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INVERSE SCATTERING FOR WAVEGUIDES

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ABSTRACT. — We study the inverse scattering problem for a waveguide $(M, \mathbf{g})$ with cylindrical ends, $M = M^c \cup \left(\cup_{\alpha=1}^{N} (\Omega^\alpha \times (0, \infty)) \right)$, where each $\Omega^\alpha \times (0, \infty)$ has a product type metric. We prove, that the physical scattering matrix, measured on just one of these ends, determines $(M, \mathbf{g})$ up to an isometry.

1. General formulation of the problem

One of the most typical geometric constructions encountered in the everyday life is that of a compound waveguide, e.g. setting of optical and electric cables, oil, gas and water pipelines, etc. Mathematically, a compound waveguide can be represented as a non-compact connected Riemannian manifold $(M, \mathbf{g})$ such that at infinity it looks like a disjoint union of the asymptotically cylindrical ends. More precisely, let X_0 be a point in M and denote by $d(X, X_0)$ the distance between X_0 and a variable point $X \in M$. Then, for large $R > 0$, the sphere $S_R(X_0) = \{X \in M : d(X, X_0) = R\}$ decomposes into a finite number, N of the $(n-1)$−dimensional compact, connected submanifolds, $(\Omega_R^\alpha, g_R^\alpha)$,

$$S_R(X_0) = \bigcup_{\alpha=1}^{N} \Omega_R^\alpha,$$

and, in a proper sense, $(\Omega_R^\alpha, g_R^\alpha) \to (\Omega^\alpha, g^\alpha)$, when $R \to \infty$. In this case, the Laplace operator, Δ_M in $L^2(M)$ (with either Dirichlet or Neumann boundary conditions on components of ∂M, if $\partial M \neq \emptyset$) is defined as the closure of the Laplace operator on C^∞−functions with compact support. It is then possible to develop a spectral and scattering theory for Δ_M, see e.g. [10], [11], [4], [13] and, in a more general setting, [19], [2], [3].

To describe the nature of the scattering theory, note that each asymptotic section $(\Omega^\alpha, g^\alpha)$, which we associate with the corresponding channel of scattering, gives rize to its own $(n-1)-$dimensional Laplacian, Δ^α (for $\partial\Omega^\alpha \neq \emptyset$ with the boundary conditions inherited from those on ∂M). Denote by $\{\lambda_m^\alpha, \phi_m^\alpha\}_{m=1}^\infty$ the eigenvalues and corresponding orthonormal eigenfunctions of Δ^α. It is then possible to establish an existence, for all but a discrete number of $k^2 \in \mathbb{R}_+$, of a generalized eigenfunction $\psi_m^\alpha(X; k^2)$ of Δ_M, associated with $\{\lambda_m^\alpha, \phi_m^\alpha\}$, when $\lambda_m^\alpha < k^2$. Physically, we can imagine the situation as follows: We send through the channel associated with Ω^α an incoming propagating wave into M that is related to the eigenfunction ϕ_m^α with $\lambda_m^\alpha < k^2$. This is the wave which behaves, when $d(X, X_0) \to \infty$, as $\exp\left(-id(X, X_0)\sqrt{k^2 - \lambda_m^\alpha}\right)\phi_m^\alpha(x^\alpha)$ (for the terminology used see [22]). Here $X = (x^\alpha, d(X, X_0))$, x^α being (local) coordinates in Ω^α, are (local) coordinates in the channel α which, for large $d(X, X_0)$, is diffeomorphic to $\Omega^\alpha \times (A, +\infty)$ with some $A > 0$. The described incoming wave propagates through M, giving rize to the wavefunction $\psi_m^\alpha(X; k^2)$, which is actually a generalised eigenfunction of the continuous spectrum for Δ_M. In general, this wavefunction goes to infinity through all N channels of M. Registering $\psi_m^\alpha(X; k^2)$ in the $\beta-$channel as $d(X, X_0) \to \infty$, we observe the asymptotic behaviour of the wave,

$$(1.1) \qquad \psi_m^\alpha(X; k^2) \sim \sum_{\lambda_l^\beta < k^2} \mathbb{S}_{ml}^{\alpha\beta}(k^2)\, e^{iy\sqrt{k^2 - \lambda_l^\beta}}\phi_l^\beta(x^\beta).$$

Here $y = d(X, X_0)$ and $X = (x^\beta, y)$, $x^\beta \in \mathbb{R}^{n-1}$ being (local) coordinates in Ω^β, are coordinates in the $\beta-$channel $\Omega^\beta \times (A, +\infty)$. This construction is valid for all but a discrete set $\sigma^s(M)$ of k^2, with $\sigma^s(M)$ consisting of the (positive) eigenvalues of the point spectrum of $-\Delta_M$ and all $k^2 = \lambda_m^\alpha$, $\alpha = 1, \ldots, N$; $m = 1, 2, \ldots$ In the following, denote by $d^\beta(k)$ the integer such that $\lambda_l^\beta \leqslant k^2$ for $l \leqslant d^\beta(k)$ and $\lambda_l^\beta > k^2$ for $l > d^\beta(k)$. The above construction associates with any $k^2 \in \mathbb{R}_+ \setminus \sigma^s(M)$ a finite-dimensional matrix, called *the physical scattering matrix*, $\mathbb{S}(k^2)$,

$$(1.2) \qquad \mathbb{S}(k^2) = [\mathbb{S}_{ml}^{\alpha\beta}(k^2)]_{\alpha,\beta\in\{1,\ldots,N\},\ m\leqslant d^\alpha(k),\ l\leqslant d^\beta(k)}.$$

Note that the dimension of $\mathbb{S}(k^2)$ is a step function with jumps at the eigenvalues of $-\Delta^\alpha$, $\alpha \in \{1, \ldots, N\}$.

A natural inverse scattering problem for a compound waveguide is *to reconstruct, up to an isometry, the manifold* $(M, \mathbf{g})$ *from its physical scattering matrix* $\mathbb{S}(k^2)$, $k^2 \in \mathbb{R}_+ \setminus \sigma^s(M)$. Moreover, as it is quite conceivable that we can not make measurements in all N channels of M or even the

number of these channels is *a priori* unknown, a more realistic inverse problem is *to reconstruct, up to an isometry, the manifold* (M, g) *from its partial physical scattering matrix* $\mathbb{S}_{ml}^{\alpha\beta}(k^2)$, *known for* $\alpha, \beta \in K \subset \{1, \dots, N\}$ *and* $k^2 \in O$, *where* O *is an infinite open subset of* $\mathbb{R}_+$ *such that* $O \cap \sigma^s(M) = \emptyset$.

Note that, making measurements on a part of the boundary at infinity of M, namely Ω^α, $\alpha \in K$, it is natural to assume that the Riemannian manifolds $(\Omega^\alpha, g^\alpha)$ and, therefore, the pairs $\{\lambda_m^\alpha, \phi_m^\alpha\}$, $\alpha \in K$, $m = 1, 2 \dots$, of eigenvalues and eigenfunctions are in our disposal.

Inverse problems in waveguides were considered from the physical point of view in [9] and from the view point similar to ours in [5], [7].

2. Waveguide with cylindrical ends

In this paper, we provide a sketch of solution to the formulated inverse problem in the special case when the manifold $(M, \mathbf{g})$ is a waveguide with cylindrical ends. This means that $(M, \mathbf{g})$ can be represented as

(2.1)
$$M = M^c \cup \left(\bigcup_{\alpha=1}^{N} (\Omega^\alpha \times (0, \infty)) \right), \quad M^c \cap (\Omega^\alpha \times (0, \infty))) = \Omega^\alpha \times (0, 1),$$

$$\Omega^\alpha \cap \Omega^\beta = \emptyset, \quad \text{for } \alpha \neq \beta.$$

Here M^c is an open, relatively compact subset of M and each $\Omega^\alpha \times (0, \infty)$ has a product structure

$$\mathbf{g}|_{\Omega^\alpha \times (0,\infty)} = \sum_{p,q=1}^{n-1} g_{pq}^\alpha(x) dx^p dx^q + dy^2,$$

where $X = (x, y)$ are the local coordinates on $\Omega^\alpha \times \mathbb{R}_+$.

Note that the spectral and scattering theory for waveguides with cylindrical ends much proceeds the general theory, see e.g. [10], [11], [17], [18], [22]. This is due to the fact that, in case of cylindrical ends, the scattering problem for $(M, \mathbf{g})$ can be considered as a perturbation in a compact domain of a direct sum of the Laplace operators,

$$\Delta_0 = \bigoplus_{\alpha=1}^{N} \left(\Delta^\alpha + \partial_y^2 \right),$$

where $y \in \mathbb{R}_+$ is the coordinate along each channel. In the case when $\partial M = \emptyset$, the domain of definition of Δ_0 is

$$\mathbb{D}(\Delta_0) = \oplus_{\alpha=1}^{N} \{ u_\alpha \in H^2(\Omega^\alpha \times \mathbb{R}_+) : \ u_\alpha|_{\Omega^\alpha \times \{0\}} = 0 \},$$

and if $\partial M \neq 0$, the Δ_0 is defined using the Dirichlet or Neumann boundary condition on $\partial \Omega^\alpha \times (0, \infty)$.

Let us return to the eigenfunctions of the continuous spectrum of Δ_M, $\psi_m^\alpha(X; k^2)$, with $m \leqslant d^\alpha(k)$, $k^2 \notin \sigma^s$, which were introduced in Section 1. Then, in $\Omega^\beta \times (0, \infty)$,

$$
\psi_m^\alpha(x^\beta, y; k^2) = e^{-iy\sqrt{k^2 - \lambda_m^\alpha}} \phi_m^\beta(x^\beta)\delta_{\alpha,\beta}
$$

$$
(2.2) \qquad\qquad + \sum_{l \leqslant d^\beta(k)} a_{ml}^{\alpha\beta}(k^2)e^{iy\sqrt{k^2 - \lambda_l^\beta}} \phi_l^\beta(x^\beta)
$$

$$
+ \sum_{l > d^\beta(k)} b_{ml}^{\alpha\beta}(k^2)e^{-y\sqrt{\lambda_l^\beta - k^2}} \phi_l^\beta(x^\beta).
$$

Comparing the above formula with the asymptotic behavior (1.2) of ψ_m^α, we see that

$$
a_{ml}^{\alpha\beta}(k^2) = \mathbb{S}_{ml}^{\alpha\beta}(k^2), \quad m \leqslant d^\alpha(k),\ l \leqslant d^\beta(k).
$$

Note that, in addition to the waves propagating towards infinity, which are represented by the first sum in the right side of (2.2), the wavefunction ψ_m^α contains, in $\Omega^\beta \times (0, \infty)$, infinitely many exponentially decaying terms, described by the second sum in the right side of (2.2). Therefore, it is possible to extend the notion of the scattering matrix from the physical one $\mathbb{S}_{ml}^{\alpha\beta}(k^2)$, $m \leqslant d^\alpha(k)$, $l \leqslant d^\beta(k)$, to the general $l \in \mathbb{Z}_+$. Namely, we first define the non-physical scattering matrix $S_{ml}^{\alpha\beta}(k^2)$ by

$$
S_{ml}^{\alpha\beta}(k^2) = a_{ml}^{\alpha\beta}(k^2) \text{ for } m \leqslant d^\alpha(k),\ l \leqslant d^\beta(k);
$$
$$
S_{ml}^{\alpha\beta}(k^2) = b_{ml}^{\alpha\beta}(k^2),\ \text{for } m \leqslant d^\alpha(k),\ l > d^\beta(k).
$$

Moreover, it is also possible to treat the case of $m > d^\alpha(k)$, that is, $\lambda_m^\alpha > k^2$. To this end, we start with an exponentially growing, as $y \to \infty$, incoming wave, $\exp{(y\sqrt{\lambda_m^\alpha - k^2})}\phi_m^\alpha(x^\alpha) = \exp{(-iy\sqrt{k^2 - \lambda_m^\alpha})}\phi_m^\alpha(x^\alpha)$. Starting from the above wave, we construct a non-physical wavefunction $\psi_m^\alpha(X; k^2)$ by using the Green function $R_{k^2}(X, X')$ of $(\Delta_M + k^2)$,

$$
(2.3) \quad \psi_m^\alpha(X; k^2) - \chi(y) \exp{(y\sqrt{\lambda_m^\alpha - k^2})}\phi_m^\alpha(x^\alpha)
$$

$$
= \int_0^\infty \int_{\Omega^\alpha} R_{k^2}(X, X')\, [\Delta_M', \chi(y')] \exp{(y'\sqrt{\lambda_m^\alpha - k^2})}\phi_m^\alpha(x^\alpha)\, dy'dx^\alpha,
$$

where $X = (x, y)$, $X' = (x', y')$. Here $\chi(y)$ is a cut-off function, equal to 0 for $y < 1/4$ and 1 for $y > 3/4$ which vanishes in $M \setminus \Omega^\alpha \times \mathbb{R}_+$, $[\Delta_M', \chi(y')]$ stands for the commutator of the above operators, and Δ_M' is the operator Δ_M in coordinates (x', y').

Next, denote the resolvent of $-\Delta_M$ by $R_z = (-\Delta_M - z)^{-1}$. It is known, see e.g. [18] for the waveguides with cylindrical ends and [13] for the general compound waveguides, that, for any $\Phi, \widetilde{\Phi} \in C_0^\infty(M)$,

$$R_{k^2}^{\Phi,\widetilde{\Phi}} := m_\Phi \circ R_{k^2} \circ m_{\widetilde{\Phi}} \in \mathcal{L}(L^2(M)),$$

where m_Φ is the multiplication operator $(m_\Phi u)(X) = \Phi(X)\, u(X)$ and $\mathcal{L}(B)$ denotes bounded operators in a Banach space B. On the other hand, due to M being a compact domain perturbation of $\bigcup_{\beta=1}^N \left(\Omega^\beta \times (0, \infty) \right)$, the operator $[\Delta'_M, \chi(y')]$ has a compact support in $\Omega^\alpha \times [1/4, 3/4]$. Thus, equation (2.3) has sense, defining the non-physical wavefunction $\psi_m^\alpha(X; k^2)$.

Using the limiting absorption principle for $R_{k^2 \pm i\varepsilon}$, $\varepsilon > 0$, see e.g [13], where R_z is the resolvent for $-\Delta_M$, we see that there are no propagating terms of the form $\exp\left(-iy\sqrt{k^2 - \lambda_l^\beta}\right)$, $l \leqslant d^\beta(k)$ due to the integral in the right side of (2.3) . Summarizing, we see that, for $y > 3/4$,

$$(2.4) \quad \psi_m^\alpha(x^\beta, y; k^2) = \exp\left(y\sqrt{\lambda_m^\alpha - k^2}\right)\phi_m^\alpha(x^\beta)\delta_{\alpha\beta}$$
$$+ \sum_{l=1}^\infty S_{ml}^{\alpha\beta}(k^2) \exp\left(iy\sqrt{k^2 - \lambda_l^\beta}\right)\phi_l^\beta(x^\beta),$$

thus defining the scattering matrix $S_{ml}^{\alpha\beta}(k^2)$ for $m > d^\alpha(k)$. In following, we denote by

$$S(k^2) = [S_{ml}^{\alpha\beta}(k^2)]_{m,l \in \mathbb{Z}_+, \, \alpha,\beta=(1,\ldots,N)}$$

this generalized, or the non-physical scattering matrix. Note that the physical scattering matrix $\mathbb{S}(k^2)$ is a finite dimensional sub-matrix of $S(k^2)$ for each $k^2 \in \mathbb{R}_+ \setminus \sigma^s(M)$.

3. Analytic properties of the non-physical scattering matrix

To analyse the dependence of $S_{ml}^{\alpha\beta}(\lambda)$, $\lambda \in \mathbb{C}$, we recall that the operator-valued function $R_\lambda^{\Phi,\widetilde{\Phi}}$ is analytic in $\mathbb{C}\setminus\mathbb{R}_+$ and has continuous limits, $R_{k^2 \pm i0}^{\Phi,\widetilde{\Phi}}$ as $\lambda \to k^2 \pm i0$ (except for $k^2 \in \sigma^s(M)$.)

Thus, equation (2.3) makes it possible to define the non-physical wavefunction $\psi_m^\alpha(X; \lambda)$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}_+$. Considered as a function of λ, $\psi_m^\alpha(\,\cdot\,; \lambda)$ is analytic in $\mathbb{C} \setminus \mathbb{R}_+$ and, moreover, there are limits

$$\psi_m^{\pm,\,\alpha}(X; k^2) = \lim_{\varepsilon \to 0} \psi_m^\alpha(X; k^2 \pm i\varepsilon).$$

Remark 3.1. — In our definition of $\psi_m^\alpha(X; k^2)$, it corresponds actually to $\psi_m^{+,\,\alpha}(X; k^2)$ with $\psi_m^{-,\,\alpha}(X; k^2) = \overline{\psi}_m^{+,\,\alpha}(X; k^2)$. Similarly, $R_{k^2}(X, X')$ in (2.3) actually coincides with $R_{k^2+i0}(X, X')$.

Therefore, when $y > 0$,

$$\Psi_{ml}^{s,\,\alpha\beta}(y; \lambda) := \left(\psi_m^\alpha(\cdot, y; \lambda) - \exp\left(-iy\sqrt{\lambda - \lambda_m}\right)\phi_m^\alpha(\cdot)\delta_{\alpha\beta},\ \phi_l^\beta(\cdot) \right)_{L^2(\Omega^\beta)}$$

satisfies, for $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$, the ordinary differential equation

$$\frac{d^2}{dy^2}\Psi_{ml}^{s,\,\alpha\beta}(y; \lambda) + (\lambda - \lambda_l^\beta)\Psi_{ml}^{s,\,\alpha\beta}(y; \lambda) = 0.$$

This and (2.4) imply that

$$\Psi_{ml}^{s,\,\alpha\beta}(y; \lambda) = S_{ml}^{\alpha\beta}(\lambda) \exp\left(iy\sqrt{\lambda - \lambda_l^\beta}\right),$$

where $S_{ml}^{\alpha\beta}(\lambda)$ is analytic in $\mathbb{C} \setminus \mathbb{R}_+$ and continuous, from above and below, up to $\mathbb{R}_+$, thus defining $S_{ml}^{\alpha\beta}(k^2 \pm i0)$. Note that $S_{ml}^{\alpha\beta}(k^2)$ in section 2 is actually $S_{ml}^{\alpha\beta}(k^2 + i0)$.

These considerations immediately imply the following lemma:

LEMMA 3.2. — *Let $\mathbb{S}(k^2)$ be given for all $k^2 \in O$, where O is an unbounded open subset of $\mathbb{R}_+$, $O \cap \sigma^s(M) = \emptyset$. Then, these data determine the non-physical scattering matrix $S_{ml}^{\alpha\beta}(\lambda)$, $m, l \in \mathbb{Z}_+$, $\alpha, \beta \in \{1, \ldots, N\}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ and also $S_{ml}^{\alpha\beta}(k^2 \pm i0)$ for all $k^2 \notin \sigma^s(M)$.*

Proof Observe that for any m, l and α, β, there is an open interval $I \subset O$ such that, for $k^2 \in I$, we have $\lambda_m^\alpha,\ \lambda_l^\beta < k^2$. Moreover, for such k^2,

$$\mathbb{S}_{ml}^{\alpha,\beta}(k^2) = S_{ml}^{\alpha\beta}(k^2 + i0).$$

By the analyticity properties of $\mathbb{S}_{ml}^{\alpha\beta}(\lambda)$, described above, this determines uniquely this matrix coefficient in $\mathbb{C} \setminus \mathbb{R}_+$ and also $S_{ml}^{\alpha\beta}(k^2 \pm i0)$ in $\mathbb{R}_+ \setminus \sigma^s(M)$. QED

Note that having only a partial physical scattering matrix S^{ph} for α, β lying in a subset K of $(1, \ldots, N)$, we can find the partial non-physical matrix $S_{ml}^{\alpha\beta}(\lambda)$ for these α, β.

4. From scattering data to the Dirichlet-to-Neumann map

In this and the next sections we consider the case when the physical scattering matrix $\mathbb{S}$ is known only in one channel of scattering $\Omega^\alpha \times (0, \infty)$, corresponding to, say, $\alpha = 1$, i.e. we are given $S_{m,l}^{\alpha\alpha}(k^2)$, $m, l \leqslant d^\alpha(k)$

with $\alpha = 1$. In this connection, in the future we skip indexes α, β writing just $\Omega = \Omega^1$, $S_{ml}(\lambda) = S_{ml}^{11}(\lambda)$, $\psi_m(X, k^2) = \psi_m^1(X, k^2)$, $\phi_m(x) = \phi_m^1(x)$, etc. We denote by $\widetilde{M}$ the domain

$$\widetilde{M} = M \setminus (\Omega \times (1, \infty)).$$

Observe that, in general, $(\widetilde{M}, \mathbf{g}|_{\widetilde{\mathbf{M}}})$ is itself a compound waveguide with $\Omega \times \{1\} \subset \partial\widetilde{M}$.

Consider the Dirichlet-to-Neumann operator $\widetilde{\Lambda}_\lambda$, $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ associated with $\widetilde{M}$,

(4.1) $\qquad \widetilde{\Lambda}_\lambda : H^{1/2}(\Omega) \to H^{-1/2}(\Omega); \quad \widetilde{\Lambda}_\lambda(f) := \partial_y u_\lambda^f|_{(\Omega \times \{1\})},$

where $u_\lambda^f(X)$ is the solution to the boundary-value problem,

$$(\widetilde{\Delta} + \lambda)u^f = 0 \quad \text{in } \widetilde{M}$$
$$u^f|_{\Omega \times \{1\}} = f,$$

where $\widetilde{\Delta}$ is the Laplace operator in $\widetilde{M}$.

Then, the partial non-physical scattering matrix $\mathbb{S}$ determines the action of $\widetilde{\Lambda}_\lambda$, for $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$, on all functions f, such that

(4.2) $\qquad f \in \mathrm{cl}_{H^{1/2}(\Omega)}\{\mathrm{Span}\,(\psi_m(x, 1; \lambda), m = 1, 2, \dots)\}.$

Indeed, for $f(x) = \psi_m(x, 1; \lambda)$, we have,

$$u_\lambda^f(X) = \psi_m(X; \lambda), \quad X \in \widetilde{M}.$$

Thus,

(4.3)
$$\partial_y u_\lambda^f(x)|_{(\Omega \times \{1\})} = -i\sqrt{\lambda - \lambda_m}\exp\left(-i\sqrt{\lambda - \lambda_m}\right)\phi_m(x)$$
$$+ \sum_{l=1}^\infty i\sqrt{\lambda - \lambda_l}\,S_{ml}(\lambda)\exp(i\sqrt{\lambda - \lambda_l})\phi_l(x).$$

Here, due to the orthogonality, $(\phi_m, \phi_l)_{H^1(\Omega)} = 0$, for $m \neq l$, and regularity $\psi_m \in H_{loc}^2(M)$, the sum in (4.3) converges in $H^1(\Omega)$. Note that we also know that

(4.4)
$$\psi_m|_{\Omega \times \{1\}} =$$
$$\exp\left(-i\sqrt{\lambda - \lambda_m}\right)\phi_m(x) + \sum_{l=1}^\infty S_{ml}(\lambda)\exp(i\sqrt{\lambda - \lambda_l})\,\phi_l(x).$$

Then the desired result that $\widetilde{\Lambda}_\lambda$ on the set (4.2) can be determined using the scattering matrix $S_{ml}(\lambda)$ follows from the continuity of $\widetilde{\Lambda}_\lambda$ from $H^{1/2}(\Omega)$ to $H^{-1/2}(\Omega)$.

Therefore, if we can show that

$$(4.5) \qquad \mathrm{cl}_{H^{1/2}(\Omega}\{\mathrm{Span}\,(\psi_m(x,1;\lambda),\, m=1,2,\dots)\} = H^{1/2}(\Omega),$$

then the partial non-physical scattering matrix $\mathbb{S}(\lambda)$ determines the Dirichlet-to-Neumann map $\widetilde{\Lambda}_\lambda$. In turn, by duality, property (4.5) is equivalent to the following result

LEMMA 4.1. — *Let* $h \in H^{-1/2}(\Omega)$. *Assume that, for some* $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$,

$$\langle h, \psi_m(\lambda)|_{\Omega \times \{1\}} \rangle = 0, \quad m = 1, 2, \dots$$

Then $h = 0$.

(In the above formula $\langle \cdot, \cdot \rangle$ stands for the sesquilinear $H^{-1/2}(\Omega) \times H^{1/2}(\Omega)$ duality.)

Proof. — Let us return to representation (2.3) which we rewrite slightly differently as

$$(4.6) \quad \begin{aligned} \psi_m(X;\lambda) &= 2i\chi(y)\sin{(y\sqrt{\lambda - \lambda_m})}\phi_m(x) \\ &+ 2i\int_0^\infty \int_\Omega R_\lambda(X,X')\,[\Delta'_M,\,\chi(y')]\sin{(y'\sqrt{\lambda - \lambda_m})}\phi_m(x')\,dy'dx'. \end{aligned}$$

Here we take into the account that the non-physical wavefunction in the "unperturbed waveguide", $M^0 = \Omega \times (0,\infty)$ with Dirichlet condition at $\Omega \times \{0\}$, which corresponds to $\{\lambda_m, \phi_m\}$, is $\sin{(y\sqrt{\lambda - \lambda_m})}\phi_m(x)$. Next, let $X = (x,y)$, $X' = (x',y')$, $X'' = (x'',y'')$ be points of M^0. Observe, that when $y'' > 1$, the Green function $R_\lambda(X,X'')$ does itself satisfy a representation similar to (4.6),

$$(4.7) \quad \begin{aligned} R_\lambda(X,X'') &= \chi(y)\,R_\lambda^0(X,X'') \\ &+ \int_0^\infty \int_\Omega R_\lambda(X,X')\,[\Delta'_M,\,\chi(y')]\,R_\lambda^0(X',X'')\,dy'dx', \end{aligned}$$

where $R_\lambda^0(X,X'')$ is the Green function in $\Omega \times (0,\infty)$ with the Dirichlet condition at $\Omega \times \{0\}$. We have, for $X = (x,y)$, $X' = (x',y')$, $y < y'$, that

$$(4.8)$$
$$R_\lambda^0(X,X') = \sum_{m=1}^\infty \sin{(y\sqrt{\lambda - \lambda_m})}\exp{(iy'\sqrt{\lambda - \lambda_m})}\,\phi_m(x)\phi_m(x').$$

Therefore, substituting (4.8) into representation (4.7), we get

$$R_\lambda(X, X'')$$

$$= \chi(y) \sum_{m=1}^{\infty} \sin\left(y\sqrt{\lambda - \lambda_m}\right) \exp\left(iy''\sqrt{\lambda - \lambda_m}\right) \phi_m(x)\phi_m(x'')$$

$$+ \int_0^\infty \int_\Omega R_\lambda(X, X') \left[\Delta'_M, \chi(y')\right]$$

$$\times \sum_{m=1}^{\infty} \sin\left(y'\sqrt{\lambda - \lambda_m}\right) \exp\left(iy''\sqrt{\lambda - \lambda_m}\right) \phi_m(x')\phi_m(x'')\, dy' dx'.$$

Comparing the above representation with (4.6), we see that, for $X = (x, y) \in \widetilde{M}$, $X'' = (x'', y'')$, $y'' > 1$,

$$(4.9) \qquad R_\lambda(X, X'') = \sum_{m=1}^{\infty} \psi_m(X; \lambda) \exp\left(iy''\sqrt{\lambda - \lambda_m}\right) \phi_m(x'').$$

Let now $h \in H^{-1/2}(\Omega)$ be orthogonal to all $\psi_m(\cdot; \lambda)|_{(\Omega \times \{1\})}$, i.e.

$$(4.10) \qquad \langle h, \psi_m|_{\Omega \times \{1\}} \rangle := \int_\Omega h(x) \overline{\psi}_m(x, 1; \lambda) dx = 0, \quad m = 1, 2, \ldots$$

Consider the equation

$$(\Delta_M + \overline{\lambda})u = h(x)\delta(y - 1) \in H^{-1}(M).$$

For $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$, it has the solution of the form of a single-layer potential,

$$(4.11) \qquad u^h(X) = \int_\Omega R_{\overline{\lambda}}(X; (x', 1))h(x')dx', \quad u^h \in H^1(M).$$

Let now $X = (x, y)$, $y > 1$. Observe that

$$R_{\overline{\lambda}}(X', X) = \overline{R}_\lambda(X, X').$$

Therefore, condition (4.10) imply that, for $y > 1$,

$$u^h(x, y) = \sum_{m=1}^{\infty} \left(\langle h, \psi_m|_{\Omega \times \{1\}} \rangle\right) \exp\left(-iy\sqrt{\lambda - \lambda_m}\right) \phi_m(x) = 0,$$

where we choose the branch of $\sqrt{z}$ to be equal to $-i|z|^{1/2}$ for $z \in \mathbb{R}_-$. As $u^h \in H^1(M)$ this means, in particular, that $u^h|_{\widetilde{M}}$ is solution to the homogeneous Dirichlet problem. As $\lambda \notin \mathbb{R}_+$, this implies that $u^h = 0$ in $\widetilde{M}$ and, therefore, $h = 0$. QED

Summarizing the above, we formulate the following result.   □

LEMMA 4.2. — *Let $(M, \mathbf{g})$ be a waveguide with cylindrical ends. Then, for any α the partial physical scattering matrices $\mathbb{S}_{ml}^{\alpha\alpha}(k^2)$, $m, l \leqslant d^\alpha(k)$, $k \in \mathbb{R}_+$ determines uniquely, for any $\lambda \in \mathbb{C} \backslash \mathbb{R}_+$, the Dirichlet-to-Neumann map $\widetilde{\Lambda}_\lambda$ of the Laplace operator, $\widetilde{\Delta}$ in $\widetilde{M}$.*

5. Main result

We are now in the position to formulate the main result of the paper. To this end, consider two waveguides, each with cylindrical ends, $(M_1, \mathbf{g}_1)$, $(M_2, \mathbf{g}_2)$. Assume that for some ends, say, those numbered by $\alpha = 1$, the boundaries of these ends at infinity, $(\Omega_1^\alpha, g_1^\alpha)$, $(\Omega_2^\alpha, g_2^\alpha)$ with $\alpha = 1$, are isometric. Next we omit α and write just $\Omega_1^1 = \Omega_1$, $\Omega_2^1 = \Omega_2$, etc. Identifying them, we write

(5.1) $$(\Omega_1, g_1) = (\Omega_2, g_2) = (\Omega, g).$$

Consider the Laplace operators Δ_{M_i} in $(M_i, \mathbf{g}_i)$, $i = 1, 2$, and the corresponding partial physical scattering matrices in channels $\Omega_1 \times (0, \infty)$, $\Omega_2 \times (0, \infty)$, namely $\mathbb{S}_{(i)}^{11}(k^2) = [\mathbb{S}_{(i),ml}^{11}(k^2)]_{m,l \leqslant d^1(k)}$.

THEOREM 5.1. — *Let $(M_1, \mathbf{g}_1)$, $(M_2, \mathbf{g}_2)$ satisfy the above conditions. Assume that, for any $k^2 \in O$, where O is an unbounded open set in $\mathbb{R}_+$ such that $O \cap \sigma^s(M_i) = \emptyset$, the physical scattering matrices corresponding to the end $\Omega \times \mathbb{R}_+$, $\mathbb{S}_{(1)}^{11}(k^2)$ and $\mathbb{S}_{(2)}^{11}(k^2)$, $k^2 \in O$, coincide. Then the manifolds $(\widetilde{M_1}, \mathbf{g}_1)$, $(\widetilde{M_2}, \mathbf{g}_2)$ are isometric. In particular, the number of scattering channels is the same, $N_1 = N_2 (= N)$ and, after a proper identification, $(\Omega_1^\alpha, g_1^\alpha)$ are isometric to $(\Omega_2^\alpha, g_2^\alpha)$, $\alpha = 1, \ldots, N$.*

Proof. — By Lemma 3.2, the conditions of the theorem imply that

$$S_{(1),ml}(k^2) = S_{(2),ml}(k^2)$$

for all $m, l \in \mathbb{Z}_+$, $k^2 \in O$. Denoting by $\psi_m^{(i)}(X, \lambda)$ the wavefunctions, both physical and non-physical, of M_i, $i = 1, 2$, and using equations (4.3), (4.4), we see that

$$\psi_m^{(1)}(x, 1; k^2) = \psi_m^{(2)}(x, 1; k^2),$$
$$\partial_y \psi_m^{(1)}(x, 1; k^2) = \partial_y \psi_m^{(2)}(x, 1; k^2);$$
$$x \in \Omega, \ k^2 \in O.$$

Due to the analyticity, with respect to λ, of $\psi_m^{(i)}(\cdot, \lambda)$, see the beginning of section 3, we see that

$$\psi_m^{(1)}(x, 1; \lambda) = \psi_m^{(2)}(x, 1; \lambda),$$
$$\partial_y \psi_m^{(1)}(x, 1; \lambda) = \partial_y \psi_m^{(2)}(x, 1; \lambda);$$
$$x \in \Omega, \ \lambda \notin \mathbb{R}_+.$$

It then follows from Lemma 4.2 that the Dirichlet-to-Neumann maps for $\widetilde{M}_1$ and $\widetilde{M}_2$ coincide,

$$(5.2) \qquad\qquad\qquad \widetilde{\Lambda}_\lambda^{(1)} = \widetilde{\Lambda}_\lambda^{(2)}.$$

Consider now the wave equations in $\widetilde{M}_i \times \mathbb{R}$,

$$(5.3) \qquad \partial_t^2 u_F^i - \widetilde{\Delta}_i u_F^i = 0, \quad u_F^i|_{t<0} = 0, \quad u_F^i|_{\Omega \times \{1\}} = F.$$

It follows from (5.2), cf. [15], that

$$(5.4) \qquad \widetilde{\Lambda}_h^{(1)}(F) = \widetilde{\Lambda}_h^{(2)}(F), \quad F \in C_0^\infty(\Omega \times \mathbb{R}_+),$$

where $\widetilde{\Lambda}_h^{(i)}$ is the hyperbolic Dirichlet-to-Neumann map,

$$\widetilde{\Lambda}^{(i)} u_h^i(F) = \partial_y u_F^i|_{(\Omega \times \{1\}) \times \mathbb{R}_+}.$$

This implies, it turn, that there exists an isometry

$$\mathbf{\Phi} : (\widetilde{M}_1, \mathbf{g}_1) \to (\widetilde{M}_2, \mathbf{g}_2), \quad \mathbf{\Phi}|_{(\Omega \times \{1\})} = \mathrm{Id}|_{(\Omega \times \{1\})},$$

see e.g. [16].

As $\widetilde{M}_i \cap (\Omega \times \mathbb{R}_+) = (\Omega \times (0, 1))$, the second equation in the above formula yields also that $(M_1, \mathbf{g}_1)$ and $(M_2, \mathbf{g}_2)$ are isometric. QED $\qquad\qquad \square$

Remark 5.2. — Instead of given isometric $\Omega_{(1)}$, $\Omega_{(2)}$, we can assume that the spectra of Δ_{M_1} and Δ_{M_2} coincide as well as the sets

$$\Phi^i := \{(\phi_1^i|_{\Omega_{(i)}}, \phi_2^i|_{\Omega_{(i)}}, \dots)\} \quad i = 1, 2,$$

cf. [1].

Remark 5.3. — The general theory [2], [3], [19] describes spectral and scattering properties of compound waveguides with asymptotically cylindrical ends. Also, constructions in [16] remain valid for the manifolds $(\widetilde{M}, \widetilde{\mathbf{g}})$ with asymptotically cylindrical ends. Therefore, the constructions of the paper, in particular Theorem 5.1, remain valid even if we do not assume that the ends of M_1 and M_2, except for $\Omega \times (0, \infty)$, are cylindrical.

Remark 5.4. — The idea to consider, in studying inverse scattering, not only the "physical" wavefunctions, but also "non-physical", exponentially growing ones, goes back to [8], which used it to study inverse quantum scattering with data given by the scattering matrix at all energies. Later, this type of exponentially growing solutions proved most effective in the study of inverse problems with fixed-frequency data, see [21], [20], [14] for the pioneering works. In this paper we return, for the waveguide inverse problem, to the original idea of Faddeev and combine it with the technique of the BC-method, see e.g. [15].

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