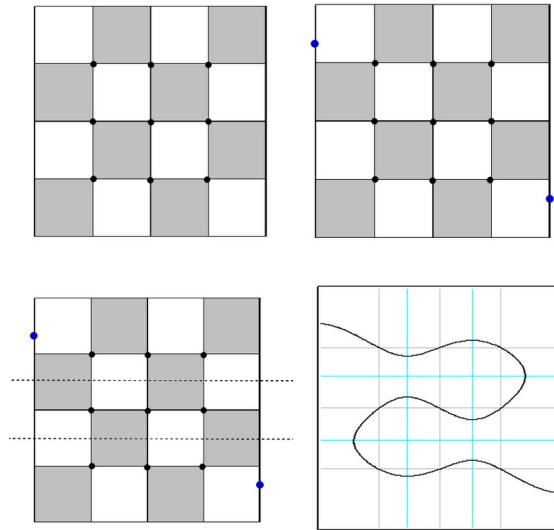
Figure 3.9. Case $\theta_c < \theta < \frac{\pi}{4}$
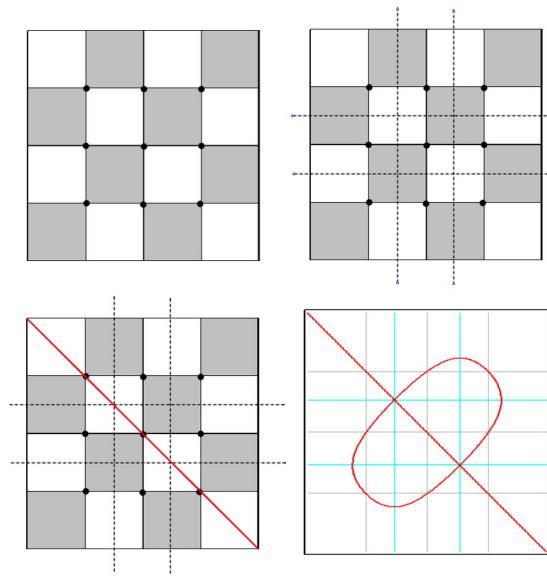

Figure 3.10. Case $\theta = \frac{\pi}{4}$

similarly. For the case $\theta = \frac{\pi}{4}$, we first notice that the nodal set contains the anti-diagonal of the square, that the only boundary critical zeros are the two vertices, and that there are exactly two interior critical zeros located on the anti-diagonal. We can then apply the above reasoning starting from one of the two interior critical zeros, following a nodal arc orthogonal to the anti-diagonal.

3.3. Stern's result for the square revisited

The above methods allow us to give a more precise version of Stern's results for the square membrane.

THEOREM 3.1. — *For $r \in \mathbb{N}^{\bullet}$, consider the family $\Phi_{1,2r}(x, y, \theta)$ of Dirichlet eigenfunctions.*

$$\Phi_{1,2r}^{\theta}(x, y) := \cos\theta \sin x \sin(2ry) + \sin\theta \sin(2rx) \sin y \,.$$

Then we have:

(1) *For $\theta = \frac{\pi}{4}$, the nodal set of Φ^{θ} consists of the anti-diagonal, and of $(r-1)$ disjoint simple closed curves containing all the fixed points, each of them crossing the diagonal at exactly two points.*

(2) *For $\theta \neq \frac{\pi}{4}$, and θ close enough to $\frac{\pi}{4}$, the nodal set of Φ^{θ} consists of a regular simple connected curve containing all the fixed points, whose extremities are located on the boundary of the square. This curve divides the square into two connected components.*

Sketch of the proof. — One first determines the nodal set of the eigenfunction $\Phi^{\frac{\pi}{4}}$. For this purpose,

- one shows that the critical zeros are the points $(q_i, \pi - q_i)$;
- the nodal set contains the anti-diagonal; starting from a critical zero, orthogonally to the anti-diagonal, one can follow the nodal set and show the existence of the $(r-1)$ simple closed curves (here, one uses a separation lemma and the local structure of the nodal set), see Figure 2.1;
- one concludes by showing that the nodal set cannot contain any other connected component, indeed such a component would contain no fixed point, leading to a contradiction (energy argument).

If $0 < |\theta - \frac{\pi}{4}| \ll 1$, the function Φ^{θ} has no interior critical zero, and only two critical zeros (of order 2) on the boundary. One can then conclude using a local analysis of the deformation of the nodal set in a neighborhood

of the critical zeros, the separation lemma (or a continuity argument), and the energy argument to exclude connected components in the interior of the small squares of the checkerboard.

Remark. — The above argument actually holds in any $J \setminus \{\frac{\pi}{4}\}$, where J is any open interval containing $\frac{\pi}{4}$ and no other critical value of the parameter θ.


Figure 3.11. Deformation of the nodal sets for $\lambda_9 = \lambda_{1,4}$

Bifurcation for eigenvalue [1,6] $\theta = 0.25\,\pi$
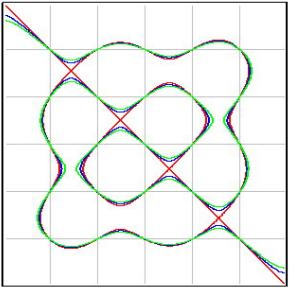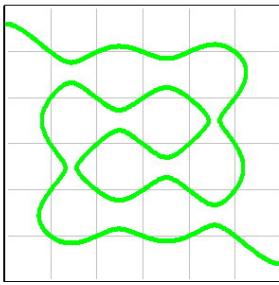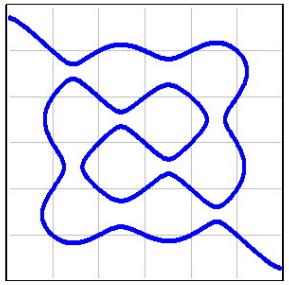
$\theta = 0.23\,\pi$ $\theta = 0.21\,\pi$

Figure 3.12. Deformation of the nodal sets for $\lambda_{23} = \lambda_{1,6}$

4. Stern's results for the sphere

The methods developed in Section 3 can be applied to the eigenfunctions of the Laplace–Beltrami operator on the two sphere $\mathbb{S}^2$.

We only illustrate the results by some figures, referring to [8] for the details. In the figures, the nodal sets are viewed in the exponential map at the north pole $(0, 0, 1)$.

4.1. The sphere $\mathbb{S}^2$, two nodal domains

Consider the family

$$\Phi_1^\theta(x, y, z) := \cos\theta\,\Im(x + iy)^\ell + \sin\theta\,Z_\ell(x, y, z)\,,$$

where Z_ℓ is the zonal spherical harmonic of degree ℓ (with respect to the z-axis).

The two functions $\Im(x + iy)^\ell$ and Z_ℓ give rise to a kind of checkerboard which can be colored according to the sign of the product (the checker patterns actually consist of spherical triangles and quadrilaterals). As in the case of the square, we have fixed points (points at which both functions vanish simultaneously) which are regular points of the family of eigenfunctions. It is easy to show that for θ positive, small enough, the function Φ_1^θ has no critical zero, so that its nodal set is a regular 1-submanifold of the sphere. To distinguish between the possible local nodal patterns, we prove a separation lemma, using the meridians which bisect the nodal domains of the function $\Im(x + iy)^\ell$ as barriers. Using the same kind of arguments as in the case of the square, one can determine the nodal set of Φ_1^θ for θ positive small enough.

Figure 4.1 illustrates the difference between odd and even degree harmonics, in the cases $\ell = 3$ and $\ell = 4$. The figures display the nodal sets of $\Im(x + iy)^\ell$ (meridians) and $Z_\ell(x, y, z)$ (latitude circles), the checkerboards, the fixed points, and the nodal set of Φ_1^θ for θ positive and small enough.
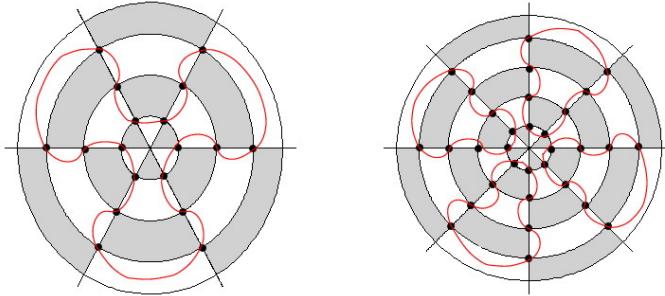

Figure 4.1. Sphere, eigenvalue $\ell(\ell + 1)$, $\ell = 3$ and 4

The simplest interesting case is $\ell = 3$. The nodal set of $\Im(x + iy)^3$ is the union of six meridians. The nodal set of $Z_3(x, y, z)$ is the union of three latitude circles. There is only one critical value θ_c of θ in the interval $]0, \frac{\pi}{3}[$. When $\theta \neq \theta_c$ in this interval, the function Φ_1^θ has no critical zero. Figure 4.2 displays the nodal set Φ_1^θ, for $\theta < \theta_c$ (left), $0 < \theta = \theta_c$ (center), and $\theta_c < \theta < \frac{\pi}{3}$ (right).
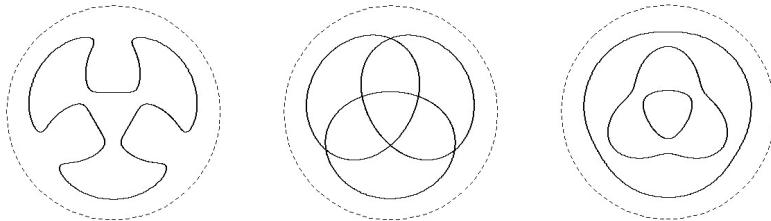
Figure 4.2. Sphere, eigenvalue $\ell(\ell+1)$, $\ell = 3$

The Figures 4.3 and 4.4 display Maple numerical computations of the nodal sets of Φ_1^θ for θ positive small enough, in the cases $\ell = 3, 4, 5$ and 6.
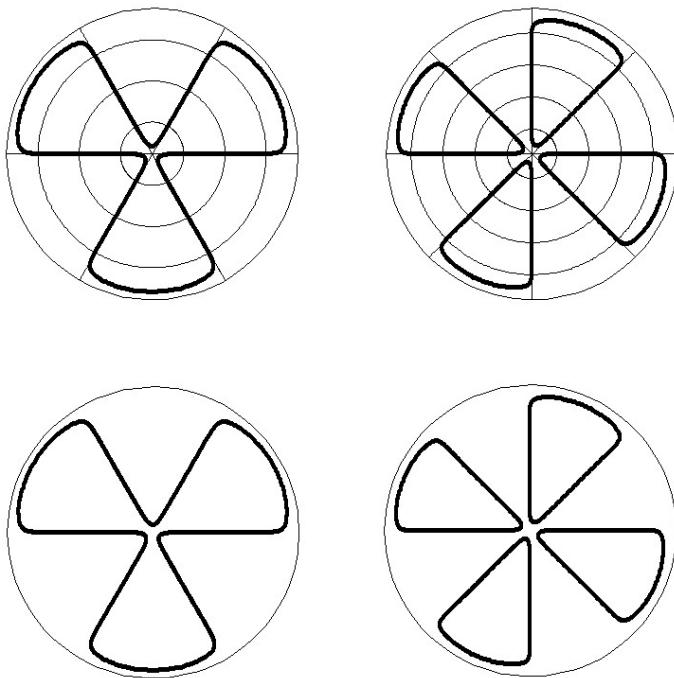

*Figure 4.3.* Sphere, eigenvalue $\ell(\ell+1)$, $\ell = 3$ and 4
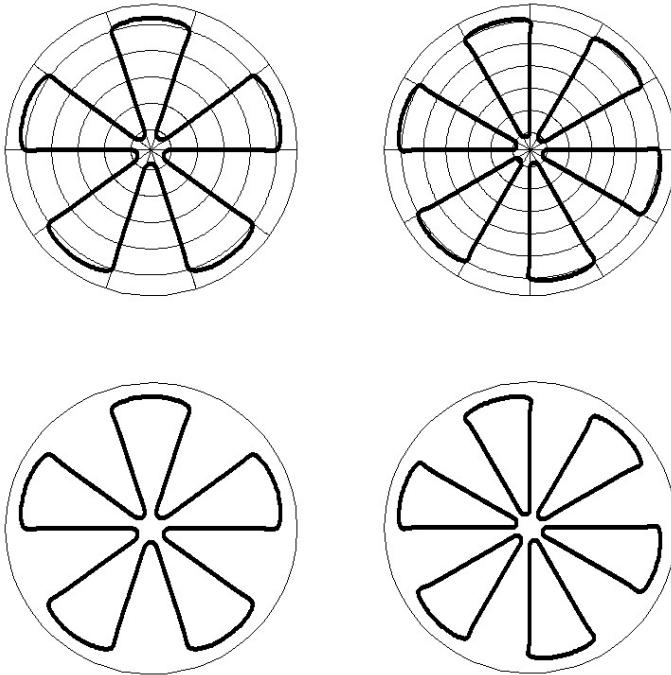
Figure 4.4. Sphere, eigenvalue $\ell(\ell+1)$, $\ell = 5$ and 6

4.2. The sphere $\mathbb{S}^2$, three nodal domains

According to H. Lewy [28, Introduction], any even spherical harmonic of positive degree has at least three nodal domains.

For $\ell = 2r$, $\alpha > 0$ small enough, and $\mu > 0$, consider the family of spherical harmonics

$$h^\mu(\vartheta, \varphi) = \sin^{2r}(\vartheta)\,\sin(2r\varphi) + \mu\,\sin\vartheta\,P'_{2r}(\cos\vartheta)\,\sin(\varphi - \alpha)\,.$$

Figure 4.5 illustrates the construction of spherical harmonics of even degree with exactly three nodal domains, in the case $\ell = 4$.

Figure 4.6 provides Maple numerical computations in the cases $\ell = 4$ and 6.
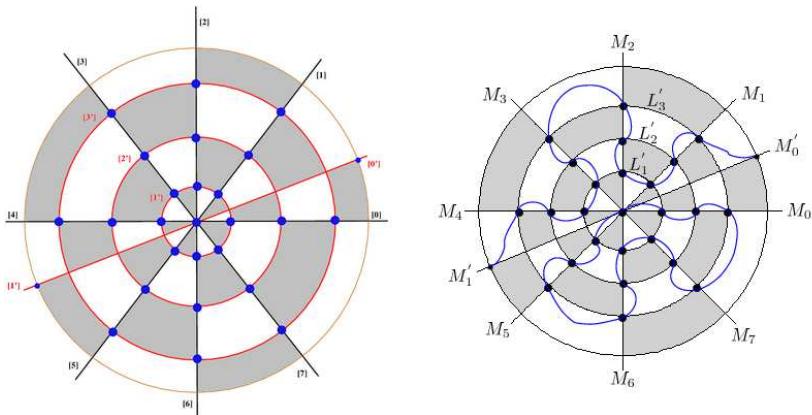
Figure 4.5. Sphere, $\ell = 4$, checkerboard and nodal set for $0 < \mu, \alpha \ll 1$

4.3. Spherical harmonics with a prescribed number of nodal domains

In her thesis, Stern states two other interesting results [4](tags [K3], [K4], families of spherical harmonics (3) and (5) respectively). They can be proved rigorously using the same arguments as in the preceding subsections.

THEOREM 4.1. — *Let d be any given integer greater than or equal to 3. If*

  (1) $d \equiv 3 \pmod{4}$, *or if*
  (2) d *is even,*

then, there exists an infinite sequence of eigenvalues of the 2-sphere, tending to infinity, and associated spherical harmonics with exactly d nodal domains.

Remark. — One can also prove that any integer d is achieved at least once as number of nodal domains by some spherical harmonics. It is however not clear whether any $d \equiv 1 \pmod{4}$ can be achieved infinitely many times for different eigenvalues.
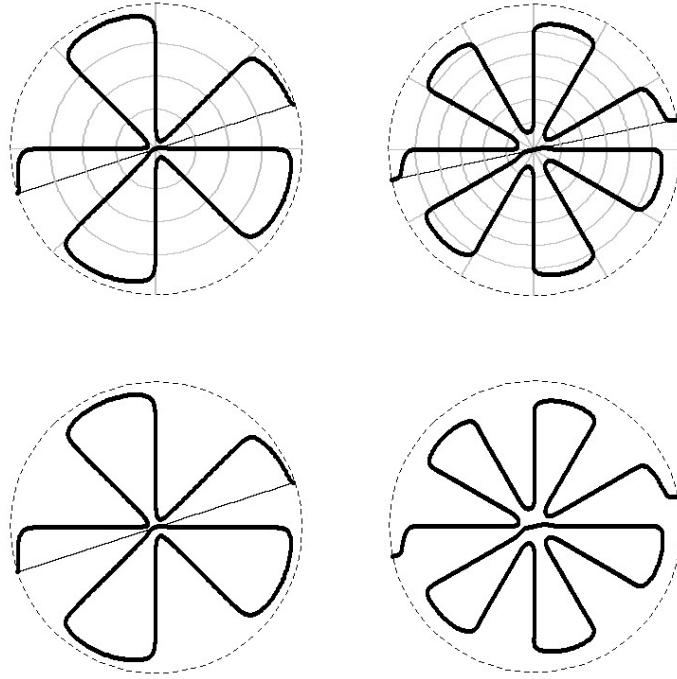
Figure 4.6. Sphere, eigenvalue $\ell(\ell+1)$, $\ell = 4, 6$

5. Spherical harmonics on $\mathbb{S}^2$, with many nodal domains

For the sphere, J. Leydold [30] has conjectured that the maximum number $\mu_m(\ell)$ of nodal domains of a spherical harmonic of degree ℓ satisfies,

$$(5.1) \qquad \mu_m(\ell) \leqslant \begin{cases} \dfrac{1}{2}(\ell+1)^2, & \text{when } \ell \text{ is odd,} \\[2ex] \dfrac{1}{2}\ell(\ell+2), & \text{when } \ell \text{ is even,} \end{cases}$$

where the values in the right-hand side correspond to decomposed spherical harmonics in spherical coordinates. He proved the conjecture for $\ell \leqslant 6$. He also constructed spherical harmonics without critical zero and "many" nodal domains (the maximum number which is conjectured, divided by two), see also [18]. The above methods give a simple approach for the construction of such examples. This is illustrated by Figure 5.1. The idea is
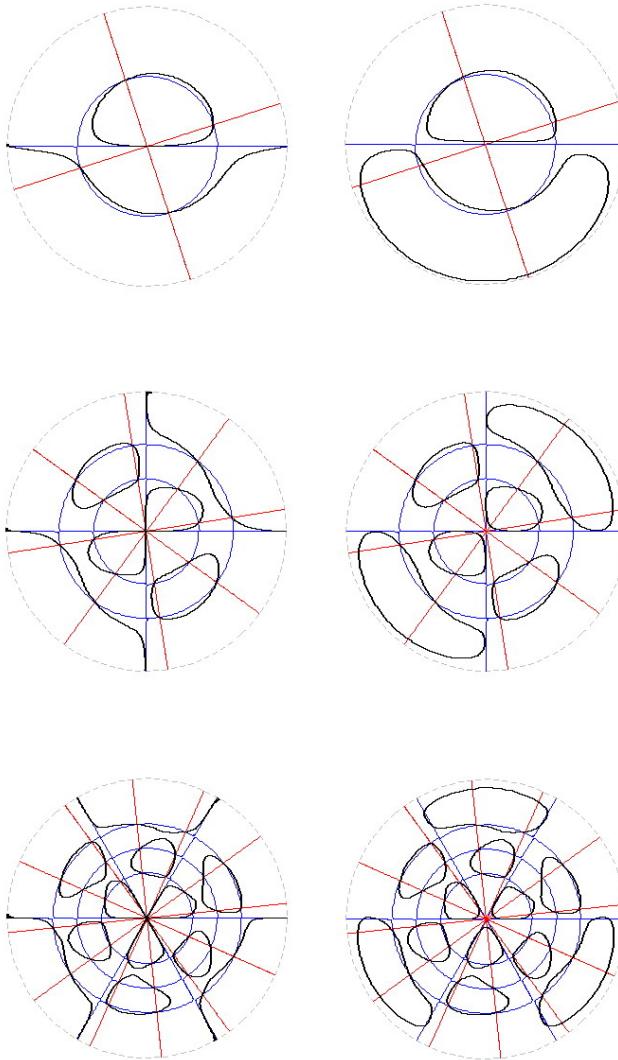
Figure 5.1. Sphere, eigenvalue $\ell(\ell+1)$, $\ell = 2m$, $m = 1, 2$ and 3

to work in spherical coordinates, and to start from a decomposed spherical harmonic whose nodal set consists of meridians and latitude circles (in blue in the figure), so that it has as many nodal domains as possible.
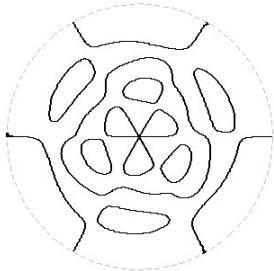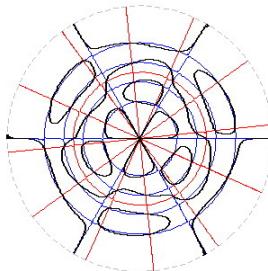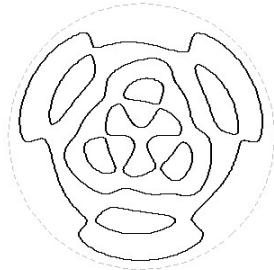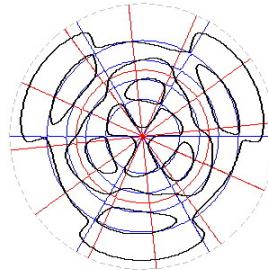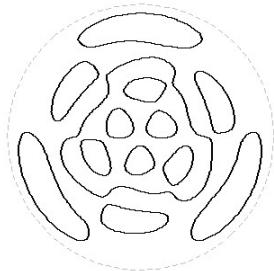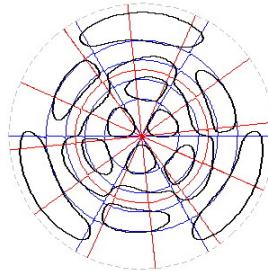
Figure 5.2. Sphere, eigenvalue $\ell(\ell + 1)$, $\ell = 4k + 3$, $k = 1$

Such a spherical harmonic has many critical zeros as well (at the north and south poles, and at the intersections between meridians and latitude circles). A first perturbation (by a spherical harmonic whose nodal set consists of meridians, in red in the figure) eliminates all the critical zeros except the poles. Another perturbation (by a zonal harmonic) eliminates

the critical zeros at the poles. The nodal sets of the perturbations appear in black in Figure 5.1 (the first column displays the nodal sets after the first perturbation, the second column displays the nodal sets after the second perturbation). Figure 5.2 illustrates another behaviour: the figures in the first line display the nodal sets after the first perturbation; the figures in the second and third lines display the nodal sets after the second perturbation, with a positive or negative perturbation parameter (the crossings open-up differently). We refer to [8] for more details.

6. The $2D$ quantum harmonic oscillator

For the 2-dimensional harmonic oscillator, one can prove results similar to the results of the previous sections. It turns out that some of these results were given by J. Leydold in his unpublished Master degree thesis [29]. In his memoir, Leydold proves the following optimal result. Let $\mathcal{H}_n$ be the n-th eigenspace of the isotropic quantum harmonic oscillator, associated with the eigenvalue $2(n+1)$. It has dimension $(n+1)$. If $n = 4k, k \geqslant 1$, there exists an eigenfunction in $\mathcal{H}_n$ with exactly three nodal domains, and this lower bound is the best possible. If $n \neq 4k$, there exists an eigenfunction in $\mathcal{H}_n$ with exactly two nodal domains.

In this context, one can describe the evolution of nodal sets for some families of eigenfunctions, see [7]. One can for example consider the family of eigenfunctions

$$\Psi_{0,n}^\theta(x,y) := \exp\left(-(x^2+y^2)/2\right) \left(\cos\theta\, H_n(x) + \sin\theta\, H_n(y)\right),$$

where n is an integer and $x, y \in \mathbb{R}^2$. One can determine the values of θ for which critical zeros occur and, as for the square and the sphere, describe how the nodal sets change along the θ-path. Figure 6.1 displays a typical situation (with $n = 7$).

Using decomposed eigenfunctions in polar coordinates, one can also construct regular eigenfunctions of the harmonic oscillator with many nodal domains, as shown in Figure 6.2, see [7].

$\theta = 0.01\,\pi$ $\theta = 0.0251153\,\pi$

$\theta = 0.04\,\pi$ $\theta = 0.0535467\,\pi$

$\theta = 0.12\,\pi$ $\theta = 0.138696\,\pi$

$\theta = 0.23\,\pi$ $\theta = 0.25\,\pi$
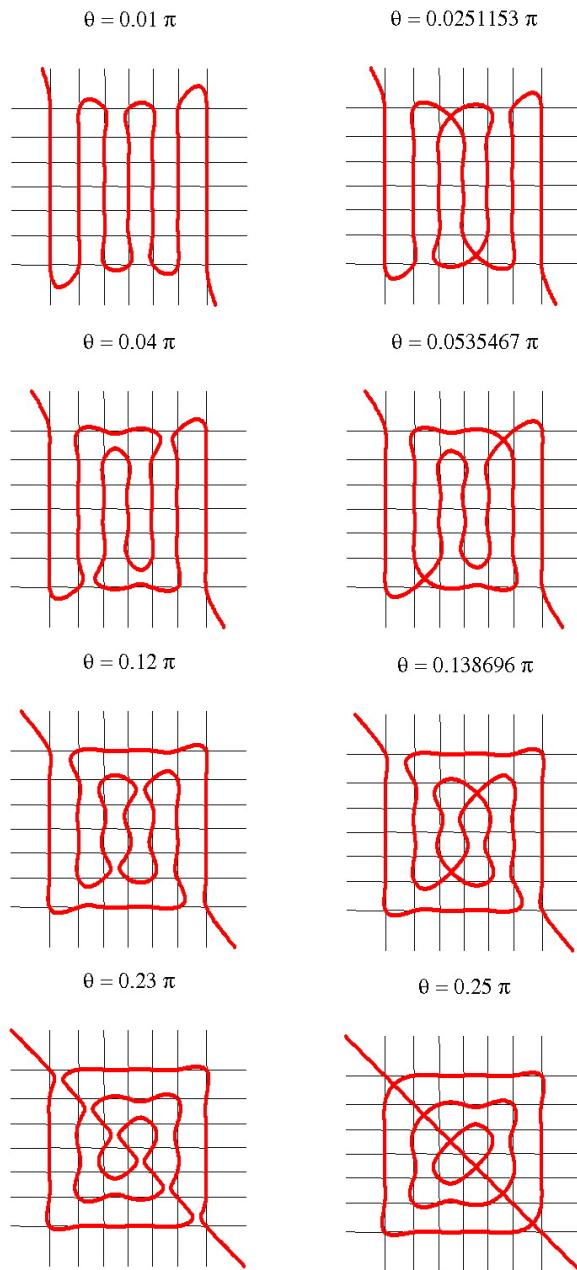
*Figure 6.1.* Harmonic oscillator, bifurcations for the family $[0, 7]$
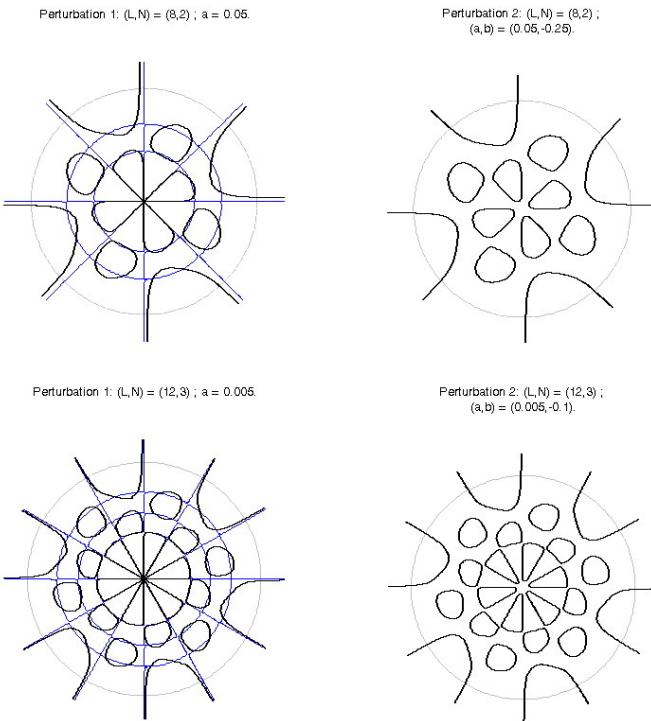
Figure 6.2. Harmonic oscillator, $n = 4k$, $k = 2, 3$

7.   **The rectangular flat 2-tori**

In this section, we consider the rectangular 2-tori, $\mathbb{T}^2_{a,b} := \mathbb{R}^2 / (a\mathbb{Z} \bigoplus b\mathbb{Z})$, with the flat metric g_0. A complete set of complex eigenfunctions is given by the family

$$\left\{ \exp\left(2i\pi \left(m\frac{x}{a} + n\frac{y}{b} \right) \right) \;\middle|\; m, n \in \mathbb{Z} \right\},$$

with associated eigenvalues $4\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$. In [19], the authors mention that the family $\sin(2\pi mx + 2\pi y)$ provides an example of an infinite sequence of eigenfunctions of $\mathbb{T}^2_{1,1}$ with exactly two nodal domains. They refer to [23], where this family is used to construct a metric g on $\mathbb{T}^2$, an infinite sequence

of eigenvalues for the Laplacian Δ_g, and associated eigenfunctions whose number of critical points remains bounded.

In this section, for the sake of completeness, we prove the following elementary result.

PROPOSITION 7.1. — *If m, n are relatively prime integers, then the eigenfunction*

$$S_{m,n}(x, y) := \sin\left(2\pi\left(m\frac{x}{a} + n\frac{y}{b}\right)\right)$$

has exactly two nodal domains, and its nodal set consists of two simple disjoint closed curves on $\mathbb{T}^2_{a,b}$. If the integers m, n have greatest common divisor $d \geqslant 2$, then the eigenfunction $S_{m,n}$ has $2d$ nodal domains, and its nodal set consists of $2d$ simple pairwise disjoint closed curves on $\mathbb{T}^2_{a,b}$.

Remarks.

  (i) Note that the eigenfunctions $\sin\left(2\pi\left(m\frac{x}{a} + n\frac{y}{b}\right)\right)$ have no critical zeros, so that their nodal sets are 1-dimensional submanifolds of $\mathbb{T}^2_{a,b}$, i.e. a union of pairwise disjoint circles.

  (ii) In [26], C. Léna proves that any non-constant eigenfunction of the flat torus $(\mathbb{T}^2_{a,b}, g_0)$ has an even number of nodal domains whenever $(a, b) = (1, 1)$ or (a, b) is such that $\left(\frac{a}{b}\right)^2$ is not rational. For Courant-sharp eigenvalues on the 2-torus, see also [26, 25]. The above proposition implies that for all even positive integer $2d$, there exist infinite sequences of eigenfunctions on the flat torus $\mathbb{T}^2_{a,b}$ with exactly $2d$ nodal domains. Note that Léna also proves that there is a flat 2-torus, and an eigenfunction on this torus with exactly three nodal domains. It is not clear whether there is an infinite sequence of eigenfunctions with exactly three nodal domains.

Proof. — It turns out that, without loss of generality, one can assume that $a = b = 1$ and that m, n are positive. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ denote the projection map. For m, n positive integers, and for $\alpha \in \mathbb{R}$, define the sets

$$\mathcal{F}_{m,n,\alpha} := \bigcup_{k \in \mathbb{Z}} \left\{(x, y) \in \mathbb{R}^2 \mid mx + ny = \alpha + k\right\} \subset \mathbb{R}^2, \text{ and}$$

$$c_{m,n,\alpha} := \pi(\mathcal{F}_{m,n,\alpha}) \subset \mathbb{T}^2.$$

CLAIM 7.2. — *For any fixed m and n, the sets $c_{m,n,\alpha}$ and $c_{m,n,\beta}$ are disjoint unless $\alpha - \beta \equiv 1 \pmod{1}$, in which case they coincide. Furthermore, the set $c_{m,n,\alpha}$ is connected if and only if m, n are relatively prime.*

Proof of the claim. — The first assertion is clear. Note that $c_{m,n,\alpha}$ is connected if and only if the parallel lines in $\mathcal{F}_{m,n\alpha}$ 'meet modulo 1', i.e.

$$\forall\, k, \ell \in \mathbb{Z}\,, \forall\, x, y \in \mathbb{R}^2 \text{ such that } mx + ny = \alpha + k\,, \ \exists\, x', y' \in \mathbb{R}^2$$
$$\text{such that } x' \equiv x \pmod 1\,, \ x' \equiv x \pmod 1 \text{ and } mx' + ny' = \alpha + \ell\,.$$

Assume the above condition is met. Take $k = 0$, and $\ell = 1$, and write $x' = x + p$ and $y' = y + q$, for $p, q \in \mathbb{Z}$. Then one finds that $mp + nq = 1$, so that m and n are relatively prime. Conversely, assume that m and n are relatively prime, so that there exist $p, q \in \mathbb{Z}$ with $mp + nq = 1$. Write $mx + ny = \alpha + k = \alpha + \ell - (\ell - k)$ and use the fact that $(\ell - k)(mp + nq) = \ell - k$ to conclude. □

CLAIM 7.3. — *If the integers m and n are relatively prime, then $c_{m,n\alpha}$ is a regular simple closed curve (a connected 1-dimensional submanifold) of $\mathbb{T}^2$.*

Proof of the claim. — Notice that the set $\mathcal{F}_{m,n,\alpha} \cap [0, 1] \times [0, 1]$ is compact (finite union of closed segments). Since π is a covering map, the claim follows. □

Case $\gcd(m, n) = 1$. — Notice that the nodal set of $S_{m,n}$ is given by

$$S_{m,n}^{-1}(\{0\}) = c_{m,n,0} \cup c_{m,n,\frac{1}{2}}\,,$$

and that these two curves are disjoint.

When $0 < \alpha < \frac{1}{2}$, the curve $c_{m,n,\alpha}$ in entirely contained in the set $\{S_{m,n} > 0\}$. Similarly, when $\frac{1}{2} < \alpha < 1$, the curve $c_{m,n,\alpha}$ in entirely contained in the set $\{S_{m,n} < 0\}$. According to Claim 7.2, this implies that the function $S_{m,n}$ has exactly two nodal domains. This proves the first assertion in Proposition 7.1.

Case $\gcd(m, n) = d \geqslant 2$. — Write $m = dm'$ and $n = dn'$. The nodal set of $S_{m,n}$ is given by the image under π of the set $\mathcal{F}_{m,n,0} \cap \mathcal{F}_{m,n,\frac{1}{2}}$. For $\beta \in \{0, \frac{1}{2}\}$ and $k \in \mathbb{Z}$, the condition

$$mx + ny = \beta + k$$

can be written as

$$m'x + n'y = \frac{\beta + j}{d} + \ell$$

for $0 \leqslant j \leqslant d-1$ and $\ell \in \mathbb{Z}$. Using Claims 7.2 and 7.3, we conclude that the nodal set of $S_{m,n}$ consists of $2d$ pairwise disjoint connected 1-submanifolds in $\mathbb{T}^2$, $c_{m',n',\frac{j}{d}}$ and $c_{m',n',\frac{1}{2}+\frac{j}{d}}$, for $0 \leqslant j \leqslant d - 1$. Sweeping the torus by curves $c_{m',n',\alpha}$ as above, we find that the function $S_{m,n}$ has $2d$ nodal domains. This proves the second assertion of Proposition 7.1. □

Remark. —   As a matter of fact, one can prove a much more general result, see [20, Section 7], in particular Lemma 7.4.

Figure 7.1 shows the nodal sets of the eigenfunction $(2\pi(mx + ny))$ respectively for the pair $(3, 2)$ (left picture) and $(6, 4)$, viewed in the fundamental domain $[0, 1] \times [0, 1] \subset \mathbb{R}^2$ of the torus $\mathbb{T}^2$. Figure 7.2 shows the nodal set of the eigenfunction $(2\pi(mx + ny))$ for the pair $(2, 1)$, represented in $\mathbb{R}^3$ through the usual map $\mathbb{T}^2 \to \mathbb{R}^3$ given by

$$(x, y) \mapsto \Big(\big(2 + \cos(2\pi y)\big) \cos(2\pi x), \big(2 + \cos(2\pi y)\big) \sin(2\pi x), \sin(2\pi y) \Big).$$
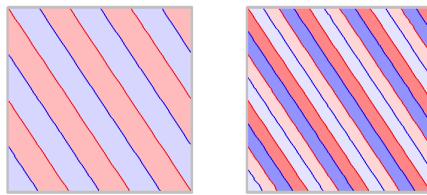

Figure 7.1. Nodal domains of $S_{3,2}$ and $S_{6,4}$, viewed in the fundamental domain of the torus $\mathbb{T}^2$
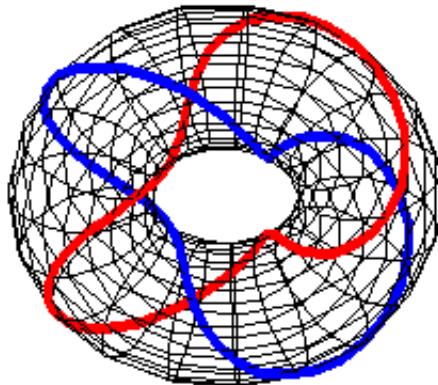

Figure 7.2. Nodal set of $S_{2,1}$, viewed in $\mathbb{R}^3$

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Pierre Bérard
Institut Fourier,
Université Grenoble Alpes, B.P.74,
38402 Saint-Martin-d'Hères Cedex (France)

Bernard Helffer
Laboratoire de Mathématiques,
Univ. Paris-Sud 11 and CNRS,
91405 Orsay Cedex (France)
 and
Laboratoire de Mathématiques Jean Leray,
Université de Nantes,
44322 Nantes (France)