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Canonical metrics on some domains of $\mathbb{C}^n$

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ABSTRACT. — The study of the existence and uniqueness of a preferred Kähler metric on a given complex manifold M is a very important area of research. In this talk we recall the main results and open questions for the most important canonical metrics (Einstein, constant scalar curvature, extremal, Kähler-Ricci solitons) in the compact and the non-compact case, then we consider a particular class of complex domains D in $\mathbb{C}^n$, the so-called Hartogs domains, which can be equipped with a natural Kaehler metric g. We show that if g is a Kähler-Einstein, constant scalar curvature, extremal or a soliton metric then (D, g) is holomorphically isometric to an open subset of the n-dimensional complex hyperbolic space. If D is bounded, we also show the same assertion under the assumption that g is a scalar multiple of the Bergman metric.

The results we present are proved in papers joint with A. Loi and A. J. Di Scala ([11], [20]).

1. Canonical metrics: existence and uniqueness

1.1. The compact case

Let M be a complex manifold, with complex structure J. Let g be a *Kähler metric* on M, i.e. a J-invariant Riemannian metric such that the associated *Kähler form* ω, defined by $\omega(v, w) = g(Jv, w)$, is closed.

In this talk we are interested in those Kähler metrics on M which are canonical in the sense that they are minima of natural geometric functionals on M or arise as limits of important geometric flows like the Kähler-Ricci flow. In order to give the precise definitions, let us fix some notations.

Let Ric_g denote the Ricci tensor of (M, g) and let ρ_ω be its *Ricci form* defined by $\rho_\omega(v, w) = \text{Ric}_g(Jv, w)$ (we will omit the subscript ω when the context is clear). If, in a chart endowed with complex coordinates $z_1, \ldots, z_n$

we have $\omega = \frac{i}{2} \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$ and $\rho = \frac{i}{2} \sum_{\alpha,\beta=1}^{n} \mathrm{Ric}_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$, the scalar curvature is given by

$$(1.1) \qquad \mathrm{scal}_\omega = \sum_{\alpha,\beta=1}^{n} g^{\bar{\beta}\alpha} \mathrm{Ric}_{\alpha\bar{\beta}},$$

where $g^{\bar{\beta}\alpha}$ are the entries of the inverse of $(g_{\alpha\bar{\beta}})$, namely

$$\sum_{\alpha=1}^{n} g^{\bar{\beta}\alpha} g_{\alpha\bar{\gamma}} = \delta_{\beta\gamma}.$$

If M is compact, we can consider the *Calabi functional* given by

$$(1.2) \qquad \mathcal{C}(\omega) = \frac{1}{V} \int_M (scal_\omega)^2 \omega^n.$$

One can see that a Kähler metric ω_0 is a critical point for C in some Kähler class if and only if the $(1,0)$ part of the gradient of $scal_{\omega_0}$ is a holomorphic field. In a chart given by coordinates $z_1, \ldots, z_n$, this amounts to say that the field $X = g_0^{\bar{j}i} \frac{\partial scal_{\omega_0}}{\partial \bar{z}_j} \frac{\partial}{\partial z_i}$ is holomorphic. If a Kähler metric has constant scalar curvature (briefly, it is a *cscK metric*) then it clearly satisfies this condition (since in this case $X = 0$), and obviously this is the unique possibility if M does not have any non-zero holomorphic fields (for example if the first Chern class $c_1(M) < 0$). In general, Calabi proved in [2] that there exist critical points of $\mathcal{C}$ which are not cscK and he christens them (nontrivial[1]) *extremal metrics*.

The main obstruction to the existence of cscK metrics is given by the *Calabi-Futaki invariant* $F(X, [\omega])$ defined by $F(X, [\omega]) = \int_M X(h) \omega^n$, where h is the solution to the elliptic PDE $\Delta h = scal_\omega - \frac{1}{V} \int_M scal_\omega \omega^n$ (see, for example, [23] for a proof that it just depends on the class $[\omega]$). More precisely, the necessary condition for such a metric to exist in the class $[\omega]$ is that $F(X, [\omega]) = 0$ for every holomorphic field X. Conversely, it is not hard to see that if this last condition is verified, then any extremal metric is cscK. It follows then that the existence of nontrivial extremal metrics is an obstruction to the existence of metrics with constant scalar curvature.

As for uniqueness, we have that any two extremal metrics in a given Kähler class are obtained one from the other via an element of the connected component $Aut_0(M)$ of the automorphism group of M (see [22] and [8]). We refer the reader interested in the subject of extremal metrics to [15].

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[1] The trivial ones being the cscK metrics.

In the case when the first Chern class $c_1(M)$ is nonpositive, the existence of cscK metrics follows from Yau's results about the existence of Kähler-Einstein metrics, i.e. the metrics satisfying the Einstein condition

$$(1.3) \qquad \rho = \lambda\omega$$

for some real number λ. The starting point to attack this equation is given by the following simple facts (see, for example, [17]):

(1) the Ricci form admits a particularly simple expression in term of the metric, namely

$$(1.4) \qquad \rho = -i\,\partial\bar{\partial}\log(\det g)$$

where ∂ and $\bar{\partial}$ denote the operators which, in any chart given by coordinates $z_1, \ldots, z_n$, act on functions as $\partial f = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i$ and $\bar{\partial}f = \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$;

(2) the cohomology class of the Ricci form is independent of the given metric: more precisely, we have $[\frac{i}{2\pi}\rho] = c_1(M)$, where $c_1(M)$ denotes the first Chern class of the manifold M.

It follows that a necessary condition for (1.3) to admit a solution is that the first Chern class is (positive or negative) definite or vanishes. Let us then make this assumption and fix a Kähler metric g_0 such that the corresponding Kähler form ω_0 belongs to $c_1(M)$, up to some scalar factor. By the $\partial\bar{\partial}$-lemma (see, for example, [29]), we have $\rho_0 = \lambda\omega_0 + i\partial\bar{\partial}f$ and $\omega = \omega_0 + i\partial\bar{\partial}\phi$ for any other Kähler form ω cohomologous to ω_0, where f and ϕ are globally defined functions on M.

It follows that, always under the assumption that M is compact, equation (1.3) can be reduced to the following scalar equation on ϕ:

$$(1.5) \qquad -\log \frac{\det(g_0 + \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k})}{\det g_0} - \lambda\phi + f = 0.$$

In his celebrated paper of 1978, Yau ([28]) proves that, if $c_1(M) \leqslant 0$, then this PDE admits a solution and then he obtains the existence of a Kähler-Einstein metric on every compact complex manifold having nonpositive first Chern class (moreover, if $c_1(M) < 0$, then this is unique up to homothety, while if $c_1(M) = 0$ there is exactly one for each positive (1,1) cohomology class with a given volume).

On the other hand, it is known that, in the case $c_1(M) > 0$, there exist manifolds which do not admit any Kähler-Einstein metric. The search for sufficient conditions or obstructions to the existence of Kähler-Einstein metrics in this case has opened a wide area of research where the most important results have been obtained by Futaki, Matsushima, Tian and others

(see, for example, [23]). When a Kähler-Einstein metric exists, anyway, by
a result of Bando and Mabuchi ([1]) it is unique up to the action of the
identity component $Aut_0(M)$ of the automorphism group of the manifold.

An alternative approach to Yau's equation in the study of existence of
Kähler-Einstein metrics consists in considering the so-called normalized
Ricci flow

$$(1.6) \qquad \begin{cases} \frac{d\omega(t)}{dt} = -\rho_{\omega(t)} + \lambda\omega(t) \\ \omega(0) = \omega_0 \end{cases}$$

which, following the same arguments as above, can be reduced to a para-
bolic PDE for which the natural questions are: does the flow locally/globally
exist? if so, does it converge to a Kähler-Einstein metric?

In [3], H.D. Cao proves that the flows admits a global solution which, in
the case $c_1(M) \leqslant 0$, converges towards a Kähler-Einstein metric, accord-
ingly with Yau's results. Conversely, Tian and Zhu ([25]) have proved that
if M is known to admit a Kähler-Einstein metric, then the flows converges
to it, even in the case $c_1(M) > 0$.

Let us also mention the result, recently proved by Cao and Zhu ([4]),
which states that if the starting metric ω_0 has strictly positive bisectional
curvature and the Futaki invariant vanishes, then Cao's method works even
in the case $c_1(M) > 0$ and the flows converges to a Kähler-Einstein metric.

The flow approach naturally leads us to consider another class of metrics
which we will include under the definition of canonical metric, namely the
soliton metrics.

By definition, a pair (ω, X) consisting of a Kähler metric ω and of a real
holomorphic field X on M is called a Kähler-Ricci soliton if it satisfies the
following equation:

$$(1.7) \qquad\qquad \rho = \lambda\omega + L_X\omega$$

for some real number $\lambda \in \mathbb{R}$, where L_X denotes the Lie derivative.

The main result showing the connection between solitons and Ricci flow
has been proved by Tian and Zhu ([25]): let M admit a Kähler-Ricci soliton
(ω, X), and let K be a maximal compact subgroup of the connected Lie
subgroup of $Aut_0(M)$ corresponding to the Lie subalgebra of all holomor-
phic fields on M (we may assume that the soliton is K-invariant). Then,
if ω_0 is a K_X-invariant metric (where K_X denotes the one-parameter sub-
group of K generated by $Im(X)$) the normalized Ricci flow starting from ω_0
converges to the soliton metric ω.

It is worth here recalling that the existence of a non-trivial soliton (i.e.
which is not a Kähler-Einstein metric) is an obstruction to the existence

of a Kähler-Einstein metric, as it can be easily seen by considering again Futaki's obstruction (see above for the definition). For some important and wide classes of complex manifolds (namely, the toric manifolds, see [27]) it has been proved that if a Kähler-Einstein metric does not exist, then the manifold admits a non-trivial soliton.

Regarding uniqueness, Tian and Zhu ([24]) have proved that if (ω, X), (ω', X') are two solitons on M, then there exists $\sigma \in Aut_0(M)$ such that $\omega = \sigma^*\omega'$ and $X = (\sigma^{-1})_* X'$.

1.2. The non-compact case

The assumption of compactness for the manifold M is essential for the validity of the methods discussed above (as they rely for example on the $\partial\bar{\partial}$-lemma or the maximum principles for parabolic and elliptic equations) or even for the definitions to make sense (the Calabi functional is defined as an integral on M). In the non-compact case we have nevertheless some remarkable results.

Concerning the existence of Kähler-Einstein metrics, let us mention that Cheng and Yau ([9]) have proved that every bounded domain of $\mathbb{C}^n$ admits a Kähler-Einstein metric (having negative curvature) provided it is smooth and strongly pseudoconvex. Let us recall that this means that the Levi form

$$L(\rho, z)(X) = \sum_{\alpha, \beta=1}^{n} \frac{\partial^2 \rho}{\partial z_\alpha \partial \bar{z}_\beta}(z) X_\alpha \bar{X}_\beta$$

is positive definite on

$$S_\rho = \{(X_1, \ldots, X_n) \in \mathbb{C}^n \mid \sum_{\alpha=1}^{n} \frac{\partial \rho}{\partial z_\alpha}(z) X_\alpha = 0\},$$

where ρ denotes any defining function of the domain.

In [7], Chau generalizes the flow approach by Cao, extending its results to the case when M is a noncompact manifold endowed with a metric g with bounded curvature and satisfying $\text{Ric}_{i\bar{j}} + g_{i\bar{j}} = \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}$ for some smooth function f (this last assumption plays in fact the role of $c_1(M) < 0$).

Finally, let us mention the results of C. LeBrun ([18]) giving explicit examples of complete Ricci-flat, non-flat metrics on $\mathbb{C}^n$.

Concerning the solitons, we refer the reader for example to [14] for the question of the existence and uniqueness in the noncompact case.

As for extremal metrics[2], the existence and uniqueness are far from being understood. For example, only recently in [5] (see also [6]), there has been shown the existence of a nontrivial extremal and complete Kähler metric in a complex one-dimensional manifold.

2. Canonical metrics on Hartogs domains

In this section we are going to look more closely to the Einstein, soliton and extremal equations for some classes of connected, open domains in $\mathbb{C}^n$.

The constructive results shown in the literature, as for instance Le Brun's example in [18] mentioned above or Calabi's first example of nontrivial, extremal metric ([2]), are mostly concerned with domains given by *rotation-invariant defining equations* and metrics given by *rotation-invariant potentials*, where we say that a function f is rotation-invariant if

$$f(z_1, \ldots, z_n) = \tilde{f}(|z_1|^2, \ldots, |z_n|^2)$$

for some real-valued function $\tilde{f}$ defined on some open domain of $\mathbb{R}^n$. The reason is that, as we shall see below, this assumption makes the PDE's arising from the canonical metric condition more symmetric, which in some cases allows us to reduce them to ordinary differential equations.

A class of domains of $\mathbb{C}^n$ which satisfy this assumption and which have been studied in the literature from different points of view are the so-called *Hartogs domains*.

DEFINITION 2.1. — *Let $x_0 \in \mathbb{R}^+ \cup \{+\infty\}$ and let $F : [0, x_0) \to (0, +\infty)$ be a decreasing continuous function, smooth on $(0, x_0)$. The* Hartogs domain $D_F \subset \mathbb{C}^n$ *associated to the function F is defined by*

$$D_F = \{(z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}^n \mid |z_0|^2 < x_0, \ |z_1|^2 + \cdots + |z_{n-1}|^2 < F(|z_0|^2)\}.$$

The most simple example of Hartogs domain is given by choosing $x_0 = 1$, $F(x) = 1 - x$, which yields the ball $B(o, 1) \subseteq \mathbb{C}^n$ centered at the origin $o \in \mathbb{C}^n$ and of radius one. One can obviously produce a lot of examples, among which we mention the *Spring domain*, corresponding to $x_0 = +\infty$, $F(x) = e^{-x}$, or domains given by functions like $F(x) = (c_1 + c_2 x)^\lambda$, where the values of λ, c_1, c_2 and x_0 are chosen in order to satisfy the assumptions of Definition 2.1.

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[2] Since we assume that M is noncompact, we define a metric to be extremal if the $(1, 0)$ part of the gradient of $scal_{\omega_0}$ is a holomorphic field.

Every Hartogs domain can be endowed with the natural form of type (1,1) given by

$$(2.1) \qquad \omega_F = \frac{i}{2} \partial \bar{\partial} \log \frac{1}{F(|z_0|^2) - |z_1|^2 - \cdots - |z_{n-1}|^2}.$$

Under the assumption that the real function xF'/F is strictly decreasing, one shows that ω_F is in fact a Kähler form.

For example, in the case when $F = 1 - x$, we have that ω_F is just the hyperbolic Kähler form, so that (D_F, ω_F) is the complex hyperbolic space $\mathbb{C}\mathbb{H}^n$.

These domains are interesting both from the mathematical and the physical point of view (see for example [12] and [19] for the study of some Riemannian properties of g_F and the Berezin quantization of (D_F, g_F), [10] and [21] for the construction of global symplectic coordinates on these domains and [26] for the construction of Kähler-Einstein metrics on Hartogs type domains on symmetric spaces).

In the context of this talk the interest for Hartogs domains comes from the fact that they yield a lot of examples of noncompact Kähler manifolds enjoying a priori different geometric properties which can be controlled by integro-differential conditions on F itself.

As an example of this fact, let us recall that in [11] it was proved that D_F is geodesically complete with respect to the Kähler metric g_F associated to the Kähler form ω_F if and only if

$$(2.2) \qquad \int_0^{\sqrt{x_0}} \sqrt{-\left(\frac{xF'}{F}\right)' |_{x=u^2}} \; du = +\infty$$

where we define $\sqrt{x_0} = +\infty$ for $x_0 = +\infty$.

Turning back to canonical metrics, we have proved that, in the class of Hartogs domains, thought as domains endowed with the metric g_F associated to the natural Kähler form ω_F, there are no examples of canonical metrics but the complex hyperbolic space. More precisely, we have the following

THEOREM 2.2. — ([20], Theorems 1.1 and 1.2) Let (D_F, g_F) be a Hartogs domain in $\mathbb{C}^n$. If (at least) one of the following assumptions is satisfied:

(1) g_F is a Kähler-Einstein metric
(2) g_F is a cscK metric
(3) g_F is an extremal metric
(4) there exists a holomorphic field X such that (g_F, X) is a Kähler-Ricci soliton

then (D_F, g_F) is holomorphically isometric to an open domain of the hyperbolic space $\mathbb{CH}^n$.

In the case when the domain is bounded (which occurs iff $x_0 < \infty$) we can consider another condition leading to the same conclusion as assumptions (1)-(4). Indeed, in this case we can consider the space $\mathcal{H}$ of the holomorphic functions $f : D_F \to \mathbb{C}$ such that $\int_{D_F} |f|^2 dz < \infty$, which is a Hilbert space with respect to the product $\langle f, g \rangle = \int_{D_F} f\bar{g} dz$. Given an orthonormal basis $\{f_i\}$ of $\mathcal{H}$, we can define the *Bergman kernel* of D_F by $K(z) = \sum_i |f_i|^2$ and the *Bergman metric* of D_F by $g_B = i\partial\bar{\partial}K$ (see, for example, [29]).

It can be verified that the Bergman metric of the ball $B(o, 1) \subseteq \mathbb{C}^n$ is, up to a real positive constant, the hyperbolic metric which, as we have remarked above, is the Hartogs metric corresponding to this domain. Like the properties given by $(1)-(4)$ above, also this one can occur only in this special case, as stated in the following

THEOREM 2.3. — *([11], Theorem 1.3) Let (D_F, g_F) be a bounded Hartogs domain in $\mathbb{C}^n$ and let g_B denote the Bergman metric of D_F. If $g_B = \lambda g_F$ for some $\lambda > 0$, then (D_F, g_F) is holomorphically isometric to an open domain of the hyperbolic space $\mathbb{CH}^n$.*

3. Sketch of the proofs of Theorems 2.2 and 2.3

Proof of Theorem 2.2 (1) and (2). —   Set

$$(3.1) \qquad A = F(|z_0|^2) - |z_1|^2 - \cdots - |z_{n-1}|^2$$

and

$$(3.2) \qquad C = F'^2(|z_0|^2)|z_0|^2 - (F''(|z_0|^2)|z_0|^2 + F'(|z_0|^2))A.$$

Then one sees that the matrix $h = (g_{\alpha\bar{\beta}})$ of the metric g_F is given by:

$$(3.3) \quad h = \frac{1}{A^2} \begin{pmatrix} C & -F'\bar{z}_0 z_1 & \dots & -F'\bar{z}_0 z_\alpha & \dots & -F'\bar{z}_0 z_{n-1} \\ -F'z_0\bar{z}_1 & A + |z_1|^2 & \dots & \bar{z}_1 z_\alpha & \dots & \bar{z}_1 z_{n-1} \\ \vdots & \vdots & & \vdots & & \vdots \\ -F'z_0\bar{z}_\alpha & z_1\bar{z}_\alpha & \dots & A + |z_\alpha|^2 & \dots & \bar{z}_\alpha z_{n-1} \\ \vdots & \vdots & & \vdots & & \vdots \\ -F'z_0\bar{z}_{n-1} & z_1\bar{z}_{n-1} & \dots & z_\alpha\bar{z}_{n-1} & \dots & A + |z_{n-1}|^2 \end{pmatrix}.$$

By the Laplace expansion along the first row, after a long but straight calculation we get

$$(3.4) \qquad \det(h) = -\frac{F^2}{A^{n+1}}\left(\frac{xF'}{F}\right)'|_{x=|z_0|^2}.$$

Let $L(x) = \frac{d}{dx}[x\frac{d}{dx}\log(xF'^2 - F(F' + F''x))]$. A straightforward computation using (3.4) and the definitions gives

$$(3.5) \qquad \mathrm{Ric}_{0\bar{0}} = -L(|z_0|^2) - (n+1)g_{0\bar{0}},$$

$$(3.6) \qquad \mathrm{Ric}_{\alpha\bar{\beta}} = -(n+1)g_{\alpha\bar{\beta}}, \quad \alpha > 0$$

so that

$$\mathrm{scal}_{g_F} = -L(|z_0|^2)g^{\bar{0}0} - (n+1)\sum_{\alpha,\beta=0}^{n-1} g^{\bar{\beta}\alpha}g_{\alpha\bar{\beta}} = -L(|z_0|^2)g^{\bar{0}0} - n(n+1),$$

which can also be written as

$$(3.7) \qquad \mathrm{scal}_{g_F} = -\frac{A}{B}FL - n(n+1)$$

where

$$B = B(|z_0|^2) = F'^2|z_0|^2 - F(F' + F''|z_0|^2).$$

Thus, scal_{g_F} is constant if and only if $\frac{AFL}{B}$ is constant. Since $A = F(|z_0|^2) - |z_1|^2 - \cdots - |z_{n-1}|^2$ depends on $z_1, \ldots, z_{n-1}$ while $\frac{LF}{B}$ depends only on z_0, it must be $L = 0$, i.e.

$$\frac{d}{dx}\left[x\frac{d}{dx}\log(xF'^2 - F(F' + F''x))\right]_{x=|z_0|^2} \equiv 0.$$

By (3.5) it is clear that this is also the condition for the metric to be Einstein. So we have proved that (1) and (2) of the theorem are equivalent to an ordinary differential equation for F.

Now, we continue as in the proof of Theorem 4.8 in [19] and conclude that $F(x) = c_1 - c_2 x$, $x = |z_0|^2$, with $c_1, c_2 > 0$, which implies that D_F is holomorphically isometric to an open subset of the complex hyperbolic space $\mathbb{C}H^n$ via the map

$$\phi : D_F \to \mathbb{C}H^n, \ (z_0, z_1, \ldots, z_{n-1}) \mapsto \left(\frac{z_0}{\sqrt{c_1/c_2}}, \frac{z_1}{\sqrt{c_1}}, \ldots, \frac{z_{n-1}}{\sqrt{c_1}}\right).$$

$\square$

Proof of Theorem 2.2 (3). — By definition, the metric g_F is an extremal metric if and only if it satisfies the following system of PDE's:

$$(3.8) \qquad \frac{\partial}{\partial\bar{z}_\gamma}\left(\sum_{\beta=0}^{n-1} g_F^{\bar{\beta}\alpha}\frac{\partial\,\mathrm{scal}_{g_F}}{\partial\bar{z}_\beta}\right) = 0,$$

for every $\alpha, \gamma = 0, \ldots, n-1$.

We are going to show that if (3.8) is satisfied then scal_{g_F} is in fact constant and hence by part (2) of the theorem (D_F, g_F) is holomorphically isometric to an open subset of $(\mathbb{C}H^n, g_{hyp})$. In order to do that, fix $i \geqslant 1$ and let us consider equation (3.8) for $\alpha = 0$, $\gamma = i$.

By deriving it with respect to $\bar{z}_i$, we get

$$2\frac{F}{B}G'z_0 z_i^2 + 2G\frac{F'}{B}z_0 z_i^2 = 0$$

where

$$G = G(|z_0|^2) = -\frac{L(|z_0|^2)F(|z_0|^2)}{B(|z_0|^2)}.$$

If $z_0 z_i \neq 0$, this reduces to $GF' + FG' = 0$ or, equivalently, $G = \frac{c}{F}$ for some constant $c \in \mathbb{R}$. The proof of (3) will thus be completed by showing that $c = 0$ since in this case $G = 0$ on the open and dense subset of D_F consisting of those points such that $z_0 z_i \neq 0$ and therefore, by (3.7), scal_{g_F} is constant on D_F. In order to prove that $c = 0$, we consider equation (3.8) for $\alpha = i$, $\gamma = i$ and apply the same argument as above, getting

$$\frac{G'F'|z_0|^2 + G(F' + F''|z_0|^2)}{B} = 0,$$

i.e. $(GF'x)' = 0, x = |z_0|^2$. Substituting $G = \frac{c}{F}$ in this equality we get $c(\frac{F'x}{F})' = 0$, and we have done since $(\frac{F'x}{F})' < 0$ (this is the condition for ω_F to be Kähler). $\qquad\square$

Proof of Theorem 2.2 (4). — We have to prove that a Kähler–Ricci soliton (g_F, X) on a Hartogs domain D_F is necessarily trivial (Notice that the automorphism group of D_F is not discrete, see also [16]).

A real holomorphic vector field X is given in the local complex coordinates $(z_0, \ldots, z_{n-1})$ by

$$(3.9) \qquad X = \sum_{k=0}^{n-1}\left(f_k \frac{\partial}{\partial z_k} + \bar{f}_k \frac{\partial}{\partial \bar{z}_k}\right),$$

for some holomorphic functions f_k, $k = 0, \ldots, n-1$.

By applying both sides of $\mathrm{Ric}_{g_F} = \lambda\, g_F + L_X g_F$ to the pair $(\frac{\partial}{\partial z_0}, \frac{\partial}{\partial \bar{z}_0})$ and taking into account (3.5) one gets:

$$(3.10) \quad -L(|z_0|^2) =$$

$$\gamma g_{0\bar{0}} + \sum_{k=0}^{n-1}\left(f_k \frac{\partial g_{0\bar{0}}}{\partial z_k} + \bar{f}_k \frac{\partial g_{0\bar{0}}}{\partial \bar{z}_k}\right) + \sum_{k=0}^{n-1}\left(\frac{\partial f_k}{\partial z_0}g_{k\bar{0}} + \frac{\partial \bar{f}_k}{\partial \bar{z}_0}g_{0\bar{k}}\right)$$

where $\gamma = \lambda + (n+1)$. By (3.3), we have

(3.11) $\quad \tilde{C} =$

$$\sum_{k=0}^{n-1} C_k(f_k\bar{z}_k + \bar{f}_k z_k) + C(\phi_0 + \bar{\phi}_0) - F' \sum_{k=1}^{n-1} \left(z_0\bar{z}_k \frac{\partial f_k}{\partial z_0} + \bar{z}_0 z_k \frac{\partial \bar{f}_k}{\partial \bar{z}_0} \right)$$

where we have set $\tilde{C} = -A^2 L - \gamma C$, $C_k = A^2 \frac{\partial g_{0\bar{0}}}{\partial x_k}$ $(x_k = |z_k|^2)$ and $\phi_0 = \frac{\partial f_0}{\partial z_0}$ (A and C are given by (3.1) and (3.2) respectively).

Now, by applying the operator $\frac{\partial^4}{\partial^2 z_i \partial^2 \bar{z}_i}$ $(i = 1, \ldots, n-1)$ to both sides of this equation and evaluating at $z_1 = \cdots = z_{n-1} = 0$ we get

(3.12) $$L = 2x\frac{F'^3}{F^3}(f_0\bar{z}_0 + \bar{f}_0 z_0) - 2x\frac{F'^2}{F^2}(\phi_i + \bar{\phi}_i),$$

where $\phi_i = \frac{\partial f_i}{\partial z_i}$.

Now, let $i = 1, \ldots, n-1$. By the same argument applied to the pair $(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i})$ one gets

(3.13) $$-\gamma F = -F'(f_0\bar{z}_0 + \bar{f}_0 z_0) + F(\phi_i + \bar{\phi}_i)$$

and

(3.14) $$0 = -\frac{F'}{F}(f_0\bar{z}_0 + \bar{f}_0 z_0) + (\phi_i + \bar{\phi}_i).$$

By comparing (3.12) with (3.14), one gets $L = 0$ and hence, by the proof of parts (1)-(2), (D_F, g_F) is holomorphically isometric to an open subset of $(\mathbb{C}H^n, g_{hyp})$, as required. (Notice that equations (3.13) and (3.14) yield $\gamma = 0$ and then one gets that X is a Killing vector field with respect to g_{hyp}, as expected). $\qquad \square$

Proof of Theorem 2.3. — Recall that the Bergman metric g_B on D_F is, by definition, the one given by the Kähler potential $\log \tilde{K}(z_0, z; z_0, z)$, where $\tilde{K}$ is the Bergman kernel of D_F.

Notice that $A = A(z_0, z) = F(|z_0|^2) - \|z\|^2$ is a local defining function (positively signed) for D_F at any boundary point (z_0, z) with $z_0 \in \mathcal{B} = \{z_0 \in \mathbb{C} \mid |z_0|^2 < x_0\}$, and such boundary points are strictly pseudoconvex. The hypothesis of the theorem and the fact that D_F is contractible means that

$$\log \tilde{K}(z_0, z) = -\lambda \log A + 2ReG(z_0, z)$$

for some holomorphic function G on D_F; here and below we will write just $\tilde{K}(z_0, z)$ for $\tilde{K}(z_0, z; z_0, z)$. By rotational symmetry of $\tilde{K}$ and A, the

pluriharmonic function $2ReG$ must depend only on $|z_0|^2$ and $\|z\|^2$, hence must be a positive constant, say μ. Thus

$$(3.15) \qquad \tilde{K}(z_0, z) = \frac{\mu}{A(z_0, z)^\lambda}.$$

On the other hand, by Fefferman's formula [13] for the boundary singularity of the Bergman kernel,

$$(3.16) \quad \tilde{K}(z_0, z) = \frac{a(z_0, z)}{A(z_0, z)^{n+1}} + b(z_0, z) \log A(z_0, z), \qquad (z_0, z) \in D_F,$$

where $a, b \in C^\infty(\mathcal{B} \times \mathbb{C}^{n-1})$ and

$$(3.17) \qquad a(z_0, z) = \frac{n!}{\pi^n} \, J[A](z_0, z),$$

for $z_0 \in \mathcal{B}$ and $\|z\|^2 = F(|z_0|^2)$ and where $J[A]$ is the Monge-Ampere determinant

$$(3.18) \qquad J[A] = (-1)^n \det \begin{pmatrix} A & \frac{\partial A}{\partial z_0} & \partial_z A \\ \frac{\partial A}{\partial \bar{z}_0} & \frac{\partial^2 A}{\partial z_0 \partial \bar{z}_0} & \partial_z(\frac{\partial A}{\partial \bar{z}_0}) \\ \partial_{\bar{z}} A & \partial_{\bar{z}}(\frac{\partial A}{\partial z_0}) & \partial_{\bar{z}} \partial_z A \end{pmatrix}.$$

A direct computation gives

$$(3.19) \qquad J[A] = -F^2 \frac{\partial^2 \log F}{\partial z_0 \partial \bar{z}_0}.$$

(which depends only on $|z_0|^2$). By comparing (3.15) with (3.16) one gets:

$$\mu = \frac{a(z_0, z) A(z_0, z)^\lambda}{A(z_0, z)^{n+1}} + b(z_0, z) A(z_0, z)^\lambda \log A(z_0, z), \qquad (z_0, z) \in D_F,$$

which evaluated at $\|z\|^2 = F(|z_0|^2)$, forces $\lambda = n + 1$. Further, by (3.17) and (3.19), the last expression gives

$$-F^2 \frac{\partial^2 \log F}{\partial z_0 \bar{\partial} z_0} = c,$$

for all $z_0 \in \mathcal{B}$ and $\|z\|^2 = F(|z_0|^2)$, where c is the negative constant given by $c = -\frac{\mu \pi^n}{n!}$ (notice that the condition $\|z\|^2 = F(|z_0|^2)$ is superfluous, since nothing there depends on z). Feeding this back into formula (3.19) one gets $J[A](z_0, z) = c$ for all $(z_0, z) \in D_F$, i.e. g_F is Kähler-Einstein, and we are done by Theorem 2.2 (1).

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