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Ugo BOSCAIN & Mario SIGALOTTI

High-order angles in almost-Riemannian geometry

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HIGH-ORDER ANGLES IN ALMOST-RIEMANNIAN GEOMETRY

Ugo Boscain & Mario Sigalotti

ABSTRACT. — Let X and Y be two smooth vector fields on a two-dimensional manifold M. If X and Y are everywhere linearly independent, then they define a Riemannian metric on M (the metric for which they are orthonormal) and they give to M the structure of metric space. If X and Y become linearly dependent somewhere on M, then the corresponding Riemannian metric has singularities, but under generic conditions the metric structure is still well defined. Metric structures that can be defined locally in this way are called almost-Riemannian structures. The main result of the paper is a generalization to almost-Riemannian structures of the Gauss-Bonnet formula for domains with piecewise-$\mathcal{C}^2$ boundary. The main feature of such formula is the presence of terms that play the role of high-order angles at the intersection points with the set of singularities.

1. Introduction

Let M be a two-dimensional smooth manifold and consider a pair of smooth vector fields X and Y on M. If the pair X, Y is Lie bracket generating, i.e., if $\mathrm{span}\{X(q), Y(q), [X,Y](q), [X,[X,Y]](q), \ldots\}$ is full-dimensional at every $q \in M$, then the control system

$$(1.1) \qquad \dot{q} = uX(q) + vY(q), \; u^2 + v^2 \leqslant 1, \; q \in M,$$

is completely controllable and the minimum-time function defines a continuous distance d on M. When X and Y are everywhere linear independent (the only possibility for this to happen is that M is parallelizable), such distance is Riemannian and it corresponds to the metric for which (X, Y) is an orthonormal moving frame.

The idea is to study the geometry obtained starting from a pair of vector fields which may become collinear. Under generic hypotheses, the set $\mathcal{Z}$

(called *singular locus*) of points of M at which X and Y are parallel is a one-dimensional embedded submanifold of M (possibly disconnected).

Metric structures that can be defined *locally* by a pair of vector fields (X, Y) through (1.1) are called almost-Riemannian structures. A notion of orientability can be introduced in a natural way for almost-Riemannian structures (see Section 2). An example of almost-Riemannian structure is provided by the Grushin plane, for which $M = \mathbb{R}^2$ and the pair of generating vector fields can be chosen as $X(x, y) = (1, 0)$ and $Y(x, y) = (0, x)$. (See [3, 4, 5, 6, 7].)

The notion of almost-Riemannian structure was introduced in [1]. That paper provides a characterization of generic almost-Riemannian structures by means of local normal forms and presents a generalization of the Gauss-Bonnet formula (for manifolds without boundary). Let M be compact and oriented, and endow it with an orientable almost-Riemannian structure. Denote by $K : M \setminus \mathcal{Z} \to \mathbb{R}$ the Gaussian curvature and by dA_s a signed volume form associated with the almost-Riemannian structure on $M \setminus \mathcal{Z}$ (see Section 4 for the precise definition).

Let $M^\varepsilon = \{q \in M \mid d(q, \mathcal{Z}) > \varepsilon\}$, where $d(\cdot, \cdot)$ is the distance globally defined by the almost-Riemannian structure on M. In [1] it was proven that, under generic assumptions and in the case in which the distribution generated by X and Y is nowhere tangent to $\mathcal{Z}$, the limit

$$(1.2) \qquad \lim_{\varepsilon \searrow 0} \int_{M_\varepsilon} K(q) dA_s$$

exists and its value is equal to $2\pi(\chi(M^+) - \chi(M^-))$, where χ denotes the Euler characteristic and M^+ (respectively, M^-) is the subset of $M \setminus \mathcal{Z}$ on which the orientation defined by dA_s coincides with (respectively, is opposite to) that of M.

In this paper we prove a sharper version of the formula presented above, for domains having piecewise-$\mathcal{C}^2$ boundary of finite length. Let U be an open connected subset of M whose boundary Γ is piecewise-$\mathcal{C}^2$ and of finite length. Let $U_\varepsilon^\pm = M_\varepsilon \cap M^\pm \cap U$. Under generic conditions, if we assume that Γ is $\mathcal{C}^2$ in a neighborhood of $\mathcal{Z}$, then the following limits exist and are finite:

$$(1.3) \qquad \int_U K dA_s = \lim_{\varepsilon \to 0} \int_{U_\varepsilon^+ \cup U_\varepsilon^-} K dA_s,$$

$$(1.4) \qquad \int_{\partial U} k_g d\sigma_s = \lim_{\varepsilon \to 0} \left(\int_{\Gamma \cap \partial U_\varepsilon^+} k_g d\sigma - \int_{\Gamma \cap \partial U_\varepsilon^-} k_g d\sigma \right),$$
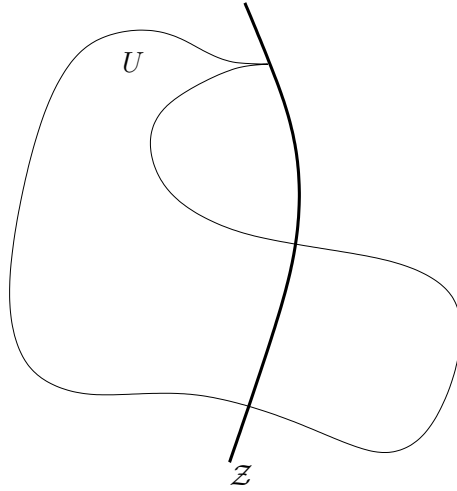
Figure 1.1. An admissible domain U.

where k_g denotes the geodesic curvature, $d\sigma$ is the Riemannian length element, and we interpret each integral $\int_{\Gamma \cap \partial U_\varepsilon^\pm} k_g d\sigma$ as the sum of the integrals along the smooth portions of $\Gamma \cap \partial U_\varepsilon^\pm$, plus the sum of the angles at the points where Γ is not $\mathcal{C}^1$.

Moreover, the following generalization of the Gauss-Bonnet formula with boundary holds true

$$\int_U K dA_s + \int_{\partial U} k_g d\sigma_s = 2\pi(\chi(U^+) - \chi(U^-)).$$

(See Section 5.)

If Γ is $\mathcal{C}^1$, but not $\mathcal{C}^2$ at the intersection points with $\mathcal{Z}$ (in particular, cusps at the singularity are allowed, see Figure 1.1), then the limits in (1.3) and (1.4) need not exist.

Nevertheless the limit

$$\lim_{\varepsilon \to 0} \left(\int_{U_\varepsilon^+ \cup U_\varepsilon^-} K dA_s + \int_{\Gamma \cap \partial U_\varepsilon^+} k_g d\sigma - \int_{\Gamma \cap \partial U_\varepsilon^-} k_g d\sigma \right)$$

does exist and is equal to

$$2\pi(\chi(U^+) - \chi(U^-)) + \sum_{i=1}^m \alpha_i,$$

where the α_i's are suitably defined high-order angles at the intersection points of Γ with $\mathcal{Z}$. The explicit expression of the α_i's can be given intrinsecally in terms of the asymptotic behavior of the geodesic curvature of Γ at the intersection points.

The paper is organized as follows: in Section 2 we introduce the general definition of two-dimensional almost-Riemannian structure. In Section 3 we recall the classification of local normal forms for generic almost-Riemannian structures. Section 4 contains the statement of the Gauss-Bonnet formula for almost-Riemannian manifolds without boundary given in [1]. Finally, in Section 5 the version of the Gauss-Bonnet formula for domains with boundary is extended to the almost-Riemannian case.

2. Almost-Riemannian structures

For every smooth manifold M denote by $\mathrm{Vec}(M)$ the set of smooth vector fields on M.

DEFINITION 2.1. — *Let M be a two-dimensional smooth manifold and consider a family*

$$\mathcal{S} = \{(\Omega^\mu, X_1^\mu, X_2^\mu)\}_{\mu \in I},$$

where $\{\Omega^\mu\}_{\mu \in I}$ is an open covering of M and, for every $\mu \in I$, $\{X_1^\mu, X_2^\mu\}$ is a family of $\mathrm{Vec}(M)$ whose restriction to Ω^μ satisfies the Lie bracket generating condition.

We say that $\mathcal{S}$ is an almost-Riemannian structure (ARS for short) if, for every $\mu, \nu \in I$ and for every $q \in \Omega^\mu \cap \Omega^\nu$, there exists an orthogonal matrix $R^{\mu,\nu}(q) = (R_{i,j}^{\mu,\nu}(q)) \in O(2)$ such that

$$(2.1) \qquad\qquad X_i^\mu(q) = \sum_{j=1}^k R_{i,j}^{\mu,\nu}(q) X_j^\nu(q).$$

We say that two ARSs $\mathcal{S}_1$ and $\mathcal{S}_2$ on M are equivalent if $\mathcal{S}_1 \cup \mathcal{S}_2$ is an ARS. Given an open subset Ω of M and a pair of vector fields (X_1, X_2), we say that (Ω, X_1, X_2) is compatible with $\mathcal{S}$ if $\mathcal{S} \cup \{(\Omega, X_1, \ldots, X_k)\}$ is equivalent to $\mathcal{S}$.

If $\mathcal{S}$ is equivalent to an ARS of the form $\{(M, X_1, X_2)\}$, i.e., for which the cardinality of I is equal to one, we say that $\mathcal{S}$ is trivializable.

If $\mathcal{S}$ admits an equivalent ARS such that each $R^{\mu,\nu}(q)$ belongs to $SO(2)$, we say that $\mathcal{S}$ is orientable.

Given an ARS $\mathcal{S}$, we define an associated distribution Δ and a quadratic form $\mathbf{G}$ on Δ by the rule

$$\Delta(q) = \text{span}\{X_1^\mu(q), \dots, X_k^\mu(q)\},$$

$$\mathbf{G}_q(v,v) = \inf\left\{\sum_{i=1}^k \alpha_i^2 \mid v = \sum_{i=1}^k \alpha_i X_i^\mu(q)\right\}, q \in \Omega^\mu, \quad v \in \Delta(q).$$

A curve $\gamma : [0,T] \to M$ is said to be *admissible* for $\mathcal{S}$ if it is Lipschitz continuous and $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$ for almost every $t \in [0,T]$. Given an admissible curve $\gamma : [0,T] \to M$, the *length of γ* is

$$l(\gamma) = \int_0^T \sqrt{\mathbf{G}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \; dt.$$

The *distance induced by $\mathcal{S}$ on M* is the function

(2.2) $d(q_0, q_1) = \inf\{l(\gamma) \mid \gamma(0) = q_0, \gamma(T) = q_1, \gamma \text{ admissible}\}.$

It is a standard fact that $l(\gamma)$ is invariant under reparameterization of the curve γ. Moreover, if an admissible curve γ minimizes the so-called *energy functional* $E(\gamma) = \int_0^T \mathbf{G}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) \; dt$ with T fixed (and fixed initial and final point) then $v = \sqrt{\mathbf{G}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}$ is constant and γ is also a minimizer of $l(\cdot)$. On the other hand a minimizer γ of $l(\cdot)$ such that v is constant is a minimizer of $E(\cdot)$ with $T = l(\gamma)/v$.

A *geodesic* for $\mathcal{S}$ is a curve $\gamma : [0,T] \to M$ such that for every sufficiently small nontrivial interval $[t_1, t_2] \subset [0,T]$, $\gamma|_{[t_1,t_2]}$ is a minimizer of $E(\cdot)$. A geodesic for which $\mathbf{G}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))$ is (constantly) equal to one is said to be parameterized by arclength.

The finiteness and the continuity of $d(\cdot, \cdot)$ with respect to the topology of M are guaranteed by the Lie bracket generating assumption on the ARS. The distance $d(\cdot, \cdot)$ gives to M the structure of metric space. The local existence of minimizing geodesics is a standard consequence of Filippov Theorem (see for instance [2]). When M is compact any two points of M are connected by a minimizing geodesic.

Notice that the problem of finding a curve minimizing the energy between two fixed points $q_0, q_1 \in M$ is naturally formulated as the optimal control problem

(2.3) $\dot{q} = \sum_{i=1}^2 u_i X_i^\mu(q) \, , u_i \in \mathbb{R} \, , \mu \in I(q) = \{\mu \in I \mid q \in \Omega^\mu\},$

(2.4) $\int_0^T \sum_{i=1}^2 u_i^2(t) \; dt \to \min, \; q(0) = q_0, q(T) = q_1.$

Here μ, u_1, u_2 are seen as controls and T is fixed. It is a standard fact that this optimal control problem is equivalent to the minimum time problem with controls u_1, u_2 satisfying $u_1^2 + u_2^2 \leqslant 1$. When the ARS is trivializable, the role of μ is empty and (2.3), (2.4) can be rewritten as a classical distributional control problem with quadratic cost

$$\dot{q} = \sum_{i=1}^{2} u_i X_i(q) \,, \ u_i \in \mathbb{R} \,, \ \int_0^T \sum_{i=1}^{2} u_i^2(t) \, dt \to \min, \ q(0) = q_0, \ q(T) = q_1.$$

Given an ARS $\mathcal{S}$, we call *singular locus* the set $\mathcal{Z} \subset M$ of points q at which the dimension of $\Delta(q)$ is equal to one. Denote by g the restriction of the quadratic form $\mathbf{G}$ on $M \setminus \mathcal{Z}$. By construction g is a Riemannian metric satisfying

$$g(X^\mu(q), X^\mu(q)) = 1, \quad g(X^\mu(q), Y^\mu(q)) = 0, \quad g(Y^\mu(q), Y^\mu(q)) = 1,$$

for every μ in I and every $q \in \Omega^\mu \setminus \mathcal{Z}$. Denote by dA the Riemannian density associated with $(M \setminus \mathcal{Z}, g)$, which coincides with $|dX^\mu \wedge dY^\mu|$ on $\Omega^\mu \setminus \mathcal{Z}$, for every $\mu \in I$. Finally, one can define on $M \setminus \mathcal{Z}$ the Gaussian curvature K associated with g, which is easily expressed in each open set $\Omega^\mu \setminus \mathcal{Z}$ through the formula (see for instance [2], equation (24.6)) $K = -(\alpha^\mu)^2 - (\beta^\mu)^2 + X^\mu \beta^\mu - Y^\mu \alpha^\mu$, where $\alpha^\mu, \beta^\mu : \Omega^\mu \setminus \mathcal{Z} \to \mathbb{R}$ are (uniquely) defined by $[X^\mu, Y^\mu] = \alpha^\mu X^\mu + \beta^\mu Y^\mu$, and $X^\mu \beta^\mu$ (respectively, $Y^\mu \alpha^\mu$) denotes the Lie derivative of β^μ with respect to X^μ (respectively, of α^μ with respect to Y^μ).

A natural tool to study the geodesics of an almost-Riemannian structure is the necessary condition for optimality given by the Pontryagin Maximum Principle (see [8]). As a result we obtain the following proposition.

PROPOSITION 2.2. — *Define on T^*M the Hamiltonian*

$$H(\lambda, q) = \frac{1}{2}(\langle \lambda, X^\mu(q) \rangle^2 + \langle \lambda, Y^\mu(q) \rangle^2), q \in \Omega^\mu, \ \lambda \in T_q^*M.$$

(Notice that H is well defined on the whole T^*M, thanks to (2.1).) Consider the minimization problem

$$(2.5) \quad \dot{q} \in \Delta(q), \int_0^T \mathbf{G}_{q(t)}(\dot{q}(t), \dot{q}(t)) dt \to \min, q(0) = M_{\mathrm{in}}, q(T) = M_{\mathrm{fin}},$$

where M_{in} and M_{fin} are two submanifolds of M and the final time $T > 0$ is fixed. Then every solution of (2.5) is the projection on M of a trajectory $(\lambda(t), q(t))$ of the Hamiltonian system associated with H satisfying $\lambda(0) \perp T_{q(0)}M_{\mathrm{in}}, \lambda(T) \perp T_{q(T)}M_{\mathrm{fin}}$, and $H(\lambda(t), q(t)) \neq 0$.

Remark 2.3. — The simple form of the statement above follows from the absence of abnormal minimizers, which follows from the Lie bracket generating assumption. As a consequence a curve is a geodesic if and only if it the projection of a normal extremal. Notice that H is constant along any given solution of the Hamiltonian system. Moreover, $H = 1/2$ if and only if $q(.)$ is parameterized by arclength.

3. Normal forms for generic ARSs

We recall in this section some results on the local characterization of generic ARSs obtained in [1].

Denote by $\mathcal{W}$ the $\mathcal{C}^2$-Whitney topology defined on $\mathrm{Vec}(M)$ and by $(\mathrm{Vec}(M), \mathcal{W})^2$ the product of two copies of $\mathrm{Vec}(M)$ endowed with the corresponding product topology. We recall that if M is compact then $\mathcal{W}$ is the standard $\mathcal{C}^2$ topology.

DEFINITION 3.1. — *A property (P) defined for ARSs is said to be generic if there exists an open and dense subset $\mathcal{O}$ of $(\mathrm{Vec}(M), \mathcal{W})^2$ such that (P) holds for every ARS admitting an atlas of local orthonormal frames whose elements belong to $\mathcal{O}$.*

Let us introduce the flag of the distribution Δ by the recursive formula

$$(3.1) \qquad \Delta_1 = \Delta, \quad \Delta_{k+1} = \Delta_k + [\Delta, \Delta_k].$$

The following proposition is a standard corollary of the transversality theorem. It formulates generic properties of a ARS in terms of the flag of the distribution Δ.

PROPOSITION 3.2. — *Let M be a two-dimensional smooth manifold. Generically, an ARS $\mathcal{S} = \{(\Omega^\mu, X^\mu, Y^\mu)\}_{\mu \in I}$ on M satisfies the following properties:*

 (i) *$\mathcal{Z}$ is an embedded one-dimensional smooth submanifold of M;*
 (ii) *The points $q \in M$ at which $\Delta_2(q)$ is one-dimensional are isolated;*
 (iii) *$\Delta_3(q) = T_q M$ for every $q \in M$.*

As a consequence of Proposition 3.2, one can classify the local normal forms of a generic ARS. See [1] for the proof.

THEOREM 3.3. — *Generically for an ARS $\mathcal{S}$, for every point $q \in M$ there exist a neighborhood U of q and a pair of vector fields (X, Y) on M*

such that (U, X, Y) is compatible with $\mathcal{S}$ and, up to a smooth change of coordinates defined on U, $q = (0,0)$ and (X, Y) has one of the forms

$$\text{(F1)} X(x, y) = (1, 0), \quad Y(x, y) = (0, e^{\phi(x,y)}),$$

$$\text{(F2)} X(x, y) = (1, 0), \quad Y(x, y) = (0, xe^{\phi(x,y)}),$$

$$\text{(F3)} X(x, y) = (1, 0), \quad Y(x, y) = (0, (y - x^2\psi(x))e^{\xi(x,y)}),$$

where ϕ, ξ and ψ are smooth real-valued functions such that $\phi(0, y) = 0$ and $\psi(0) \neq 0$.

DEFINITION 3.4. — *Let $\mathcal{S}$ be a ARS and assume that the generic conditions **(i)**, **(ii)**, **(iii)** of Proposition 3.2 hold true. A point $q \in M$ is said to be an ordinary point if $\Delta(q) = T_q M$, hence, if $\mathcal{S}$ is locally described by (F1). We call q a Grushin point if $\Delta(q)$ is one-dimensional and $\Delta_2(q) = T_q M$, i.e., if the local description (F2) applies. Finally, if $\Delta(q) = \Delta_2(q)$ is of dimension one and $\Delta_3(q) = T_q M$ we say that q is a tangency point and $\mathcal{S}$ can be described near q by the normal form (F3).*

The local behavior of g, K, and dA close to ordinary, Grushin, and tangency points is described by the following lemma.

LEMMA 3.5. — *Let $X(x, y) = (1, 0)$ and $Y(x, y) = (0, f(x, y))$ be two smooth vector fields on $\mathbb{R}^2$. Let $D = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \neq 0\}$ and g be the Riemannian metric on D having (X, Y) as an orthonormal frame. Denote by K the curvature of g and by dA the Riemannian density. We have*

$$g = dx^2 + \frac{1}{f^2}dy^2, \quad K = \frac{-2\,(\partial_x f)^2 + f\,\partial_x^2 f}{f^2}, \quad dA = \frac{1}{|f|}\,dx\,dy.$$

4. A Gauss-Bonnet-like formula for manifolds without boundary

Let M be an orientable two-dimensional manifold and let $\mathcal{S}$ be an orientable ARS on M. Chose a positive oriented atlas of orthonormal frames $\{(\Omega^\mu, X^\mu, Y^\mu)\}_{\mu \in I}$. Then there exists a two-form dA_s on $M \setminus \mathcal{Z}$ such that $dA_s = dX^\mu \wedge dY^\mu$ on $\Omega^\mu \setminus \mathcal{Z}$ for every $\mu \in I$.

Fix now an orientation Ξ of M. Recall that the choice of Ξ determines uniquely a notion of integration on $M \setminus \mathcal{Z}$ with respect to the form dA_s. More precisely, given a dA-integrable function f on $\Omega \subset M$, if for every $q \in \Omega$, Ξ and dA_s define the same orientation at q (i.e. if $\Xi(q) = \alpha dA_s(q)$ with $\alpha > 0$), then

$$\int_\Omega f\,dA_s = \int_{(\Omega, \Xi)} f dA_s = \int_\Omega f|dA_s| = \int_\Omega f dA.$$

Let
$$M^{\pm} = \{q \in \Omega^{\mu} \setminus \mathcal{Z} \mid \mu \in I, \pm \Xi(X^{\mu}, Y^{\nu})(q) > 0\}.$$
Then $\int_{\Omega} f dA_s = \pm \int_{\Omega} f dA$ if $\Omega \subset M^{\pm}$.

For every $\varepsilon > 0$ let $M_{\varepsilon} = \{q \in M \mid d(q, \mathcal{Z}) > \varepsilon\}$, where $d(\cdot, \cdot)$ is the almost-Riemannian distance (see equation (2.2)). We say that K is $\mathcal{S}$-integrable if $\lim_{\varepsilon \to 0} \int_{M_{\varepsilon}} K \, dA_s$ exists and is finite. In this case we denote such limit by $\int K dA_s$.

THEOREM 4.1. — *Let M be a compact oriented two-dimensional manifold without boundary. For a generic oriented ARS on M such that no tangency point exists, K is $\mathcal{S}$-integrable and*
$$\int K dA_s = 2\pi(\chi(M^{+}) - \chi(M^{-})),$$
where χ denotes the Euler characteristic.

A proof of Theorem 4.1 can be found in [1]. For a generic trivializable ARS without tangency points one can show, thanks to topological considerations (see [1]), that $\chi(M^{+}) = \chi(M^{-})$. As a consequence, one derives the following result.

COROLLARY 4.2. — *Let M be a compact oriented two-dimensional manifold without boundary. For a generic trivializable ARS on M without tangency points we have $\int K dA_s = 0$.*

Remark 4.3. — In the results stated above, the hypothesis that there are not tangency points seems to be essential. Technically, the difficulty comes when one tries to integrate the Hamiltonian system given by the Pontryagin Maximum Principle applied to a system written in the normal form (F3).

It is interesting to notice that the hypotheses of Corollary 4.2 are never empty, independently of M. Indeed:

LEMMA 4.4. — *Every compact orientable two-dimensional manifold admits a trivializable ARS satisfying the generic conditions of Proposition 3.2 and having no tangency points.*

5. A Gauss-Bonnet formula on domains with boundary

The Gauss-Bonnet formula on domains with boundary can also be generalized to almost-Riemannian structures without tangency points.

Definition 5.1. — *Let U be an open bounded connected subset of M. We say that U is an admissible domain if $\bar{U}$ contains only ordinary and Grushin points and if the boundary Γ of U is the union of the supports of a finite set of curves $\gamma^1, \ldots, \gamma^m$ satisfying the following conditions: each $\gamma^l : [0, T^l] \to M$ is $\mathcal{C}^2$ on the closed interval $[0, T^l]$; each γ^l is admissible and is parameterized by arclength (in particular Γ has finite length); each γ^l is oriented according to the orientation on Γ induced by the orientation of M.*

Theorem 5.2. — *Let M be an orientable smooth two-dimensional manifold and $\mathcal{S}$ be an orientable ARS on M. Let U be an admissible domain of M and denote by $\gamma^1, \ldots, \gamma^m$ the parameterizations of the boundary Γ of U as in Definition 5.1. For every $\varepsilon > 0$ define $M_\varepsilon^\pm = M^\pm \cap M_\varepsilon$ and $U_\varepsilon^\pm = M_\varepsilon^\pm \cap U$. Let $t_1^j, \ldots, t_{l_j}^j$ be the times at which γ^j crosses $\mathcal{Z}$. Associate to each t_l^j the quantity $\Sigma^+(t_l^j)$ as follows: if for $\varepsilon > 0$ small enough the support of $\gamma^j|_{(t_l^j, t_l^j + \varepsilon)}$ lies in M^+ then $\Sigma^+(t_l^j) = 1$, if it lies in M^-, then $\Sigma^+(t_l^j) = -1$. Similarly, if for $\varepsilon > 0$ small enough the support of $\gamma^j|_{(t_l^j - \varepsilon, t_l^j)}$ lies in M^+ then we set $\Sigma^-(t_l^j) = 1$, if it lies in M^-, then $\Sigma^-(t_l^j) = -1$. Denote by $k_g^j(t)$ the geodesic curvature of γ^j at the point $\gamma^j(t)$. Define, in addition,*

$$\Upsilon(\Xi) = \frac{\Xi}{2\sqrt{1 - \Xi^2}} + \arccos(\Xi)$$

and

$$\Xi^\pm(t_l^j) = \lim_{t \to t_l^j \pm} k_g^j(t)|t - t_l^j|, \quad \alpha(t_l^j) = \Sigma^-(t_l^j)\Upsilon(\Xi^-(t_l^j)) + \Sigma^+(t_l^j)\Upsilon(\Xi^+(t_l^j)).$$

Then

$$(5.1) \quad \lim_{\varepsilon \to 0} \left(\int_{U_\varepsilon^+ \cup U_\varepsilon^-} K dA_s + \int_{\Gamma \cap \partial U_\varepsilon^+} k_g d\sigma - \int_{\Gamma \cap \partial U_\varepsilon^-} k_g d\sigma \right) =$$

$$2\pi(\chi(U^+) - \chi(U^-)) - \sum_{j=1}^m \sum_{l=1}^{l_j} \alpha(t_l^j),$$

when we interpret each integral $\int_{\Gamma \cap \partial U_\varepsilon^\pm} k_g d\sigma$ as the sum of the integrals along the smooth portions of $\Gamma \cap \partial U_\varepsilon^\pm$, plus the sum of the angles at the points where Γ is not $\mathcal{C}^1$.

If, moreover, Γ is $\mathcal{C}^2$ in a neighborhood of $\mathcal{Z}$, then

$$(5.2) \quad \int_U K dA_s + \int_{\partial U} k_g d\sigma_s = 2\pi(\chi(U^+) - \chi(U^-)),$$

where

$$(5.3) \qquad \int_U K dA_s = \lim_{\varepsilon \to 0} \int_{U_\varepsilon^+ \cup U_\varepsilon^-} K dA_s,$$

$$(5.4) \qquad \int_{\partial U} k_g d\sigma_s = \lim_{\varepsilon \to 0} \left(\int_{\Gamma \cap \partial U_\varepsilon^+} k_g d\sigma - \int_{\Gamma \cap \partial U_\varepsilon^-} k_g d\sigma \right).$$

The existence of all the objects introduced in the statement above is motivated in the proof of the theorem.

Proof. — First notice that, according to the hypotheses of the theorem, the set $\Gamma \cap \mathcal{Z}$ is necessarily finite. Indeed, assume by contradiction that a connected component of Γ intersects $\mathcal{Z}$ infinitely many times. Then there exists a sequence $(t_l)_{l \in \mathbb{N}}$ such that $\gamma^j(t_l) \in \mathcal{Z}$ and $t_l \to t_\infty$ as $l \to \infty$. Since the length of γ^j is finite and $\dot{\gamma}^j(t) \neq 0$ for every t, it easily turns out that γ^j is not $\mathcal{C}^2$ in a (half-)neighborhood of t_∞.   □

By suitably cutting U in subdomains and because of mutual cancelations, we can assume without loss of generality that U is simply connected, that Γ is $\mathcal{C}^1$ and intersects $\mathcal{Z}$ twice, and that $U \setminus \mathcal{Z}$ has two connected components (see Figure 5).

Take $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ the sets U_ε^+ and U_ε^- are diffeomorphic to U^+ and U^-, respectively.

By applying the standard Gauss-Bonnet formula to U_ε^+ and U_ε^-. We obtain the equalities

$$\int_{U_\varepsilon^\pm} K dA + \int_{\Gamma \cap \partial U_\varepsilon^\pm} k_g d\sigma + \int_{\partial M_\varepsilon \cap \partial U_\varepsilon^\pm} k_g d\sigma + \theta_\varepsilon^{1,\pm} + \theta_\varepsilon^{2,\pm} =$$
$$2\pi \chi(U_\varepsilon^\pm) = 2\pi \chi(U^\pm)$$

where the angles $\theta_\varepsilon^{1,\pm}$ and $\theta_\varepsilon^{2,\pm}$ are defined as in Figure 5.

Therefore,

$$\int_{U_\varepsilon^+ \cup U_\varepsilon^-} K dA_s + \int_{\Gamma \cap \partial U_\varepsilon^+} k_g d\sigma - \int_{\Gamma \cap \partial U_\varepsilon^-} k_g d\sigma =$$
$$(5.5) \qquad 2\pi(\chi(U^+) - \chi(U^-)) + \theta_\varepsilon^{1,-} + \theta_\varepsilon^{2,-} - \theta_\varepsilon^{1,+} - \theta_\varepsilon^{2,+} +$$
$$+ \int_{\partial M_\varepsilon \cap \partial U_\varepsilon^-} k_g d\sigma - \int_{\partial M_\varepsilon \cap \partial U_\varepsilon^+} k_g d\sigma.$$

Let Z^1 and Z^2 be the intersection points of Γ with $\mathcal{Z}$, as in Figure 5. Define $Z_\varepsilon^{1,\pm}$ and $Z_\varepsilon^{2,\pm}$ as the intersections of Γ with $\partial M_\varepsilon^\pm$. Let, moreover, $Y_\varepsilon^{1,\pm}$ and $Y_\varepsilon^{2,\pm}$ be the points of $\partial M_\varepsilon^\pm$ at distance ε from Z^1 and Z^2, respectively. (The definition is well-posed for ε small enough, see [1, Lemma 1].)
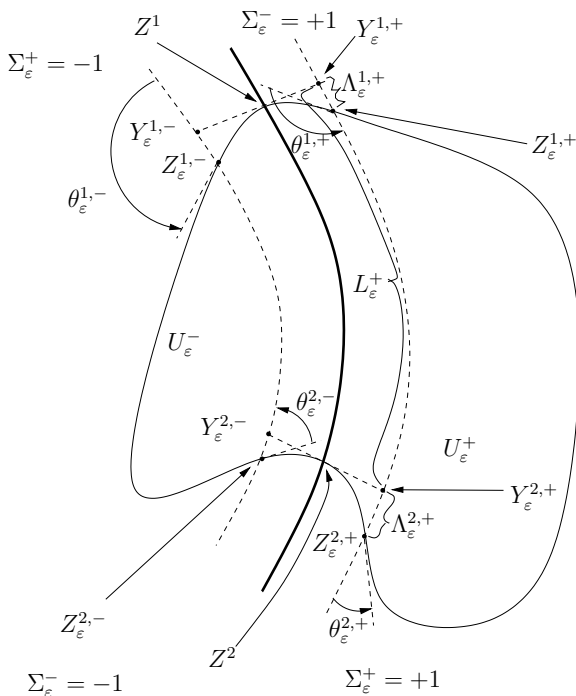
Figure 5.1. The domain U.

Denote by $L_\varepsilon^\pm$ the oriented portion of $\partial M_\varepsilon^\pm$ going from $Y_\varepsilon^{1,\pm}$ to $Y_\varepsilon^{2,\pm}$, by $\Lambda_\varepsilon^{1,\pm}$ the oriented portion of $\partial M_\varepsilon^\pm$ going from $Z_\varepsilon^{1,\pm}$ to $Y_\varepsilon^{1,\pm}$, and by $\Lambda_\varepsilon^{2,\pm}$ the oriented portion of $\partial M_\varepsilon^\pm$ going from $Y_\varepsilon^{2,\pm}$ to $Z_\varepsilon^{2,\pm}$.

It was proven in [1, Section 5.2] that

$$\lim_{\varepsilon \to 0} \left(\int_{L_\varepsilon^+} k_g d\sigma + \int_{L_\varepsilon^-} k_g d\sigma \right) = 0.$$

Hence, (5.5) rewrites

$$\int_{U_\varepsilon^+ \cup U_\varepsilon^-} K dA_s + \int_{\Gamma \cap \partial U_\varepsilon^+} k_g d\sigma - \int_{\Gamma \cap \partial U_\varepsilon^-} k_g d\sigma =$$
$$2\pi(\chi(U^+) - \chi(U^-)) + \theta_\varepsilon^{1,-} + \theta_\varepsilon^{2,-} - \theta_\varepsilon^{1,+} - \theta_\varepsilon^{2,+}$$
$$- \int_{\Lambda_\varepsilon^{1,+}} k_g d\sigma - \int_{\Lambda_\varepsilon^{2,+}} k_g d\sigma - \int_{\Lambda_\varepsilon^{1,-}} k_g d\sigma - \int_{\Lambda_\varepsilon^{2,-}} k_g d\sigma + o(1)$$

for $\varepsilon \to 0$.

Let us compute now the asymptotic behavior of $-\int_{\Lambda_\varepsilon^{2,+}} k_g d\sigma - \theta_\varepsilon^{2,+}$. Fix a system of coordinates in a neighborhood of Z^2 such that $Z^2 = (0,0)$ and an orthonormal basis for $\mathcal{S}$ is given by the pair of vector fields $(1,0)$ and $(0, x e^{\phi(x,y)})$. In this system of coordinates $\Gamma \cap M^+$ can be locally parameterized by a curve of the type $t \mapsto (t, c(t))$ with $c(t) = (1/2)\xi t^2 + o(t^2)$. Although such curve is not parameterized by arclength, we have that the norm of its derivative tends to one as $t \searrow 0$. The relation between $\xi = \ddot{c}(0+)$ and $\Xi^+(0)$ can be obtained by directly computing the geodesic curvature of $t \mapsto (t, c(t))$. One gets

$$(5.6) \qquad \Xi^+(0) = -\frac{\xi}{\sqrt{1+\xi^2}}.$$

(In particular, $\Xi^+(0)$ is well defined and finite.)

A similar computation shows that

$$\int_{\Lambda_\varepsilon^{2,+}} k_g d\sigma = -\frac{\xi}{2} + o(1)$$

for $\varepsilon \to 0$. Inverting (5.6) one gets

$$\int_{\Lambda_\varepsilon^{2,+}} k_g d\sigma = \frac{\Xi^+(0)}{2\sqrt{1 - (\Xi^+(0))^2}} + o(1).$$

Finally, the angle $\theta_\varepsilon^{2,+}$ is computed using the Riemannian metric defined by $\mathcal{S}$ on M^+ and has the expression

$$\theta_\varepsilon^{2,+} = \arccos\left(-\frac{\dot{c}(\varepsilon)}{\varepsilon e^{\phi(\varepsilon, c(\varepsilon))}\sqrt{1 + \frac{\dot{c}(\varepsilon)^2}{\varepsilon^2 e^{2\phi(\varepsilon, c(\varepsilon))}}}}\right)$$

$$= \arccos\left(-\frac{\xi}{\sqrt{1+\xi^2}}\right) + o(1) = \arccos\left(\Xi^+(0)\right) + o(1)$$

for $\varepsilon \to 0$. Therefore,

$$\lim_{\varepsilon \to 0}\left(-\int_{\Lambda_\varepsilon^{2,+}} k_g d\sigma - \theta_\varepsilon^{2,+}\right) = -\Upsilon(\Xi^+(0)).$$

Similarly one computes the limits as ε goes to zero of $-\theta_\varepsilon^{1,+} - \int_{\Lambda_\varepsilon^{1,+}} k_g d\sigma$, $\theta_\varepsilon^{1,-} - \int_{\Lambda_\varepsilon^{1,-}} k_g d\sigma$, and $\theta_\varepsilon^{2,-} - \int_{\Lambda_\varepsilon^{2,-}} k_g d\sigma$, and obtains (5.1).

The second part of the statement follows as a particular case of what has just been proven. The existence and finiteness of the limits (5.3) and (5.4) is a consequence of the existence and finiteness of $\Xi^\pm$ and of the fact that in the $\mathcal{C}^2$-case $\Xi^+ = \Xi^-$.

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Ugo BOSCAIN
SISSA-ISAS
Via Beirut 2-4, 34014 Trieste (Italy)
and
Université de Bourgogne
LE2i, CNRS UMR5158
9, avenue Alain Savary
BP 47870
21078 DIJON cedex (France)
boscain@sissa.it

Mario SIGALOTTI
Institut Élie Cartan, UMR 7502
INRIA/Nancy-Université/CNRS
POB 239
54506 Vandœuvre-lès-Nancy (France)
Mario.Sigalotti@inria.fr